Fermi National Accelerator Laboratory

FERMILAB-PUB-92/256-T
September 1992

The Vector Equivalence Technique

ERAN YEHUDAI*

Theory Group, M.S. 106

Fermi National Accelerator Laboratory

P.O. Box 500, Batavia, Illinois 60510

ABSTRACT

We present the Vector Equivalence technique. This technique allows a simple and systematic calculating of Feynman diagrams involving massive fermions at the matrix element level. As its name suggests, the technique allows two Lorentz four-vectors to serve as an equivalent of two external fermions. In further calculations, traces involving these vectors replace the matrix element with the external fermions. The technique can be conveniently used for both symbolic and numeric calculations.

Submitted to *Physical Review D*

* Internet: eran@fnth06.fnal.gov

1. Introduction

Calculations of Feynman diagrams with external fermions occur frequently in particle phenomenology. The traditional calculation technique calls for squaring the amplitude while summing over polarizations. This method has the advantage that the final expression involves only dot-products of Lorentz vectors and, possibly, contraction with the Levi-Civita tensor. A major disadvantage of this method is that the number of terms in the result grows as the square of the number of terms in the amplitude. Both in tree-level and in higher order calculations, this can be a severe shortcoming.

Several authors have proposed methods for calculating the matrix element without squaring [1, 2]. We propose yet another such method. Its main advantage is that it gives, much like the traditional method, a relatively simple symbolic expression of $\mathcal{M}$ even when massive fermions are involved. Unlike other methods, one can perform calculations with free Lorentz indices. A similar technique has been implemented using the symbolic language Form.3

Generally speaking, the method entails substituting for each pair of external fermions, two complex Lorentz vectors, corresponding to the vector and pseudo-vector currents. Any amplitude involving the two fermions can then be rewritten as a trace involving the various four-vectors (and free Lorentz indices) in the problem and these two new four-vectors.

The Vector Equivalence technique was first described, and used extensively, in ref. 6. The technique can easily be combined with computerized packages for symbolic manipulation of the Dirac algebra [4, 5]. The Vector Equivalence technique is already implemented in the package described in ref. 4, and can easily be added to other packages.

This paper proceeds as follows. In the next section we derive the Vector Equivalence technique. We describe how to use the two currents to rewrite arbitrary amplitudes, and quote some useful identities. In sec. 3 we give an example for the use of the Vector Equivalence technique. We use it to calculate the helicity amplitude for the process $e^+e^- \to W^+W^-$ in a model which includes excited neutrinos [7]. The excited neutrinos couple to the electrons via a magnetic dipole transition. The methods described in refs. 1-2 cannot be simply used to derive this result. In sec. 4 we present our conclusions. In order to calculate the actual vector currents, one has to resort to an explicit representation of the spinors. In the appendix we describe, for completeness, one such representation, closely following ref. 1.

2. The Vector Equivalence Technique

In calculating a Feynman diagram with external fermions, one encounters objects of the form

$$\mathcal{M} = \bar{u}(p,s)\Gamma u(p',s'), \tag{2.1}$$

where p and p' are the momenta of the external fermions, s and s' are their helicities, and Γ is an arbitrary string of Dirac gamma matrices. For simplicity, we are only referring to fermions (as opposed to anti-fermions) in this derivation. For the purpose of this discussion, an anti-fermion with mass m behaves exactly like a fermion with mass $-m$. Additionally, we suppress the reference to s and s' in the derivation.

The traditional method calls for squaring $\mathcal{M}$ while summing over fermion helicities:

$$\sum_{s,s'} |\mathcal{M}|^2 = \sum_{s,s'} \mathrm{tr}\left\{\bar{u}(p,s)\Gamma u(p',s')\bar{u}(p',s')\bar{\Gamma}u(p,s)\right\} = \mathrm{tr}\left\{(\not{p}+m)\Gamma(\not{p}'+m')\bar{\Gamma}\right\}, \tag{2.2}$$

where $\gamma^0\bar{\Gamma} = \Gamma^{R^*}\gamma^0$ and m and m' are the masses of p and p' respectively. This method is advantageous in that the final result is expressed in terms of easy-to-calculate Lorentz invariants. However, it becomes cumbersome as the number of terms in $\mathcal{M}$ increases.

