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Padé Approximants and the R and R_τ Ratios in Perturbative QCD.

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ABSTRACT

We recently proposed a new method to estimate coefficients, in a given order of perturbative quantum field theory, without actually evaluating all of the Feynman Diagrams which occur in this order. Here we consider the R and R_τ ratios in perturbative QCD, in the general MS-type scheme, described by the parameter t. For $t = 0$ ($\overline{MS}$ scheme), although the method works well for R_τ it does not for R. However, due to a remarkable relation which is satisfied by the coefficients, the method works well for R, for larger values of t. It works well for R_τ for all values of t. This is true for all values of N_f ($0 \leq N_f \leq 6$).

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Perturbative Quantum Field Theory (PQFT) seems to describe nature very well, as manifested in the Standard Model of high energy physics. However there is as yet, no way to estimate, in a given order, the result for the coefficient, without the brute force evaluation of all of the Feynman diagrams contributing in this order. An attempt to do this was made recently by West[1] in the case of the R ratio in perturbative QCD.

$$R = \frac{\sigma_{tot}(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} \tag{1}$$

Although this method worked well for $N_f = 5$, where N_f is the number of fermions (quarks), it failed[2] for other values of N_f. He is now attempting to calculate corrections to his result[3].

Recently we proposed[4] a new method to estimate coefficients, in a given order of PQFT, without actually evaluating all the Feynman Diagrams which occur in this order. Our method makes use of Padé Approximants, with which we can predict the next term S_{n+m+1}, in the perturbation series S, given by

$$S = S_0 + S_1 x + \ldots + S_{n+m} x^{n+m} \tag{2}$$

The results which we used are:

$$\begin{aligned} I \quad S_3 &= S_2^2/S_1 \\ S_4 &= S_3^2/S_2 \\ II \quad S_4 &= 2S_2S_3/S_1 - S_2^3/S_1^2, \; S_0 = 0 \\ III \; S_4 &= \frac{2S_1S_2S_3 - S_0S_3^2 - S_2^3}{S_1^2 - S_0S_2} \end{aligned} \tag{3}$$

We applied these results to various perturbation series in QED and QCD. Our predictions agreed very well with the known results. Furthermore we were able to predict the next unknown term (NT) and the next-next (second) unknown term (NNT). Our method works best for ordinary series, positive definite, negative definite or oscillating series. For other (unusual) series, although the method still works, it requires more terms and does not seem to work as well as for ordinary series. Eqs (3) ensure that a positive-definite series remains positive definite, a negative-definite series remains negative definite and an oscillating series remains oscillating. One can tell if the method will work well in a given perturbative series by testing to see if a condition is well satisfied or not. That condition is

$$A + A^{-1} = 2 \tag{4}$$

$$\text{where } A = \frac{S_1 S_3}{S_2^2} \tag{5}$$

In this paper we will consider the R ratio and the R_τ ratio in perturbative QCD. They are defined as follows:

$$R_\tau = \frac{\Gamma(\tau \to \nu + hadrons)}{\Gamma(\tau \to e\nu\,\bar{\nu})} \tag{6}$$

and R is given by eq (1).

We first consider R in the general MS - type scheme given by the parameter t.

$$\Lambda_t = e^{-t/2}\Lambda_{\overline{MS}} \tag{7}$$

Obviously t = 0 corresponds to the $\overline{MS}$ scheme, $t = \ln 4\pi - \gamma = 1.95$ represents the MS scheme, t = 1.0 for the G scheme and t = 4 ζ (3) - 11 / 2 = -.692 yields our $\widetilde{\mathrm{MS}}$ scheme[5]. The scale-dependent R (in the general MS-type scheme) is given by

$$R = 3 \sum Q_f^2 \, R(t) - 1.24 \left(\sum Q_f\right)^2 x^3 \tag{8}$$

$$\begin{aligned} \text{where } R(t) &= 1 + x + x^2[(1.9857 + 2.75t) \\ &- N_f(.1153 + .1667t)] + x^3[(-6.6369 \\ &+ 17.2964t + 7.5625t^2) - N_f(1.2001 \\ &+ 2.0877t + .9167t^2) + N_f^2(-.0052 \\ &+ .0384t + .0278t^2)] \end{aligned} \tag{9}$$

where $x = \frac{\alpha_s}{\pi}$ and N_f is the number of fermions (quarks). We neglect the second term in eq (8) as it is small in all cases of interest.

