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Nonequilibrium Neutrino Statistical Mechanics in the Expanding Universe

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Abstract. We study neutrino decoupling in the early Universe ($t \sim$ sec, $T \sim$ MeV). In particular, we compute the distortions in the ν_e and ν_μ/ν_τ phase-space distributions that arise in the standard cosmology due to $e^\pm$ annihilations. These distortions are nonthermal, with the effective neutrino temperature increasing with energy, approaching an 0.7% increase for electron neutrinos and an 0.3% increase for mu/tau neutrinos at the highest energies, and correspond to an increase in the energy density of ν_e's of about 1.2% and in the energy density of ν_μ/ν_τ's of about 0.5% (roughly one additional relic neutrino per cm^{-3} per species). (The distortion for electron-neutrinos is larger than that for mu/tau neutrinos because electron neutrinos couple to $e^\pm$'s through both charged- and neutral-current interactions.) Our results graphically illustrate that neutrino decoupling is a continuous process which is momentum dependent. The distortions in the neutrino spectra affect primordial nucleosynthesis in three ways; due to subtle cancellations, only a tiny increase in the primordial ^{4}He abundance, $\Delta Y \simeq 1 - 2 \times 10^{-4}$, is predicted.

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I. Introduction

Much of the history of the Universe is well described by equilibrium thermodynamics. However, if thermal equilibrium were the entire story, the Universe today would be a very boring place. A number of crucial departures from equilibrium have taken place during the history of the Universe: photon decoupling, primordial nucleosynthesis, baryogenesis, and perhaps even an inflationary phase transition (see e.g., Ref. [1]). The departure from equilibrium that we address here involves neutrinos and the weak interactions.

Around a second after the bang the rates for the weak interactions that keep neutrinos in thermal contact with the electromagnetic plasma ($e^{\pm}$'s and γ's), $\nu + e^{\pm} \leftrightarrow \nu + e^{\pm}$, $\nu + \bar{\nu} \leftrightarrow e^{-} + e^{+}$, $\nu + \nu \leftrightarrow \nu + \nu$, and $\nu + \bar{\nu} \leftrightarrow \nu + \bar{\nu}$, as well as those that keep the neutron-to-proton ratio tracking its equilibrium value, $\nu_e + n \leftrightarrow p + e^{-}$, $\bar{\nu}_e + p \leftrightarrow n + e^{+}$, and to a lesser degree, $n \leftrightarrow p + e^{-} + \bar{\nu}_e$, become ineffective; i.e., interaction rate per particle Γ becomes less than the expansion rate of the Universe H. The outcome of primordial nucleosynthesis depends crucially upon this: Were it not for the fact that the neutron-to-proton ratio ceased to track its equilibrium value and "froze out" at a value of about 0.2 when the temperature of the Universe was about 0.1 MeV, the neutron abundance would have been negligibly small by the time that nucleosynthesis commenced ($T \sim 0.07\,\mathrm{MeV}$), and essentially no nucleosynthesis would have taken place.

According to the standard treatment neutrinos decouple ($T \sim 2\,\mathrm{MeV}$ for ν_e, $T \sim 3 - 4\,\mathrm{MeV}$ for ν_μ and ν_τ) before the $e^{\pm}$ pairs annihilate ($T \sim m_e/3 \sim 0.1\,\mathrm{MeV}$), and thus do not share in the entropy transfer from $e^{\pm}$ pairs to photons that heats photons relative to neutrinos. This is why the neutrino temperature is expected to be less than the photon temperature today. To be specific, after neutrinos decouple their temperature varies as the inverse of the cosmic-scale factor $R(t)$; entropy conservation implies that the photon temperature varies as $g_*^{-1/3}R^{-1}(t)$ (g_* is the number of degrees of freedom in thermal equilibrium with the photons). Because g_* drops from 11/2 before $e^{\pm}$ pairs annihilate to 2 after, the photon temperature is today predicted to be larger than the neutrino temperature: $T/T_\gamma = (4/11)^{1/3}$ (see e.g., Refs. [1,2]).

Because neutrino decoupling occurs only slightly before the $e^{\pm}$ pairs "disappear," neutrinos (especially electron neutrinos which stay in thermal contact slightly longer because of their charged-current interactions) will share to a small degree in the entropy transfer, so that their "temperature" is expected to be slightly higher than the estimate above [3]. Further, because neutrino cross sections are very energy dependent, varying as energy squared, one also expects the degree of heating to depend upon neutrino momentum, which inevitably leads to a spectral distortion of the neutrino phase-space distributions. In previous work [3], authors have studied the "integrated effect" of the slight heating by $e^{\pm}$ annihilations, estimating that the neutrino energy density is increased by about 1%. In this paper we compute the evolution of the neutrino phase-space distribution functions during decoupling to study the effect of this heating in detail.

The primary motivation for our detailed investigation of neutrino decoupling is primordial nucleosynthesis. The yield of ^{4}He is sensitive to the phase-space distribution of neutrinos, as they play an integral role in determining when the weak interactions that interconvert neutrons and protons freeze out, which in turn determines the value of the neutron fraction at the time of nucleosynthesis. (The primordial mass fraction of ^{4}He is given by about twice the neutron fraction.) Since the accuracy to which the primordial ^{4}He is known is improving, with recent estimates being given to three significant figures [4,5], we decided to carefully study the decoupling of all three neutrino species by numerically evolving the Boltzmann equations that govern their phase-space distributions.

We find that the slight heating provided by $e^{\pm}$ annihilations increases the energy density in electron neutrinos over the "canonical" estimate by about 1.2% and by about 0.5% for mu/tau neutrinos (this corresponds to roughly one additional relic neutrino per cm^{-3} per species). We also find that due to the back reaction of neutrino heating the increase in the number of photons per comoving volume since before $e^{\pm}$ annihilations is about 0.5% less than the canonical prediction of 11/4. (By canonical, we mean assuming that neutrinos are not heated by $e^{\pm}$ annihilations.) The neutrino phase-space distortions we find are nonthermal: The effective neutrino temperature rises with neutrino energy, where $T_{\rm eff} \equiv -p/\ln f_\nu(p)$, where $f_\nu(p)$ is the phase-space distribution, assuming Maxwell-Boltzmann statistics. Our results very clearly illustrate neutrino decoupling is not an instantaneous event and is momentum dependent.

The distorted neutrino distributions affect the primordial ^{4}He abundance in three ways: The first two effects involve changes in the weak-interaction rates that regulate the neutron fraction, while the third involves the number of neutron decays after the neutron fraction "freezes out." The first two effects nearly cancel, leading to a change in the ^{4}He mass fraction of the order of 10^{-5}; thus, the third effect dominates. It leads to a change in the primordial mass fraction of ^{4}He of about $1-2\times10^{-4}$, which, at present, is not large enough to be of significance. As we discuss, were it not for cancellations, the change in the primordial ^{4}He abundance would have been an order of magnitude larger.

The outline of our paper is as follows. In the next Section we derive the Boltzmann equations that govern the phase-space distributions of neutrinos in the expanding Universe, and from it the equations that govern small perturbations from the canonical thermal distribution with temperature decreasing as $R^{-1}(t)$. In Section III we numerically calculate the small distortions in the neutrino spectra that develop due to slight heating by $e^{\pm}$ annihilations, and in Section IV we compute their effect on primordial nucleosynthesis. We end with some concluding remarks in the final Section. The details of evaluating the numerous nine-dimensional phase-space integrals that arise, some useful identities for Maxwell-Boltzmann statistics, and a detailed discussion of the back reaction of neutrino heating on the photon temperature are relegated to the Appendix.

II. Boltzmann Equations

Our starting point is the Boltzmann equation that governs the evolution of the phase-space distribution of a neutrino species (or any particle species) in the expanding Universe. For simplicity, we assume all phase-space distribution functions are independent of spatial coordinates (homogeneity) and use Maxwell-Boltzmann statistics. Homogeneity in the early Universe is well justified, and because we are not interested in neutrino degeneracy (or Bose condensation) the use of Maxwell-Boltzmann statistics should be adequate. The time evolution of the neutrino distribution function $f_a(E,t)$ in the FRW cosmology is

$$E\frac{\partial f_a}{\partial t} - H|\vec{p}|^2\frac{\partial f_a}{\partial E} = -\frac{1}{2}\sum_{\text{processes}}\int d\Pi_1 d\Pi_2 d\Pi_3 (2\pi)^4\delta^4(p_a+p_1-p_2-p_3) \times S|\mathcal{M}_{a+1\leftrightarrow 2+3}|^2\left[f_a f_1 - f_2 f_3\right]; \tag{2.1}$$

where $H \equiv \dot{R}/R$ is the expansion rate of the Universe, $d\Pi_i \equiv d^3p_i/2E_i(2\pi)^3$ is the Lorentz-invariant phase-space volume element, and for simplicity we have only displayed $2 \leftrightarrow 2$ processes in the collision term (for more details concerning Eq. (2.1), see e.g., Refs. [1,6]). The quantity $S|\mathcal{M}|^2$ is the matrix-element squared for the process $a+1 \leftrightarrow 2+3$ (CP-invariance is assumed), summed over the spin states of all particles except the a, times a symmetry factor, 1/2! for identical particles in the initial or final states. Throughout we shall use units where $\hbar = k_B = c = 1$.

If we specialize to ultrarelativistic particles, as we will in our study of neutrino decoupling, we can simplify Eq. (2.1). In the expanding Universe the momentum of any freely propagating particle red shifts as $R(t)^{-1}$; for massless particles, energies also red shift as $R(t)^{-1}$. In dealing with ultrarelativistic particles it is thus useful to introduce momenta that are scaled by the expansion: $\tilde{p} \equiv R(t)p$ ($\tilde{p}$ corresponds to the covariant components of the four momentum in the conformal frame). For massless particles all components of the rescaled four momentum $\tilde{p}$ remain constant. *For simplicity of notation we will henceforth not explicitly include tildes over four momenta when we employ this rescaling.* (By simple dimensional analysis it will always be clear when we have used the rescaling.) In terms of the rescaled momenta, the Boltzmann equation simplifies to

$$\tilde{E}_a\frac{\partial f_a(\tilde{E}_a,t)}{\partial t} = -\frac{1}{2}R^{-5}\sum_{\text{processes}}\int d\Pi_1 d\Pi_2 d\Pi_3 (2\pi)^4\delta^4(\tilde{p}_a+\tilde{p}_1-\tilde{p}_2-\tilde{p}_3) \times |\tilde{\mathcal{M}}_{a+1\leftrightarrow 2+3}|^2\left[f_a f_1 - f_2 f_3\right]; \tag{2.2}$$

where all momenta (including those in the matrix-element squared) are now rescaled momenta. The advantages of this rescaling are now manifest: The $|\vec{p}|^2 H\partial f_a/\partial E$ term drops out, and in the absence of interactions the solution to Eq. (2.2) is just $f_a(\tilde{E}_a,t) = f_a(\tilde{E}_a,t_0)$ (t_0 is some initial time). This, of course, is well known: The momentum distribution of a massless, noninteracting species just red shifts with the expansion; if the original distribution was thermal, then the distribution remains thermal, albeit with a temperature that varies as $R^{-1}(t)$. In our treatment of neutrino decoupling we will use the "unperturbed" neutrino temperature ($\equiv T$) as the independent variable; since $T \propto R^{-1}$ we can simplify further by taking $RT = 1$, so that $\tilde{p} = p_i/T$ and $R^{-5} = T^5$.

Now let us apply this formalism to the decoupling of neutrinos. Around the time of neutrino decoupling ($T \sim$ MeV, $t \sim$ sec), the reactions that keep neutrinos in thermal contact with the electromagnetic plasma and other neutrino species are $2 \leftrightarrow 2$ scattering and annihilation processes that involve neutrinos/antineutrinos and electrons/positrons. (Neutrino-nucleon interactions are extremely unimportant because of the scarcity of nucleons, only about one nucleon per 10^9 electrons, positrons, neutrinos, and antineutrinos.)

Scattering and annihilation processes involving electrons and positrons can "heat" neutrinos, $\nu + e^{\pm} \leftrightarrow \nu + e^{\pm}$ and $\nu + \bar{\nu} \leftrightarrow e^- + e^+$; while scattering and annihilation processes involving only neutrinos can only thermalize the neutrino distributions, e.g., $\nu_e + \nu_\mu \leftrightarrow \nu_e + \nu_\mu$ or $\nu_e + \bar{\nu}_e \leftrightarrow \nu_\tau + \bar{\nu}_\tau$. All the annihilation and scattering processes involving electron neutrinos and their matrix-elements squared times symmetry factors are displayed in Table 1 [7]; the analogous compilation for mu/tau neutrinos is given in Table 2. In addition, our notation is explained in the Tables and illustrated in Fig. 1.

The mu- and tau-neutrino phase-space distribution functions are identical, but not equal to that of electron neutrino, since electron neutrinos have both neutral- and charged-current interactions. We shall assume that the chemical potentials of all lepton species are very small $|\mu| \ll T$ (as is known for $e^{\pm}$'s and is expected for all the neutrino species); this implies that the phase-space distribution functions of particles and their antiparticles are identical. This and the fact that the ν_μ and ν_τ distributions are identical means that we need only track the phase-space distribution functions of electron and muon neutrinos.

We are now ready to derive the Boltzmann equations that govern the small distortions to the neutrino phase-space distribution functions that develop due to $e^{\pm}$ heating. Around the time that "neutrinos decouple," the temperature of the electromagnetic plasma begins to decrease more slowly than $R^{-1}(t)$, as $e^{\pm}$ pairs become fewer in number and transfer their entropy to photons and the remaining $e^{\pm}$ pairs. If neutrinos had completely decoupled by this time, their temperature would simply decrease as $R^{-1}(t)$ and would be dropping relative to the photon temperature. It is this small temperature difference that "drives" residual neutrino-electron/positron interactions to heat the neutrinos. By calculating how well neutrinos are able to track the "rising" photon temperature, we are able to follow the process of neutrino decoupling.

With these facts in mind, we write the phase-space distribution functions as

$$f_{\nu_e}(p,t) = f_0(p) + \Delta_{\nu_e}(p,t), \qquad f_{\nu_\mu} = f_0(p) + \Delta_{\nu_\mu}(p,t); \tag{2.3}$$

$$f_{e^\pm}(p,t) = \exp(-p/T_\gamma) = \exp[-p(1-\delta)/T] = f_0(p)[1 + (p/T)\delta(t) + \cdots +]; \tag{2.4}$$

where we take $T \equiv R^{-1}(t)$ so that $f_0(p) \equiv \exp(-p/T)$ is the unperturbed neutrino phase-space distribution, $\Delta_{\nu_e}(p,t)$ is the small perturbation to the electron-neutrino distribution caused by slight $e^{\pm}$ heating, $\Delta_{\nu_\mu}(p,t)$ is the small perturbation to the mu/tau-neutrino phase-space distribution, $\delta(t) \equiv T_\gamma/T - 1$ is the photon-neutrino temperature difference. Further, by writing $f_{e^\pm}(p,t) = \exp(-p/T_\gamma)$ we assume that the electromagnetic plasma is always in thermal equilibrium; because of rapid electromagnetic interactions between electrons, positrons, and photons this is a very good approximation. In Eq. (2.4) we have expanded to lowest order in the neutrino-photon temperature difference since we will work to first order in the small quantities Δ_{ν_e}, Δ_{ν_μ}, and δ.

As we show in the Appendix, for Maxwell-Boltzmann statistics, when neutrinos *do not* share in the heat released by $e^{\pm}$ annihilations, the ratio of the photon and neutrino temperature is given by

$$\frac{T_{0\gamma}}{T} = \left[\frac{3}{1+\left[z^3K_1(z)+4z^2K_2(z)\right]/4}\right]^{1/3}; \qquad (2.5a)$$

$$\delta_0(t) \equiv \frac{T_{0\gamma}}{T} - 1; \qquad (2.5b)$$

$$\delta_0(t) \longrightarrow \frac{1}{36}\left(\frac{m_e}{T}\right)^2; \qquad (2.5c)$$

where $T_{0\gamma}$ is the photon temperature when the back reaction of neutrino heating is neglected, $z = m_e/T_{0\gamma}$, $m_e = 0.511\,\mathrm{MeV}$ is the mass of the electron, and the limit shown is for $z \to 0$. While the back reaction of neutrino heating on the temperature of the electromagnetic plasma is a small effect ($\delta T_\gamma/T \approx -2\times 10^{-3}$), it is formally first order in Δ_i, and so must be taken into account. In the Appendix we show that the change in the photon temperature due to back reaction is

$$\delta T_\gamma = -\frac{\delta\rho_\nu}{d\rho_{\mathrm{EM}}/dT_\gamma}; \qquad (2.6)$$

where ρ_ν is the small change in the energy density in neutrinos due to heating by $e^{\pm}$ annihilations—which is of order Δ_i—and given by

$$\rho_\nu = \sum_{i=e,\mu,\tau} 2\int p\, d^3p\, \Delta_i(p,t)/(2\pi)^3. \qquad (2.7)$$

Finally, the photon-neutrino temperature difference to the desired order is given by

$$\delta(t) = \delta_0(t) + \delta T_\gamma/T; \qquad (2.8)$$

and is shown around the epoch of nucleosynthesis in Fig. 2.