We start by rewriting

$$\mathcal{M} = \bar{u}(p)\Gamma u(p') = \mathrm{tr}\left\{\Gamma u(p')\bar{u}(p)\right\}.$$

Next, express $u(p')\bar{u}(p)$ in terms of an orthogonal basis $\left\{\Gamma^{(i)}\right\}$ of the four dimensional Dirac space. This basis obeys the orthonormality relation

$$\mathrm{tr}\left\{\Gamma^{(i)}\Gamma^{(j)^R}\right\} = \delta^{ij}.$$

In terms of such a basis, one can write

$$u(p')\bar{u}(p) = \sum_{(i)} V^{(i)}\Gamma^{(i)}.$$

The coefficients $V^{(i)}$ can be calculated using a projection:

$$V^{(i)} = \mathrm{tr}\left\{u(p')\bar{u}(p)\Gamma^{(i)^R}\right\} = \bar{u}(p)\Gamma^{(i)^R}u(p'). \tag{2.3}$$

Given $V^{(i)}$, $\mathcal{M}$ may be written as

$$\mathcal{M} = \bar{u}(p)\Gamma u(p') = \sum_{(i)} V^{(i)}\mathrm{tr}\left\{\Gamma\Gamma^{(i)}\right\}. \tag{2.4}$$

This equation can be simplified if we consider the fact that p and p' represent on-shell fermions obeying the Dirac equation [8]:

$$\bar{u}(p)(\not{p} - m) = (\not{p}' - m')u(p') = 0.$$

For any $\Gamma^{(i)}$ we can write

$$0 = \bar{u}(p)(\not{p} - m){\Gamma^{(i)}}^R u(p') = \sum_{(j)} V^{(j)} \mathrm{tr}\left\{(\not{p} - m){\Gamma^{(i)}}^R \Gamma^{(j)}\right\},$$
$$0 = \bar{u}(p){\Gamma^{(i)}}^R (\not{p}' - m')u(p') = \sum_{(j)} V^{(j)} \mathrm{tr}\left\{{\Gamma^{(i)}}^R (\not{p}' - m')\Gamma^{(j)}\right\},$$

or

$$\begin{aligned} mV^{(i)} &= \sum_{(j)} V^{(j)} \mathrm{tr}\left\{\not{p}{\Gamma^{(i)}}^R \Gamma^{(j)}\right\}, \\ m'V^{(i)} &= \sum_{(j)} V^{(j)} \mathrm{tr}\left\{{\Gamma^{(i)}}^R \not{p}' \Gamma^{(j)}\right\}. \end{aligned} \tag{2.5}$$

Let us now consider a particular choice for the basis $\left\{\Gamma^{(i)}\right\}$, namely

$$\left\{\Gamma^{(i)}\right\} = \left\{\frac{1}{2}, \frac{\gamma^\mu}{2}, \frac{\gamma^{\mu\nu}}{2\sqrt{2}}, \frac{\gamma^5\gamma^\mu}{2}, \frac{\gamma^5}{2}\right\},$$

where $\gamma^{\mu\nu} = (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)/2$. The corresponding $\left\{V^{(i)}\right\}$ are

$$\left\{V^{(i)}\right\} = \left\{U, V^\mu, W^{\mu\nu}, V_5^\mu, U_5\right\}.$$

Equation (2.4) then takes the form

$$\bar{u}(p)\Gamma u(p') = \frac{1}{2}\mathrm{tr}\left\{\Gamma(U + \not{V} + \frac{1}{\sqrt{2}}W_{\mu\nu}\gamma^{\mu\nu} + \gamma^5 \not{V}_5 + U_5\gamma^5)\right\}. \tag{2.6}$$

The string Γ of equation (2.1) can always be written as a sum $\Gamma = \Gamma_{\mathrm{odd}} + \Gamma_{\mathrm{even}}$ where Γ_{odd} and Γ_{even} contain an even and an odd number of gamma matrices respectively. Equation (2.6) can be broken into

$$\bar{u}(p)\Gamma_{\rm odd}u(p') = \frac{1}{2}{\rm tr}\left\{\Gamma_{\rm odd}(\not{V} + \gamma^5 \not{V}_5)\right\}, \tag{2.7}$$

$$\bar{u}(p)\Gamma_{\rm even}u(p') = \frac{1}{2}{\rm tr}\left\{\Gamma_{\rm even}(U + U_5\gamma^5 + \frac{1}{\sqrt{2}}W_{\mu\nu}\gamma^{\mu\nu})\right\}. \tag{2.8}$$