Our results for t = 2, 4 and 10 are shown in Tables I, II and III, respectively. It can be seen that the method works very well and we can predict the NT and the NNT terms. For small t, however, the x^3 term is negative, as can be seen from eq. (9), we have an unusual series and the method does not work. The NNT terms from II and III of eqs. (3) agree very well with those from I and so are not listed in our Tables. In Figures 1 & 2. we plot the estimated and exact terms as a function of t for two representative values of N_f ($N_f = 1$ and $N_f = 5$, respectively). It can be seen that the agreement is excellent for $t > 1$ and improves as t increases. The reason for this behavior can be seen as follows.

From I of eqs. (4) and eq (9) we obtain

$$\begin{aligned} S_4 = S_3^2/S_2 &= 3.943 + 10.92t + 7.5625t^2 \\ &- N_f(.458 + 1.2962t + .9167t^2) \\ &+ N_f^2(.0133 + .0384t + .0278t^2) \end{aligned} \tag{10}$$

The exact result is given by the x^3 term in eq. (9). It can be seen by comparing this term with eq (10) that the t^2, t^2N_f, $t^2N_f^2$ and tN_f^2 coefficients agree. In fact, this agreement is exact! Now we understand why the estimate and the exact result agree so well for large t.

We now turn to R_τ. In the general MS-type scheme R_τ^{pert} is given by[5]

$$R_\tau^{pert} = 3R_\tau(t)$$

$$\begin{aligned} \text{where } R_\tau(t) &= 1 + x + x^2[(6.3399 + 2.75t) \\ &- N_f(.3792 + .1667t)] + x^3[(48.5832 \\ &+ 41.2443t + 7.5625t^2) - N_f(7.8795 \\ &+ 4.9905t + .9167t^2) + N_f^2(.1579 \\ &+ .1264t + .0278t^2)] \end{aligned} \tag{11}$$

The results for t = 0, 4 and 10 are shown in Tables IV, V and VI, respectively. It can be seen that the method works very well and we can predict the NT and the NNT terms. The NNT terms from II and III of eq (4) agree very well with those from I and so are not listed in our

Tables. In figures 3 and 4 we plot the estimated and exact terms as a function of t for two representative values of N_f ($N_f = 1$ and $N_f = 5$, respectively). It can be seen that in this case, the agreement is excellent, even for $t = 0$, and, again, improves as t increases. Again, we can see why we get this behavior.

From I of eqs (4) and eq (11) we obtain

$$\begin{aligned} S_4 = S_3^2/S_2 &= 40.1943 + 34.8695t \\ &+ 7.5625t^2 - N_f(4.8082 + 4.1989t \\ &+ .9167t^2) + N_f^2(.1438 + .1264t \\ &+ .0278t^2)] \end{aligned} \qquad (12)$$

The exact result is given by the x^3 term of eq (11). It can be seen that again the t^2, t^2N_f, $t^2N_f^2$ and tN_f^2 coefficients agree. Again this agreement is exact! Moreover the t^2, t^2N_f and $t^2N_f^2$ of eq (10) and eq (12) also agree exactly!

In conclusion, we have shown how one can accurately estimate coefficients of PQFT. In this paper we have considered the R ratio and the R_τ ratio of PQCD in the general MS-type scheme. In our previous paper we have shown that the method works well for a_μ - a_e, a_e, a_μ, R_τ for $N_f = 3$ and $t = 0$ and the QCD beta-function for $N_f = 1$, 3 and 5, where a_μ and a_e are the anomalous magnetic moments of the muon and the electron, respectively.

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References

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TABLE CAPTIONS

TABLE I Results for R(t) for the estimated (first column) and the exact (if known) coefficients (second column) for t = 2. NT refers to the next (unknown term and NNT refers to the next-next (second unknown) term. I, II and III refer to eqs (3).

TABLE II Results for R(t) for the estimated (first column) and the exact (if known) coefficients (second column) for t = 4. NT refers to the next (unknown term and NNT refers to the next-next (second unknown) term. I, II and III refer to eqs (3).

