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Knots and Links of Nonabelian String

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Abstract

We consider a class of closed stringlike configurations, called *essentially knotted*, which occur in certain (3+1)-dimensional Yang-Mills–Higgs theories where a compact gauge group G breaks to a finite, nonabelian subgroup H. Such objects are labelled by more than one flux element in H, and cannot be deformed into the vacuum without overlapping the cores of two segments of string carrying noncommuting fluxes. Analogous results for multicomponent links of string are also given. Our analysis is performed within a general topological framework for discussing the knotting and linking of strings in spontaneously broken gauge theories.

It is well known that spontaneously broken gauge theories can have topological defects [1]. For example, if a connected group G breaks to a subgroup H which is *not* path-connected, then such a model possesses vortices in (2+1)-dimensions, or strings in (3+1)-dimensions. (For most of our discussion below, we will assume that H is finite.) These topological objects have generated much recent interest, especially when the unbroken subgroup H is nonabelian [2–7]. In this paper, we study stringlike Yang-Mills–Higgs configurations which form nontrivial knots and links in three-dimensional space.

First, consider a (2+1)-dimensional gauge theory with gauge group G spontaneously broken, via the Higgs mechanism, to a nontrivial finite subgroup H. We work in temporal gauge ($A_0 = 0$), and assume that the compact, connected Lie group G is simply connected, that is, the fundamental group $\pi_1(G)$ is trivial. On the circle S^1 at spatial infinity, the Higgs field must lie in the coset space ("vacuum manifold") G/H. Since $\pi_1(G/H) = H$, the theory possesses topologically stable vortices of finite energy, labelled by $h \in H$. Here h represents the homotopy class, relative to a fixed basepoint $y_0 \in S^1$, of the map from S^1 to G/H defined by the Higgs field at infinity. Equivalently, h is the Wilson loop of the gauge field, in a faithful representation of G, along a contour (based at y_0) going once around the vortex. If two vortex configurations are labelled by conjugate group elements h_1 and $h_2 = h^{-1}h_1h$, then one can be transformed into the other by a global h gauge transformation.

In a (3+1)-dimensional gauge theory with G broken to H as above, there will exist topologically stable strings of infinite extent and energy. Finite energy configurations which are closed loops of string can also be formed in these theories—such objects will generally be unstable, preferring to collapse to a point. We can always imagine, however, adding additional interactions which make the string repel itself at short distances, allowing stable loops [6]. Our purpose is to classify such loops of string. More precisely, given a fixed closed curve K in $\mathbb{R}^3$, we will determine the number of distinct gauge-Higgs configurations whose energy density is localized[1] on K. (Two configurations that are related by a static gauge transformation which is trivial at spatial infinity will be considered equivalent.) If K is unknotted, this number is equal to the order of the finite group H, which we denote by $|H|$. By contrast, we will see that if the closed loop K forms a nontrivial knot in $\mathbb{R}^3$, then the elements of H may not be sufficient to label the distinct configurations. More

[1] We are working with idealized strings of zero thickness (K is one-dimensional).

generally, we will consider configurations which are arbitrary n-component links (that is, links of n closed strings), $n \geq 1$.

In order to construct the gauge-Higgs configurations whose energy density is localized on a link L, it will be necessary to consider the fundamental group $\pi_1(\mathbb{R}^3 - L)$. A generic element of $\pi_1(\mathbb{R}^3 - L)$ is a (homotopy class of a) loop starting at a *fixed* point y_0 (which we choose to be at spatial infinity) and winding around various portions of the link L before returning to y_0. Given any L-configuration as above, we assign to each element $\ell \in \pi_1(\mathbb{R}^3 - L)$ the Wilson loop $W(\ell) \in H$ of the corresponding gauge field. The resulting map from $\pi_1(\mathbb{R}^3 - L)$ to H is a homomorphism. Conversely, every homomorphism $W : \pi_1(\mathbb{R}^3 - L) \rightarrow H$ gives rise to an L-configuration by considering the elements $W(\ell)$ as Wilson loops[2]. Therefore, there is a 1-1 correspondence between these homomorphisms and gauge-Higgs strings of type L. As an example, let L be a trivial knot K_0 which we take as the unit circle in the x-y plane. Clearly, $\pi_1(\mathbb{R}^3 - K_0)$ is isomorphic to the integers $\mathbb{Z}$. The integer p corresponds to a loop which circles through K_0 p times (in the same sense). One constructs a homomorphism $W : \mathbb{Z} \rightarrow H$ by sending the integer 1 to any $h \in H$; the flux element of the associated string is h. Thus, the number of K_0-configurations is equal to $|H|$. (Note that a 180^o rotation of the circle K_0 about any of its diameters turns a string with flux h into one with flux h^{-1}.) To show that more can happen when L is nontrivial, we must better understand the general structure of $\pi_1(\mathbb{R}^3 - L)$, for which it will be necessary to use certain aspects of the mathematical theory of knots and links which we now review.

