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On the Production of Scalar and Tensor Perturbations in Inflationary Models

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ABSTRACT

Scalar (density) and tensor (gravity-wave) perturbations provide the basis for the fundamental observable consequences of inflation, including CBR anisotropy and the fluctuations that seed structure formation. These perturbations are nearly scale invariant (Harrison-Zel'dovich spectrum), though slight deviation from scale invariance ("tilt") can have significant consequences for both CBR anisotropy and structure formation. In particular, a slightly tilted spectrum of scalar perturbations may improve the agreement of the cold dark matter scenario with the present observational data. The amplitude and spectrum of the scalar and tensor perturbations depend upon the shape of the inflationary potential in the small interval where the scalar field responsible for inflation was between about 46 and 54 e-folds before the end of inflation. By expanding the inflationary potential in a Taylor series in this interval we show that the amplitude of the perturbations and the power-law slope of their spectra can be expressed in terms of the value of the potential 50 e-folds before the end of inflation, V_{50}, the steepness of the potential, $x_{50} \equiv m_{\rm Pl}V'_{50}/V_{50}$, and the rate of change of the steepness, x'_{50} (prime denotes derivative with respect to the scalar field). In addition, the power-law index of the cosmic-scale factor at this time, $q_{50} \equiv [d\ln R/d\ln t]_{50} \simeq 16\pi/x_{50}^2$. (Formally, our results for the perturbation amplitudes and spectral indices are

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accurate to lowest order in the deviation from scale invariance.) In general, the deviation from scale invariance is such to enhance fluctuations on large scales, and is only significant for steep potentials, large x_{50}, or potentials with rapidly changing steepness, large x'_{50}. In the latter case, only the spectrum of scalar perturbations is significantly tilted. Steep potentials are characterized by large tensor-mode contribution to the quadrupole CBR temperature anisotropy, similar tilt in both scalar and tensor perturbations, and slower expansion rate, i.e., smaller q_{50}. Measurements of the amplitude and tilt of the scalar and tensor perturbations over determine V_{50}, x_{50}, and x'_{50}, and can in principle be used to infer these quantities as well as testing the inflationary hypothesis. Our formalism has its limitations; it is not applicable to potentials with unusual features in the region that affects astrophysical scales.

1 Introduction

In inflationary Universe models [1, 2] scalar (density) and tensor (gravity-wave) metric perturbations arise due to de Sitter-space produced quantum fluctuations. The production of both density perturbations [3] and gravity-wave perturbations [4] have been well studied and are by now well understood. Very roughly, the quantum fluctuations on a given length scale become classical metric perturbations when that scale (Fourier mode) crosses outside the Hubble radius during inflation, that is, when $\lambda_{\rm phys} = R\lambda \sim H^{-1}$. Here R is the cosmic-scale factor, λ is the comoving wavelength of the Fourier mode, and H is the Hubble parameter. The scales of astrophysical interest, say from galaxy-size perturbations of 1 Mpc to the present Hubble scale of 10^4 Mpc, cross outside the Hubble radius about 50 or so e-foldings in the scale factor before the end of inflation, over an a span of about $\ln(10^4) \sim 8$ e-folds. In the post-inflationary Universe scalar-mode perturbations re-enter the Hubble radius with an amplitude that is approximately scale invariant: $(\delta\rho/\rho)_{\rm HOR} \sim [V^{3/2}/V' m_{\rm Pl}{}^3]\lambda_{\rm Mpc}^{\alpha_S}$, $|\alpha_S| \ll 1$; $\alpha_S = 0$ corresponds to the Harrison-Zel'dovich scale-invariant spectrum [5]. Likewise, the tensor-mode perturbations re-enter the Hubble radius with a dimensionless amplitude (gravitational-wave strain) that is approximately scale invariant: $h_{\rm GW} \sim [V^{1/2}/m_{\rm Pl}{}^2]\lambda_{\rm Mpc}^{\alpha_T}$, $|\alpha_T| \ll 1$. Here $V(\phi)$ is the inflationary potential, $m_{\rm Pl} = 1.22 \times 10^{19}$ GeV is the Planck mass, and $\lambda_{\rm Mpc} = \lambda/$ Mpc. The power-law indices α_S and α_T are related to the those frequently used to characterize the power spectra of the perturbations: $n = 1 - 2\alpha_S$ and $n_T = -2\alpha_T$.

The scalar perturbations provide the primeval density fluctuations that seed structure formation in cold dark matter scenarios, and so their amplitude and spectrum are of crucial importance. Both scalar- and tensor-mode perturbations can lead to temperature fluctuations in the cosmic background radiation (CBR), as briefly summarized in the Appendix. On angular scales much larger than a degree, the scale subtended by the Hubble radius at decoupling, the scalar and tensor contributions are inseparable: on smaller angular scales the contribution of the tensor-mode perturbations becomes subdominant and the angular dependence of CBR anisotropy can in principle be used to separate the scalar and tensor contributions. The detection of anisotropy in the CBR on angular scales greater than about 10° by the DMR instrument on COBE [6] spurred interest as to the mix of scalar and tensor contribution to large-angle CBR anisotropy [7], and what can be learned about inflationary models [8]. The purpose of this paper is to relate the amplitude and spectrum of the scalar and tensor perturbations to the shape of the inflationary potential. Some of the issues, e.g., deviations from scale invariance [9] and the relative contributions of scalar and tensor perturbations to the quadrupole anisotropy [8], have been addressed elsewhere; in addition to extending previous work in several important regards, we have attempted to concisely and clearly relate specific properties of the inflationary potential to the potentially measurable features of the metric perturbations [10].

In the next Section we briefly review slow-rollover inflation and the production of metric perturbations; since all the observable effects of these perturbations involve the shape of the potential over an interval of only about 8 e-folds around 50 e-folds before the end of inflation, we expand the potential about this point in terms of its value, V_{50}, its steepness, $x_{50} \equiv [m_{\rm Pl}V'/V]_{50}$, and the change in its steepness, x'_{50} (prime denotes derivation with respect to the scalar field). We show that the amplitude and scale dependence of scalar and tensor perturbations, quantified as α_S and α_T, are simply related to these quantities, and further that the rate of growth of the cosmic-scale factor around 50 e-folds before the end of inflation is related to the steepness of the potential. In Section III we apply our formalism to four different types of inflationary potentials, and draw some general conclusions. The deviations from scale invariance tend to enhance large-scale perturbations. The models that have significant deviation from scale invariance involve either steep potentials or potentials with rapidly changing steepness. In the latter case, only the scalar perturbations are tilted significantly. In the case of steep potentials, the scalar and tensor perturbations are tilted by a similar amount. The relative contributions of the scalar and tensor perturbations to the quadrupole CBR anisotropy is related to steepness of the potential (and hence the deviation from scale invariance): Large tensor contribution implies significant deviation from scale invariance, and slower expansion rate during inflation. In Section IV we finish with some concluding remarks.

2 Inflationary Perturbations

All viable models of inflation are of the slow-rollover variety, or can be recast as such [2, 11]. In slow-rollover inflation a scalar field that is initially displaced from the minimum of its potential rolls slowly to that minimum, and as it does the cosmic-scale factor grows very rapidly; the Universe is said to inflate. Once the scalar field reaches the minimum of the potential it oscillates about it, so that the large potential energy has been converted into coherent scalar-field oscillations, corresponding to a condensate of non-relativistic scalar particles. The eventual decay of these particles into lighter particle states and their subsequent thermalization lead to the reheating of the Universe to a temperature $T_{\rm RH} \simeq \sqrt{\Gamma m_{\rm Pl}}$, where Γ is the decay width of the scalar particle [12, 11]. Quantum fluctuations in the scalar field driving inflation lead to scalar metric perturbations (referred to density or curvature perturbations) [3], while quantum fluctuations in the metric itself lead to tensor metric perturbations (gravity waves) [4]; isocurvature perturbations can arise due to quantum fluctuations in other massless fields, e.g., the axion field, if it exists [13].