As stated, our analysis is to lowest order in all small quantities; that is, in expanding $[f_a f_1 - f_2 f_3]$ in Eq. (2.1) we keep only terms that are linear in $\delta(t)$, $\Delta_{\nu_e}(p,t)$, or $\Delta_{\nu_\mu}(p,t)$. To illustrate, consider the terms that arise from the scattering processes $\nu_e(p) + e^-(q) \leftrightarrow \nu_e(p') + e^-(q')$:

$$f_{\nu_e}(p)f_{e^\pm}(q) - f_{\nu_e}(p')f_{e^\pm}(q') = f_0(q)\Delta_{\nu_e}(p,t) - f_0(q')\Delta_{\nu_e}(p',t) + \delta(t)f_0(p)f_0(q)(p-p')/T + \cdots\ ; \qquad (2.9)$$

where the zeroth-order terms cancel by energy conservation. It is now simple to write down the Boltzmann equation governing the electron-neutrino distortion:

$$(p/T)\dot{\Delta}_{\nu_e}(p,t) = 4G_F^2T^5\left[-A_e(p,t)\Delta_{\nu_e}(p,t) + B_e(p,t)\delta(t) + C_e(p,t) + C_e'(p,t)\right]; \quad (2.10a)$$

$$A_e = \int d\Lambda f_0(q)\left[(a+b+3)s^2 + bt^2 + (2a+b+8)u^2\right]; \quad (2.10b)$$

$$B_e = f_0(p)\int d\Lambda f_0(q)\left[(bt^2+au^2)(p+q)/T + (a+b)(s^2+u^2)(p-p')/T\right]; \quad (2.10c)$$

$$C_e = \int d\Lambda\Big\{-f_0(p)\Delta_{\nu_e}(q,t)[s^2+bt^2+(a+6)u^2] + f_0(q')\Delta_{\nu_e}(p',t)[(a+b+3)s^2 + (a+b+6)u^2] + f_0(p')\Delta_{\nu_e}(q',t)[s^2+4u^2]\Big\}; \quad (2.10d)$$

$$C_e' = \int d\Lambda\Big[-f_0(p)\Delta_{\nu_\mu}(q,t)(2s^2+2u^2) + f_0(q')\Delta_{\nu_\mu}(p',t)(2u^2) + f_0(p')\Delta_{\nu_\mu}(q',t)(2s^2+4u^2)\Big]; \quad (2.10e)$$

where $d\Lambda \equiv d\Pi_q d\Pi_{p'} d\Pi_{q'}(2\pi)^4\delta^4(p+q-p'-q')$ is a nine(!)-dimensional phase-space volume element, $s=(p+q)^2$, $t=(p-p')^2$, $u=(p-q')^2$, $a=(2\sin^2\theta_W+1)^2 \simeq 2.13$, $b=4\sin^2\theta_W \simeq 0.212$, the weak mixing angle $\sin^2\theta_W \simeq 0.23$, and the Fermi constant $G_F = 1.17\times10^{-5}\,\mathrm{GeV}^{-2}$. For purposes of numerically integrating this equation, it is useful to write $\dot{\Delta}_{\nu_e}(p,t) = H\partial\Delta_{\nu_e}(p,t)/\partial\ln T^{-1}$, where the expansion rate $H(T) = 1.67g_*^{1/2}T^2/m_{\rm Pl}$, and $4G_F^2T^5/H \simeq 1.2(T/\,\mathrm{MeV})^3$ ($g_* \simeq 12$ and $m_{\rm Pl} = 1.22\times10^{19}\,\mathrm{GeV}$).

The analogous Boltzmann equation governing the mu/tau-neutrino distortion is:

$$(p/T)\dot{\Delta}_{\nu_\mu}(p,t) = 4G_F^2T^5\left[-A_\mu(p,t)\Delta_{\nu_\mu}(p,t) + B_\mu(p,t)\delta(t) + C_\mu(p,t) + C_\mu'(p,t)\right]; \quad (2.11a)$$

$$A_\mu = \int d\Lambda f_0(q)\left[(b+c+3)s^2 + bt^2 + (b+2c+8)u^2\right]; \quad (2.11b)$$

$$B_\mu = f_0(p)\int d\Lambda f_0(q)\left[(bt^2+cu^2)(p+q)/T + (b+c)(s^2+u^2)(p-p')/T\right]; \quad (2.11c)$$

$$C_\mu = \int d\Lambda\Big\{-f_0(p)\Delta_{\nu_\mu}(q,t)[2s^2+bt^2+(c+8)u^2] + f_0(q')\Delta_{\nu_\mu}(p',t)[(b+c+3)s^2 + (b+c+7)u^2] + f_0(p')\Delta_{\nu_\mu}(q',t)[2s^2+6u^2]\Big\}; \quad (2.11d)$$

$$C_\mu' = C_e'/2; \quad (2.11e)$$

where $c = (2\sin^2\theta_W - 1)^2 \simeq 0.292$.

The four different types of terms in Eqs. (2.10) and (2.11) arise from the expansion of $[f_af_1 - f_2f_3]$ as noted above. Their physical significance is manifest: The "A terms" represent damping (i.e., disappearance of a neutrino of energy p) and arise from all the scattering and annihilation processes, e.g., $\nu(p) + e^- \to \nu + e^-$; the "$B$ terms" represent the heating of neutrinos through interactions with $e^\pm$'s and arise from the scattering and annihilation processes involving electrons and positrons, cf. Tables 1 and 2; the "C terms" represent scattering interactions that simply change the momentum of a neutrino from p' to p, e.g., $\nu(p') + e^- \to \nu(p) + e^-$, and hence involve an integration over $\Delta(p', t)$; and the "C' terms" are analogous to the C terms except that they involve the interaction of electron neutrinos with mu/tau neutrinos or vice versa, e.g., $\nu_\mu(p') + \bar{\nu}_\mu \to \nu_e(p) + \bar{\nu}_e$.

It is a straightforward, but arduous, task to evaluate the coefficients A_i, B_i, C_i, and C_i'. Needless to say the C and C' terms are the most challenging to compute. The details of these calculations are left to the Appendix. The coefficients are given by

$$A_e = \frac{(p/T)^2}{3\pi^3}[5a + 5b + 17]; \tag{2.12a}$$

$$A_\mu = \frac{(p/T)^2}{3\pi^3}[5b + 5c + 17]; \tag{2.12b}$$

$$B_e = (a + b)\frac{(p/T)^2 e^{-p/T}}{\pi^3}\left(\frac{11}{12}\frac{p}{T} - 1\right); \tag{2.13a}$$

$$B_\mu = (b + c)\frac{(p/T)^2 e^{-p/T}}{\pi^3}\left(\frac{11}{12}\frac{p}{T} - 1\right); \tag{2.13b}$$

$$\begin{aligned} C_e = &-\frac{(p/T)^2 e^{-p/T}}{18\pi^3 T^4}(a + b + 9)\int_0^\infty dq\, q^3\, \Delta_{\nu_e}(q, t) + \frac{e^{-p/2T}}{64\pi^3 (p/T) T}\int_0^\infty dq'\, e^{q'/2T}\Delta_{\nu_e}(q', t) \\ &\times \{(a + b + 10)g_1(p, q') + (2a + 2b + 12)g_2(p, q') + (2a + 2b + 10)g_3(p, q')\}; \end{aligned} \tag{2.14a}$$

$$\begin{aligned} C_\mu = &-\frac{(p/T)^2 e^{-p/T}}{18\pi^3 T^4}(b + c + 14)\int_0^\infty dq\, q^3\, \Delta_{\nu_\mu}(q, t) + \frac{e^{-p/2T}}{64\pi^3 (p/T) T}\int_0^\infty dq'\, e^{q'/2T}\Delta_{\nu_\mu}(q', t) \\ &\times \{(b + c + 13)g_1(p, q') + (2b + 2c + 14)g_2(p, q') + (2b + 2c + 12)g_3(p, q')\}; \end{aligned} \tag{2.14b}$$

$$\begin{aligned} C_e' = &-\frac{4(p/T)^2 e^{-p/T}}{9\pi^3 T^4}\int_0^\infty dq\, q^3\, \Delta_{\nu_\mu}(q, t) + \frac{e^{-p/2T}}{32\pi^3 (p/T) T}\int_0^\infty dq'\, e^{q'/2T}\Delta_{\nu_\mu}(q', t) \\ &\times \{3g_1(p, q') + 2g_2(p, q') + 2g_3(p, q')\}; \end{aligned} \tag{2.15a}$$

$$C_\mu' = C_e'/2; \tag{2.15b}$$

where the functions $g_i(p, q')$ are defined in the Appendix, cf. Eqs. (A.21) through (A.23), and shown in Fig. 11.

By comparing the source terms, Eqs. (2.13a) and (2.13b), we can see that electron neutrinos are heated more than mu/tau neutrinos: The coefficient of B_e is $(a + b) \simeq 2.34$ vs. $(b + c) \simeq 0.502$ for B_μ, while the coefficient of the damping term for electron neutrinos is only somewhat larger than that for mu neutrinos, $(5a + 5b + 17) \simeq 28.7$ for A_e vs.

$(5b+5c+17) \simeq 19.5$ for A_μ. Equations (2.10) and (2.11) provide the master equations for our work.

Finally, consider the distortions to neutrino phase-space distribution functions that would arise due to the decay of a massive, nonrelativistic particle species, $X \to \nu_i \bar{\nu}_i$. In this situation we neglect the small difference between the neutrino and photon temperatures, and as above write the neutrino distribution functions as the unperturbed piece, $f_0(p) = e^{-p/T}$, plus a small perturbation, $\Delta_i(p,t)$. The Boltzmann equations governing $\Delta_{\nu_e}(p,t)$ and $\Delta_{\nu_\mu}(p,t)$ in this circumstance are obtained from Eqs. (2.10) and (2.11) above by dropping the B terms and adding a source term from decays:

$$(p/T)\dot{\Delta}_i(p,t) = 4G_F^2 T^5 \left[-A_i(p,t)\Delta_i(p,t) + C_i(p,t) + C_i'(p,t)\right] + \frac{4\pi^2 n_X r_i}{T^2(m_X/T)H\tau}\delta(p/T - m_X/2T). \tag{2.16}$$

Here T is the common photon/neutrino temperature which is assumed to vary as $R^{-1}(t)$, r_i is the branching ratio of the massive neutrino to neutrino species $i = e, \mu$, τ is the lifetime of the massive particle species, and n_X is its abundance (number density), which of course decreases as $e^{-t/\tau}$. In deriving these equations we have assumed that the mu/tau distributions are identical; if $r_\mu \neq r_\tau$, this will no longer be true and a separate equation must be derived for tau neutrinos, which can be done in a straightforward manner. The effect of the decays of a massive particle species on the neutrino distributions will be addressed elsewhere [8].

III. Distortions of the Neutrino Distributions due to $e^\pm$ Annihilations

The master equations, Eqs. (2.10) and (2.11), are coupled, partial integro-differential equations, which are very "stiff" at high temperatures because of rapid neutrino-interaction rates, quantified by $G_F^2 T^5/H \sim (T/\,\mathrm{MeV})^3$. To integrate these equations, we have transformed them into $2N$ coupled, first-order integro-differential equations by imposing a grid of size N on neutrino momentum (more precisely, on p/T). For the results shown here $N = 60$, spanning $p/T = 1/3$ to 20 in intervals of 1/3. We then applied standard techniques for integrating stiff, first-order differential equations; see e.g., Ref. [9]. The actual numerical integrations proceeded uneventfully.

Let us begin with the standard treatment; the neutrino species have identical Fermi-Dirac distributions characterized by temperature T which, since neutrino decoupling ($T \sim$ MeV), has decreased as R^{-1}. In the present treatment, this corresponds to Δ_{ν_e}, $\Delta_{\nu_\mu} = 0$. Before discussing our results let us imagine a world in which the Fermi constant is very large, so that when $T \sim 1\,\mathrm{MeV}$ neutrinos are still strongly coupled to the electromagnetic plasma. In this hypothetical world, neutrinos would share in the electron-positron entropy transfer, which occurs at $T \sim m_e/3$, and the neutrino temperature would always be equal to the photon temperature. Returning to the definitions of Δ_{ν_e} and Δ_{ν_μ}, cf. Eqs. (2.3) and (2.4), we see that continued equilibrium with the electromagnetic plasma would imply that $\Delta_{\nu_e} = \Delta_{\nu_\mu} = \delta(t)(p/T)e^{-p/T}$ (to first order). The real Universe lies somewhere between these two extremes: At very early times ($T \gg 1\,\text{MeV}$), neutrino-interaction rates are sufficiently large so that neutrinos are tightly coupled to the electromagnetic plasma. At late times ($T \ll 1\,\text{MeV}$), neutrino interaction rates are quite small, and the neutrino distributions "freeze out." Our numerical treatment of decoupling allows us to follow the continuous change from the high-temperature, "tightly coupled" regime to the low-temperature, "decoupled" regime.

With these limits in mind, consider Fig. 3, which shows Δ_{ν_e} as a function of p/T for $T = 8\,\text{MeV}$, $4\,\text{MeV}$, and $1\,\text{MeV}$. Also shown is $(p/T)e^{-p/T}\delta(t)$, the form Δ_{ν_e} would take if electron neutrinos remained tightly coupled to the electromagnetic plasma. We see that at highest temperatures and for large neutrino momenta, Δ_{ν_e} does indeed "track." However, even at a temperature of $8\,\text{MeV}$, neutrinos with small momenta have already began to decouple; indeed, for the smallest neutrino momenta, Δ_{ν_e} is negative, corresponding to the fact that low-energy neutrinos are scattered up to higher momenta, thereby depleting low-momenta neutrinos. As the temperature drops, even for the largest momenta, Δ_{ν_e} cannot keep pace with the rising (relative) temperature of the electromagnetic plasma, and Δ_{ν_e} levels off. Figure 3 makes clear the fact that decoupling is not an instantaneous event.

It is also instructive to follow the time evolution of the neutrino distortions for several values of p/T. To that end, we define the effective temperature of the neutrino distribution:

$$T_{\text{eff}} \equiv \frac{-p}{\ln f_\nu(p,t)} = \frac{-p}{\ln[e^{-p/T} + \Delta_i(p,t)]} \simeq T\left[1 + (T/p)e^{p/T}\Delta_i(p/T,)\right]; \qquad (3.1)$$

for a Maxwell-Boltzmann equilibrium distribution, i.e., $\Delta_i = 0$, $T_{\text{eff}} = T$. Note that T_{eff} is a function of both time and momentum. Based upon the discussion above, we expect that at early times $T_{\text{eff}} = T_\gamma$ for large values of p/T, while T_{eff} should be between T and T_γ for smaller values of p/T. In Figs. 4 and 5 we show the evolution of $(T_{\text{eff}} - T)/T$ for electron and muon neutrinos and $p/T = 3, 5, 10$, and 15. Figures 4 and 5 illustrate very clearly the fact that neutrino decoupling is momentum dependent. Note too, most of the distortion develops by the time that the temperature has dropped to about $0.5\,\text{MeV}$, justifying our neglect of the electron mass in deriving the master equations.

Finally, in Figs. 6 and 7 we show the perturbations to the neutrino energy densities that arise,

$$\frac{\delta\rho_{\nu_i}}{\rho_{\nu_i}} = \frac{2\int p\,d^3p\,\Delta_i/(2\pi)^3}{\rho_{\nu_i}}. \qquad (3.2)$$

For electron neutrinos $\delta\rho_{\nu_e}/\rho_{\nu_e}$ approaches about 1.2%, while for mu/tau neutrinos it approaches about 0.5%. In Fig. 7 we show the effect that mu and tau neutrinos have on the distortion that arises in electron neutrinos: In the absence of any coupling of electron neutrinos to mu/tau neutrinos, the distortion to electron neutrinos is about 20% larger.

IV. Helium Synthesis

We have identified three ways in which the perturbations to the neutrino distributions affect the yields of primordial nucleosynthesis: The first two involve changes in the weak-interaction rates that govern the neutron fraction—due to the distorted electron-neutrino distribution and due to the decreased photon temperature—while the third involves the change in the number of neutron decays from the time the neutron fraction "freezes out" ($T \sim 0.1\,\mathrm{MeV}$) until the onset of nucleosynthesis ($T \sim 0.07\,\mathrm{MeV}$), because of the increased energy density in neutrinos (and faster expansion rate).

The changes in the primordial abundances are very small, and only the ^{4}He abundance is known well enough for its change to be of interest. We can obtain a reasonable estimate for the change in the ^{4}He abundance by simply following the evolution of the neutron fraction ($\equiv X_n$) since the mass fraction of ^{4}He synthesized ($\equiv Y$) is given by twice the neutron fraction at the epoch when nucleosynthesis commences ($T \sim 0.07\,\mathrm{MeV}$):

$$Y \simeq 2X_n(T = 0.07\,\mathrm{MeV}). \tag{4.1}$$

The Boltzmann equation governing the neutron fraction can be written as [10]

$$\begin{aligned} \dot{X}_n &= -X_n\lambda_{np} + (1 - X_n)\lambda_{pn} \\ &= -\lambda X_n + \lambda_{pn}; \end{aligned} \tag{4.2}$$

where $\lambda \equiv \lambda_{np} + \lambda_{pn}$, λ_{pn} is proton-to-neutron conversion rate (per particle), and λ_{np} is the neutron-to-proton conversion rate (per particle). Since we will only use this equation at early times ($t \ll \tau_n$), we can neglect, for the moment, neutron decays and the nuclear reactions that eventually gobble up all the neutrons into the light nuclei.