A convenient presentation (a set of generators and defining relations) for the group $\pi_1(\mathbb{R}^3 - L)$, the *Wirtinger presentation* [8], can be obtained as follows. First, choose an orientation for each component of the link L in question, and then consider a regular planar projection of this oriented version of L with indicated over- and under-crossings (see fig. 1). Any given link possesses numerous different planar projections, related by the so-called Reidemeister moves [8]. Each projection will yield, by the procedure defined below, a presentation of $\pi_1(\mathbb{R}^3 - L)$. Any one of these will do for our purposes, and in what follows we fix a specific projection P_L. In P_L there will be a number, say m, of disjoint line segments or arcs. If n is the number of components of L, then $m \geq n$. There will be a

[2] Two conjugate homomorphisms, W_1 and $W_2 = h^{-1} W_1 h$, yield gauge-Higgs fields related by a global h gauge transformation. See [3] for a similar homotopy-theoretic approach to vortices.

generator of $\pi_1(\mathbb{R}^3 - L)$ associated with each such arc. Call these generators x_i, $1 \le i \le m$, and label each arc in the diagram P_L by an x_i as in fig. 1. The generator x_i represents the homotopy class of a loop (with basepoint y_0) which encircles arc i once, and passes in front of every other arc. We choose the orientation of the loop relative to that of the arc i to obey a right hand rule. Each crossing A of P_L yields a relation between those generators which are associated with the arcs incident on A. This relation can always be put in the form $x_k = x_i^{-1} x_j x_i$, where several of the indices i, j, k may be the same depending on the global nature of the diagram. In fig. 2 we show a generic oriented crossing and its corresponding relation. The number of crossings in a link diagram is always less than or equal to the number of arcs, so we end up with a presentation of $\pi_1(\mathbb{R}^3 - L)$ having m generators x_i and $q \le m$ relations R_j. We write this as $\pi_1(\mathbb{R}^3 - L) = \langle\, x_i \mid R_j \,\rangle$. One can show that any particular relation in the above set is a consequence of all the others taken together, so that we may delete any one of the R_j's from our presentation.

As an example, we reconsider the unknot K_0 which projects to a simple circle. Here we have a single generator x, and since there are no crossings, there are no relations. We thus recover $\pi_1(\mathbb{R}^3 - K_0) = \langle\, x \mid \emptyset \,\rangle = \mathbb{Z}$. A second, less trivial example is the trefoil knot T shown in fig. 1. Using the procedure above we find three generators x_1, x_2 and x_3 subject to the relations $x_1 = x_3^{-1} x_2 x_3$, $x_2 = x_1^{-1} x_3 x_1$ and $x_3 = x_2^{-1} x_1 x_2$. As alluded to earlier, the last relation (for example) is a consequence of the first two, so that we may write our presentation as

$$\pi_1(\mathbb{R}^3 - T) = \langle\, x_1, x_2, x_3 \mid x_1 = x_3^{-1} x_2 x_3,\ x_2 = x_1^{-1} x_3 x_1 \,\rangle. \tag{1}$$

This can be simplified further by eliminating x_3 in favor of x_1 and x_2 using the above (deleted) relation. After some algebra, we obtain

$$\pi_1(\mathbb{R}^3 - T) = \langle\, x_1, x_2 \mid x_1 x_2 x_1 = x_2 x_1 x_2 \,\rangle. \tag{2}$$

This infinite, nonabelian group is known as the braid group B_3 [8]. When using the presentation (2) to find homomorphisms from $\pi_1(\mathbb{R}^3 - T)$ into H and to discuss the resulting string configurations, one must remember that the flux in the arc labelled x_3 in fig. 1 is determined by $x_3 = x_2^{-1} x_1 x_2$.