We assume that the scalar field driving inflation is minimally coupled so that its stress-energy tensor takes the canonical form,

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \mathcal{L}g_{\mu\nu}; \tag{1}$$

where the Lagrangian density of the scalar field $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$. If we make the usual assumption that the scalar field ϕ is spatially homogeneous, or at least so over a Hubble radius, the stress-energy tensor takes the perfect-fluid form with energy density, $\rho = \frac{1}{2}\dot\phi^2 + V(\phi)$, and isotropic pressure, $p = \frac{1}{2}\dot\phi^2 - V(\phi)$. The classical equations of motion for ϕ can be obtained from the first law of thermodynamics, $d(R^3\rho) = -pdR^3$, or by taking the four-divergence of $T^{\mu\nu}$:

$$\ddot\phi + 3H\dot\phi + V'(\phi) = 0; \tag{2}$$

the $\Gamma\dot{\phi}$ term responsible for reheating has been omitted since we shall only be interested in the slow-rollover phase. In addition, there is the Friedmann equation, which governs the expansion of the Universe,

$$H^2 = \frac{8\pi}{3{m_{\rm Pl}}^2}\left(V(\phi) + \frac{1}{2}\dot{\phi}^2\right) \simeq \frac{8\pi V(\phi)}{3{m_{\rm Pl}}^2}; \tag{3}$$

where we assume that the contribution of all other forms of energy density, e.g., radiation and kinetic energy of the scalar field, and the curvature term (k/R^2) are negligible. The justification for discussing inflation in the context of a flat FRW model with a homogeneous scalar field driving inflation are discussed at length in Ref. [2]; including the ϕ kinetic term increases the righthand side of Eq. (3) by a factor of $(1 + x^2/48\pi)$, a small correction for viable models.

Later in this Section and in the Appendix we will be more precise about the amplitude of density perturbations and gravitational waves; for now it will suffice to give the characteristic amplitude of each:

$$(\delta\rho/\rho)_{\rm HOR,\lambda} = c_S\left(\frac{V^{3/2}}{{m_{\rm Pl}}^3 V'}\right)_1; \tag{4}$$

$$h_{\rm HOR,\lambda} = c_T\left(\frac{V^{1/2}}{{m_{\rm Pl}}^2}\right)_1; \tag{5}$$

where $(\delta\rho/\rho)_{\rm HOR,\lambda}$ is the amplitude of the density perturbation on the scale λ when it crosses the Hubble radius during the post-inflation epoch, $h_{\rm HOR,\lambda}$ is the dimensionless amplitude of the gravitational wave perturbation on the scale λ when it crosses the Hubble radius, and c_S, c_T are numerical constants of order unity. Subscript 1 indicates that the quantity involving the scalar potential is to be evaluated when the scale in question crossed outside the horizon during the inflationary era.

[Two small points: in Eq. (4) we got ahead of ourselves and used the slow-roll approximation (see below) to rewrite the fundamental expression, $(\delta\rho/\rho)_{\rm HOR,\lambda} \simeq (V/{m_{\rm Pl}}^2\dot{\phi})_1$ [3], in terms of the potential only. While we shall always mean "cross outside" or "inside the Hubble radius," we will occasionally slip and say, "cross outside" or "inside the horizon" instead.]

Eqs. (2-5) are the fundamental equations that govern inflation and the production of metric perturbations. It proves very useful to recast these equations using the scalar field as the independent variable; we then express the scalar and tensor perturbations in terms of the value of the potential, its steepness, and the rate of change of its steepness when the interesting scales crossed outside the Hubble radius during inflation, about 50 e-folds in scale factor before the end of inflation, defined by

$$V_{50} \equiv V(\phi_{50}); \qquad x_{50} \equiv \frac{m_{\rm Pl}V'(\phi_{50})}{V(\phi_{50})}; \qquad x'_{50} = \frac{m_{\rm Pl}V''(\phi_{50})}{V(\phi_{50})} - \frac{m_{\rm Pl}[V'(\phi_{50})]^2}{V^2(\phi_{50})}.$$

And as we shall discuss, we will work to lowest order in the deviations from scale invariance, α_S and α_T, which corresponds to order x_{50}^2, $m_{\rm Pl}x'_{50}$. Terms involving higher-order derivatives of the potential lead to corrections that are higher-order in the deviation from scale invariance.

To evaluate these three quantities 50 e-folds before the end of inflation we must find the value of the scalar field at this time. During the inflationary phase the $\ddot{\phi}$ term is negligible (the motion of ϕ is friction dominated), and Eq. (2) becomes

$$\dot{\phi} \simeq \frac{-V'(\phi)}{3H}; \tag{6}$$

this is known as the slow-roll approximation [9]. (The corrections to the slow-roll approximation are $\mathcal{O}(\alpha_i)$ for the amplitude of perturbations, and $\mathcal{O}(\alpha_i^2)$ for the power-law indices themselves. There are models where the slow-roll approximation cannot be used at all: e.g., a potential where during the crucial 8 e-folds the scalar field rolls uphill, "powered" by the velocity it had when it hit the incline.)

The conditions that must be satisfied in order that $\ddot{\phi}$ be negligible are:

$$|V''| \; < \; 9H^2 \simeq 24\pi V/{m_{\rm Pl}}^2; \tag{7}$$

$$|x| \equiv |V'm_{\rm Pl}/V| \; < \; \sqrt{48\pi}. \tag{8}$$

The end of the slow roll occurs when either or both of these inequalities are saturated, at a value of ϕ denoted by $\phi_{\rm end}$. Since $H \equiv \dot{R}/R$, or $H dt = d\ln R$, it follows that

$$d\ln R = \frac{8\pi}{{m_{\rm Pl}}^2}\frac{V(\phi)d\phi}{-V'(\phi)} = -\frac{8\pi d\phi}{m_{\rm Pl}\, x}. \tag{9}$$

Now express the cosmic-scale factor in terms of is value at the end of inflation, $R_{\rm end}$, and the number of e-foldings before the end of inflation, $N(\phi)$,

$$R = \exp[-N(\phi)]\, R_{\rm end}.$$

The quantity $N(\phi)$ is a time-like variable whose value at the end of inflation is zero and whose evolution is governed by

$$\frac{dN}{d\phi} = \frac{8\pi}{m_{\rm Pl}\, x}. \tag{10}$$

Using Eq. (10) we can compute the value of the scalar field 50 e-folds before the end of inflation ($\equiv \phi_{50}$); the values of V_{50}, x_{50}, and x'_{50} follow directly.

As ϕ rolls down its potential during inflation its energy density decreases, and so the growth in the scale factor is not exponential. By using the fact that the stress-energy of the scalar field takes the perfect-fluid form, we can solve for evolution of the cosmic-scale factor. Recall, for the equation of state $p = \gamma\rho$, the scale factor grows as $R \propto t^q$, where $q = 2/3(1+\gamma)$. Here,

$$\gamma = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V} = \frac{x^2 - 48\pi}{x^2 + 48\pi}; \tag{11}$$

$$q = \frac{1}{3} + \frac{16\pi}{x^2}. \tag{12}$$

Since the steepness of the potential can change during inflation, γ is not in general constant; the power-law index q is more precisely the logarithmic rate of the change of the logarithm of the scale factor, $q = d\ln R/d\ln t$.

When the steepness parameter is small, corresponding to a very flat potential, γ is close to -1 and the scale factor grows as a very large power of time. To solve the horizon problem the expansion must be "superluminal" ($\ddot{R} > 0$), corresponding to $q > 1$, which requires that $x^2 < 24\pi$. Since $\frac{1}{2}\dot{\phi}^2/V = x^2/48\pi$, this implies that $\frac{1}{2}\dot{\phi}^2/V(\phi) < \frac{1}{2}$, justifying neglect of the scalar-field kinetic energy in computing the expansion rate for all but the steepest potentials. (In fact there are much stronger constraints; the COBE DMR data imply that $n \gtrsim 0.5$, which restricts $x_{50}^2 \lesssim 4\pi$, $\frac{1}{2}\dot{\phi}^2/V \lesssim \frac{1}{12}$, and $q \gtrsim 4$.)