If both fermions are massless, the string Γ of equation (2.1) has to contain an odd number of gamma matrices, and we therefore have $U = U_5 = W^{\mu\nu} = 0$. Let us assume then that $m \neq 0$. Substituting $\Gamma^{(i)} = \gamma^{\mu\nu}/2\sqrt{2}$ into equation (2.5) gives

$$\begin{aligned} W^{\mu\nu} &= \frac{1}{4\sqrt{2}m}{\rm tr}\left\{\not{p}\gamma^{\nu\mu}(\not{V} + \gamma^5 \not{V}_5)\right\} = \frac{1}{\sqrt{2}m}(p^\nu V^\mu - p^\mu V^\nu + i\epsilon^{\mu\nu p V_5}), \\ &= \frac{1}{4\sqrt{2}m'}{\rm tr}\left\{\gamma^{\nu\mu} \not{p}'(\not{V} + \gamma^5 \not{V}_5)\right\} = \frac{1}{\sqrt{2}m'}(p'^\mu V^\nu - p'^\nu V^\mu + i\epsilon^{\mu\nu p' V_5}), \end{aligned} \tag{2.9}$$

where $\epsilon^{\mu\nu V_5 p}$ is a shorthand for $\epsilon^{\mu\nu\alpha\beta}V_{5\alpha}p_\beta$. Similarly, substituting U and U_5 for $\Gamma^{(i)}$ gives

$$\begin{aligned} U &= \frac{1}{4m}{\rm tr}\left\{\not{p}(\not{V} + \gamma^5 \not{V}_5)\right\} = \frac{V\cdot p}{m} = \frac{V\cdot p'}{m'} \\ U_5 &= \frac{1}{4m}{\rm tr}\left\{\not{p}\gamma^5(\not{V} + \gamma^5 \not{V}_5)\right\} = \frac{V_5\cdot p}{m} = -\frac{V_5\cdot p'}{m'}. \end{aligned} \tag{2.10}$$

Using equation (2.9) and (2.10), equation (2.8) takes the form:

$$\bar{u}(p)\Gamma_{\rm even}u(p') = \frac{1}{2m}{\rm tr}\left\{\Gamma_{\rm even}(\not{V} + \gamma^5 \not{V}_5)\not{p}\right\} \tag{2.11}$$

or

$$\bar{u}(p)\Gamma_{\rm even}u(p') = \frac{1}{2m'}{\rm tr}\left\{\Gamma_{\rm even}\not{p}'(\not{V} + \gamma^5 \not{V}_5)\right\} \tag{2.12}$$

Equations (2.7) and (2.11) (or (2.12)) are all one needs to calculate the generic matrix element $\mathcal{M}$ of equation (2.1) in terms of the two four-vectors V^μ and V_5^μ.

V^μ and V_5^μ depend on the four-vectors p and p' and the helicities s and s'. Since we do not implicitly sum over fermion helicity, this summation has to be carried out explicitly.

When the fermions are involved in chiral interactions such as electro-weak interactions, it is often more convenient to use a chiral basis for the Dirac space:

$$\left\{\Gamma^{(i)}\right\} = \left\{\frac{P_{L,R}}{\sqrt{2}}, \frac{\gamma^\mu P_{L,R}}{\sqrt{2}}, \frac{\gamma^{\mu\nu}}{2\sqrt{2}}\right\}, \tag{2.13}$$

where $P_L = (1-\gamma^5)/2$ and $P_R = (1+\gamma^5)/2$. The corresponding four-vectors V_L and V_R are related to V and V_5 via

$$\begin{aligned} V_R &= \frac{1}{\sqrt{2}}(V - V_5), \qquad V_L = \frac{1}{\sqrt{2}}(V + V_5), \\ V &= \frac{V_L + V_R}{\sqrt{2}}, \qquad V_5 = \frac{V_L - V_R}{\sqrt{2}}. \end{aligned} \tag{2.14}$$