The solution to Eq. (4.2) is simple to write down:

$$X_n(t) = \int_0^t dt' \lambda_{pn}(t') f(t, t'); \tag{4.3}$$

where the integrating factor $f(t,t') = \exp[-\int_{t'}^t du\lambda(u)]$. The evolution of the neutron fraction is shown in Fig. 8. At early times it decreases, tracking its decreasing equilibrium abundance; eventually, the weak interactions that interconvert neutrons and protons become ineffective and the neutron fraction freezes out ($T_f \sim 0.1\,\mathrm{MeV}$). Light-element synthesis does not begin until the temperature drops to about $0.07\,\mathrm{MeV}$ (see Refs. [1, 2, 6]; from the time that X_n freezes out until nucleosynthesis commences, the neutron fraction decreases by about a factor of about 2/3 due to free neutron decays. For our estimates of the change in ^{4}He production we will take

$$Y \simeq 2\left(\frac{2}{3}X_n(T_f \simeq 0.1\,\mathrm{MeV})\right) \simeq 1.33X_n(T_f).$$

We will again consider the effects of neutron decays at the end of this section.

The proton-to-neutron conversion rate λ_{pn} is comprised of two terms, that for $p+e^- \rightarrow n+\nu_e$ and that for $p+\bar{\nu}_e \rightarrow n+e^+$:

$$\lambda_{pn} = \frac{1}{\lambda_0 \tau_n}\Big\{\int_Q^\infty dE(E^2-m_e^2)^{1/2}(E-Q)^2 E e^{-E/T_\gamma} + \int_{m_e}^\infty dE(E^2-m_e^2)^{1/2}(E+Q)^2 E f_{\nu_e}(E+Q)\Big\}; \tag{4.4}$$

where $Q = 1.293\,\mathrm{MeV}$ is the neutron-proton mass difference, $m_e = 0.511\,\mathrm{MeV}$ is the electron mass, T_γ is the photon temperature, and $f_{\nu_e}(E)$ is the electron-neutrino phase-space distribution. The quantity $\lambda_0 \equiv \int_{m_e}^Q dE(E^2-m_e^2)^{1/2}(E-Q)^2E$ and τ_n is the neutron mean lifetime. (Note, for simplicity and consistency we have continued our use of Maxwell-Boltzmann statistics.) The neutron-to-proton conversion rate λ_{np} is likewise comprised of two terms, that for $n+\nu_e \rightarrow p+e^-$ and that for $n+e^+ \rightarrow p+\bar{\nu}_e$:

$$\lambda_{np} = \frac{1}{\lambda_0 \tau_n}\Big\{\int_Q^\infty dE(E^2-m_e^2)^{1/2}(E-Q)^2 E f_{\nu_e}(E-Q) + \int_{m_e}^\infty dE(E^2-m_e^2)^{1/2}(E+Q)^2 E e^{-E/T_\gamma}\Big\}. \tag{4.5}$$

We are interested in the small change in the neutron fraction ($\equiv \delta X_n$) at freeze out that arises due to distortions in the neutrino distribution functions. In solving Eq. (4.2) it is most convenient to use a temperature as the independent variable, rather than time, since all the rates depend upon temperature. We find it simplest to use $z \equiv \ln R = \ln T^{-1}$ as the independent variable; recall that T is the neutrino temperature in the absence of heating by $e^\pm$ annihilations (in the Appendix we discuss an alternate choice, photon temperature).

In order to relate $\dot{X}_n$ to $X_n{}' \equiv dX_n/dz$ we must compute dz/dt:

$$\frac{dz}{dt} = H; \qquad H^2 = \frac{8\pi G\rho}{3}. \tag{4.6}$$

Further, to compute the expansion rate as a function of T, we must compute the total energy density as a function of T: $\rho(T) = \rho_{\mathrm{EM}}(T) + \rho_\nu(T)$. The first term, the energy density in the electromagnetic plasma, is only a function of T_γ, given by its equilibrium value, because rapid electromagnetic interactions keep the electromagnetic plasma in thermal equilibrium. However, we need ρ_{EM} as a function of T, not T_γ. To this end we write

$$T_\gamma(T) = T_{0\gamma}(T) + \delta T_\gamma(T); \qquad \rho_{\mathrm{EM}}(T) = \rho_{\mathrm{EM}}(T_{0\gamma}) + \delta\rho_{\mathrm{EM}}(T); \tag{4.7}$$

where $T_{0\gamma}$ is the photon temperature at a given value of the scale factor in the absence of neutrino heating by $e^\pm$ annihilations, δT_γ (<0) is the change in the photon temperature at a given value of the scale factor due to back reaction from neutrino heating, and $\delta\rho_{\mathrm{EM}}$ is the change in the electromagnetic energy density due to this back reaction. Because the electromagnetic plasma is in thermal equilibrium it follows that $\delta\rho_{\mathrm{EM}} = (d\rho_{\mathrm{EM}}/dT_\gamma)\delta T_\gamma$.

Similarly, we denote the total neutrino energy density as

$$\rho_\nu(T) = 18T^4/\pi^2 + \delta\rho_\nu(T); \tag{4.8}$$

where the first term is the neutrino energy density in the absence of heating by $e^\pm$ annihilations and the second term is the perturbation to the neutrino energy density,

$$\delta\rho_\nu = \sum_{i=e,\mu,\tau} 2\int p\, d^3p\, \Delta_i(p,t)/(2\pi)^3. \tag{4.9}$$

In the Appendix we derive the fact that $\delta\rho_{\rm EM} = -\delta\rho_\nu$, which implies that the total energy density—and expansion rate—at a given value of the scale factor is unchanged by the effect of the slight heating of neutrinos by $e^\pm$ annihilations: At fixed value of the scale factor neutrino heating by $e^\pm$ annihilations leads to slightly more energy density in neutrinos and slightly less energy density in the electromagnetic plasma. While this result seems obvious, it actually is not. Had we used the photon temperature as the independent variable, we would have found that $H(T_\gamma)$—that is the expansion rate at fixed photon temperature—is increased by the slight heating of neutrinos by $e^\pm$ annihilations for the simple reason that $\rho_{\rm EM}(T_\gamma)$ remains the same while ρ_ν increases. In addition, dz/dt (where now $z = \ln T_\gamma^{-1}$) is not simply equal to the expansion rate (see the Appendix) and changes when neutrino heating is taken into account.

We can now identify the two effects of neutrino heating on the evolution of X_n. Neither involve the expansion rate, since as a function of the scale factor it does not change; both involve perturbations to the rates λ_{pn} and λ_{np}: (i) the perturbation due to the slight decrease in the photon temperature, $\delta T_\gamma = -\delta\rho_\nu/(d\rho_{\rm EM}/dT_\gamma)$; and (ii) the perturbation due to the distorted electron-neutrino distribution, $\Delta_{\nu_e}(p,t)$.

By expanding Eq. (4.3) to first order in the changes in the weak-interaction rates we obtain the change in the neutron fraction:

$$\delta X_n(t) = \int_{-\infty}^{z(t)} dz' \frac{\lambda_{pn}(z')}{H(z')} f(z,z') \left[\frac{\delta\lambda_{pn}(z')}{\lambda_{pn}(z')} - \int_{z'}^{z} dz'' \delta\lambda(z'')/H(z'')\right]; \tag{4.10}$$

$$\delta\lambda_{pn}(t) = \frac{1}{\lambda_0\tau_n}\Big\{\int_Q^\infty dE(E^2-m_e^2)^{1/2}(E-Q)^2(E^2\delta T_\gamma/T^2)e^{-E/T} + \int_{m_e}^\infty dE(E^2-m_e^2)^{1/2}(E+Q)^2 E\Delta_{\nu_e}(E+Q,t)\Big\}; \tag{4.11}$$

$$\delta\lambda_{np}(t) = \frac{1}{\lambda_0\tau_n}\Big\{\int_Q^\infty dE(E^2-m_e^2)^{1/2}(E-Q)^2 E\Delta_{\nu_e}(E-Q,t); + \int_{m_e}^\infty dE(E^2-m_e^2)^{1/2}(E+Q)^2(E^2\delta T_\gamma/T^2)e^{-E/T}\Big\}; \tag{4.12}$$

where $z \equiv \ln T^{-1}$, $\delta T_\gamma(t) = -\delta\rho_\nu/(d\rho_{\rm EM}/dT_\gamma)$ is the perturbation to the photon temperature, and $\delta\lambda(t) \equiv \delta\lambda_{pn}(t) + \delta\lambda_{np}(t)$. Because the change in the neutron fraction is linear in the perturbed rates, we can compute separately the change due to δT_γ, denoted by δX_γ, and that due to Δ_{ν_e}, denoted by δX_ν. The evolution of $\delta X_\gamma(t)$ and $\delta X_\nu(t)$ are shown in Fig. 9.

First, consider δX_ν. At early times δX_ν is positive, and at late times δX_ν is negative. To understand this, let us consider the perturbed version of Eq. (4.2),

$$\delta \dot{X}_n = -X_n \delta\lambda_{np} + (1 - X_n)\delta\lambda_{pn} - \lambda \delta X_n; \tag{4.13}$$

where here the perturbed rates only take into account the change due to Δ_{ν_e}. The source terms that drive δX_n involve the difference between $\delta\lambda_{pn}$ times the proton fraction ($= 1 - X_n$) and $\delta\lambda_{np}$ times the neutron fraction; the final term can only decrease the neutron fraction. At early times the perturbation to the neutron fraction grows because the ratio $\delta\lambda_{pn}/\delta\lambda_{np}$ is slightly larger than $X_n/(1 - X_n)$. This is because the distortion in the electron-neutrino distribution is larger for high momentum. (In the limit that the distortion only involved neutrino momenta much greater than the neutron-proton mass difference, $\delta\lambda_{pn}/\delta\lambda_{np} \rightarrow 1$.) However, as the temperature drops, the neutron-proton mass difference becomes an insurmountable energy barrier and $\delta\lambda_{pn}/\delta\lambda_{np}$ becomes less than $X_n/(1 - X_n)$, and the source term becomes negative, so that δX_n decreases. Eventually, all the interactions that interconvert neutrons and protons become ineffective and δX_ν reaches an asymptotic value of about -1.1×10^{-4}. The predicted change in ^{4}He production is

$$\Delta Y_\nu \simeq 1.33 \delta X_\nu(T_f) \simeq -1.5 \times 10^{-4}.$$

The evolution of δX_ν is shown in Fig. 9. In Fig. 10 we show the interplay between the production term, $(1 - X_n)\delta\lambda_{pn}$, and the destruction term, $X_n \delta\lambda_{np}$. Note that each of these two terms separately would be expected to produce δX_ν of order 10^{-3}; it is their cancellation that reduces δX_ν to order 10^{-4}.

There is yet another cancellation(!): The effect of the slightly lower photon temperature is to increase the freeze-out value of the neutron fraction, by about 1.0×10^{-4}. The reason is simple to understand: The weak rates are very temperature dependent, varying as T_γ^5; decreasing the photon temperature decreases the rates for two of the processes that interconvert neutrons and protons ($e^- + p \rightarrow n + \nu_e$ and $e^+ + n \rightarrow p + \bar{\nu}_e$), thereby causing the neutron fraction to freeze out earlier and at a higher value. The increase in ^{4}He production due to the decreased photon temperature is

$$\Delta Y_\gamma \simeq 1.33 X_\gamma(T_f) \simeq 1.3 \times 10^{-4}.$$

The change in ^{4}He production when both the decreased photon temperature and perturbed electron-neutrino distribution are taken into account is

$$\Delta Y_\gamma + \Delta Y_\nu \simeq \mathcal{O}(10^{-5}). \tag{4.14}$$

The net change in very tiny because ΔY_ν and ΔY_γ almost equal and opposite. This is not completely unexpected. The coupling of neutrinos to the electromagnetic plasma leads to a slight increase in the neutrino energy density and a corresponding decrease in the energy density of the electromagnetic plasma (at fixed value of the scale factor). Since the rates that regulate the neutron fraction involve both incoming electrons/positrons and incoming electron-neutrinos/antineutrinos, they depend upon both the $e^{\pm}$ energy density and electron-neutrino energy density. Thus, the net change in the rates that interconvert neutrons and protons tend to cancel because the $e^{\pm}$ energy density decreases and the electron-neutrino energy density increases by an equal amount.

Were it not for all the cancellations, the change in the ^{4}He mass fraction due to the changes in the weak rates could well have been as large as 10^{-3} or so, which would have been much more interesting. Because of the approximations we have made—Maxwell-Boltzmann statistics and neglect of the electron mass in computing $\Delta_{\nu_e}(p,t)$—the actual values for ΔY_γ and ΔY_ν could well differ from those we obtain by 10% or so, which could lead to a larger net change in ^{4}He production. In any case, because both ΔY_γ and ΔY_ν are in magnitude order 10^{-4}, the largest value one could imagine for their sum is order 10^{-4}.

Finally, we return to the fraction of neutrons that decay from the time that the neutron fraction freezes out until nucleosynthesis commences. Nucleosynthesis commences when the *photon temperature* is about 0.07 MeV. (The photon temperature is the relevant parameter as neutrinos play no role in the actual onset of light-element synthesis.) The age of the Universe at this epoch determines the fraction of neutrons that decay. Since the Universe is radiation dominated, the age of the Universe $t_{\rm nuc} = \frac{1}{2}H^{-1}(T_\gamma \simeq 0.07\,{\rm MeV})$, which in turn is determined by the total energy density at a photon temperature of about 0.07 MeV. As we discuss in the Appendix, at *fixed photon temperature*, the total energy density is increased by amount equal to $2\delta\rho_\nu$; thus, the age of the Universe when nucleosynthesis commences is decreased by $\delta t_{\rm nuc}/t_{\rm nuc} \simeq -\delta\rho_\nu/\rho$, which decreases the number of neutron decays and increases ^{4}He production.

Without modifying the standard nucleosynthesis code to take into account neutrino heating in detail, it is difficult to make a definitive statement about the size of this effect. We can however estimate it. To wit, consider the well known effect of additional neutrino species on the predicted ^{4}He abundance: $\Delta Y_{1\nu} \simeq 0.012$ (see e.g., Refs. [1, 6, 10]). About two-thirds of this increase is due to the freezing out of the neutron fraction at a higher value and about one-third is due to the earlier onset of nucleosynthesis (owing to the greater energy density of the Universe). In this case, the perturbation to the energy density of the Universe is one full additional neutrino species; in the present circumstance, the perturbation to the energy density of the Universe is about 4% of an additional neutrino species. Based on this, we estimate the increase in ^{4}He production due to the earlier onset of nucleosynthesis to be

$$\Delta Y_{\delta t} \simeq \frac{0.012 \cdot 0.04}{3} \simeq 1.5 \times 10^{-4}; \tag{4.15}$$

which is much larger than the change in ^{4}He yield due to the change in rates, $\Delta Y_\gamma + \Delta Y_\nu \simeq$

$\mathcal{O}(10^{-5})$. That is, because the two competing changes to the ^{4}He abundance involving the rates that govern the neutron fraction almost cancel, the largest effect is due to the change in the fraction of neutrons that decay—and it is the most difficult effect to compute.

In sum, we can be confident that the change in ^{4}He production due to the slight heating of neutrinos by $e^{\pm}$ annihilations is small, $|\Delta Y| \ll 10^{-3}$; without recourse to modifying the standard nucleosynthesis code to include neutrino heating we can only estimate the change to be of the order of $1 - 2 \times 10^{-4}$.

V. Concluding Remarks

We have studied neutrino decoupling in the early Universe by numerically solving the Boltzmann equations that govern the neutrino phase-space distribution functions. We find that due to the slight heating of neutrinos by $e^{\pm}$ annihilations the current energy density of electron neutrinos is about 1.2% larger than the standard estimate, and that of mu/tau neutrinos is about 0.5% larger. This corresponds to roughly one additional relic neutrino per cm^{-3} per species (about 10^{85} additional neutrinos in the observable Universe!). Likewise, slightly less of the entropy in $e^{\pm}$ pairs is transferred to photons, so that the increase in the number of photons per comoving volume since before $e^{\pm}$ annihilations is about 0.5% less than the canonical factor of 11/4 [12].

Our work illustrates that decoupling is not an instantaneous event, and further, that it is momentum dependent. The distortions to the neutrino distributions are nonthermal: The perturbation to the effective neutrino temperature rises with momentum to almost 0.7% for electron neutrinos and about 0.3% for mu/tau neutrinos. This is explained by the fact that neutrino cross sections vary as energy squared, so that the high-momentum neutrinos remain in thermal contact with the electromagnetic plasma longer.