TABLE III Results for R(t) for the estimated (first column) and the exact (if known) coefficients (second column) for t = 10. NT refers to the next (unknown term and NNT refers to the next-next (second unknown) term. I, II and III refer to eqs (3).

TABLE IV Results for $R_r(t)$ for the estimated (first column) and the exact (if known) coefficients (second column) for t = 0. NT refers to the next (unknown term and NNT refers to the next-next (second unknown) term. I, II and III refer to eqs (3).

TABLE V Results for $R_r(t)$ for the estimated (first column) and the exact (if known) coefficients (second column) for t = 4. NT refers to the next (unknown term and NNT refers to the next-next (second unknown) term. I, II and III refer to eqs (3).

TABLE VI Results for $R_r(t)$ for the estimated (first column) and the exact (if known) coefficients (second column) for t = 10. NT refers to the next (unknown term and NNT refers to the next-next (second unknown) term. I, II and III refer to eqs (3).

TABLE I

For R
$t = 2.0$

$N_f = 0$ $A + A^{-1} = 2.0014$

I	56.04	58.21
	452.59	NT
	3519.14	NNT
II	451.96	NT
III	452.68	NT

$N_f = 1$ $A + A^{-1} = 2.0000$

I	49.52	49.35
	346.04	NT
	2426.56	NNT
II	346.03	NT
III	346.04	NT

$N_f = 2$ $A + A^{-1} = 2.0037$

I	43.41	40.85
	253.32	NT
	1570.76	NNT
II	252.33	NT
III	253.49	NT

$N_f = 3$ $A + A^{-1} = 2.0200$

I	37.69	32.72
	174.42	NT
	929.40	NNT
II	170.40	NT
III	175.20	NT

$N_f = 4$ $A + A^{-1} = 2.0682$

I	32.39	24.96
	109.49	NT
	480.23	NNT
II	99.80	NT
III	111.55	NT

$N_f = 5$ $A + A^{-1} = 2.2037$

I	27.48	17.56
	58.85	NT
	197.18	NNT
II	40.09	NT
III	63.27	NT

$N_f = 6$ $A + A^{-1} = 2.6399$

I	22.98	10.53
	23.14	NT
	50.86	NNT
II	-9.16	NT
III	31.66	NT

TABLE II

For R
t = 4.