In terms of V_L and V_R, equations (2.7), (2.11) and (2.12) take the form

$$\begin{aligned} \bar{u}(p)\Gamma_{\text{odd}}u(p') &= \frac{1}{\sqrt{2}}\text{tr}\left\{\Gamma_{\text{odd}}(\not{V}_R P_R + \not{V}_L P_L)\right\}, \\ \bar{u}(p)\Gamma_{\text{even}}u(p') &= \frac{1}{\sqrt{2}m}\text{tr}\left\{\Gamma_{\text{even}}(\not{V}_R P_R + \not{V}_L P_L)\,\not{p}\right\}, \\ \bar{u}(p)\Gamma_{\text{even}}u(p') &= \frac{1}{\sqrt{2}m'}\text{tr}\left\{\Gamma_{\text{even}}\,\not{p}'(\not{V}_R P_R + \not{V}_L P_L)\right\}. \end{aligned} \tag{2.15}$$

The entire derivation thus far did not depend on any particular representation of gamma matrices or spinors. In order to express the four-vectors V and V_5 (or V_L and V_R) in terms of the fermion momenta p and p', one needs to settle on a particular representation. For the specific cases of V, V_5 and V_λ, equation (2.3) gives

$$
\begin{aligned}
V^\mu &= \frac{1}{2}\bar{u}(p,s)\gamma^\mu u(p',s'),\\
V_5^\mu &= \frac{1}{2}\bar{u}(p,s)\gamma^\mu\gamma^5 u(p',s'),\\
V_\lambda^\mu &= \frac{1}{\sqrt{2}}\bar{u}(p,s)\gamma^\mu P_\lambda u(p',s').
\end{aligned}
\tag{2.16}
$$

coupled with a specific representation, these equations form a prescription for calculating V, V_5 and V_λ. Calculating V_λ is particularly convenient if one chooses a chiral representation for the spinors, such as the one described in the appendix.

Finally, we would like to collect several identities involving the V's which can be used in simplifying and verifying calculations. From equation (2.9) one gets

$$
i\epsilon^{\mu\nu\alpha V_5}\left(\frac{p}{m}-\frac{p'}{m'}\right)_\alpha = \left(\frac{p}{m}+\frac{p'}{m'}\right)^\mu V^\nu - \left(\frac{p}{m}+\frac{p'}{m'}\right)^\nu V^\mu .
$$

From equations (2.10) follows:

$$
V\cdot\left(\frac{p}{m}-\frac{p'}{m'}\right)=0, \qquad V_5\cdot\left(\frac{p}{m}+\frac{p'}{m'}\right)=0.
$$

When squaring an expression involving the V's, one can make use of equation (2.2) to arrive at the following identities:

$$
\begin{aligned}
&\sum_{ss'} V_\mu V_\nu^* = p_\mu p'_\nu + p_\nu p'_\mu - ((p\cdot p') - mm')g_{\mu\nu}\\
&\sum_{ss'} V_\mu^5 V^{5*}_{\ \nu} = p_\mu p'_\nu + p_\nu p'_\mu - ((p\cdot p') + mm')g_{\mu\nu}\\
&\sum_{ss'} i\epsilon^{\mu\nu\alpha\beta} V_\alpha^5 V_\beta^* = 2(p^\mu p'^\nu - p^\nu p'^\mu).
\end{aligned}
$$

3. Example: $e^+e^- \to W^+W^-$ with Excited Fermions

In this section we present one calculation carried out with the Vector Equivalence technique. We chose to calculate one helicity amplitude for the process $e^+e^- \to W^+W^-$ in a model which extends the Standard Model by including an excited neutrino. The excited neutrino is massive, and couples to the W and electron via a magnetic transition [7]. The relevant effective Lagrangian is:

$$\mathcal{L}_{\text{eff}} = \frac{e}{\Lambda}\bar{\nu}^* \sigma^{\mu\nu}(c - d\gamma_5)e\partial_\mu W_\nu + \text{h.c.}, \tag{3.1}$$

where Λ is the compositeness scale. While the electron mass can normally be neglected in high-energy collisions, we keep it finite to illustrate the treatment of massive external fermions.