The perturbations to the neutrino distributions affect the primordial synthesis of ^{4}He (and the other light elements) in three ways. The first two effects involve changes in the rates of the weak-interactions that control the neutron fraction: due to the distorted electron-neutrino spectrum, and due to the slightly lower photon temperature (at fixed value of the scale factor) because of the back reaction of neutrino heating. These two effects are of opposite sign, and their net effect is a very tiny decrease in the predicted ^{4}He abundance, $\Delta Y_\gamma + \Delta Y_\nu \simeq \mathcal{O}(10^{-5})$. Were it not for the fact that these two effects nearly cancel, their net effect could have been an order of magnitude larger. The third effect is due to the increased energy density in neutrinos, which hastens the onset of nucleosynthesis, decreasing the fraction of neutrons that decay after the neutron fraction freezes out and increasing the mass fraction of ^{4}He by about $1 - 2 \times 10^{-4}$. Our estimate of the change in ^{4}He production due to all three effects is dominated by the third,

$$\Delta Y = \Delta Y_\gamma + \Delta Y_\nu + \Delta Y_{\delta t} \simeq \Delta Y_{\delta t} \simeq 1 - 2 \times 10^{-4}. \tag{5.1}$$

Finally, we should remind the reader of the approximations that we made. Throughout we have used Maxwell-Boltzmann statistics, and in computing the distortions to the neutrino distributions we have neglected the electron mass. The largest effect of neutrino heating by $e^{\pm}$ annihilations is for large neutrino momenta, where the use of Maxwell-Boltzmann statistics is a good approximation. Likewise, since the perturbations to the neutrino distributions develop at temperatures greater than the electron mass, the neglect of the electron mass is justified. In addition, we have, in an ad hoc way, reduced the rates in our master equations by a factor of $n_e(m_e \neq 0)/n_e(m_e = 0)$ to account for the Boltzmann suppression of $e^{\pm}$ pairs; the changes in our results were not significant. Because of the approximations used, the fact that two of the three effects almost cancel, and the fact that we can only estimate the most important effect, we state our estimate for the change in the primordial production of ^{4}He to one significant figure.

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References

[1] See e.g., E.W. Kolb and M.S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, CA, 1990), Ch. 5.

[2] See e.g., S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), Ch. 15.

[3] D.A. Dicus et al., *Phys. Rev. D* **26**, 2694 (1982). Also see, M.A. Herrera and S. Hacyan, *Astrophys. J.* **336**, 539 (1989); N.C. Rana and B. Mitra, *Phys. Rev. D* **44**, 393 (1991).

[4] B.E.J. Pagel, *Physica Scripta* **T36**, 7 (1991).

[5] T.P. Walker et al., *Astrophys. J.* **376**, 51 (1991).

[6] J. Bernstein, *Kinetic Theory in the Expanding Universe* (Cambridge University Press, Cambridge, 1988).

[7] For discussion of the calculation of the various matrix elements see e.g., G. 't Hooft, *Phys. Lett. B* **37**, 195 (1971); D.A. Dicus, *Phys. Rev. D* **6**, 941 (1972); C. Quigg, *Gauge Theories of the Strong, Weak, and Electromagnetic Interactions* (Addison-Wesley, Redwood City, CA, 1983), Ch. 6.

[8] Work in preparation (1992).

[9] W.H. Press, B.P. Flannery, S.A. Teukolsky, and W.T. Vetterling, *Numerical Recipes: The Art of Scientific Computing* (Cambridge University Press, Cambridge, 1985).

[10] See e.g., R.A. Alpher, J.W. Follin, and R.C. Herman, *Phys. Rev.* **92**, 1347 (1953); Refs. [1,2]; or J. Bernstein, L. Brown, and G. Feinberg, *Rev. Mod. Phys.* **61**, 25 (1989).

[11] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).

[12] Because essentially all of the entropy transfer to photons occurs before nucleosynthesis commences in earnest ($T \sim 0.07\,\mathrm{MeV}$), the slight decrease in the amount by which the number of photons per comoving volume increases due to $e^{\pm}$ annihilations does not alter the value of the baryon-to-photon ratio at nucleosynthesis relative to its present value. If it did, there would have been yet another effect of neutrino heating on nucleosynthesis since the ^{4}He yield depends upon the value of the baryon-to-photon ratio at the time of nucleosynthesis.

Appendix: Mathematical Details

(a) Phase-space integrations

The purpose of this part of the Appendix is to outline the calculation of the coefficients A_i, B_i, C_i, and C_i' in the Boltzmann equations that govern the perturbations in the neutrino phase-space distribution functions, Eqs. (2.10) and (2.11). To begin, recall that

$$d\Lambda = d\Pi_q d\Pi_{p'} d\Pi_{q'} (2\pi)^4 \delta^4(p+q-p'-q') = \frac{1}{256\pi^5}\frac{d^3q}{q}\frac{d^3p'}{p'}\frac{d^3q'}{q'}\delta^4(p+q-p'-q'). \quad (A.1)$$

Several of the terms can be evaluated by exploiting the Lorentz invariance of portions of the integrand, carrying out the $d\Pi_{p'}$ and $d\Pi_{q'}$ integrations in the CM frame, and the $d\Pi_q$ integration in the FRW frame. Those integrals are:

$$\int d\Lambda s^2 f_0(q) = \frac{p^2T^4}{\pi^3}; \quad (A.2)$$

$$\int d\Lambda t^2 f_0(q) = \int d\Lambda u^2 f_0(q) = \frac{p^2T^4}{3\pi^2}; \quad (A.3)$$

$$\int d\Lambda s^2 f_0(q)\left(\frac{p+q}{T}\right) = \frac{p^2T^4}{\pi^3}\left(\frac{p}{T}+4\right); \quad (A.4)$$

$$\int d\Lambda t^2 f_0(q)\left(\frac{p+q}{T}\right) = \int d\Lambda u^2 f_0(q)\left(\frac{p+q}{T}\right) = \frac{p^2T^4}{3\pi^3}\left(\frac{p}{T}+4\right); \quad (A.5)$$

$$f_0(p)\int d\Lambda s^2 \Delta_i(q,t) = \frac{p^2e^{-p/T}}{6\pi^3}\int_0^\infty q^3 dq \Delta_i(q,t); \quad (A.6)$$

$$f_0(p)\int d\Lambda t^2 \Delta_i(q,t) = \frac{p^2e^{-p/T}}{18\pi^3}\int_0^\infty q^3 dq \Delta_i(q,t); \quad (A.7)$$

$$f_0(p)\int d\Lambda u^2 \Delta_i(q,t) = \frac{p^2e^{-p/T}}{18\pi^3}\int_0^\infty q^3 dq \Delta_i(q,t); \quad (A.8)$$

where $f_0(p) \equiv e^{-p/T}$ and we have not used rescaled momenta.

For many of the terms this trick cannot be used and one must carry out all of the integrations in the FRW frame. For these, the key is choosing the order of integration. In the first case we use the three-momentum part of the energy-momentum delta function to carry out the $d\Pi_{q'}$ integration. After doing so we can express $d\Lambda$ as

$$d\Lambda = \frac{1}{256\pi^5}\frac{d^3q}{q}\frac{\delta(\mu-\mu_0)dp'd\mu d\phi}{|\vec{p}+\vec{q}|}; \quad (A.9)$$

where the energy part of the energy-momentum delta function has been rewritten as an angular delta function, $\mu = \cos\theta$, θ is the angle between $\vec{p}'$ and $(\vec{p}+\vec{q})$, and ϕ is the angle between the plane defined by $\vec{p}$ and $\vec{q}$ and that defined by $\vec{p}'$ and $\vec{q}'$. The quantity μ_0 is given by

$$\mu_0 = \frac{\vec{p}\cdot\vec{q} + p'(p+q) - pq}{p'|\vec{p}+\vec{q}|}; \quad (A.10)$$

the limits of the dp' integration are $p_- \le p' \le p_+$, where

$$p_\pm = \frac{(p+q) \pm |\vec{p}+\vec{q}|}{2}. \qquad (A.11)$$

The use of this technique allows us to evaluate the following:

$$\int d\Lambda s^2 f_0(q)\left(\frac{p'}{T}\right) = \frac{p^2T^4}{2\pi^3}\left(\frac{p}{T}+4\right); \qquad (A.12)$$

For the most taxing integrations we must use the momentum part of the energy-momentum delta function to carry out the $d\Pi_q$ integration; after doing so we can express $d\Lambda$ as

$$d\Lambda = \frac{1}{128\pi^4 p} dq' dy dp' d\phi d\mu \delta(\mu - \mu_0); \qquad (A.13)$$

where now $y = |\vec{p} - \vec{q}'|$, $\mu = \cos\theta$, and θ is the angle between $\vec{p}'$ and $(\vec{p} - \vec{q}')$, and ϕ is the angle between the plane defined by $\vec{p}$ and $-\vec{q}'$ and that defined by $\vec{p}'$ and $-\vec{q}$. The quantity

$$\mu_0 = \frac{y^2 - (p-q')^2 + 2p'(p-q')}{2p'y}. \qquad (A.14)$$

The limits of integration for dp' are $[y + (p-q')]/2$ to ∞, and those for dy are $|(p-q')|$ to $(p+q')$. Using this representation for $d\Lambda$ we can evaluate the remaining phase-space integrals:

$$\int d\Lambda u^2 f_0(q)\left(\frac{p-p'}{T}\right) = \frac{p^3T^3}{12\pi^3} - \frac{p^2T^4}{3\pi^3}; \qquad (A.15)$$

$$\begin{aligned}\int d\Lambda u^2 f_0(q')\Delta(p',t) &= \int d\Lambda t^2 f_0(p')\Delta(q',t) \\ &= \int d\Lambda (s^2+u^2+2us) f_0(p')\Delta(q',t);\end{aligned} \qquad (A.16)$$

$$\int d\Lambda s^2 f_0(q')\Delta(p',t) = \int d\Lambda s^2 f_0(p')\Delta(q',t); \qquad (A.17)$$

$$\int d\Lambda u^2 f_0(p')\Delta(q',t) = \frac{T^5 e^{-p/2T}}{64\pi^3(p/T)} \int_0^\infty dq' e^{q'/2T} g_1(p,q')\Delta(q',t); \qquad (A.18)$$

$$\int d\Lambda us f_0(p')\Delta(q',t) = \frac{T^5 e^{-p/2T}}{64\pi^3(p/T)} \int_0^\infty dq' e^{q'/2T} g_2(p,q')\Delta(q',t); \qquad (A.19)$$

$$\int d\Lambda s^2 f_0(p')\Delta(q',t) = \frac{T^5 e^{-p/2T}}{64\pi^3(p/T)} \int_0^\infty dq' e^{q'/2T} g_3(p,q')\Delta(q',t); \qquad (A.20)$$

where

$$g_1(p,q') = \int_{|p-q'|/T}^{(p+q')/T} dy\, e^{-y/2}[v^2 - y^2]^2; \qquad (A.21)$$

$$\begin{aligned}g_2(p,q') = &\int_{|p-q'|/T}^{(p+q')/T} dy\, e^{-y/2}\left[v^2 - y^2\right] \\ &\times \left\{y^2/2 + wy/2 + (w - v^2/2) - wv^2/2y - wv^2/y^2\right\};\end{aligned} \qquad (A.22)$$

$$g_3(p,q') = \int_{|p-q'|/T}^{(p+q')/T} dy\, e^{-y/2}\Big\{(v^2-y^2)^2[1/8 - 3wv/4y^2 + w^2v^2/8y^4] - (v^2-y^2)[y/2 + (1+v/2)]\left[(2x-3v/2) - (2x+v)wv/y^2 + w^2v^3/2y^4\right] + [y^2/4 + (1+v/2)y + (2+v^2/4)]\left[(6x^2 - 4xv + v^2/2) + (v-4x)wv^2/y^2 + w^2v^4/2y^4\right]\Big\}; \qquad (A.23)$$

here $v = (p-q')/T$, $w = (p+q')/T$, and $x = p/T$. In principle, the functions $g_i(p,q')$ can be evaluated in closed form in terms of elementary functions; however, since terms (A.18) through (A.20) must be evaluated numerically in any case, we have simply constructed a a look-up table for each. In the limit that $q' \gg p \gg T$ the $g_i(p,q')$ are simple to evaluate:

$$g_1(p,q'), -g_2(p,q'), g_3(p,q') \longrightarrow 64(q'/T)^2\, e^{-q'/2T} e^{p/2T}. \qquad (A.24)$$

The functions $g_i(p,q')$ are shown in Fig. 9.

(b) Maxwell-Boltzmann statistics

Here we review a few fundamental relationships for Maxwell-Boltzmann statistics that prove useful. In the absence of a chemical potential the equilibrium phase-space distribution function $f = e^{-E/T}$, from which it follows that the equilibrium number density n, energy density ρ, and pressure $\mathcal{P}$ of a Maxwell-Boltzmann are

$$n = \frac{g}{2\pi^2}\int_0^\infty p^2 dp e^{-E/T} = \frac{gm^3}{2\pi^2} K_2(z)/z \longrightarrow \frac{gT^3}{\pi^2}; \qquad (A.25a)$$

$$\rho = \frac{g}{2\pi^2}\int_0^\infty E p^2 dp e^{-E/T} = \frac{gm^4}{2\pi^2}\left[K_1(z)/z + 3K_2(z)/z^2\right] \longrightarrow \frac{3gT^4}{\pi^2}; \qquad (A.25b)$$

$$\mathcal{P} = \frac{g}{6\pi^2}\int_0^\infty \frac{p^4 dp}{E} e^{-E/T} = \frac{gm^4}{2\pi^2} K_2(z)/z^2 \longrightarrow \frac{\rho}{3}; \qquad (A.25c)$$

where g is the number of internal degrees of freedom of the species, $z \equiv m/T$, the $K_i(z)$ are modified Bessel functions (see e.g., Ref. [11]), and the limits shown correspond to $m/T \to 0$.

The evolution of the ratio of the photon and neutrino temperatures when one assumes that neutrinos do not share in the entropy transfer from $e^\pm$'s (canonical case) is simple to compute. The constancy of the electromagnetic entropy per comoving volume $S \equiv R^3 s$ [$s = (\rho + p)/T$ is the entropy density] implies that $S = R^3[\rho_\gamma + \mathcal{P}_\gamma + \rho_{e^\pm} + \mathcal{P}_{e^\pm}]/T =$const; using the fact that $T \propto R^{-1}(t)$, it follows

$$\frac{T_{0\gamma}}{T} = \left[\frac{3}{1 + \left[z^3K_1(z) + 4z^2K_2(z)\right]/4}\right]^{1/3} \longrightarrow 1 + \frac{1}{36}\left(\frac{m_e}{T}\right)^2; \qquad (A.26)$$

where $T_{0\gamma}$ is the photon temperature in the absence of the back reaction of neutrino heating, $z = m_e/T_\gamma$, $m_e = 0.511$ MeV is the mass of the electron, and the limit shown is for $z \to 0$. At low temperatures ($T \ll m_e$), the use of Maxwell-Boltzmann statistics leads to the prediction $T_\gamma/T = 3^{1/3} \simeq 1.44$, rather than the canonical prediction, $(11/4)^{1/3} \simeq 1.40$. The difference of the photon and neutrino temperatures around the time of nucleosynthesis is shown in Fig. 2.

(c) "Back reaction" of neutrino heating on the photon temperature

In computing the effect of the slight heating of neutrinos by $e^\pm$ annihilations on primordial nucleosynthesis in Section IV we needed to consider the back reaction of neutrino heating on the electromagnetic plasma. In doing so, we must be careful in defining all quantities, especially perturbations (perturbation with respect to what?). To begin, it proves very useful to use a temperature as the independent variable since all rates depend upon temperature rather than time. We find it most convenient to use the inverse of the cosmic-scale factor, denoted by $T \equiv R^{-1}(t)$, as the independent variable. In the absence of the slight heating of neutrinos by $e^\pm$ annihilations the neutrino temperature varies as the inverse of the cosmic-scale factor; hence T is the unperturbed neutrino temperature. At the end, we shall briefly discuss using the photon temperature as the independent variable.