$N_f = 0$ $A + A^{-1} = 2.0072$

I	168.63	183.55
	2594.40	NT
	36671.04	NNT
II	2577.26	NT
III	2595.83	NT

$N_f = 1$ $A + A^{-1} = 2.0051$

I	148.93	159.92
	2095.74	NT
	27463.97	NNT
II	2085.84	NT
III	2096.63	NT

$N_f = 2$ $A + A^{-1} = 2.0020$

I	130.45	137.49
	1654.97	NT
	19921.52	NNT
II	1650.63	NNT
III	1655.38	NT

$N_f = 3$ $A + A^{-1} = 2.0007$

I	113.20	116.23
	1269.82	NT
	13872.55	NNT
II	1268.96	NT
III	1269.91	NT

$N_f = 4$ $A + A^{-1} = 2.0001$

I	97.17	96.17
	938.21	NT
	9153.12	NNT
II	938.11	NT
III	938.22	NT

$N_f = 5$ $A + A^{-1} = 2.0040$

I	82.36	77.29
	658.22	NT
	5605.67	NNT
II	655.39	NT
III	658.57	NT

$N_f = 6$ $A + A^{-1} = 2.0206$

I	68.78	59.60
	428.26	NT
	3077.53	NNT
II	418.10	NT
III	429.65	NT

TABLE III

For R
t = 10.0

$N_f = 0$ $A + A^{-1} = 2.0035$

I	869.41	922.58
	28866.48	NT
	903202.49	NNT
II	28770.60	NT
III	28869.85	NT

$N_f = 1$ $A + A^{-1} = 2.0032$

I	767.48	811.99
	23799.95	NT
	697563.86	NNT
II	23727.94	NT
III	23802.13	NT

$N_f = 2$ $A + A^{-1} = 2.0027$

I	671.90	707.72
	19322.67	NT
	527562.67	NNT
II	19273.19	NT
III	19324.66	NT

$N_f = 3$ $A + A^{-1} = 2.0021$

I	582.68	609.76
	15403.14	NT
	389095.43	NNT
II	15372.75	NT
III	15404.45	NT

$N_f = 4$ $A + A^{-1} = 2.0013$

I	499.81	518.13
	12008.06	NT
	278296.24	NNT
II	11993.05	NT
III	12008.76	NT

$N_f = 5$ $A + A^{-1} = 2.0005$

I	423.30	432.81
	9104.89	NT
	191536.15	NNT
II	9100.49	NT
III	9105.12	NT

$N_f = 6$ $A + A^{-1} = 2.0000$

I	353.14	353.81
	6661.51	NT
	125422.00	NNT
II	6661.49	NT
III	6661.51	NT

TABLE IV

For R_r
t = 0.0

$N_f = 0$ $A + A^{-1} = 2.0360$

I	40.19	48.58
	372.30	NT
	2852.95	NNT
II	361.20	NT
III	374.38	NT

$N_f = 1$ $A + A^{-1} = 2.0196$

I	35.53	40.86
	280.11	NT
	1920.22	NNT
II	275.34	NNT
III	281.07	NT

$N_f = 2$ $A + A^{-1} = 2.0051$

I	31.15	33.46
	200.54	NT
	1202.02	NNT
II	199.59	NT
III	200.74	NT

$N_f = 3$ $A + A^{-1} = 2.0007$

I	27.06	26.37
	133.62	NT
	677.22	NNT
II	133.53	NT
III	133.65	NT

$N_f = 4$ $A + A^{-1} = 2.0296$

I	23.26	19.59
	79.58	NT
	323.26	NNT
II	76.79	NT
III	80.31	NT

$N_f = 5$ $A + A^{-1} = 2.1687$

I	19.75	13.13
	38.81	NT
	114.71	NNT
II	28.97	NT
III	41.67	NT