Next, let us relate the size of a given scale to when that scale crosses outside the Hubble radius during inflation, specified by $N_1(\lambda)$, the number of e-folds before the end of inflation. The physical size of a perturbation is related to its comoving size, $\lambda_{\rm phys} = R\lambda$; with the usual convention, $R_{\rm today} = 1$, the comoving size is the physical size today. When the scale λ crosses outside the Hubble radius $R_1\lambda = H_1^{-1}$. We then assume that: (1) at the end of inflation the energy density is $\mathcal{M}^4 \simeq V(\phi_{\rm end})$; (2) inflation is followed by a period where the energy density of the Universe is dominated by coherent scalar-field oscillations which decrease as R^{-3}; and (3) when value of the scale factor is $R_{\rm RH}$ the Universe reheats to a temperature $T_{\rm RH} \simeq \sqrt{m_{\rm Pl}\Gamma}$ and expands adiabatically thereafter. The "matching equation" that relates λ and $N_1(\lambda)$ is:

$$\lambda = \frac{R_{\rm today}}{R_1} H_1^{-1} = \frac{R_{\rm today}}{R_{\rm RH}} \frac{R_{\rm RH}}{R_{\rm end}} \frac{R_{\rm end}}{R_1} H_1^{-1}. \tag{13}$$

Adiabatic expansion since reheating implies $R_{\rm today}/R_{\rm RH} \simeq T_{\rm RH}/2.73\,{\rm K}$; and the decay of the coherent scalar-field oscillations implies $(R_{\rm RH}/R_{\rm end})^3 = (\mathcal{M}/T_{\rm RH})^4$. If we define $\bar{q} = \ln(R_{\rm end}/R_1)/\ln(t_{\rm end}/t_1)$, the mean power-law index, it follows that $(R_{\rm end}/R_1)H_1^{-1} = \exp[N_1(\bar{q}-1)/\bar{q}]H_{\rm end}^{-1}$, and Eq. (13) becomes

$$N_1(\lambda) = \frac{\bar{q}}{\bar{q}-1}\left[48 + \ln\lambda_{\rm Mpc} + \frac{2}{3}\ln(\mathcal{M}/10^{14}\,{\rm GeV}) + \frac{1}{3}\ln(T_{\rm RH}/10^{14}\,{\rm GeV})\right]; \tag{14}$$

In the case of perfect reheating, which probably only applies to first-order inflation, $T_{\rm RH} \simeq \mathcal{M}$.

The scales of astrophysical interest today range roughly from that of galaxy size, $\lambda \sim$ Mpc, to the present Hubble scale, $H_0^{-1} \sim 10^4$ Mpc; up to the logarithmic corrections these scales crossed outside the horizon between about $N_1(\lambda) \sim 48$ and $N_1(\lambda) \simeq 56$ e-folds before the end of inflation. *That is, the interval of inflation that determines its all observable consequences covers only about 8 e-folds.*

Except in the case of strict power-law inflation q varies during inflation: this means that the $(R_{\rm end}/R_1)H_1^{-1}$ factor in Eq. (13) cannot be written in closed form. Taking account of this, the matching equation becomes a differential equation,

$$\frac{d\ln\lambda_{\rm Mpc}}{dN_1}=\frac{q(N_1)-1}{q(N_1)}; \tag{15}$$

subject to the "boundary condition:" $\ln\lambda_{\rm Mpc}=-48-\frac{4}{3}\ln(\mathcal{M}/10^{14}\,{\rm GeV})+\frac{1}{3}\ln(T_{\rm RH}/10^{14}\,{\rm GeV})$ for $N_1=0$, the matching relation for the mode that crossed outside the Hubble radius at the end of inflation. Equation (15) allows one to obtain the precise expression for when a given scale crossed outside the Hubble radius during inflation. To actually solve this equation, one would need to supplement it with the expressions $dN/d\phi=8\pi/m_{\rm Pl}x$ and $q=16\pi/x^2$. For our purposes we need only know: (1) The scales of astrophysical interest correspond to $N_1\sim$ "50 ± 4," where for definiteness we will throughout take this to be an equality sign. (2) The expansion of Eq. (15) about $N_1=50$,

$$\Delta N_1(\lambda)=\left(\frac{q_{50}-1}{q_{50}}\right)\Delta\ln\lambda_{\rm Mpc}; \tag{16}$$

which, with the aid of Eq. (10), implies that

$$\Delta\phi=\left(\frac{q_{50}-1}{q_{50}}\right)\frac{x_{50}}{8\pi}\Delta\lambda_{\rm Mpc}. \tag{17}$$

We are now ready to express the perturbations in terms of V_{50}, x_{50}, and x'_{50}. First, we must solve for the value of ϕ, 50 e-folds before the end of inflation. To do so we use Eq. (10),

$$N(\phi_{50})=50=\frac{8\pi}{m_{\rm Pl}{}^2}\int_{\phi_{\rm end}}^{\phi_{50}}\frac{V d\phi}{V'}. \tag{18}$$

Next, with the help of Eq. (17) we expand the potential V and its steepness x about ϕ_{50}:

$$V\simeq V_{50}+V'_{50}(\phi-\phi_{50})=V_{50}\left[1+\frac{x_{50}^2}{8\pi}\left(\frac{q_{50}}{q_{50}-1}\right)\Delta\ln\lambda_{\rm Mpc}\right]; \tag{19}$$

$$x\simeq x_{50}+x'_{50}(\phi-\phi_{50})=x_{50}\left[1+\frac{m_{\rm Pl}x'_{50}}{8\pi}\left(\frac{q_{50}}{q_{50}-1}\right)\Delta\ln\lambda_{\rm Mpc}\right]; \tag{20}$$

of course these expansions only make sense for potentials that are smooth. We note that additional terms in either expansion are $\mathcal{O}(\alpha_i^2)$ and beyond the accuracy we are seeking.

Now recall the equations for the amplitude of the scalar and tensor perturbations,

$$(\delta\rho/\rho)_{\mathrm{HOR},\lambda} = c_S \left(\frac{V^{1/2}}{m_{\mathrm{Pl}}{}^2 x}\right)_1 ; \tag{21}$$

$$h_{\mathrm{HOR},\lambda} = c_T \left(\frac{V^{1/2}}{m_{\mathrm{Pl}}{}^2}\right)_1 ; \tag{22}$$

where subscript 1 means that the quantities are to be evaluated where the scale λ crossed outside the Hubble radius, $N_1(\lambda)$ e-folds before the end of inflation. The origin of any deviation from scale invariance is clear: For tensor perturbations it arises due to the variation of the potential; and for scalar perturbations it arises due to the variation of both the potential and its steepness.

Using Eqs. (16-22) it is now simple to calculate the power-law exponents α_S and α_T that quantify the deviations from scale invariance,

$$\alpha_T = \frac{x_{50}^2}{16\pi}\frac{q_{50}}{q_{50}-1} \simeq \frac{x_{50}^2}{16\pi}; \tag{23}$$

$$\alpha_S = \alpha_T - \frac{m_{\mathrm{Pl}}x'_{50}}{8\pi}\frac{q_{50}}{q_{50}-1} \simeq \frac{x_{50}^2}{16\pi} - \frac{m_{\mathrm{Pl}}x'_{50}}{8\pi}; \tag{24}$$

where

$$q_{50} = \frac{1}{3} + \frac{16\pi}{x_{50}^2} \simeq \frac{16\pi}{x_{50}^2}; \tag{25}$$

$$h_{\mathrm{HOR},\lambda} = c_T \left(\frac{V_{50}^{1/2}}{m_{\mathrm{Pl}}{}^2}\right) \lambda_{\mathrm{Mpc}}^{\alpha_T}; \tag{26}$$

$$(\delta\rho/\rho)_{\mathrm{HOR},\lambda} = c_S \left(\frac{V_{50}^{1/2}}{x_{50} m_{\mathrm{Pl}}{}^2}\right) \lambda_{\mathrm{Mpc}}^{\alpha_S}. \tag{27}$$

The spectral indices α_i are defined as. $\alpha_S = [d\ln(\delta\rho/\rho)_{\mathrm{HOR},\lambda}/d\ln\lambda_{\mathrm{Mpc}}]_{50}$ and $\alpha_T = [d\ln h_{\mathrm{HOR},\lambda}/d\ln\lambda_{\mathrm{Mpc}}]_{50}$. and in general vary slowly with scale.