The matrix element for the process is given by

$$\begin{aligned}\mathcal{M}^{\lambda^-\lambda^+}_{\sigma^-\sigma^+} = {}& \epsilon^{\mu *}(p_3,\lambda^-)\epsilon^{\nu *}(p_4,\lambda^+)\times \\ & \left(\mathcal{M}^{\mu\nu}_{\sigma^-\sigma^+}(\nu) + \mathcal{M}^{\mu\nu}_{\sigma^-\sigma^+}(\gamma) + \mathcal{M}^{\mu\nu}_{\sigma^-\sigma^+}(Z) + \mathcal{M}^{\mu\nu}_{\sigma^-\sigma^+}(H) + \mathcal{M}^{\mu\nu}_{\sigma^-\sigma^+}(\nu^*)\right),\end{aligned} \tag{3.2}$$

where σ^-, σ^+, λ^- and λ^+ are the helicities of the electron, positron, W^- and W^+ respectively,

$$\mathcal{M}^{\mu\nu}_{\sigma^+\sigma^-}(\nu) = \bar{v}_{\sigma^+}(p_2)\left(\frac{ig}{\sqrt{2}}\gamma_\nu P_L\right)\frac{i(\not{p}_1 - \not{p}_3 + m_e)}{t - m_e^2}\left(\frac{ig}{\sqrt{2}}\gamma_\mu P_L\right)u_{\sigma^-}(p_1)$$

$$\mathcal{M}^{\mu\nu}_{\sigma^+\sigma^-}(\gamma) = \bar{v}_{\sigma^+}(p_2)(-ie\gamma_\alpha)u_{\sigma^-}(p_1)\left(\frac{-ig^{\alpha\tau}}{s}\right)(i\Gamma^{\gamma}_{\tau\mu\nu})$$

$$\mathcal{M}^{\mu\nu}_{\sigma^+\sigma^-}(Z) = \bar{v}_{\sigma^+}(p_2)(ie\gamma_\alpha(g_0 + g_L P_L))u_{\sigma^-}(p_1)\left(\frac{-ig^{\alpha\tau}}{s - m_Z^2}\right)(i\Gamma^{Z}_{\tau\mu\nu}) \tag{3.3}$$

$$\mathcal{M}^{\mu\nu}_{\sigma^+\sigma^-}(H) = \bar{v}_{\sigma^+}(p_2)\left(\frac{-igm_e}{2m_W}\right)u_{\sigma^-}(p_1)\frac{i}{s - m_H^2}(im_W g g^{\mu\nu})$$

$$\mathcal{M}^{\mu\nu}_{\sigma^+\sigma^-}(\nu^*) = \bar{v}_{\sigma^+}(p_2)\sigma^{\nu\beta}\frac{ie}{\Lambda}(c - d\gamma_5)\left(\frac{i(\not{p}_3 - \not{p}_1 + m_{\nu^*})}{t - m_{\nu^*}^2}\right)\times$$
$$\sigma^{\mu\alpha}\frac{ie}{\Lambda}(c - d\gamma_5)u_{\sigma^-}(p_1)p_3^\alpha p_4^\beta.$$

Here $\epsilon^{\mu *}(p_3, \lambda^-)$ and $\epsilon^{\nu *}(p_4\lambda^+)$ are the polarization vectors of the W^- and W^+ respectively, $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$,

$$g_0 = \frac{\sin\theta_W}{\cos\theta_W}, \qquad g_L = -\frac{1}{2\sin\theta_W\cos\theta_W}, \tag{3.4}$$

and

$$\Gamma_V^{\tau\mu\nu} = g_V\left((p_3 - p_4)^\tau g^{\mu\nu} + 2p_4^\mu g^{\nu\tau} - 2p_3^\nu g^{\mu\tau}\right), \tag{3.5}$$

with $g_\gamma = e$ and $g_Z = e\cot\theta_W$.

In the e^+e^- center-of-mass frame, the momenta in the process take the following values:

$$
\begin{aligned}
p_1 &= \frac{\sqrt{s}}{2}(0,0,\beta_e,1) \qquad & p_3 &= \frac{\sqrt{s}}{2}(\beta_W \sin\theta, 0, \beta_W \cos\theta, 1) \\
p_2 &= \frac{\sqrt{s}}{2}(0,0,-\beta_e,1) \qquad & p_4 &= \frac{\sqrt{s}}{2}(-\beta_W \sin\theta, 0, -\beta_W \cos\theta, 1),
\end{aligned}
\tag{3.6}
$$

where $\beta_e = \sqrt{1-4m_e^2/s}$ and $\beta_W = \sqrt{1-4{m_w}^2/s}$. The W polarization vectors are:

$$
\begin{aligned}
\epsilon^*(p_3,\pm) &= \frac{1}{\sqrt{2}}(\cos\theta, \pm i, -\sin\theta, 0) \qquad & \epsilon^*(p_4,\pm) &= \frac{1}{\sqrt{2}}(\cos\theta, \mp i, -\sin\theta, 0) \\
\epsilon^*(p_3,0) &= \frac{\sqrt{s}}{2m_w}(\sin\theta, 0, \cos\theta, \beta_W) \qquad & \epsilon^*(p_4,0) &= \frac{\sqrt{s}}{2m_w}(-\sin\theta, 0, -\cos\theta, \beta_W).
\end{aligned}
\tag{3.7}
$$

The vectors $V^{\sigma^+\sigma^-}$ and $V_5^{\sigma^+\sigma^-}$ are given in table 1.