$N_f = 6$ $A + A^{-1} = 2.7865$

I	16.52	6.99
	12.02	NT
	20.68	NNT
II	-10.33	NT
III	19.32	NT

TABLE V

For R_τ
t = 4.0

$N_f = 0$ $A + A^{-1} = 2.0114$

I	300.67	334.56
	6455.09	NT
	124546.18	NNT
II	6388.86	NT
III	6459.15	NT

$N_f = 1$ $A + A^{-1} = 2.0098$

I	265.49	293.16
	5274.54	NT
	94899.53	NNT
II	5227.55	NT
III	5277.16	NT

$N_f = 2$ $A + A^{-1} = 2.0078$

I	232.50	253.98
	4230.35	NT
	70462.65	NNT
II	4200.09	NT
III	4232.47	NT

$N_f = 3$ $A + A^{-1} = 2.0054$

I	201.69	217.01
	3315.96	NT
	50668.77	NNT
II	3299.44	NT
III	3317.21	NT

$N_f = 4$ $A + A^{-1} = 2.0027$

I	173.08	182.26
	2524.96	NT
	34980.14	NNT
II	2518.55	NT
III	2525.49	NT

$N_f = 5$ $A + A^{-1} = 2.0004$

I	146.65	149.72
	1851.16	NT
	22887.44	NNT
II	1850.38	NT
III	1851.23	NT

$N_f = 6$ $A + A^{-1} = 2.0006$

I	122.41	119.44
	1288.70	NT
	13908.26	NNT
II	1287.88	NT
III	1288.78	NT

TABLE VI

For R_τ
t = 10.0

$N_f = 0$ $A + A^{-1} = 2.0037$

I	1145.14	1217.28
	43787.40	NT
	1575104.06	NNT
II	43633.63	NT
III	43792.09	NT

$N_f = 1$ $A + A^{-1} = 2.0035$

I	1010.84	1072.02
	36146.61	NT
	1218795.65	NNT
II	36028.87	NT
III	36150.44	NT

$N_f = 2$ $A + A^{-1} = 2.0031$

I	884.91	935.17
	29399.17	NT
	924224.42	NNT
II	29314.25	NT
III	29402.13	NT

$N_f = 3$ $A + A^{-1} = 2.0025$

I	767.36	806.73
	23493.95	NT
	684201.46	NNT
II	23438.00	NT
III	23496.25	NT

$N_f = 4$ $A + A^{-1} = 2.0018$

I	658.18	686.69
	18380.02	NT
	491962.60	NNT
II	18348.35	NT
III	18381.30	NT

$N_f = 5$ $A + A^{-1} = 2.0010$

I	557.38	575.05
	14006.75	NT
	341167.80	NNT
II	13993.52	NT
III	14007.33	NT

$N_f = 6$ $A + A^{-1} = 2.0002$

I	464.95	471.82
	10323.93	NT
	225899.95	NNT
II	10321.74	NT
III	10324.04	NT

Figure Captions

Fig 1 The exact (EXA) and the estimated (EST) coefficients vs t for the x^3 coefficient of $R(t)$ for $N_f = 1$.

Fig 2 The exact (EXA) and the estimated (EST) coefficients vs t for the x^3 coefficient of $R(t)$ for $N_f = 5$.

Fig 3 The exact (EXA) and the estimated (EST) coefficients vs t for the x^3 coefficient of $R_\tau(t)$ for $N_f = 1$.

Fig 4 The exact (EXA) and the estimated (EST) coefficients vs t for the x^3 coefficient of $R_\tau(t)$ for $N_f = 5$.