Note too that the deviations from scale invariance, quantified by α_S and α_T, are of the order of x_{50}^2, $m_{\rm Pl}x'_{50}$. In the expressions above we retained only lowest-order terms in $\mathcal{O}(\alpha_i)$. The next-order contributions to the spectral indices are $\mathcal{O}(\alpha_i^2)$; those to the amplitudes are $\mathcal{O}(\alpha_i)$ and are given in the Appendix. The justification for truncating the expansion at lowest order is that the deviations from scale invariance are expected to be small.

As we discuss in more detail in the Appendix, our more intuitive power-law indices α_S, α_T are related to the indices that are usually used to describe the power spectra of scalar and tensor perturbations, $P_S(k) = |\delta_k|^2 = Ak^n$ and $P_T(k) = |h_k|^2 = A_T k^{n_T}$,

$$n = 1 - 2\alpha_S = 1 - \frac{x_{50}^2}{8\pi} + \frac{m_{\rm Pl}x'_{50}}{4\pi}; \tag{28}$$

$$n_T = -2\alpha_T = -\frac{x_{50}^2}{8\pi}. \tag{29}$$

(30)

Finally, let us be more specific about the amplitude of the scalar and tensor perturbations; in particular, for small α_S, α_T the contributions of each to the quadrupole CBR temperature anisotropy:

$$\left(\frac{\Delta T}{T_0}\right)^2_{Q-S} \approx \frac{32\pi}{45}\frac{V_{50}}{m_{\rm Pl}{}^4 x_{50}^2}; \tag{31}$$

$$\left(\frac{\Delta T}{T_0}\right)^2_{Q-T} \approx 0.61\frac{V_{50}}{m_{\rm Pl}{}^4}; \tag{32}$$

$$\frac{T}{S} \equiv \frac{(\Delta T/T_0)^2_{Q-T}}{(\Delta T/T_0)^2_{Q-S}} \approx 0.28x_{50}^2; \tag{33}$$

where expressions have been evaluated to lowest order in x_{50}^2 and $m_{\rm Pl}x'_{50}$. These quantities represent the ensemble averages of the scalar and tensor contributions to the quadrupole temperature anisotropy, which in terms of the spherical-harmonic expansion of the CBR temperature anisotropy on the sky are given by $5\langle|a_{2m}|^2\rangle/4\pi$. Further, the scalar and tensor contributions to the *measured* quadrupole anisotropy add in quadrature, and are subject to "cosmic variance." (Cosmic variance refers to the dispersion in the values measured by different observers in the Universe.) We refer the reader to the Appendix for more details.

Before going on to specific models, let us make some general remarks. The steepness parameter x_{50}^2 must be less than about 24π to ensure superluminal expansion. For "steep" potentials, the expansion rate is "slow," i.e., q_{50} closer to unity, the gravity-wave contribution to the quadrupole CBR temperature anisotropy becomes comparable to, or greater than, that of density perturbations, and both scalar and tensor perturbations exhibit significant deviations from scale invariance. For "flat" potentials, i.e., small x_{50}, the expansion rate is "fast," i.e., $q_{50} \gg 1$, the gravity-wave contribution to the quadrupole CBR temperature anisotropy is much smaller than that of density perturbations, and the tensor perturbations are scale invariant. Unless the steepness of the potential changes rapidly, i.e., large x'_{50}, the scalar perturbations are also scale invariant.

3 Worked Examples

In this Section we apply the formalism developed in the previous Section to four specific models. So that we can, where appropriate, solve numerically for model parameters, we will: (1) Assume that the astrophysically interesting scales crossed outside the horizon 50 e-folds before the end of inflation; and (2) Use the COBE DMR quadrupole measurement, $\langle(\Delta T)_{Q-S}^2\rangle^{1/2} \approx 16\mu\mathrm{K}$ [6], to normalize the scalar perturbations; using Eq. (31) this implies

$$V_{50} \approx 1.6 \times 10^{-11}\, {m_{\rm Pl}}^4\, x_{50}^2. \tag{34}$$

We remind the reader that it is entirely possible that a significant portion of the quadrupole anisotropy is due to tensor-mode perturbations. It is, of course, straightforward to change "50" to the number appropriate to a specific model, or to normalize the perturbations another way.

3.1 Exponential potentials

There are a class of models that can be described in terms of an exponential potential,

$$V(\phi) = V_0 \exp(-\beta\phi/m_{\rm Pl}). \tag{35}$$

This type of potential was first invoked in the context of power-law inflation [14], and has recently received renewed interest in the context of extended inflation [15]. In the simplest model of extended, or first-order, inflation, that based upon the Brans-Dicke-Jordan theory of gravity [16], β is related to the Brans-Dicke parameter: $\beta^2 = 64\pi/(2\omega+3)$.

For such a potential the slow-roll conditions are satisfied provided that $\beta^2 \lesssim 24\pi$; thus inflation does not end until the potential changes shape, or in the case of extended inflation, until the phase transition takes place. In either case we can relate ϕ_{50} to $\phi_{\rm end}$,

$$N(\phi_{50}) = 50 = \frac{8\pi}{{m_{\rm Pl}}^2}\int_{\phi_{50}}^{\phi_{\rm end}} \frac{V d\phi}{-V'}; \qquad \Rightarrow \quad \phi_{50} = \phi_{\rm end} - 50\beta/8\pi. \tag{36}$$

Since $\phi_{\rm end}$ is in effect arbitrary, the overall normalization of the potential is irrelevant. The two other parameters, x_{50} and x'_{50}, are easy to compute:

$$x_{50} = -\beta; \qquad x'_{50} = 0. \tag{37}$$

Using the COBE DMR normalization, we can relate V_{50} and β:

$$V_{50} = 1.6 \times 10^{-11}\, {m_{\rm Pl}}^4 \beta^2. \tag{38}$$

Further, we can compute q, α_S, α_T, and T/S:

$$q = 16\pi/\beta^2; \qquad T/S = 0.28\beta^2; \qquad \alpha_T = \alpha_S = 1/(q-1) \simeq \beta^2/16\pi. \tag{39}$$

Note, for the exponential potential, q, $\alpha_T = \alpha_S$ are independent of epoch. In the case of extended inflation, $\alpha_S = \alpha_T = 4/(2\omega+3)$; since ω must be less than about 20 [17], this implies significant tilt: $\alpha_S = \alpha_T \gtrsim 0.1$.

3.2 Chaotic inflation

These models are based upon a very simple potential:

$$V(\phi) = a\phi^b; \tag{40}$$

$b = 4$ corresponds to Linde's original model of chaotic inflation and a is dimensionless [18], and $b = 2$ is a model based upon a massive scalar field and $m^2 = 2a$ [19]. In these models ϕ is initially displaced from $\phi = 0$, and inflation occurs as ϕ slowly rolls to the origin. The value of $\phi_{\rm end}$ is easily found: $\phi^2_{\rm end} = b(b-1){m_{\rm Pl}}^2/24\pi$, and

$$N(\phi_{50}) = 50 \quad = \quad \frac{8\pi}{{m_{\rm Pl}}^2}\int_{\phi_{\rm end}}^{\phi_{50}} \frac{V d\phi}{V'}; \tag{41}$$

$$\Rightarrow \quad \phi^2_{50}/{m_{\rm Pl}}^2 = 50b/4\pi + b^2/48\pi \simeq 50b/4\pi; \tag{42}$$

the value of ϕ_{50} is a few times the Planck mass.