Electromagnetic interactions occur very rapidly around the time of nucleosynthesis; hence, it is a good approximation to assume that the electromagnetic plasma ($e^\pm$ pairs and photons) is always in thermal equilibrium. We denote the value of the photon temperature at a given value of the cosmic-scale factor by $T_\gamma(T)$. Thermal equilibrium implies that the energy density of the electromagnetic plasma $\rho_{\rm EM}$ is only a function of T_γ, given by $\rho_\gamma + \rho_{e^\pm}$, where $\rho_\gamma = 6T_\gamma/\pi^2$ and $\rho_{e^\pm}$ is given by Eq. (A.25b) with $g = 4$. Likewise, the entropy density associated with the electromagnetic plasma $s_{\rm EM}$ is only a function of T_γ, given by $(\rho_\gamma + \mathcal{P}_\gamma)/T_\gamma + (\rho_{e^\pm} + \mathcal{P}_{e^\pm})/T_\gamma$. Since T is the independent variable, we must express the photon temperature in terms of T,

$$T_\gamma = T_{0\gamma}(T) + \delta T_\gamma(T); \qquad (A.27)$$

where $T_{0\gamma}(T)$ is the photon temperature in the absence of neutrino heating by $e^\pm$ annihilations (the evolution of $T_{0\gamma}/T$ and $\delta T_\gamma/T$ are shown in Fig. 2). Using these definitions and expanding $s_{\rm EM}$ and $\rho_{\rm EM}$ to first order in δT_γ it follows that

$$\rho_{\rm EM}(T_\gamma) = \rho_{\rm EM}(T_{0\gamma}) + \delta\rho_{\rm EM}(T); \qquad s_{\rm EM}(T_\gamma) = s_{\rm EM}(T_{0\gamma}) + \delta s_{\rm EM}(T); \qquad (A.28a)$$

$$\delta\rho_{\rm EM}(T) = \frac{d\rho_{\rm EM}}{dT}\delta T_\gamma; \qquad \delta s_{\rm EM}(T) = \frac{ds_{\rm EM}}{dT}\delta T_\gamma. \qquad (A.28b)$$

Finally, we write the energy density in neutrinos (summed over all three species) as

$$\rho_\nu(T) = \rho_{0\nu}(T) + \delta\rho_\nu(T); \qquad (A.29)$$

where $\rho_{0\nu}(T) = 18T^4/\pi^2$ is the energy density in neutrinos in the absence of the slight heating by $e^\pm$ annihilations, and

$$\delta\rho_\nu = \sum_{i=e,\mu,\tau} 2\int \Delta_i(p,t)p\, d^3p/(2\pi)^3.$$

We define the entropy density associated with neutrinos,

$$
\begin{aligned}
s_\nu(T) &\equiv \frac{\rho_\nu + \mathcal{P}_\nu}{T}; \\
&\equiv s_{0\nu}(T) + \delta s_\nu(T); \\
&= \frac{4}{3}\frac{\rho_{0\nu}}{T} + \frac{4}{3}\frac{\delta\rho_\nu}{T};
\end{aligned} \tag{A.30}
$$

where the final expression follows from the fact that neutrinos are ultrarelativistic, so that $\mathcal{P}_\nu = \rho_\nu/3$ always. While $s_{0\nu}$ is the entropy density associated with neutrinos in the absence of heating by $e^\pm$ pairs, the expression for δs_ν is merely a useful definition.

Our goal is to solve for δT_γ in terms of $\delta\rho_\nu$. Recall, δT_γ is the temperature difference at a fixed value of the cosmic-scale factor when the slight heating of neutrinos is taken into account; intuition strongly suggests that δT_γ is negative—which is what we find. The starting point is the first-law of thermodynamics,

$$
d[R^3\rho] = -\mathcal{P}dR^3; \tag{A.31a}
$$

$$
d[(\rho + \mathcal{P})/T^3] = d\mathcal{P}/T^3; \tag{A.31b}
$$

where the second, more useful expression follows from the first and fact that $R(t) \equiv T^{-1}$, $\rho = \rho_{\rm EM} + \rho_\nu$ is the total energy density, and $\mathcal{P} = \mathcal{P}_{\rm EM} + \mathcal{P}_\nu$ is the total pressure. There is another very useful identity that applies to a system (or subsystem) that is in thermal equilibrium:

$$
\frac{d\mathcal{P}}{dT} = \frac{\rho + \mathcal{P}}{T}. \tag{A.32}
$$

Expression (A.32) always applies to the electromagnetic plasma and to neutrinos in the absence of $e^\pm$ heating, because $\rho_{0\nu} \propto T^4$.

In the absence of neutrino heating by $e^\pm$ annihilations the entropy per comoving volume ($S \propto R^3 s$) in the electromagnetic plasma and in the neutrinos are separately conserved:

$$
d[R^3 s_{\rm EM}(T_{0\gamma})] \quad = \quad d[R^3 s_{0\nu}] \quad = \quad 0. \tag{A.33}
$$

Using these facts, and Eqs. (A.31b) and (A.32), it is straightforward to obtain the relations that we are seeking,

$$
\delta\rho_{\rm EM}(T) = -\delta\rho_\nu(T); \tag{A.34a}
$$

$$
\delta T_\gamma(T) = -\frac{\delta\rho_\nu(T)}{d\rho_{\rm EM}(T_\gamma)/dT_\gamma}; \tag{A.34b}
$$

where the expression $d\rho_{\rm EM}/dT_\gamma$ is obtained by differentiating the equilibrium expression for $\rho_{\rm EM}$:

$$
\frac{d\rho_{\rm EM}}{dT_\gamma} = \frac{d}{dT_\gamma}\left\{\frac{6T_\gamma^4}{\pi^2} + \frac{2m_e^3 T_\gamma}{\pi^2}[K_1(m_e/T) + 3K_2(m_e/T_\gamma)/(m_e/T_\gamma)]\right\}. \tag{A.35}
$$

Through essentially all of the epoch of interest $e^{\pm}$'s are relativistic. This implies $\rho_{e^\pm} \approx 12T_\gamma^4/\pi^2$, so that $\rho_{\rm EM} \approx 18T_\gamma^4/\pi^2$ and $\delta T_\gamma/T_\gamma \approx -\frac{1}{2}(\delta\rho_\nu/\rho)$, which is of the order of -0.2%. While the expressions we obtained for $\delta\rho_{\rm EM}$ and δT_γ took a bit of effort to derive, their physical content is simple to understand: The energy delivered to neutrinos by $e^{\pm}$ annihilations is taken away from the electromagnetic plasma.

According to the usual treatment, the number of photons per comoving volume, $N_\gamma = R^3 n_\gamma$, increases by a factor of 3 (11/4 when the proper statistics are used) from the epoch before $e^{\pm}$ annihilations take place until the present epoch. When the slight heating of neutrinos is taken into account the factor is slightly less than 3, and is given by

$$\begin{aligned}
\frac{N_\gamma(\text{today})}{N_\gamma(t \ll \text{sec})} &= \frac{R^3(\text{today})T_\gamma^3(\text{today})}{R^3(t \ll \text{sec})T_\gamma^3(t \ll \text{sec})}; \\
&= \left(\frac{T_{0\gamma}}{T}\right)^3 \left(1 + \frac{\delta T_\gamma}{T_{0\gamma}}\right)^3 \bigg|_{\text{today}}; \\
&= 3\left(1 - \frac{3^{2/3}}{4}\frac{\delta\rho_\nu}{\rho_\nu}\right)\bigg|_{\text{today}}; \\
&\simeq 3\left[1 - \mathcal{O}(0.5\%)\right]; \qquad (A.36)
\end{aligned}$$

where we have used the fact that today, $\rho_{\rm EM} = 6T_\gamma^4/\pi^2$ and $d\rho_{\rm EM}/dT_\gamma = 4\rho_{\rm EM}/T_\gamma$. Since the asymptotic value of $\delta\rho_\nu/\rho_\nu$ is about 0.7%, due to the slight sharing of the entropy transfer of $e^{\pm}$ pairs with neutrinos, the number of photons per comoving volume has increased since before $e^{\pm}$ annihilations by about 0.5% less than the canonical estimate.

Finally, let us end this part of the Appendix by briefly discussing how things change—and become much more complicated—when one uses the photon temperature T_γ as the independent variable rather than $T = R^{-1}(t)$. With this choice, there is no perturbation to the photon temperature, nor to the energy density in the electromagnetic plasma. However, the energy density in neutrinos is now more difficult to compute. It is given by

$$\rho_\nu(T_\gamma) = 18T^4(T_\gamma)/\pi^2 + \delta\rho_\nu; \qquad (A.37)$$

where the first term is the energy density in neutrinos in the absence of heating by $e^{\pm}$ annihilations, and the second term is the perturbation due to heating. Here $T(T_\gamma)$ is the value of the neutrino temperature (in the absence of heating), evaluated at photon temperature T_γ; it is related to T_γ by Eq. (2.8),

$$T(T_\gamma) = (1-\delta)T_\gamma = (1-\delta_0)T_\gamma - \delta T_\gamma = (1-\delta_0)\left[1 - \frac{\delta T_\gamma}{T}\right]T_\gamma. \qquad (A.38)$$

Thus, the energy density in neutrinos is given by

$$\rho_\nu = \rho_{0\nu}\left(1 - 4\frac{\delta T_\gamma}{T}\right) + \delta\rho_\nu \approx \rho_{0\nu} + 2\delta\rho_\nu; \qquad (A.39)$$

where $\rho_{0\nu} = 18[(1-\delta_0)T_\gamma]^4/\pi^2$ is the functional form that applies in the absence of neutrino heating, and in the final expression we have used $\delta T_\gamma/T \approx -\frac{1}{4}(\delta\rho_\nu/\rho_\nu) \simeq -2\times 10^{-3}$. The change in the total energy density *at fixed photon temperature* is approximately twice that due to the distortion of the neutrino spectra alone.

The expansion rate at a given value of the photon temperature is related to the total energy density, $H^2(T_\gamma) = 8\pi G\rho(T_\gamma)/3$. When neutrino heating is taken into account, the energy density at a fixed value of the photon temperature is

$$\rho(T_\gamma) = \rho_{\rm EM}(T_\gamma) + \rho_{0\nu}(T_\gamma) + 2\delta\rho_\nu(T_\gamma); \tag{A.40}$$

which is larger by the amount $2\delta\rho_\nu$ than in the absence of neutrino heating: The expansion rate *at fixed photon temperature* is increased by neutrino heating:

$$\frac{\delta H}{H} \approx \frac{\delta\rho_\nu}{\rho}.$$

Recall, there is no change in the expansion rate *at fixed value of the scale factor* due to neutrino heating. The difference in these two results is explained by the fact that when neutrino heating is taken into account, the value of the photon temperature at a given value of the scale factor is smaller.

As we discuss in Section IV, in order to transform the time derivative in the rate equation that governs the neutron fraction into a derivative with respect to the independent variable $z \equiv \ln T_\gamma^{-1}$ we must compute the quantity dz/dt. Using the first-law of thermodynamics and the following definitions

$$g_{*\rho}(T_\gamma) \equiv \frac{\rho}{3T_\gamma^4/\pi^2}; \qquad g_{*p}(T_\gamma) \equiv \frac{\mathcal{P}}{T_\gamma^4/\pi^2};$$

it is straightforward to show that

$$\frac{dz}{dt} = \frac{H\,(3/4 + g_{*p}/4g_{*\rho})}{1 - d\ln g_{*\rho}/dz}; \tag{A.41}$$

which is approximately equal to the expansion rate—and changes when the slight heating of neutrinos due to $e^\pm$ annihilations is taken into account. For comparison, when we defined $z \equiv \ln T^{-1}$, $dz/dt = H(z)$, which does not change due to the slight heating of neutrinos by $e^\pm$ annihilations.

Finally, consider the small change in dz/dt when neutrino heating is taken into account; it is simple to show that

$$\frac{\delta(dz/dt)}{dz/dt} \approx \frac{\delta\rho_\nu}{\rho} + 2\left(\frac{\rho_\nu}{\rho}\right)\frac{d}{dz}\left(\frac{\delta\rho_\nu}{\rho_\nu}\right). \tag{A.42}$$

The first term is just due to the change in the expansion rate (and of course is positive); the second term is an additional term, which is also positive. If we were to use T_γ as the independent variable in calculating the small change in the neutron fraction due to neutrino heating, there would be three effects: first, that due to the change in the weak-interaction rates from the higher neutrino temperature at fixed photon temperature, $T(T_\gamma) = (1 - \delta)(1 - \delta T_\gamma / T_\gamma) T_\gamma$; second, that due to the change in the weak-interaction rates from the distortion of the electron-neutrino distribution; and third, that due to the change in dz/dt. Of course, in the final analysis, the value obtained for the change in the ^{4}He abundance must agree with that computed by using $\ln T^{-1}$ as the independent variable.

Table 1: Scattering and annihilation processes involving electron neutrinos; the four momentum of the incoming electron neutrino is denoted by p; the four momentum of the other incoming particle is q; the four momentum of the outgoing ν_e (or lepton) is p'; and the four momentum of the outgoing anti-lepton is q' (see Fig. 1). Muon and tau neutrinos are denoted by ν_i ($i = \mu, \tau$). The invariants s, t, and u are defined by: $s = (p+q)^2 \simeq 2p \cdot q$, which is the total energy squared in the center-of-mass frame (CM); $t = (p-p')^2 \simeq -2p \cdot p'$ is the four-momentum transfer between the incoming electron neutrino and outgoing lepton; and $u = (p-q')^2 \simeq -2p \cdot q'$ is the four-momentum transfer between the incoming electron neutrino and outgoing anti-lepton. In computing the matrix-elements squared we have assumed that all leptons are ultrarelativistic, which implies that $s+t+u \simeq 0$; $G_F \simeq 1.17 \times 10^{-5}\,\mathrm{GeV}^{-2}$ is the Fermi constant, $a = (2\sin^2\theta_W + 1)^2 \simeq 2.13$, $b = (2\sin^2\theta_W)^2 \simeq 0.212$, and $\sin^2\theta_W \simeq 0.23$. Both neutral- and charged-current interactions have been included.

PROCESS	$S \sum_{\mathrm{spin}} \lvert\mathcal{M}\rvert^2$
Annihilation	
$\nu_e + \bar{\nu}_e \to e^- + e^+$	$8G_F^2(bt^2 + au^2)$
$\nu_e + \bar{\nu}_e \to \nu_i + \bar{\nu}_i$	$8G_F^2 u^2$
Scattering	
$\nu_e + e^- \to \nu_e + e^-$	$8G_F^2(as^2 + bu^2)$
$\nu_e + e^+ \to \nu_e + e^+$	$8G_F^2(bs^2 + au^2)$
$\nu_e + \nu_e \to \nu_e + \nu_e$	$8G_F^2 s^2$
$\nu_e + \bar{\nu}_e \to \nu_e + \bar{\nu}_e$	$8G_F^2(4u^2)$
$\nu_e + \nu_i \to \nu_e + \nu_i$	$8G_F^2 s^2$
$\nu_e + \bar{\nu}_i \to \nu_e + \bar{\nu}_i$	$8G_F^2 u^2$

Table 2: Same as Table 1, except for mu/tau neutrinos. The four momenta are denoted in the analogous manner: p is the four momentum of the incoming ν_i; q is the four momentum of the other incoming particle; p' is the four momentum of the outgoing ν_i (or lepton); and q' is the four momentum of the outgoing particle that scatters with the ν_i (or anti-lepton) (see Fig. 1); $s = (p+q)^2$, $t = (p-p')^2$, $u = (p-q')^2$, $i, j = \mu, \tau$, $i \neq j$, and $c = (2\sin^2\theta_W - 1)^2 \simeq 0.292$.

PROCESS	$S\sum_{\text{spin}} \lvert\mathcal{M}\rvert^2$
Annihilation	
$\nu_i + \bar{\nu}_i \to e^- + e^+$	$8G_F^2(bt^2 + cu^2)$
$\nu_i + \bar{\nu}_i \to \nu_e + \bar{\nu}_e$	$8G_F^2u^2$
$\nu_i + \bar{\nu}_i \to \nu_j + \bar{\nu}_j$	$8G_F^2u^2$
Scattering	
$\nu_i + e^- \to \nu_i + e^-$	$8G_F^2(cs^2 + bu^2)$
$\nu_i + e^+ \to \nu_i + e^+$	$8G_F^2(bs^2 + cu^2)$
$\nu_i + \nu_e \to \nu_i + \nu_e$	$8G_F^2s^2$
$\nu_i + \bar{\nu}_e \to \nu_i + \bar{\nu}_e$	$8G_F^2u^2$
$\nu_i + \nu_i \to \nu_i + \nu_i$	$8G_F^2s^2$
$\nu_i + \nu_j \to \nu_i + \nu_j$	$8G_F^2s^2$
$\nu_i + \bar{\nu}_i \to \nu_i + \bar{\nu}_i$	$8G_F^2(4u^2)$
$\nu_i + \bar{\nu}_j \to \nu_i + \bar{\nu}_j$	$8G_F^2u^2$

Figure Captions

Fig. 1: The labeling of four momenta for neutrino interactions, cf. Tables 1 and 2, and our definitions of the Mandelstam variables s, t, and u.

Fig. 2: The evolution of the temperature difference between neutrinos and photons, $\delta_0(t) \equiv (T_{0\gamma} - T)/T$, assuming that neutrinos *do not* participate in the $e^{\pm}$ entropy transfer (solid curve), and taking into account the slight back reaction of neutrino heating on the photon temperature, $\delta(t) = (T_\gamma - T)/T$ (broken curve). The small correction to the photon temperature, $\delta T_\gamma/T$ is also shown. Note since we have used Maxwell-Boltzmann statistics, today $T_\gamma/T = 3^{1/3}$ rather than $(11/4)^{1/3}$.

Fig. 3: The perturbation to the electron-neutrino phase-space distribution, $\Delta_{\nu_e}(p/T, t)$, for $T = 8\,\mathrm{MeV}$, $4\,\mathrm{MeV}$, and $1\,\mathrm{MeV}$. The broken curves show the perturbation that would result if electron neutrinos maintained good thermal contact with the electromagnetic plasma, in which case $\Delta_{\nu_e} = (p/T)e^{-p/T}\delta$. For very small values of p/T, Δ_{ν_e} is negative.