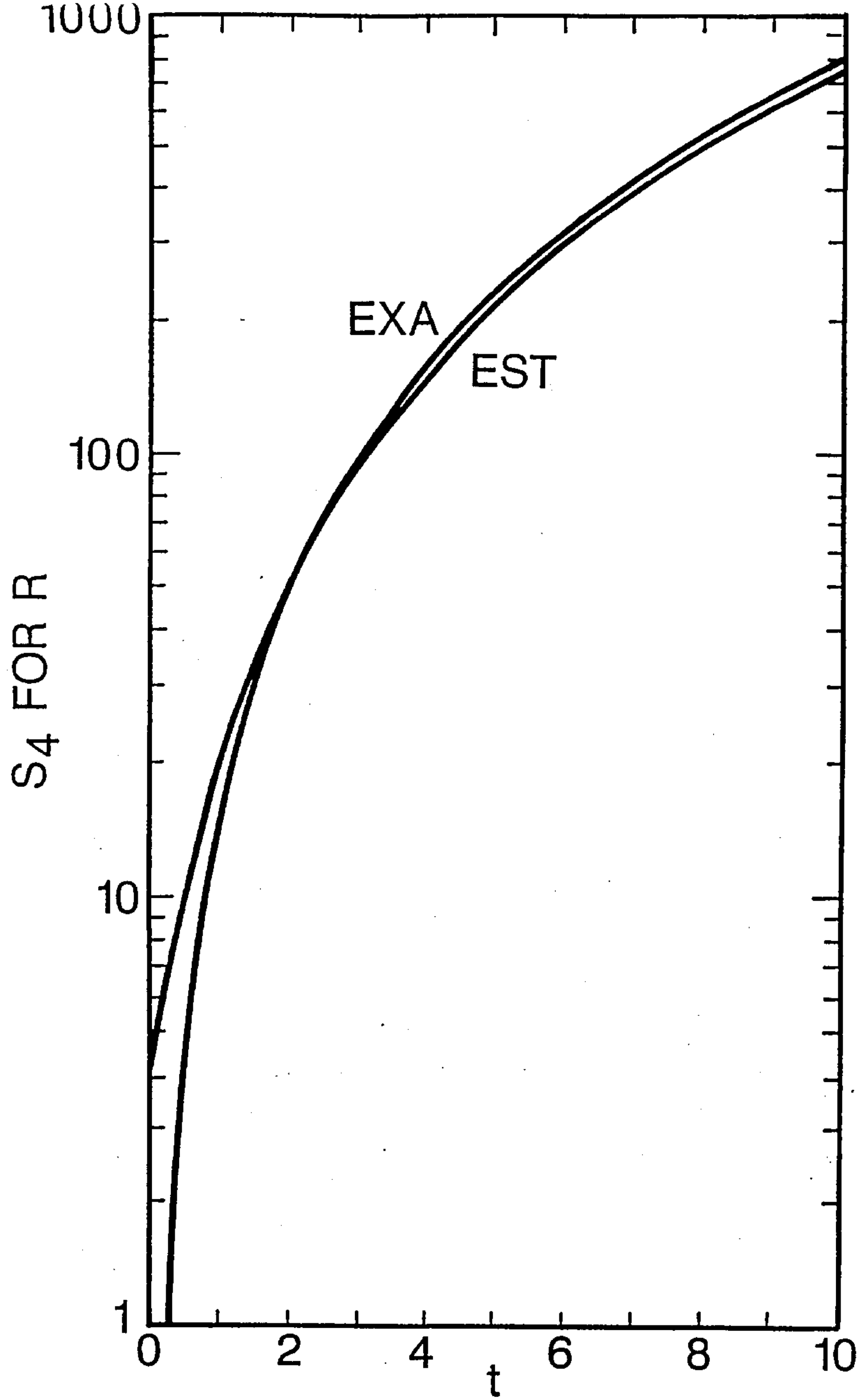

Figure 1