For purposes of illustration consider $b = 4$; $\phi_{\rm end} = m_{\rm Pl}/\sqrt{2\pi} \simeq 0.4m_{\rm Pl}$, $\phi_{50} \simeq 4m_{\rm Pl}$, $\phi_{46} \simeq 3.84m_{\rm Pl}$, and $\phi_{54} \simeq 4.16m_{\rm Pl}$. In order to have sufficient inflation the initial value of ϕ must exceed about $4.2m_{\rm Pl}$; inflation ends when $\phi \approx 0.4m_{\rm Pl}$; and the scales of astrophysical interest cross outside the horizon over an interval $\Delta\phi \simeq 0.3m_{\rm Pl}$.

The values of the potential, its steepness, and the change in steepness are easily found,

$$V_{50} = a\,{m_{\rm Pl}}^b \left(\frac{50b}{4\pi}\right)^{b/2}; \qquad x_{50} = \sqrt{\frac{4\pi b}{50}}; \qquad m_{\rm Pl}x'_{50} = \frac{-4\pi}{50}; \tag{43}$$

$$q_{50} = 200/b; \qquad T/S = 0.07b; \qquad \alpha_T \simeq b/200; \qquad \alpha_S = \alpha_T + 0.01. \tag{44}$$

Unless b is very large, scalar perturbations dominate tensor perturbations [20], α_T, α_S are very small, and q is very large. Further, when α_T, α_S become significant, they are equal. Using the COBE DMR normalization we find:

$$a = 1.6 \times 10^{-11} b^{1-b/2} (4\pi/50)^{b/2+1}\, {m_{\rm Pl}}^{4-b}. \tag{45}$$

For the two special cases of interest: $b = 4$, $a = 6.4 \times 10^{-14}$; and $b = 2$, $m^2 \equiv 2a = 2.0 \times 10^{-12}{m_{\rm Pl}}^2$.

3.3 New inflation

These models entail a very flat potential where the scalar field rolls from $\phi \approx 0$ to the minimum of the potential at $\phi = \sigma$. The original models of slow-rollover inflation [21] were based upon potentials of the Coleman-Weinberg form

$$V(\phi) = B\sigma^4/2 + B\phi^4\left[\ln(\phi^2/\sigma^2) - \frac{1}{2}\right]; \tag{46}$$

where B is a very small dimensionless coupling constant. Other very flat potentials also work (e.g., $V = V_0 - \alpha\phi^4 + \beta\phi^6$ [9]). As before we first solve for ϕ_{50}:

$$N(\phi_{50}) = 50 = \frac{8\pi}{m_{\rm Pl}{}^2}\int_{\phi_{\rm end}}^{\phi_{50}} \frac{Vd\phi}{V'}; \qquad \Rightarrow \quad \phi_{50}^2 = \frac{\pi\sigma^4}{100|\ln(\phi_{50}^2/\sigma^2)|m_{\rm Pl}{}^2}; \tag{47}$$

where the precise value of $\phi_{\rm end}$ is not relevant, only the fact that it is much larger than ϕ_{50}. Provided that $\sigma \lesssim m_{\rm Pl}$, both ϕ_{50} and $\phi_{\rm end}$ are much less than σ; we then find

$$V_{50} \simeq B\sigma^4/2; \qquad x_{50} \simeq -\frac{(\pi/25)^{3/2}}{\sqrt{|\ln(\phi_{50}^2/\sigma^2|)}}\left(\frac{\sigma}{m_{\rm Pl}}\right)^2 \ll 1; \tag{48}$$

$$m_{\rm Pl}x'_{50} \simeq -24\pi/100; \qquad q_{50} \simeq \frac{2.5\times 10^5|\ln(\phi_{50}^2/\sigma^2)|}{\pi^2}\left(\frac{m_{\rm Pl}}{\sigma}\right)^4 \gg 1; \tag{49}$$

$$\alpha_S \simeq \frac{1}{q_{50}} \ll 1; \qquad \alpha_T = \alpha_S + 0.03; \qquad \frac{T}{S} \simeq \frac{6\times 10^{-4}}{|\ln(\phi_{50}^2/\sigma^2)|}\left(\frac{\sigma}{m_{\rm Pl}}\right)^4. \tag{50}$$

Provided that $\sigma \lesssim m_{\rm Pl}$, x_{50} is very small, implying that q is very large, the gravitational-wave and density perturbations are very nearly scale invariant, and T/S is small. Finally, using the COBE DMR normalization, we can determine the dimensionless coupling constant B:

$$B \simeq 6\times 10^{-14}/|\ln(\phi_{50}^2/\sigma^2)| \approx 3\times 10^{-15}. \tag{51}$$

3.4 Natural inflation

This model is based upon a potential of the form [22]

$$V(\phi) = \Lambda^4 \left[1 + \cos(\phi/f)\right]. \tag{52}$$

The flatness of the potential (and requisite small couplings) arise because the ϕ particle is a pseudo-Nambu-Goldstone boson (f is the scale of spontaneous symmetry breaking and Λ is the scale of explicit symmetry breaking; in the limit that $\Lambda \to 0$ the ϕ particle is a massless Nambu-Goldstone boson). It is a simple matter to show that $\phi_{\rm end}$ is of the order of πf.

This potential is difficult to analyze in general; however, there are two limiting regimes: (i) $f \gg m_{\rm Pl}$; and (ii) $f \lesssim m_{\rm Pl}$ [9]. In the first regime, the 50 or so relevant e-folds take place close to the minimum of the potential, $\sigma = \pi f$, and inflation can be analyzed by expanding the potential about $\phi = \sigma$,

$$V(\psi) \simeq m^2\psi^2/2; \tag{53}$$

$$m^2 = \Lambda^4/f^2; \qquad \psi = \phi - \sigma. \tag{54}$$

In this regime natural inflation is equivalent to chaotic inflation with $m^2 = \Lambda^4/f^2 \simeq 2 \times 10^{-12} {m_{\rm Pl}}^2$.

In the second regime, $f \lesssim m_{\rm Pl}$, inflation takes place when $\phi \lesssim \pi f$, so that we can make the following approximations: $V \simeq 2\Lambda^4$ and $V' = -\Lambda^4\phi/f^2$. Taking $\phi_{\rm end} \sim \pi f$, we can solve for $N(\phi)$:

$$N(\phi) = \frac{8\pi}{{m_{\rm Pl}}^2}\int_\phi^{\pi f} \frac{V d\phi}{-V'} \simeq \frac{16\pi {m_{\rm Pl}}^2}{f^2} \ln(\pi f/\phi); \tag{55}$$

from which it is clear that achieving 50 e-folds of inflation places a lower bound to f, very roughly $f \gtrsim m_{\rm Pl}/3$ [9, 22].