Fig. 4: The evolution of the effective neutrino temperature, $(T_{\rm eff} - T)/T$, for neutrino momenta $p/T = 3, 5, 10, 15$. The photon-neutrino temperature difference $\delta(t) \equiv (T_\gamma - T)/T$ is also shown (broken curve). "Electron-neutrino decoupling" occurs at a temperature of around $2\,\mathrm{MeV}$, though these curves very graphically illustrate that the decoupling process is not instantaneous and is momentum dependent.

Fig. 5: Same as Fig. 4, except for mu/tau neutrinos. "Decoupling" for mu/tau neutrinos occurs at a temperature between $3\,\mathrm{MeV}$ and $4\,\mathrm{MeV}$.

Fig. 6: The evolution of $\delta\rho_\nu/\rho_\nu$ for electron neutrinos (solid curve) and mu/tau neutrinos (broken curve). Asymptotically, $\delta\rho_{\nu_e}/\rho_{\nu_e} \to 1.2\%$, and $\delta\rho_{\nu_\mu}/\rho_{\nu_\mu} \to 0.5\%$. This corresponds to roughly one additional relic neutrino per cm^{-3} per species.

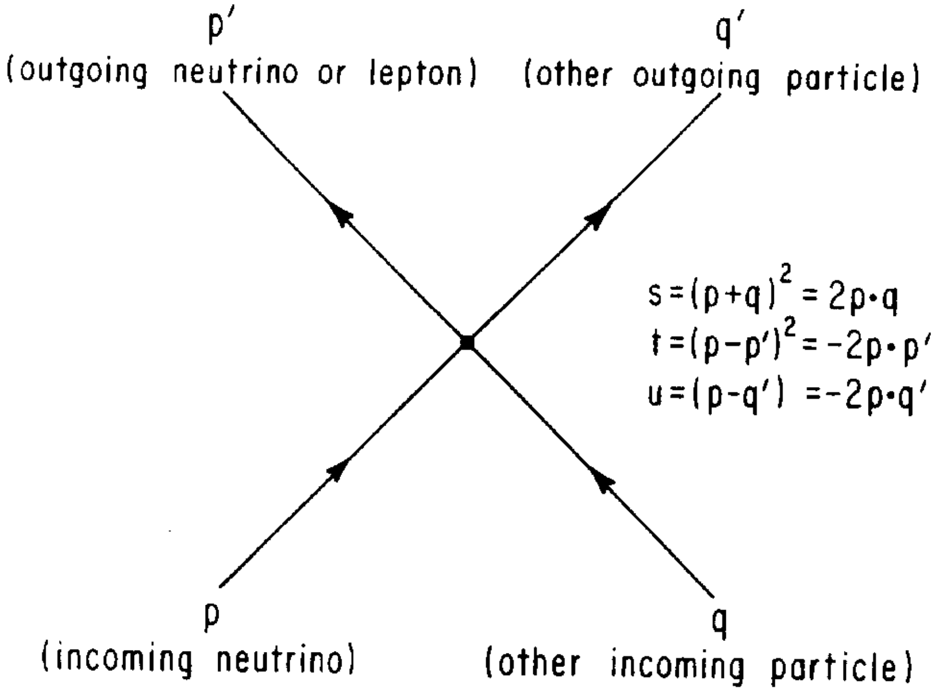
Fig. 7: The evolution of $\delta\rho_{\nu_e}/\rho_{\nu_e}$ with (solid curve) and without (broken curve) the coupling of ν_e's to ν_μ's and ν_τ's. The coupling of the mu/tau neutrinos to the electron neutrinos does not significantly alter the heating of electron neutrinos by $e^{\pm}$ annihilations.

Fig. 8: The evolution of the neutron fraction in the standard scenario; at a temperature of about $0.1\,\mathrm{MeV}$ the neutron fraction has frozen out at a value of about 0.2.

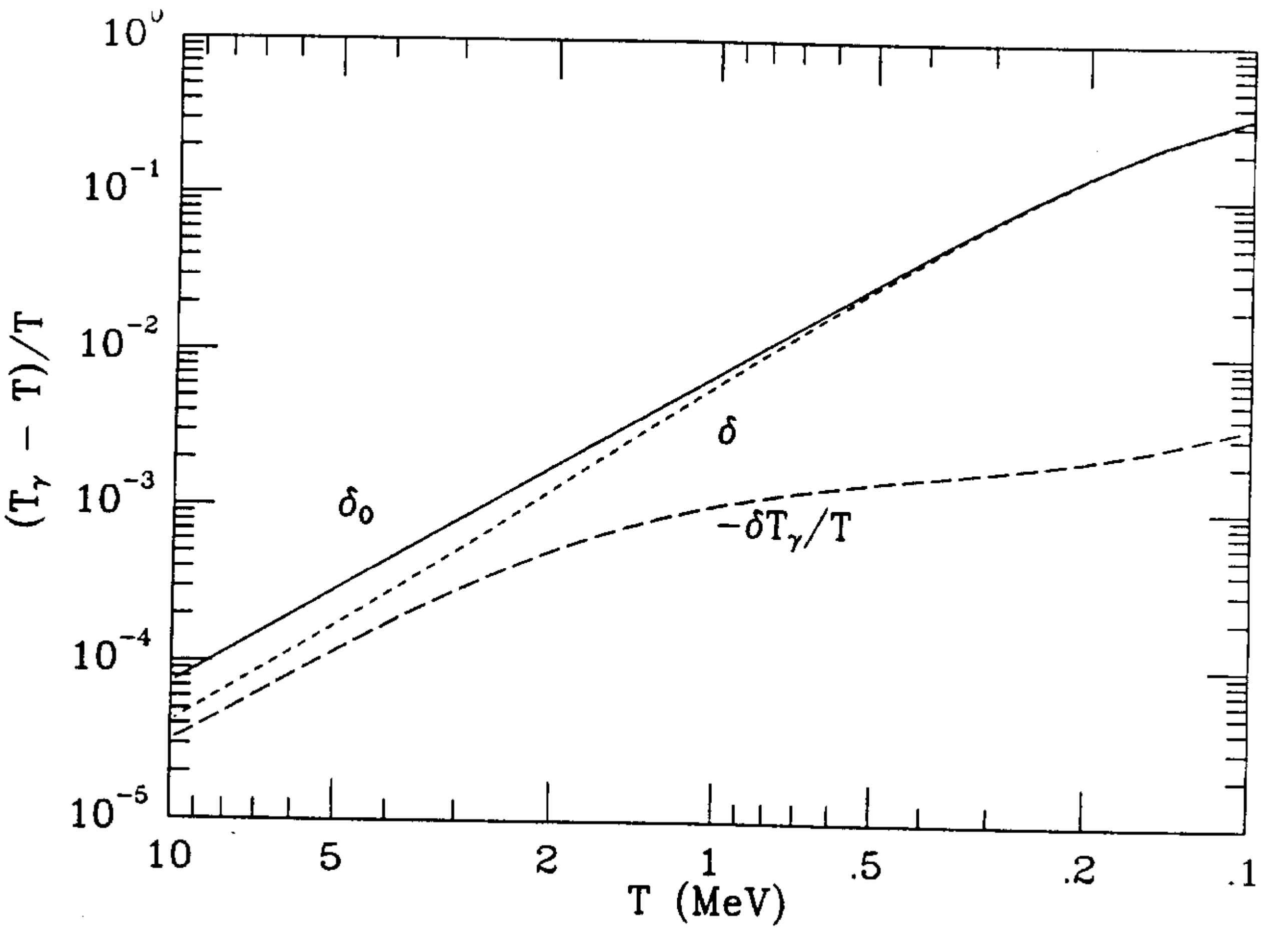
Fig. 9: The evolution of $\delta X_\gamma(t)$, the change due to the decrease in the photon temperature, and $\delta X_\nu(t)$, the change due to the distortion in the electron-neutrino distribution.

Fig. 10: The neutron production, $(1 - X_n)\delta\lambda_{pn}$, and destruction, $X_n\delta\lambda_{np}$, terms in Eq. (4.13).

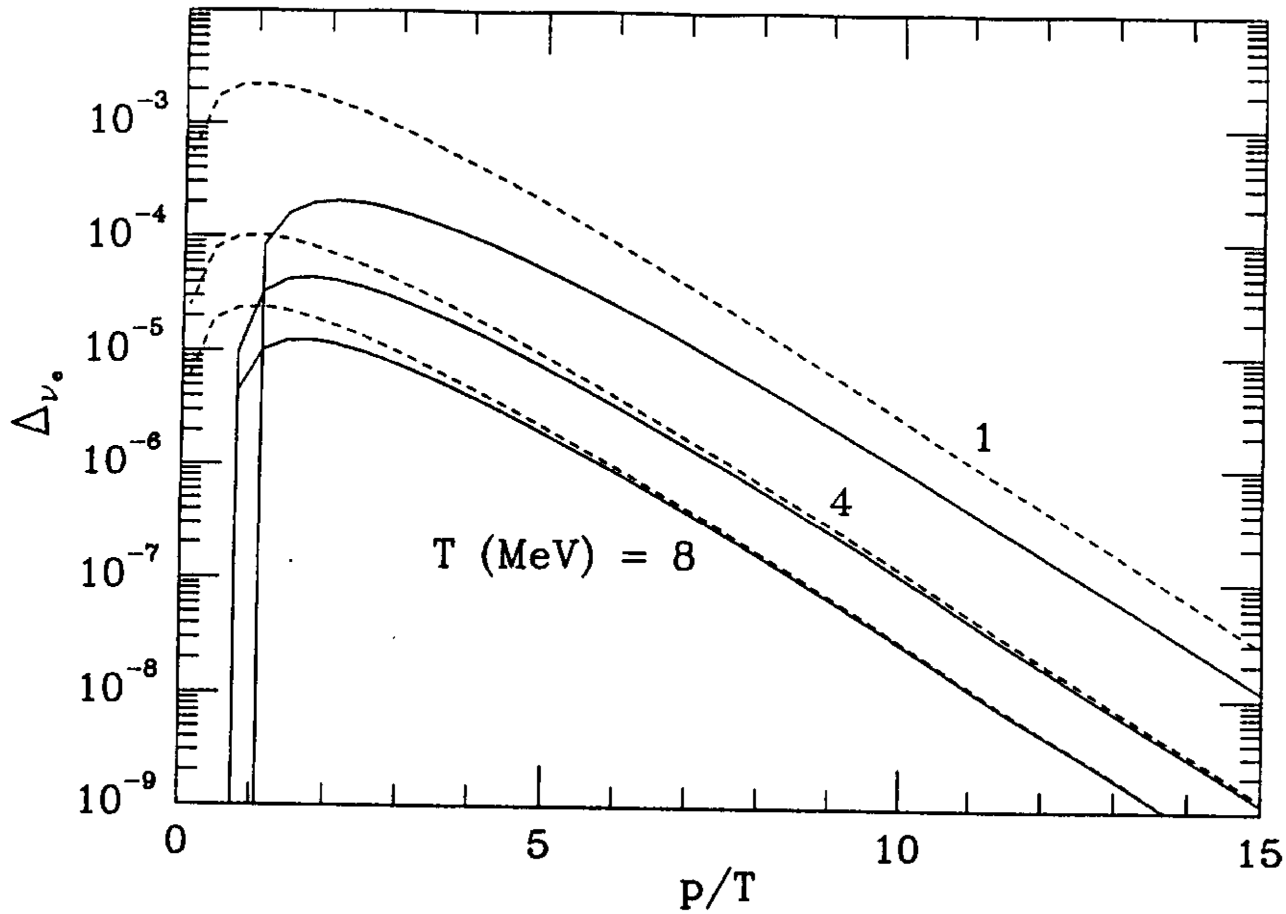
Fig. 11: The functions $g_i(p, q')$, cf. Eqs. (A21) - (A.24): (a) $g_1(p, q')/64(q'/T)^2 e^{(p-q')/2T}$; (b) $-g_2(p, q')/64(q'/T)^2 e^{(p-q')/2T}$; and (c) $g_3(p, q')/64(q'/T)^2 e^{(p-q')/2T}$.


p′
(outgoing neutrino or lepton)
q′
(other outgoing particle)
s=(p+q)^2=2p·q
t=(p−p′)^2=−2p·p′
u=(p−q′) =−2p·q′
p
(incoming neutrino)
q
(other incoming particle)