Table 1. The vectors $V^{\sigma^+\sigma^-}$ and $V_5^{\sigma^+\sigma^-}$

$(\sigma^+\sigma^-)$	$V^{\sigma^+\sigma^-}$	$V_5^{\sigma^+\sigma^-}$
$(++)$	$(0,0,0,-m_e/\sqrt{s})$	$(0,0,0,-m_e/\sqrt{s})$
$(+-)$	$(\beta_e\sqrt{s}/2, -i\beta_e\sqrt{s}/2, 0, 0)$	$(\beta_e\sqrt{s}/2, -i\beta_e\sqrt{s}/2, 0, 0)$
$(-+)$	$(-\beta_e\sqrt{s}/2, -i\beta_e\sqrt{s}/2, 0, 0)$	$(-\beta_e\sqrt{s}/2, -i\beta_e\sqrt{s}/2, 0, 0)$
$(--)$	$(0,0,m_e/\sqrt{s},0)$	$(0,0,0,-m_e/\sqrt{s})$

Applying eqns. (2.7) and (2.11) to eqn. (3.3) gives (dropping the $\sigma^+\sigma^-$ for convinience):

$$
\begin{aligned}
\mathcal{M}^{\mu\nu}(\nu) &= -i\frac{g^2}{4(t-m_e^2)}\mathrm{tr}\left\{\gamma^\nu P_L(\not p_1 - \not p_3)\gamma^\mu(\not V + \not V^5\gamma_5)\right\} \\
&= i\frac{g^2}{\sqrt{2}(t-m_e^2)}\big(- i\epsilon^{\mu\nu\alpha\beta}(p_1-p_3)_\alpha V_{L\beta} + (p_1-p_3)\cdot V_L g^{\mu\nu} \\
&- (p_1-p_3)^\mu V_L{}^\nu - (p_1-p_3)^\nu V_L^\mu\big)
\end{aligned}
$$

$$
\mathcal{M}^{\mu\nu}(\gamma) = -i\frac{2e}{s}\Gamma^\gamma_{\mu\nu\tau}V^\tau
$$

$$
\mathcal{M}^{\mu\nu}(Z) = -i\frac{2e}{s}\Gamma^Z_{\mu\nu\tau}(g_0V^\tau + \sqrt{2}g_L V_R^\tau) \tag{3.8}
$$

$$
\mathcal{M}^{\mu\nu}(H) = i\frac{g^2(p_2\cdot V)g^{\mu\nu}}{4(s-m_H^2)}
$$

$$
\begin{aligned}
\mathcal{M}^{\mu\nu}(\nu^*) = \frac{-ie^2p_3^\alpha p_4^\beta}{2\Lambda^2(t-m_{\nu^*}^2)}\Big((c^2-d^2)\mathrm{tr}\left\{\sigma^{\nu\beta}(\not p_3 - \not p_1)\sigma^{\mu\alpha}(\not V + \not V^5\gamma_5)\right\} - \\
\frac{m_{\nu^*}}{m_e}\mathrm{tr}\left\{(c^2+d^2-2cd\gamma_5)\sigma^{\nu\beta}\sigma^{\mu\alpha}(\not V + \not V^5\gamma_5)\,\not p_2\right\}\Big).
\end{aligned}
$$

Equation (3.2) together with equations (3.5)-(3.8) allow a straight-forward, if lengthy, calculation of the various helicity amplitudes.

4. Conclusion

To summarize, we have developed a technique for calculating Feynman amplitudes involving (possibly massive) fermions. The technique uses two (complex) four-vectors V and V_5 which depend on the fermion momenta and helicities. Equations (2.7) and (2.11) contain the prescription for expressing any Feynman amplitude as a trace involving these two four-vectors.