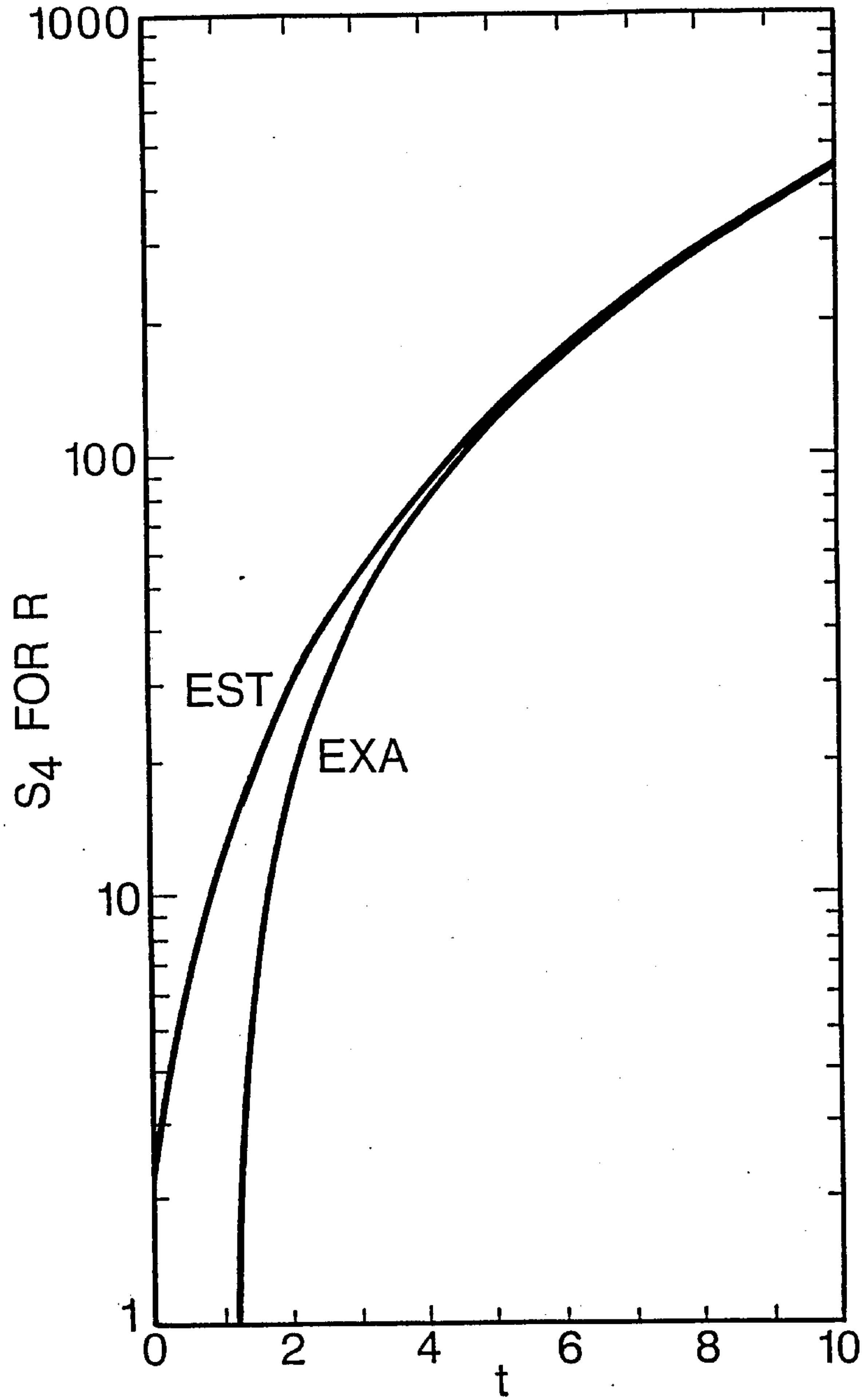

Figure 2

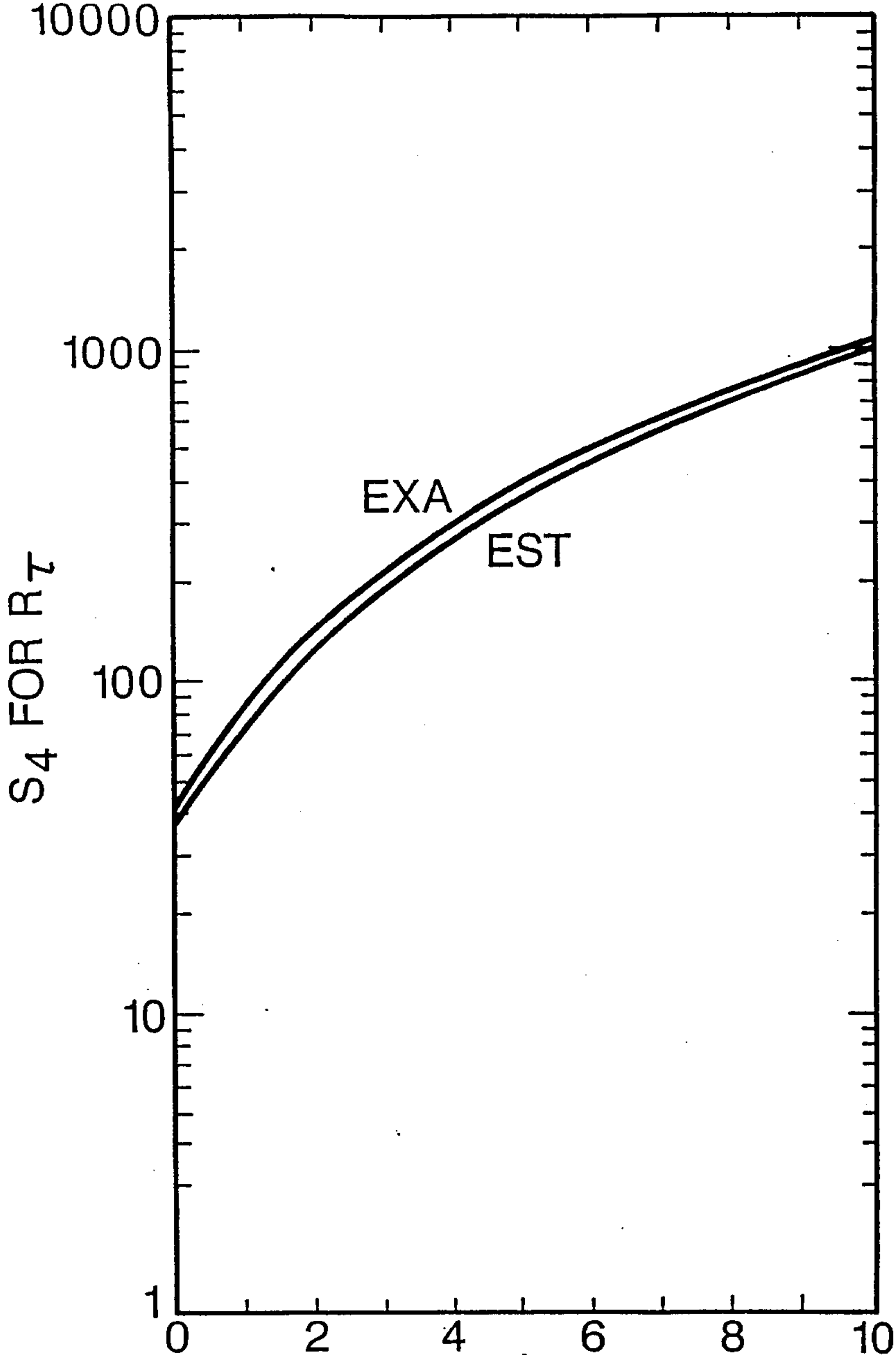