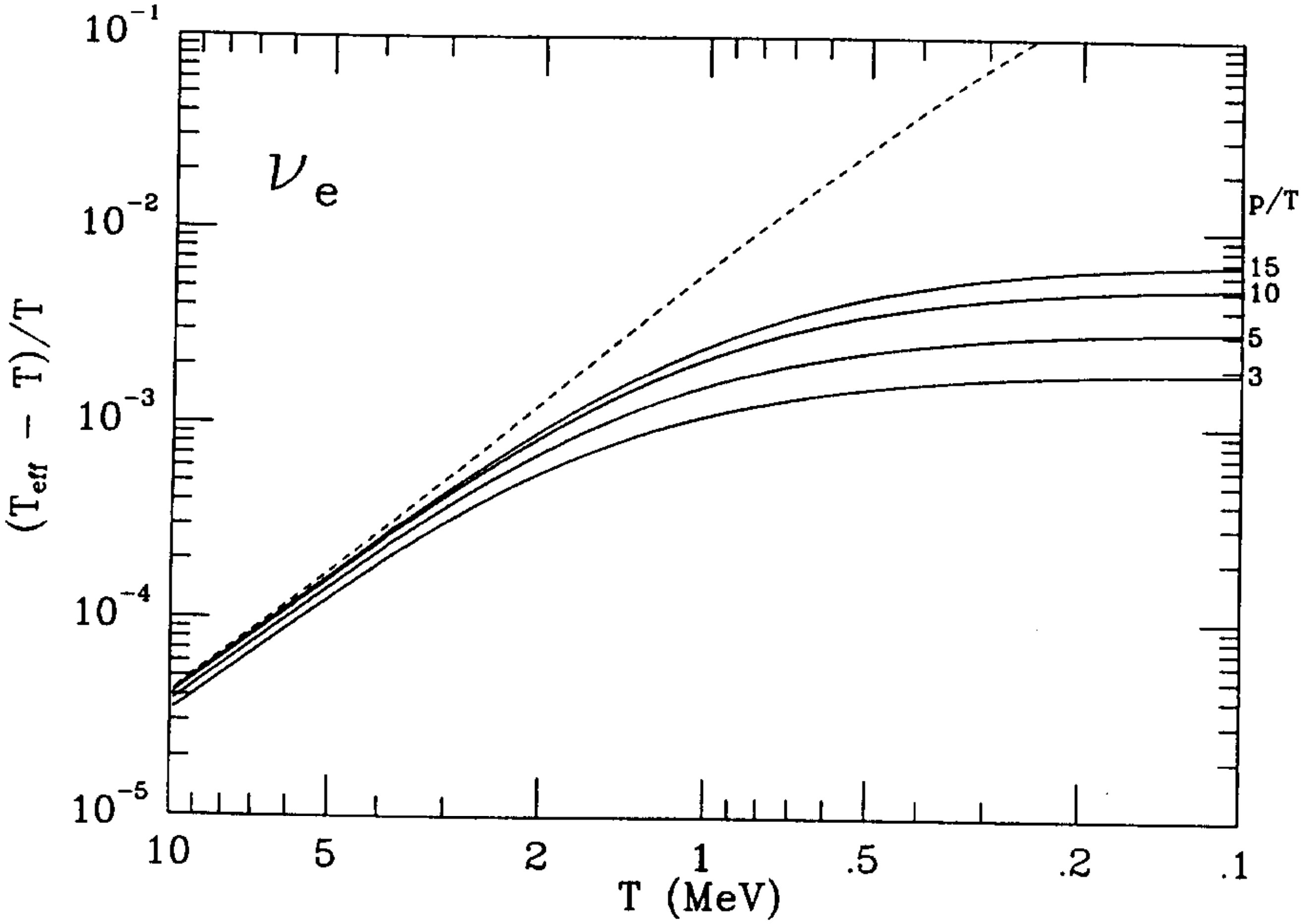
-FIG 1-

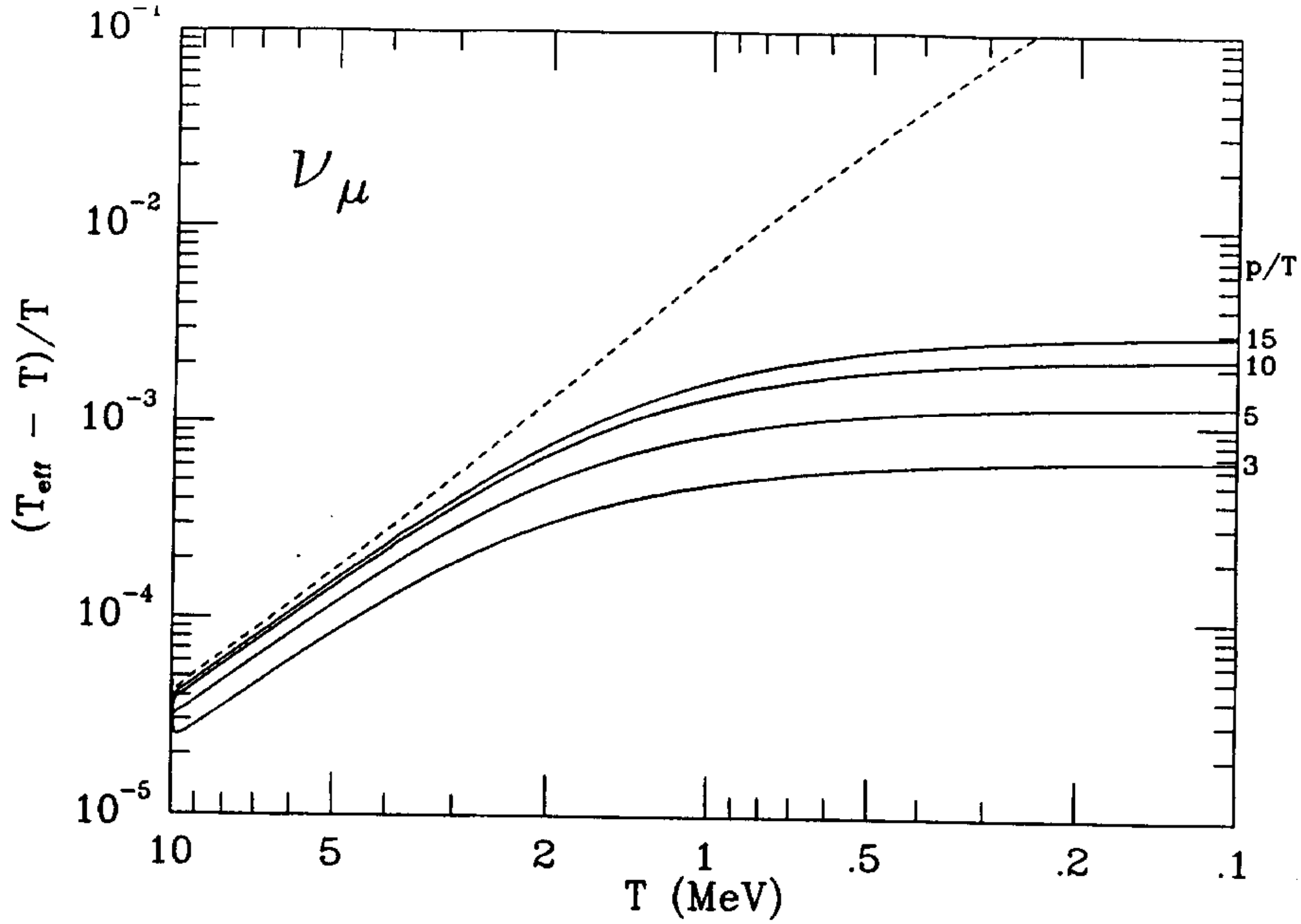

10^0
10^-1
10^-2
10^-3
10^-4
10^-5
(T_γ − T)/T
δ_0
δ
−δT_γ/T
10
5
2
1
.5
.2
.1
T (MeV)

-FIG 2-

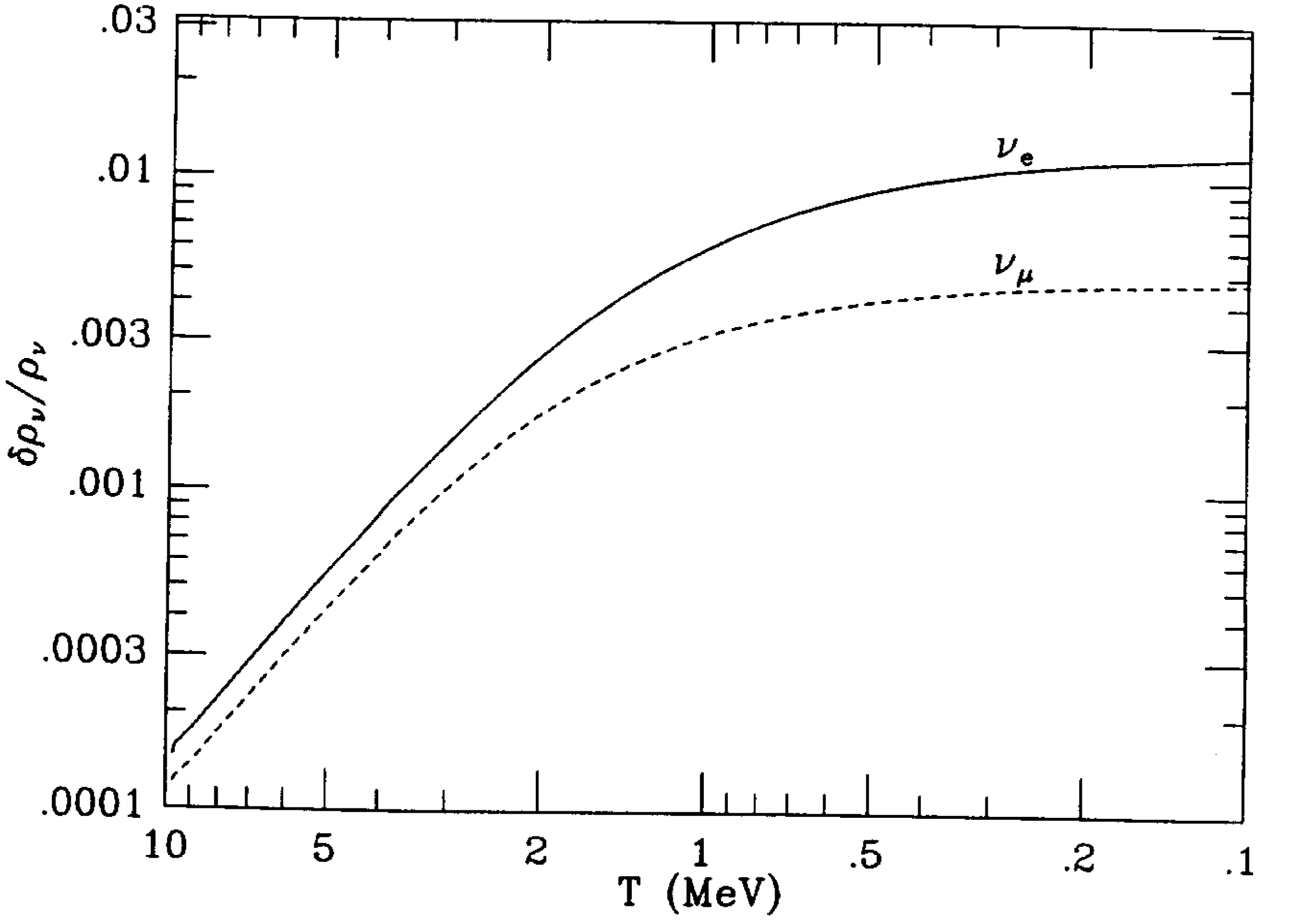

10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
Δ_{ν_e}
1
4
T (MeV) = 8
0
5
10
15
p/T

-FIG 3-

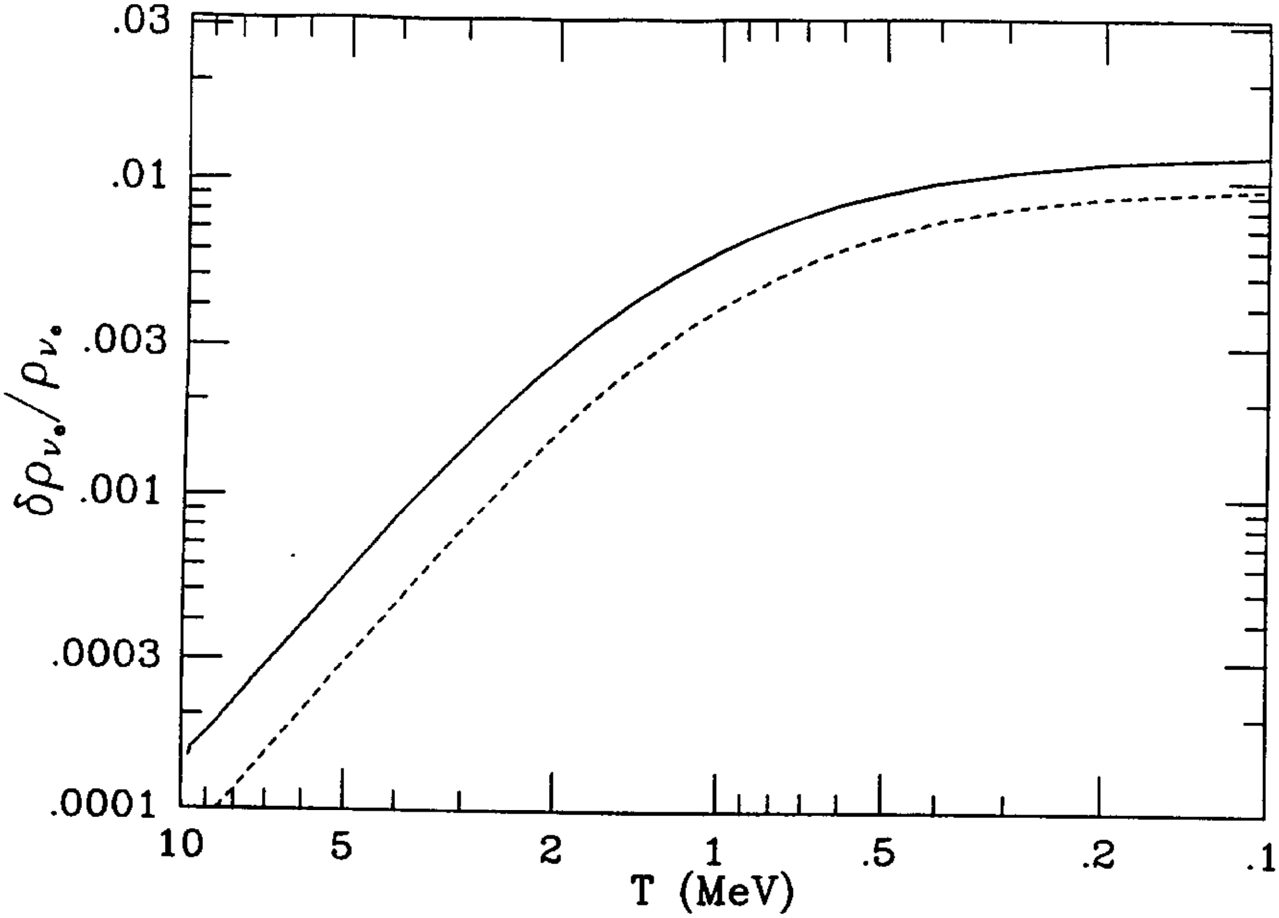

10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
ν_e
$(T_{eff} - T)/T$
p/T
15
10
5
3
10
5
2
1
.5
.2
.1
T (MeV)



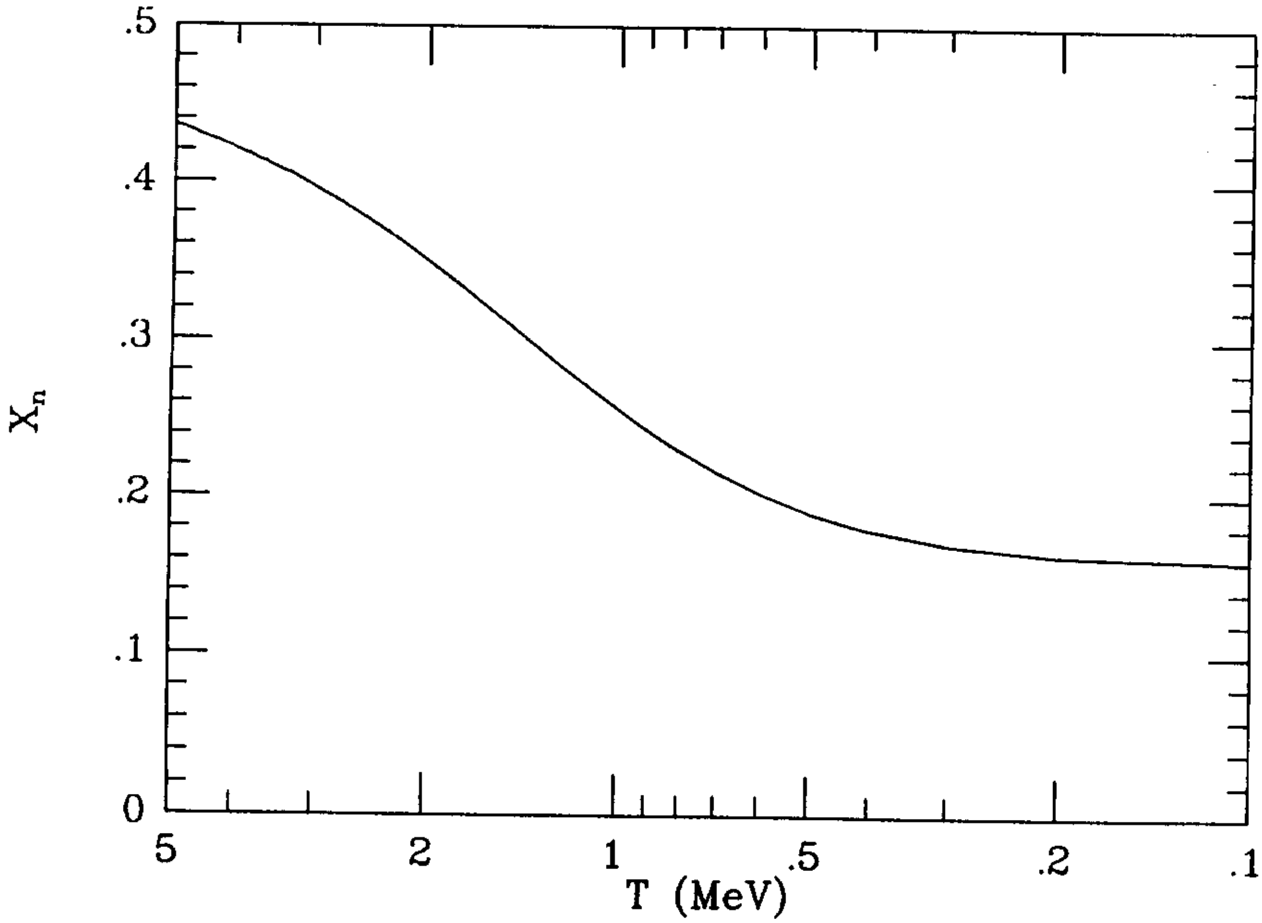
-FIG5-

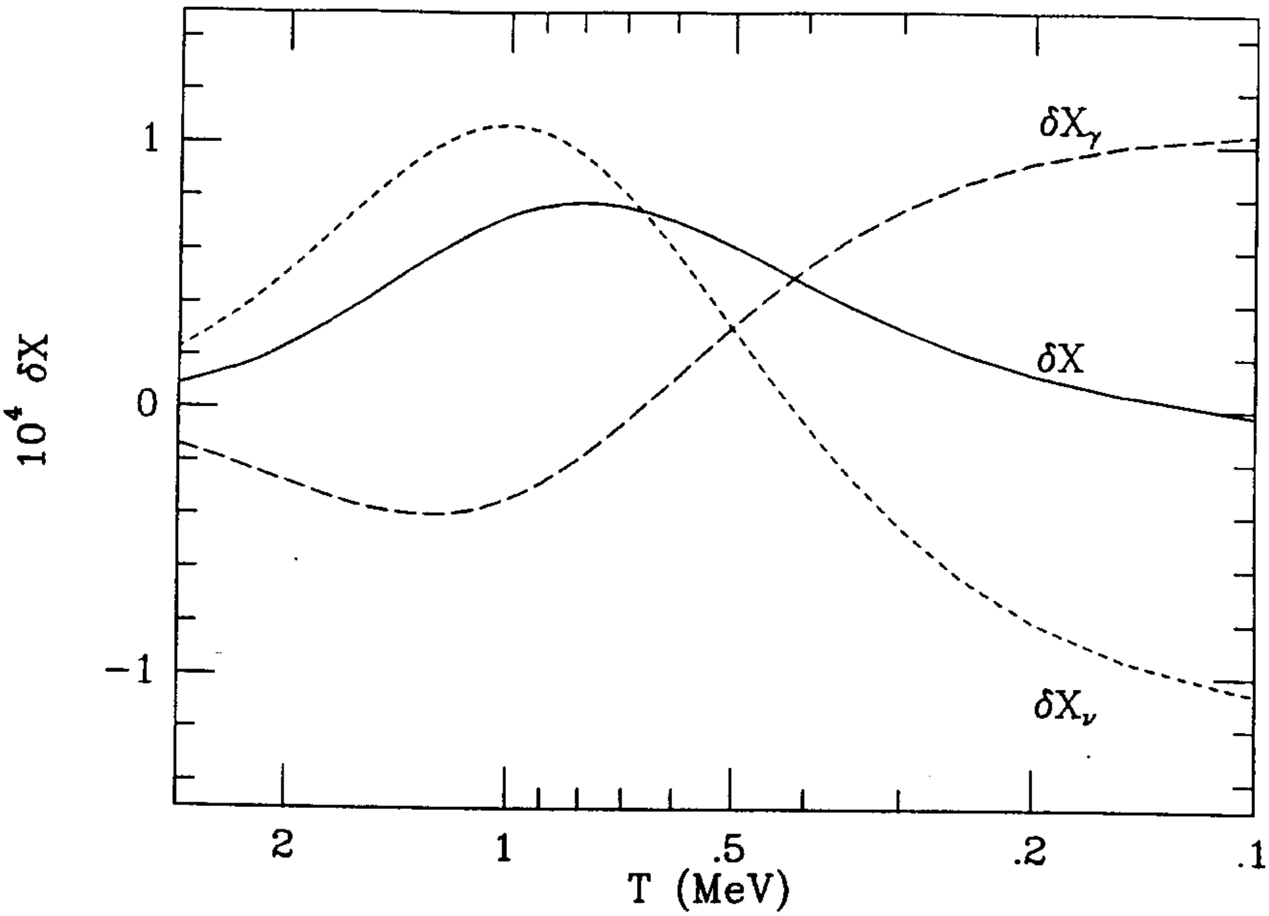


-FIG6-


.03
.01
.003
.001
.0003
.0001
$\delta\rho_{\nu_e}/\rho_{\nu_e}$
10
5
2
1
.5
.2
.1
T (MeV)