In addition to the calculation of tree-level amplitudes with massless or massive fermions, the method can also be used in the calculation quantities arising in loop calculations provided the spinors can be taken to be in 4 dimensions.

The Vector Equivalence technique easily lends itself to computerized evaluation of helicity amplitudes. The HIP package [4] implements the method symbolically.

APPENDIX

Expressing the four-vectors V and V_5 in terms of the fermion momenta can only be done in the context of a specific spinor representation. For completeness we provide a full description of one such representation. Our description closely follows that of reference 1.

The gamma matrices are given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma_+^\mu \\ \sigma_-^\mu & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A.1}$$

where $\sigma_\pm^\mu = (1, \pm\sigma)$, and

$$\sigma = (\sigma^1, \sigma^2, \sigma^3) = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]. \tag{A.2}$$

The spinors $u(p,\lambda)$ and $v(p,\lambda)$ are given by

$$u(p,\lambda) = \begin{pmatrix} u(p,\lambda)_- \\ u(p,\lambda)_+ \end{pmatrix}, \qquad \bar{u}(p,\lambda) = (\, u(p,\lambda)_+^\dagger \quad u(p,\lambda)_-^\dagger \,),$$
$$v(p,\lambda) = \begin{pmatrix} v(p,\lambda)_- \\ v(p,\lambda)_+ \end{pmatrix}, \qquad \bar{v}(p,\lambda) = (\, v(p,\lambda)_+^\dagger \quad v(p,\lambda)_-^\dagger \,). \tag{A.3}$$

The explicit components of $u_\pm$ and $v_\pm$ are given by

$$u(p,\lambda)_\pm = \omega_\pm(p)\chi_\lambda(p).\backslash v(p,\lambda)_\pm = \pm\lambda\omega_\lambda(p)\chi_{-\lambda}(p), \tag{A.4}$$

where $\omega_\pm(p) = \sqrt{E \pm |\mathbf{p}|}$ and $\chi_\pm(p)$ are the helicity eigenstates

$$\frac{\sigma \cdot \mathbf{p}}{|\mathbf{p}|}\chi_\lambda(p) = \lambda\chi_\lambda(p), \tag{A.5}$$

and are given by

$$\chi_+(p) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| + p_z)}} \begin{pmatrix} |\mathbf{p}| + p_z \\ p_x + ip_y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+\cos\theta} \\ \sqrt{1-\cos\theta}e^{i\phi} \end{pmatrix}$$
$$\chi_-(p) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| + p_z)}} \begin{pmatrix} -p_x + ip_y \\ |\mathbf{p}| + p_z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sqrt{1-\cos\theta}e^{-i\phi} \\ \sqrt{1+\cos\theta} \end{pmatrix} \tag{A.6}$$

For an arbitrary momentum $p^\mu = (E, \mathbf{p})$ where

$$\mathbf{p} = (p_x, p_y, p_z) = (\sin\theta\cos\phi|\mathbf{p}|,\ \sin\theta\sin\phi|\mathbf{p}|,\ \cos\theta|\mathbf{p}|)\,. \tag{A.7}$$

In the special case of $\theta = \pi$ ($p_z = -|\mathbf{p}|$) we use

$$\chi_+(p) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \chi_-(p) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{A.8}$$

The equations in this appendix, together with eqn. (2.16) can be used to calculation V, V_5 and V_λ in terms of the fermion momenta and helicities. The result of the calculation is:

$$V = \frac{1}{4}\left(|\mathbf{p}|\,|\mathbf{p}'|\,p_0\,p_0'\right)^{-1/2}\tilde{V},$$
$$V^5 = \frac{s}{4}\left(|\mathbf{p}|\,|\mathbf{p}'|\,p_0\,p_0'\right)^{-1/2}\tilde{V}^5 \tag{A.9}$$

$$\left.\begin{aligned}
\tilde{V}_x &= (-p_-p_-' + p_+p_+')(p_0'p_\perp + p_0p_\perp') \\
\tilde{V}_y &= is(-p_-p_-' + p_+p_+')(p_0'p_\perp - p_0p_\perp') \\
\tilde{V}_z &= (p_-p_-' - p_+p_+')(p_\perp p_\perp' - p_0p_0') \\
\tilde{V}_t &= (p_-p_-' + p_+p_+')(p_\perp p_\perp' + p_0p_0')
\end{aligned}\right\} \quad (s = s'), \tag{A.10}$$