Figure 3

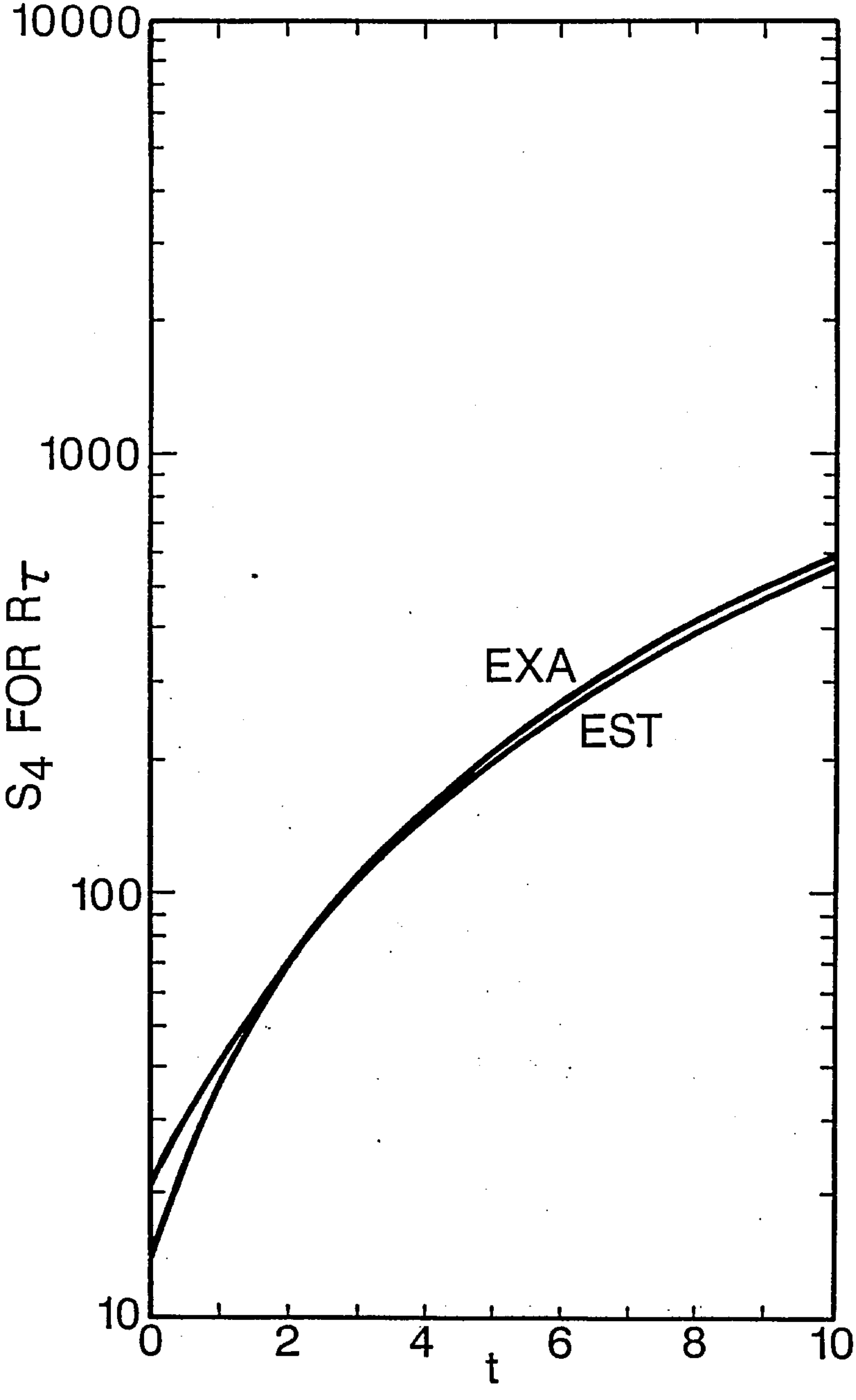

Figure 4