Now we can solve for ϕ_{50}, V_{50}, x_{50}, and x'_{50}:

$$\phi_{50}/\pi f \simeq \exp(-50{m_{\rm Pl}}^2/16\pi f^2) \lesssim \mathcal{O}(0.1); \qquad V_{50} \simeq 2\Lambda^4; \tag{56}$$

$$x_{50} \simeq \frac{1}{2}\frac{m_{\rm Pl}}{f}\frac{\phi_{50}}{f} \lesssim \mathcal{O}(0.1); \qquad x'_{50} \simeq -\frac{1}{2}\left(\frac{m_{\rm Pl}}{f}\right)^2. \tag{57}$$

Using the COBE DMR normalization, we can relate Λ to $f/m_{\rm Pl}$:

$$\Lambda/m_{\rm Pl} = 6.7 \times 10^{-4}\sqrt{\frac{m_{\rm Pl}}{f}}\exp(-25{m_{\rm Pl}}^2/16\pi f^2). \tag{58}$$

Further, we can solve for T/S, α_T, and α_S:

$$\frac{T}{S} \simeq 0.07\left(\frac{m_{\rm Pl}}{f}\right)^2\left(\frac{\phi_{50}}{f}\right)^2 \lesssim \mathcal{O}(0.1); \tag{59}$$

$$\alpha_T = \frac{1}{16\pi}\frac{q_{50}}{q_{50}-1}\left(\frac{1}{4}\frac{{m_{\rm Pl}}^2}{f^2}\frac{\phi_{50}^2}{f^2}\right) \approx \frac{1}{64\pi}\left(\frac{m_{\rm Pl}}{f}\right)^2\left(\frac{\phi_{50}}{f}\right)^2 \ll 0.1; \tag{60}$$

$$\alpha_S = \frac{1}{16\pi}\frac{q_{50}}{q_{50}-1}\left(\frac{1}{4}\frac{{m_{\rm Pl}}^2}{f^2}\frac{\phi_{50}^2}{f^2} + \frac{{m_{\rm Pl}}^2}{f^2}\right) \approx \frac{1}{16\pi}\left(\frac{m_{\rm Pl}}{f}\right)^2; \tag{61}$$

$$q_{50} = 64\pi\left(\frac{f}{m_{\rm Pl}}\right)^2\left(\frac{f}{\phi_{50}}\right)^2 \gg 1. \tag{62}$$

Regime (ii) provides the exception to the rule that $\alpha_S \approx \alpha_T$ and large α_S implies large T/S. For example, taking $f = m_{\rm Pl}/2$, we find:

$$\phi_{50}/f \sim 0.06; \qquad x_{50} \sim 0.06; \qquad x'_{50} = -2; \qquad q_{50} \sim 10^4; \tag{63}$$

$$\alpha_T \sim 10^{-4}; \qquad \alpha_S \sim 0.08; \qquad T/S \sim 10^{-3}. \tag{64}$$

The gravitational-wave perturbations are very nearly scale invariant, while the density perturbations deviate significantly from scale invariance. We note that this regime (ii), i.e., $f \lesssim m_{\rm Pl}$, occupies only a tiny fraction of parameter space because f must be greater than about $m_{\rm Pl}/3$ to achieve sufficient inflation; further, regime (ii) is "fine tuned" and "unnatural" in the sense that the required value of Λ is exponentially sensitive to the value of $f/m_{\rm Pl}$.

Finally, we note that the results for regime (ii) apply to any inflationary model whose Taylor expansion in the inflationary region is similar; e.g., $V(\phi) = -m^2\phi^2 + \lambda\phi^4$, which was originally analyzed in Ref. [9].

4 Concluding Remarks

Beyond the generic prediction of a flat Universe and its important consequences for the matter content of the Universe, namely that most of the matter in the Universe is nonbaryonic [23], the observable consequences of inflation are tied to density and gravity-wave perturbations. (In models of first-order inflation vacuum-bubble collisions provide a very potent source of short-wavelength gravity waves [24].) The amplitude and spectrum of these perturbations depend upon the shape of the inflationary potential in the narrow interval where the scalar field was around "50 ± 4" e-folds before the end of inflation. By expanding the potential about this interval in terms of its value, V_{50}, its steepness, $x_{50} = [m_{\rm Pl}V'/V]_{50}$, and the rate of change of its steepness, x'_{50}, we have expressed the amplitudes and power-law indices of the scalar and tensor metric perturbations in terms of these three quantities to lowest order in the deviations from scale invariance. Measurements of the amplitudes and spectral indices of the density and gravity-wave perturbations determine—in fact over determine—V_{50}, x_{50}, and x'_{50}, and, in principle, such measurements allow one to both infer the shape of the inflationary potential and to test the consistency of the inflationary hypothesis [25].

There are limitations to our formalism; it is not applicable to potentials that are not "smooth" or have inclines in the region that affects astrophysically interesting scales. This includes potentials with "specially engineered" bumps and wiggles [26].

To summarize the general features of our results. In all examples the deviations from scale invariance enhance perturbations on large scales. The only potentials that have significant deviations from scale invariance are very steep or have rapidly changing steepness. In the former case, both the scalar and tensor perturbations are tilted by a similar amount; in the latter case, only the scalar perturbations are tilted.

For "steep" potentials, the expansion rate is "slow," i.e., q_{50} close to unity, the gravity-wave contribution to the CBR quadrupole anisotropy becomes comparable to, or greater than, that of density perturbations, and both scalar and tensor perturbations are tilted significantly. For flat potentials, i.e., small x_{50}, the expansion rate is "fast," i.e., $q_{50} \gg 1$, the gravity-wave contribution to the CBR quadrupole is much smaller than that of density perturbations, and unless the steepness of the potential changes significantly, large x'_{50}, both spectra very nearly scale invariant; if the steepness of the potential changes rapidly, the spectrum of scalar perturbations can be tilted significantly. The models that permit significant deviations from scale invariance involve exponential or cosine potentials. The former by virtue of their steepness, the latter by virtue of the rapid variation of their steepness.

Only recently has the deviation of the metric perturbations from the scale-invariant Harrison-Zel'dovich form drawn intense scrutiny, though their deviation from scale invariance has been noted since the very beginning [3, 9]. This new interest traces in part to the growing body of observational data that are putting the cold dark matter scenario to the test: The COBE DMR result, together with a host of other observations, *may* be inconsistent with the simplest version of cold dark matter, that with scale-invariant density perturbations ($\alpha_S = 0$). (Then again, the problems may disappear.) A slight deviation from scale invariance or tilt, $\alpha_S \simeq 0.08$ or $n \approx 0.84$, seems to improve concordance with the observational data by reducing the amplitude of scalar perturbations on small scales [27]. Of the models analyzed here, only two permit significant tilt, those based on exponential potentials, which include the very attractive extended-inflation models, and natural inflation. The former are also characterized by significant tensor contribution to the quadrupole anisotropy, while the latter are not; a separation of the tensor and scalar contributions could cleanly distinguish between these two types of models. And in that regard, measurements of CBR anisotropy on angular scales of less than a few degrees will play a crucial role.

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A Appendix

In Section II we were purposefully vague when discussing the amplitudes of the scalar and tensor modes, except when specifying their contributions to the quadrupole CBR temperature anisotropy; in fact, the spectral indices α_S and α_T, together with the scalar and tensor contributions to the CBR quadrupole serve to provide all the information necessary. In this Appendix we fill in more of the details about the metric perturbations.

The scalar and tensor metric perturbations are expanded in harmonic functions, in the flat Universe predicted by inflation, plane waves,

$$h(\mathbf{x},t) = \frac{1}{(2\pi)^3}\int d^3k\, h^i_{\mathbf{k}}(t)\,\varepsilon^i_{\mu\nu}\, e^{-i\mathbf{k}\cdot\mathbf{x}}; \qquad (65)$$

$$\frac{\delta\rho(\mathbf{x},t)}{\rho} = \frac{1}{(2\pi)^3}\int d^3k\, \delta_{\mathbf{k}}(t)\, e^{-i\mathbf{k}\cdot\mathbf{x}}; \qquad (66)$$

where $\varepsilon^i_{\mu\nu}$ is the polarization tensor for the gravity-wave modes, and $i = +$, $\times$ are the two polarization states. Everything of interest can be computed in terms of $h^i_{\mathbf{k}}$ and $\delta_{\mathbf{k}}$. For example, the *rms* mass fluctuation in a sphere of radius r is obtained in terms of the window function for a sphere and the power spectrum $P_S(k) \equiv \langle|\delta_{\mathbf{k}}|^2\rangle$ (see below),

$$\langle(\delta M/M)^2\rangle = \frac{9}{2\pi^2 r^2}\int_0^\infty [j_1(kr)]^2\, P_S(k)dk; \qquad (67)$$

where $j_1(x)$ is the spherical Bessel function of first order. If $P_S(k)$ is a power law, it follows roughly that $(\delta M/M)^2 \sim k^3|\delta_{\mathbf{k}}|^2$, evaluated on the scale $k = r^{-1}$. This is what we meant by $(\delta\rho/\rho)_{\mathrm{HOR},\lambda}$: the *rms* mass fluctuation on the scale λ when it crossed inside the horizon. Likewise, by $h_{\mathrm{HOR},\lambda}$ we meant the *rms* strain on the scale λ as it crossed inside the Hubble radius, $(h_{\mathrm{HOR},\lambda})^2 \sim k^3|h^i_{\mathbf{k}}|^2$.