$$\left.\begin{aligned}
\tilde{V}_x &= (p_-p_+' + p_+p_-')(-p_\perp p_\perp' + p_0p_0') \\
\tilde{V}_y &= -is(p_-p_+' + p_+p_-')(p_\perp p_\perp' + p_0p_0') \\
\tilde{V}_z &= -(p_-p_+' + p_+p_-')(p_0'p_\perp + p_0p_\perp') \\
\tilde{V}_t &= (-p_-p_+' + p_+p_-')(p_0'p_\perp - p_0p_\perp')
\end{aligned}\right\} \quad (s = -s'), \tag{A.11}$$

$$\left.\begin{aligned}
\tilde{V}_x^5 &= (p_-p_-' + p_+p_+')(p_0'p_\perp + p_0p_\perp') \\
\tilde{V}_y^5 &= is(p_-p_-' + p_+p_+')(p_0'p_\perp - p_0p_\perp') \\
\tilde{V}_z^5 &= -(p_-p_-' + p_+p_+')(p_\perp p_\perp' - p_0p_0') \\
\tilde{V}_t^5 &= -(p_-p_-' - p_+p_+')(p_\perp p_\perp' + p_0p_0')
\end{aligned}\right\} \quad (s = s'), \tag{A.12}$$

$$\left.\begin{aligned}
\tilde{V}_x^5 &= (p_-p_+' - p_+p_-')(p_\perp p_\perp' - p_0p_0') \\
\tilde{V}_y^5 &= -is(-p_-p_+' + p_+p_-')(p_\perp p_\perp' + p_0p_0') \\
\tilde{V}_z^5 &= (p_-p_+' - p_+p_-')(p_0'p_\perp + p_0p_\perp') \\
\tilde{V}_t^5 &= (p_-p_+' + p_+p_-')(p_0'p_\perp - p_0p_\perp')
\end{aligned}\right\} \quad (s = -s'), \tag{A.13}$$

where $s^{(\prime)}$ is the helicity of $p^{(\prime)}$ ($-$helicity for an anti-fermion), $c^{(\prime)}$ is 1 (-1) for an (anti) fermion,

$$
\begin{aligned}
|\mathbf{p}^{(\prime)}| &= \left[p_x{}^{(\prime)2} + p_y{}^{(\prime)2} + p_z{}^{(\prime)2}\right]^{1/2}, \\
p_0{}^{(\prime)} &= |\mathbf{p}^{(\prime)}| + p_z{}^{(\prime)}, \\
p_-{}^{(\prime)} &= \left[E^{(\prime)} - |\mathbf{p}^{(\prime)}|\right]^{1/2}, \\
p_+{}^{(\prime)} &= s^{(\prime)}c^{(\prime)}\left[E^{(\prime)} + |\mathbf{p}^{(\prime)}|\right]^{1/2}, \\
p_\perp &= p_x - isp_y, \quad \text{and} \qquad p'_\perp = p'_x + is'p'_y.
\end{aligned}
\tag{A.14}
$$

In the limit $p_0{}^{(\prime)}, p_\perp{}^{(\prime)} \to 0$, one should take $p_\perp{}^{(\prime)}/\sqrt{p_0{}^{(\prime)}} \to \sqrt{2|\mathbf{p}^{(\prime)}|}$.

REFERENCES

[1] K. Hagiwara and D. Zeppenfeld, *Nucl. Phys.* **B274** (1986), 1.

[2] H. W. Fearing and R. B. Silbar, *Phys. Rev.* **D6** (1972), 471;
R. Kleiss, W. J. Stirling, *Nucl. Phys.* **B262** (235), 1985.

[3] J. A. M. Vermaseren, Report No. KEK-Th-326 (1992).

[4] A. Hsieh and E. Yehudai, *Comp. Phys.* **6** (1992), 253;
E. Yehudai,Report No. Fermilab-Pub-92/22-T (1992)

[5] R. Mertig, M. Böhm and A. Denner, *Comp. Phys. Comm.* **64** (1991), 345.

[6] E. Yehudai, Report No. SLAC-Report-383, 1991 (Ph.D. dissertation).

[7] K. Hagiwara, S. Komamiya and D. Zeppenfeld, *Z. Phys.* **C29** (1985), 115.

[8] I would like to thank J. Vermaseren for his help in pointing out the usefulness of these identities.