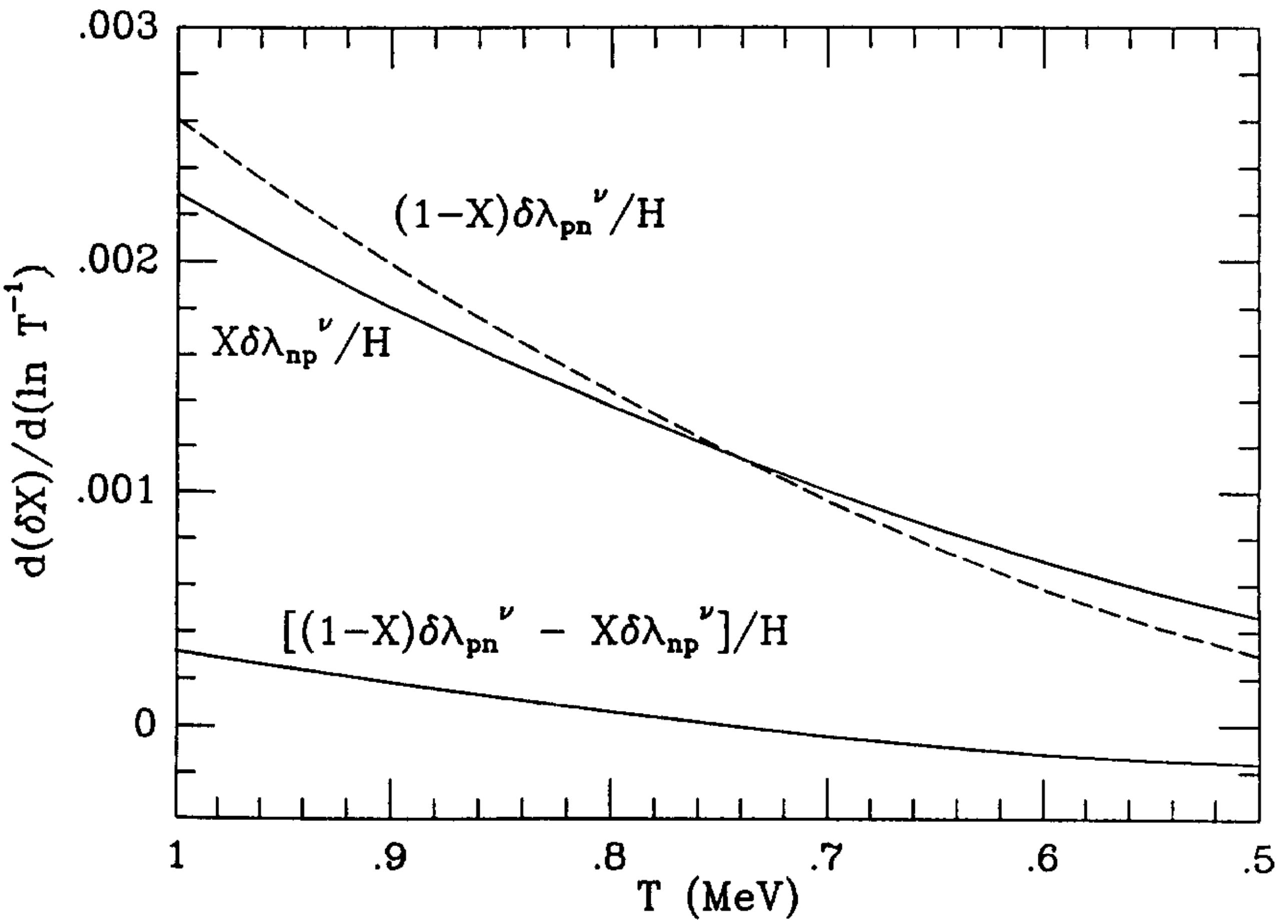
-FIG7-


.5
.4
.3
.2
.1
0
X_n
5
2
1
.5
.2
.1
T (MeV)



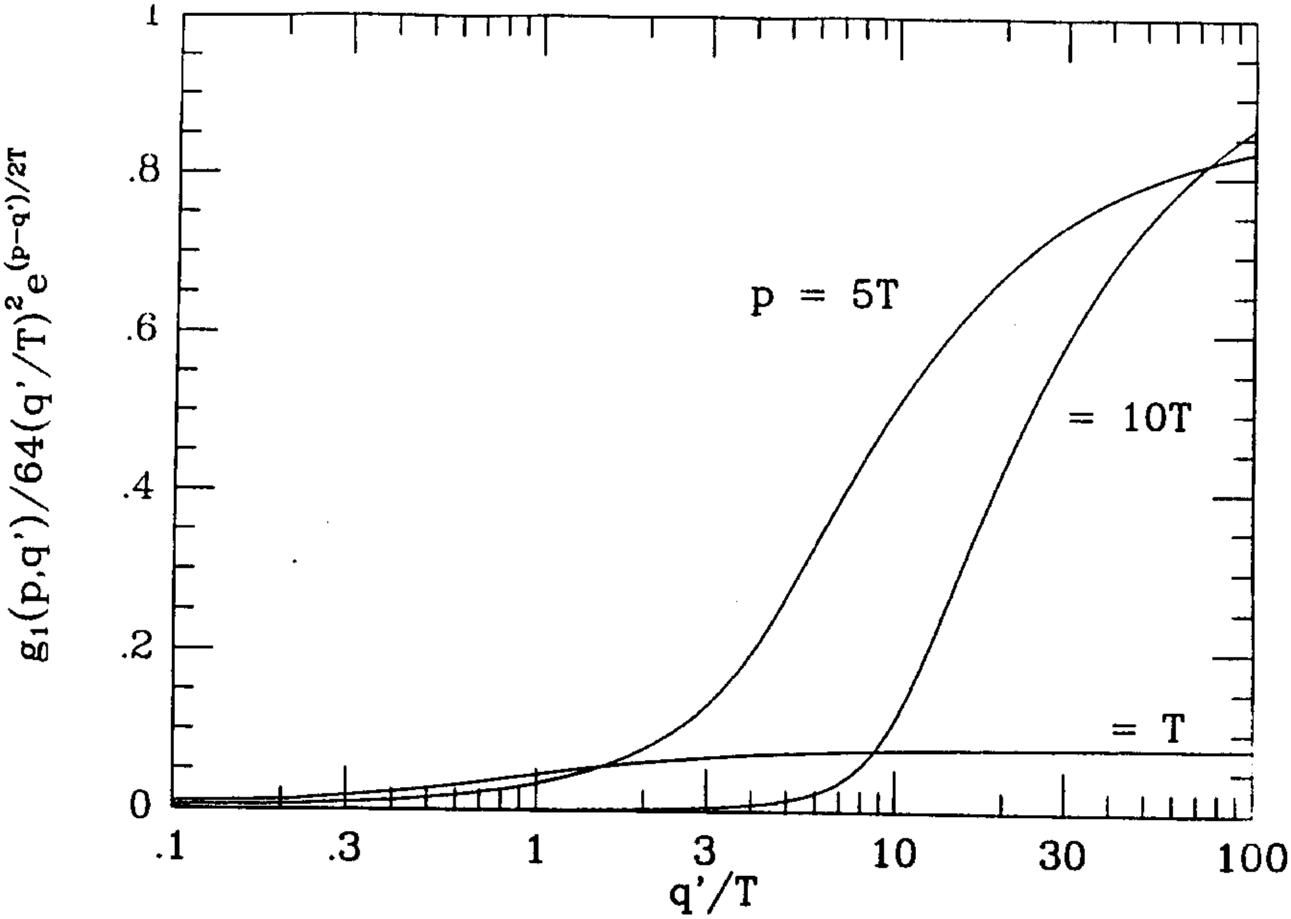
δX_γ
δX
δX_ν
$10^4\ \delta X$
1
0
−1
2
1
.5
.2
.1
T (MeV)

– FIG 9 –

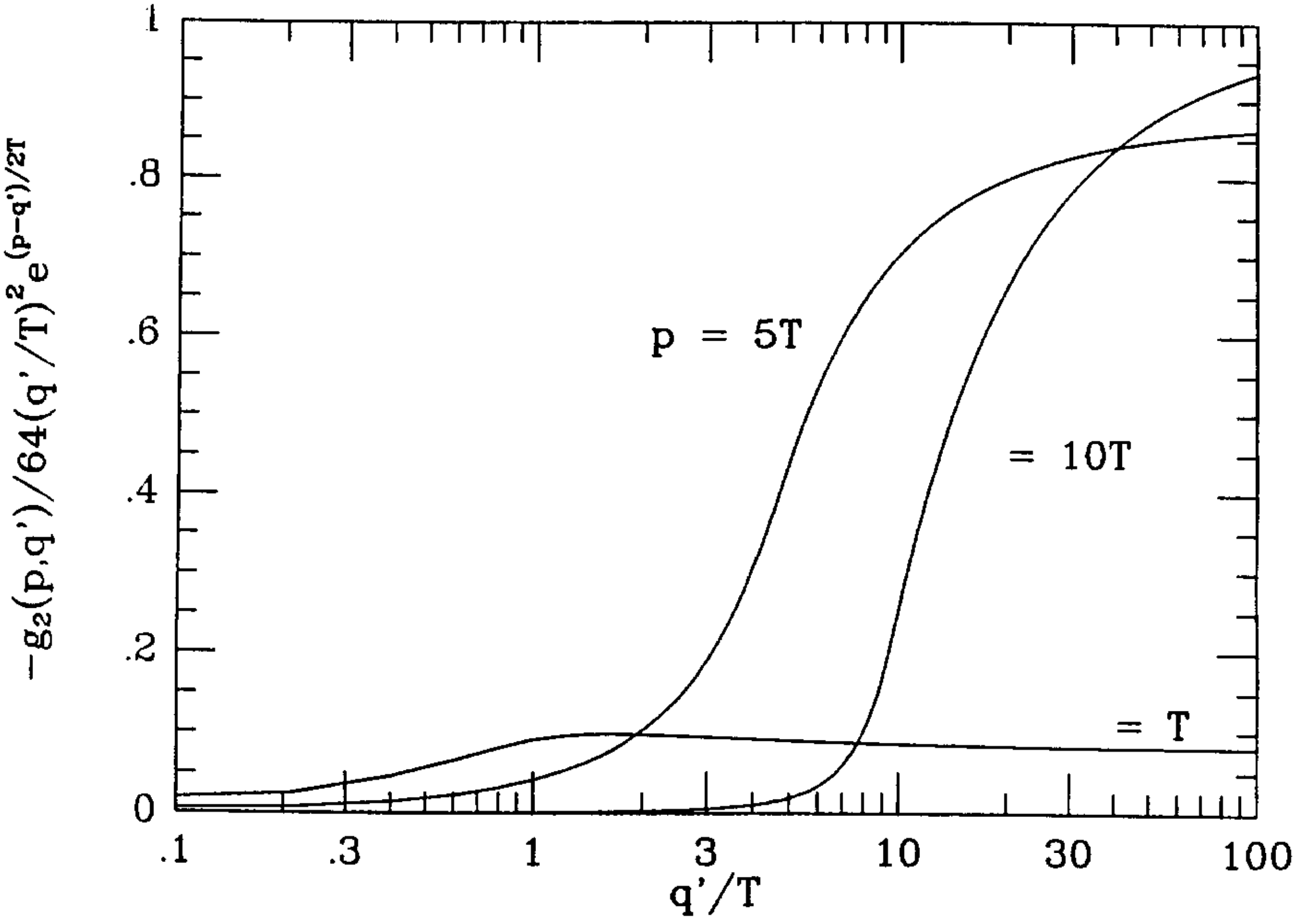


$(1-X)\delta\lambda_{pn}{}^{\nu}/H$
$X\delta\lambda_{np}{}^{\nu}/H$
$[(1-X)\delta\lambda_{pn}{}^{\nu} - X\delta\lambda_{np}{}^{\nu}]/H$
$d(\delta X)/d(\ln T^{-1})$
.003
.002
.001
0
1
.9
.8
.7
.6
.5
T (MeV)

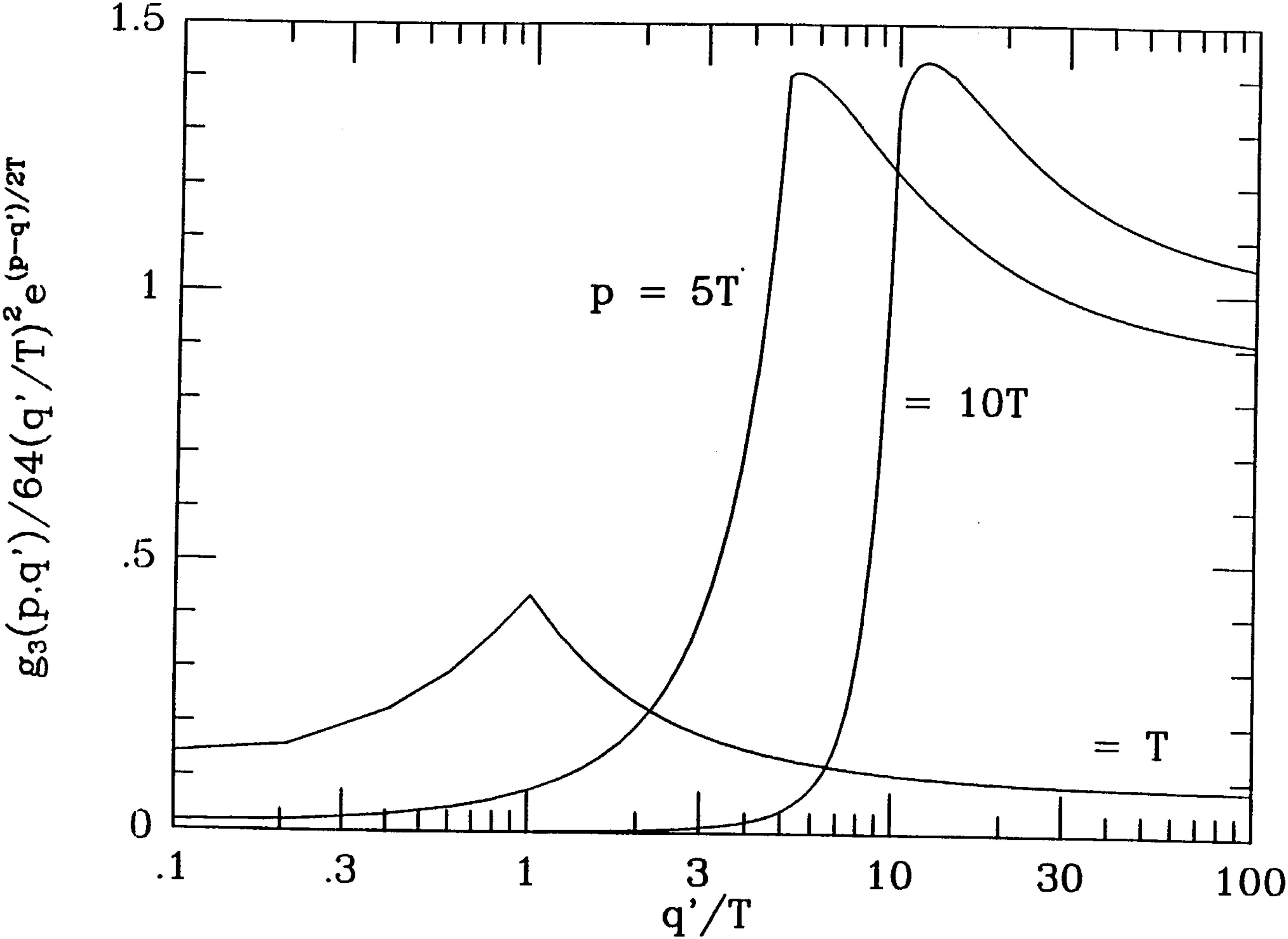
– FIG 10 –



-FIG 11a-



-FIG 11b-



—FIG 11c—