In the previous discussions we have chosen to specify the metric perturbations for the different Fourier modes when they crossed inside the horizon, rather than at a common time. We did so because scale invariance is made manifest, as the scale independence of the metric perturbations at post-inflation horizon crossing. Further, in the case of scalar perturbations

$(\delta\rho/\rho)_{\rm HOR}$ is up to a numerical factor the fluctuation in the Newtonian potential, and, by specifying the scalar perturbations at horizon crossing, we avoid the discussion of scalar perturbations on superhorizon scales, which is beset by the subtleties associated with the gauge noninvariance of $\delta_{\bf k}$.

It is, however, necessary to specify the perturbations at a common time to carry out most calculations; e.g., an N-body simulation of structure formation or the calculation of CBR anisotropy. To do so, one has to take account of the evolution of the perturbations after they enter the horizon. After entering the horizon tensor perturbations behave like gravitons, with $h_{\bf k}$ decreasing as R^{-1} and the energy density associated with a given mode, $\rho_k \sim {m_{\rm Pl}}^2 k_{\rm phys}^2 k^3 |h_{\bf k}|^2$, decreasing as R^{-4}. The evolution of scalar perturbations is slightly more complicated; modes that enter the horizon while the Universe is still radiation dominated remain essentially constant until the Universe becomes matter dominated (growing only logarithmically); modes that enter the horizon after the Universe becomes matter dominated grow as the scale factor. (The gauge noninvariance of $\delta_{\bf k}$ is not an important issue for subhorizon size modes; here a Newtonian analysis suffices, and there is only one growing mode, corresponding to a density perturbation.)

The method for characterizing the scalar perturbations is by now standard: The spectrum of perturbations is specified at the present epoch (assuming linear growth for all scales); the spectrum at earlier epochs can be obtained by multiplying $\delta_{\bf k}$ by $R(t)/R_{\rm today}$. First, it should be noted that $\delta_{\bf k}$ is a gaussian, random variable with statistical expectation

$$\langle \delta_{\bf k}\, \delta_{\bf q} \rangle = P_S(k) \delta^{(3)}({\bf k} - {\bf q}); \tag{68}$$

where the power spectrum today is written as

$$P_S(k) \equiv A k^n T(k)^2; \tag{69}$$

$n = 1 - 2\alpha_S$ (= 1 for scale-invariant perturbations), and $T(k)$ is the "transfer function" which encodes the information about the post-horizon crossing evolution of each mode and depends upon the matter content of the Universe, e.g., baryons plus cold dark matter, hot dark matter, warm dark matter, and so on. The transfer function is defined so that $T(k) \to 1$ for $k \to 0$

(long-wavelength perturbations); an analytic approximation to the cold dark matter transfer function is given by [28]

$$T(k) = \frac{\ln(1+2.34q)/2.34q}{[1+(3.89q)+(16.1q)^2+(5.46q)^3+(6.71q)^4]^{1/4}}; \qquad (70)$$

where $q = k/(\Omega h^2\,\mathrm{Mpc}^{-1})$. The overall normalization factor

$$A = \frac{1024\pi^3}{75 H_0^{3+n}} \frac{V_{50}}{m_{\rm Pl}{}^4 x_{50}^2} \frac{[1+\frac{7}{6}n_T - \frac{1}{3}(n-1)]\left\{\Gamma[\frac{3}{2}-\frac{1}{2}(n-1)]\right\}^2}{2^{n-1}[\Gamma(\frac{3}{2})]^2} k_{50}^{1-n}; \quad (71)$$

where the $\mathcal{O}(\alpha_i)$ correction to A has been included [30]. The quantity $n_T = -2\alpha_T = -x_{50}^2/8\pi$, $n-1 = -2\alpha_S = n_T + x_{50}'/4\pi$, k_{50} is the comoving wavenumber of the scale that crossed outside the horizon 50 e-folds before the end of inflation. All the formulas below simplify if this scale corresponds to the present horizon scale, specifically, $k_{50} = H_0/2$. [Eq. (71) can be simplified by expanding $\Gamma(\frac{3}{2}+x) = \Gamma(3/2)[1+x(2-2\ln 2-\gamma)]$, valid for $|x| \ll 1$; $\gamma \simeq 0.577$ is Euler's constant.]

From this expression it is simple to compute the Sachs-Wolfe contribution of scalar perturbations to the CBR temperature anisotropy; on angular scales much greater than about 1° (corresponding to multipoles $l \ll 100$) it is the dominant contribution. If we expand the CBR temperature on the sky in spherical harmonics,

$$\frac{\delta T(\theta,\phi)}{T_0} = \sum_{l\geq 2, m=-l}^{l=\infty, m=l} a_{lm} Y_{lm}(\theta,\phi); \qquad (72)$$

where $T_0 = 2.73\,\mathrm{K}$ is the CBR temperature today, then the ensemble expectation for the multipole coefficients is given by

$$\langle |a_{lm}|^2 \rangle = \frac{H_0^4}{2\pi} \int_0^\infty k^{-2} P_S(k)\, |j_l(kr_0)|^2\, dk; \qquad (73)$$

$$\simeq \frac{A H_0^{3+n} r_0^{1-n}}{16} \frac{\Gamma(l+\frac{1}{2}n-\frac{1}{2})\Gamma(3-n)}{\Gamma(l-\frac{1}{2}n+\frac{5}{2})[\Gamma(2-\frac{1}{2}n)]^2}; \qquad (74)$$

where $r_0 \approx 2H_0^{-1}$ is the comoving distance to the last scattering surface, and this expression is for the Sachs-Wolfe contribution from scalar perturbations only. For n not too different from one, the ensemble expectation for the quadrupole CBR temperature anisotropy is

$$\left(\frac{\Delta T}{T_0}\right)^2_{Q-S} \equiv \frac{5|a_{2m}|^2}{4\pi} \approx \frac{32\pi}{45} \frac{V_{50}}{m_{\rm Pl}{}^4 x_{50}^2} (k_{50}r_0)^{1-n}. \tag{75}$$

(By choosing $k_{50} = r_0^{-1} = \frac{1}{2}H_0$, the last factor becomes unity.)

The procedure for specifying the tensor modes is similar, cf. Refs. [31, 32]. For the modes that enter the horizon after the Universe becomes matter dominated, $k \lesssim 0.1h^2\,{\rm Mpc}$, which are the only modes that contribute significantly to CBR anisotropy on angular scales greater than a degree,

$$h^i_{\bf k}(\tau) = a^i({\bf k}) \left(\frac{3j_1(k\tau)}{k\tau}\right); \tag{76}$$

where $\tau = r_0(t/t_0)^{1/3}$ is conformal time. [For the modes that enter the horizon during the radiation-dominated era, $k \gtrsim 0.1h^2\,{\rm Mpc}^{-1}$, the factor $3j_1(k\tau)/k\tau$ is replaced by $j_0(k\tau)$ for the remainder of the radiation era. In either case, the factor involving the spherical Bessel function quantifies the fact that tensor perturbations remain constant while outside the horizon, and after horizon crossing decrease as R^{-1}.]

The tensor perturbations too are characterized by a gaussian, random variable, here written as $a^i({\bf k})$; the statistical expectation

$$\langle h^i_{\bf k} h^j_{\bf q}\rangle = P_T(k)\delta^{(3)}({\bf k}-{\bf q})\delta_{ij}; \tag{77}$$

where the power spectrum

$$P_T(k) = A_T k^{n_T-3}\left[\frac{3j_1(k\tau)}{k\tau}\right]^2; \tag{78}$$

$$A_T = \frac{8}{3\pi}\frac{V_{50}}{m_{\rm Pl}{}^4}\frac{(1+\frac{5}{6}n_T)[\Gamma(\frac{3}{2}-\frac{1}{2}n_T)]^2}{2^{n_T}[\Gamma(\frac{3}{2})]^2}k_{50}^{-n_T}; \tag{79}$$

where the $\mathcal{O}(\alpha_i)$ correction to A_T has been included. Note that $n_T = -2\alpha_T$ is zero for scale-invariant perturbations.

Finally, the contribution of tensor perturbations to the multipole amplitudes, which arise solely due to the Sachs-Wolfe effect [29, 31, 32], is given by

$$\langle|a_{lm}|^2\rangle \simeq 36\pi^2 \frac{\Gamma(l+3)}{\Gamma(l-1)} \int_0^\infty k^{n_T+1} A_T \, |F_l(k)|^2 \, dk; \tag{80}$$

where

$$F_l(k) = -\int_{r_D}^{r_0} dr \, \frac{j_2(kr)}{kr} \left[\frac{j_l(kr_0 - kr)}{(kr_0 - kr)^2}\right]; \tag{81}$$

and $r_D = r_0/(1+z_D)^{1/2} \approx r_0/35$ is the comoving distance to the horizon at decoupling (= conformal time at decoupling). Equation (80) is approximate in that very short wavelength modes, $kr_0 \gg 100$, have not been properly taken into account; since their contribution is very small, the error made is insignificant. (Further, for $l \gtrsim 1000$, the finite thickness of the last-scattering surface must be taken into account.)

The tensor contribution to the quadrupole CBR temperature anisotropy for n_T not too different from zero is

$$\left(\frac{\Delta T}{T_0}\right)^2_{Q-T} \equiv \frac{5|a_{2m}|^2}{4\pi} \simeq 0.61 \frac{V_{50}}{m_{\rm Pl}{}^4} (k_{50}r_0)^{-n_T}; \tag{82}$$

where the integrals in the previous expressions have been evaluated numerically.

The scalar and tensor contributions to a given multipole are dominated by wavenumbers $kr_0 \sim l$. For scale-invariant perturbations ($n-1 = n_T = 0$) and small l, both the scalar and tensor contributions to $(l+\frac{1}{2})^2\langle|a_{lm}|^2\rangle$ are approximately constant. The contribution of scalar perturbations to $(l+\frac{1}{2})^2\langle|a_{lm}|^2\rangle$ begins to decrease for $l \sim 150$ because the scalar contribution to these multipoles is dominated by modes that entered the horizon before matter domination (and hence are suppressed by the transfer function). The contribution of tensor modes to $(l+\frac{1}{2})^2\langle|a_{lm}|^2\rangle$ begins to decrease for $l \sim 30$ because the tensor contribution to these multipoles is dominated by modes that entered the horizon before decoupling (and hence decayed as R^{-1} until decoupling). All of this is illustrated in Fig. 1.

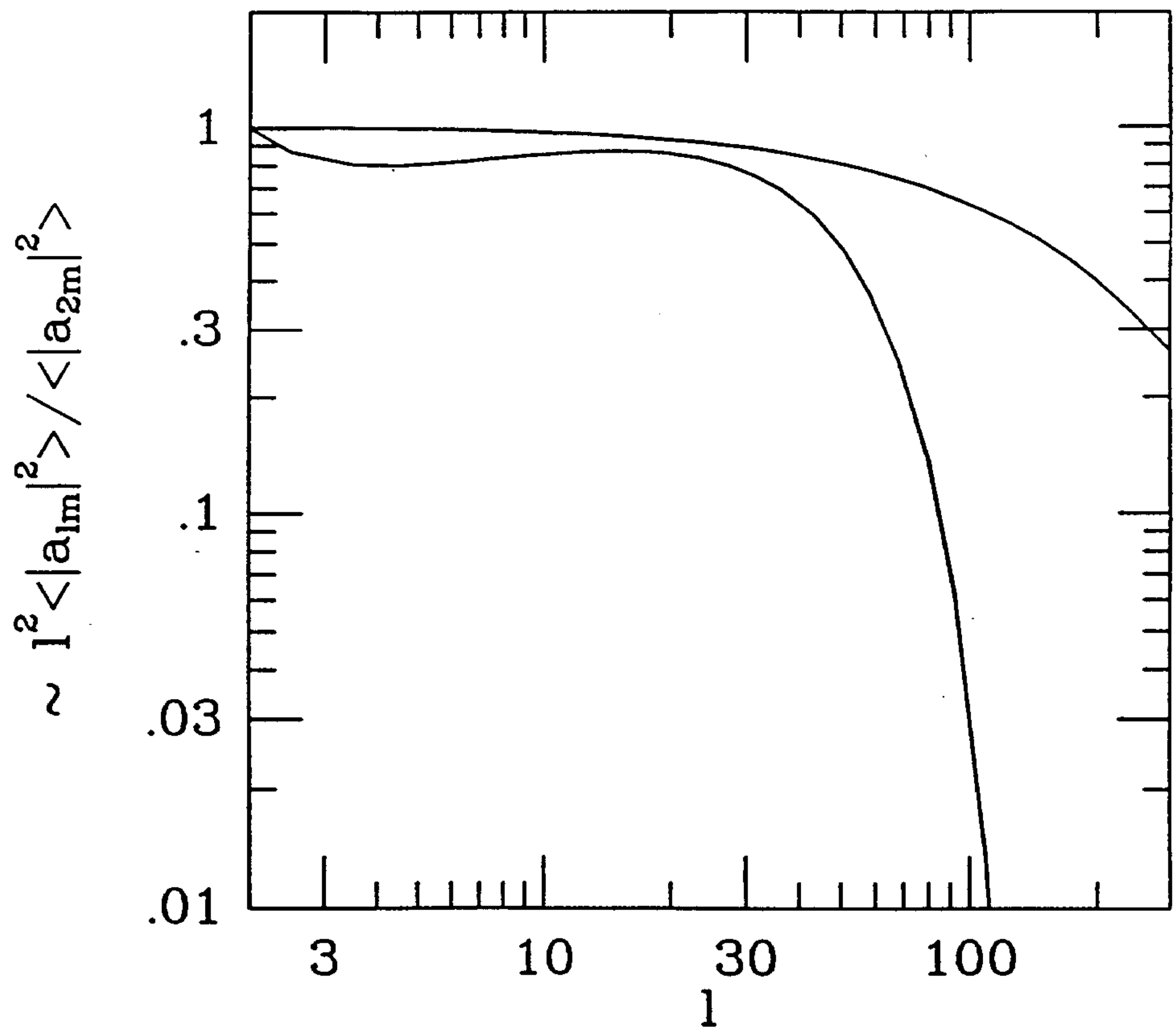

Figure 1 The (normalized) contributions of the scalar and tensor perturbations to the multipole moments; more precisely, $l(l+1)\langle|a_{lm}|^2\rangle/6\langle|a_{2m}|^2\rangle$ for the scalar contribution and $l(l+\frac{1}{2})\langle|a_{lm}|^2\rangle/5\langle|a_{2m}|^2\rangle$ for the tensor contribution. The curve for tensor contributions begins decreasing for $l \sim 30$. These results were obtained by numerically integrating Eqs. (73, 80) for $n-1=n_T=0$, $z_{\rm DEC}=1000$, and the cold dark matter transfer function (with $h=\frac{1}{2}$), cf. Eq. (70).