We now turn to some two component links. First, consider a trivial link L_0 along with a projection $P^{(1)}_{L_0}$ given by two disjoint circles as in fig. 3. In this case we have two generators, one for each component, and no relations. The resulting nonabelian group is the free group F_2 on two generators, x_1 and x_2;

$$\pi_1(\mathbb{R}^3 - L_0) = \langle\ x_1, x_2 \mid \emptyset\ \rangle. \tag{3}$$

The simplest nontrivial link is the Hopf link L_1 shown in fig. 4. Again we have two generators, but now two relations as well coming from the crossings in the diagram. Both of these relations say the same thing, namely, that the two generators commute. So we have

$$\pi_1(\mathbb{R}^3 - L_1) = \langle\ x_1, x_2 \mid x_1 = x_2^{-1} x_1 x_2\ \rangle. \tag{4}$$

This abelian group is isomorphic to the direct product $\mathbb{Z} \times \mathbb{Z}$. It is straightforward to treat more complicated examples. In all cases, the group $\pi_1(\mathbb{R}^3 - L)$ is infinite and torsion-free.

Calculating the link groups $\pi_1(\mathbb{R}^3 - L)$ is only half the story. For the purpose of classifying string configurations in the above gauge theories, we still must study the homomorphisms of these groups into the appropriate finite group H. These are constructed by sending the generators x_i to specific elements of H such that the defining relations in $\pi_1(\mathbb{R}^3 - L)$ are satisfied. For any knot (i.e., one-component link) K, one can always construct a homomorphism by sending all of the generators to a single arbitrary element $h \in H$. The relations in $\pi_1(\mathbb{R}^3 - K)$ are then automatically satisfied. Such homomorphisms are in 1-1 correspondence with the elements of H. These are the "expected" maps whose existence we have already seen at the level of the trivial knot K_0. However, there may be other homomorphisms of $\pi_1(\mathbb{R}^3 - K)$ into H. A gauge-Higgs field associated with one of these new maps has no analogue in the unknotted case. We will call these configurations *essentially knotted.* In an essentially knotted configuration, the H-flux in the string changes discontinuously (at three or more crossings) as one traverses a diagram P_K for the relevant knot K. The Wirtinger relations between the fluxes in P_K can be understood by transforming to a singular gauge in which the gauge potential is only nonzero on a compact surface whose boundary is K. Also, choose this surface to emanate from the string in a direction perpendicular to the plane of projection of the diagram, and away from the viewer. At each crossing, the string which crosses under punctures the portion of the surface coming from the string which crosses over. As it pierces the surface, the H-flux of the under-crossing string will be conjugated by that of the over-crossing string. Though the H-representatives of distinct Wirtinger generators of $\pi_1(\mathbb{R}^3 - K)$ are, in general, different from each other for an essentially knotted string, we see that they must all be in the same conjugacy class of H.

For multicomponent links, it is still true that sending all the Wirtinger generators of $\pi_1(\mathbb{R}^3 - L)$ to a single element $h \in H$ yields a homomorphism. In the resulting string picture the H-flux is constant in any component of the link, and the same flux h runs through each component. One can also attempt to give the different components distinct fluxes, while still keeping the flux constant in any fixed component. In general, there will be strong constraints on the choice of these flux elements. If we take a trivial two-component link L_0 with projection $P_{L_0}^{(1)}$ (fig. 3), then $\pi_1(\mathbb{R}^3 - L_0)$ is presented as in (3) and we may assign any two fluxes to the two components and obtain a homomorphism. If L_0 also possesses the overlapping planar projection $P_{L_0}^{(2)}$ shown in fig. 3, then the appropriate Wirtinger presentation of $\pi_1(\mathbb{R}^3 - L_0)$ for this choice is

$$\pi_1(\mathbb{R}^3 - L_0) = \langle\ x_1, x_2, x_3 \mid x_3 = x_2^{-1} x_1 x_2\ \rangle. \tag{5}$$

(This is, of course, still isomorphic to the free group F_2.) One can again assign any elements h_1 and h_2 to the generators x_1 and x_2 respectively. However the flux through arc 3, which belongs to the same component as arc 1, will be different from h_1 unless we have $h_1 h_2 = h_2 h_1$. Note that the two descriptions of L_0-configurations provided by $P_{L_0}^{(1)}$ and $P_{L_0}^{(2)}$ are indeed equivalent, even though they look somewhat different. In particular, using either projection we find that there are $|H|^2$ L_0-configurations. As another example, consider the Hopf link L_1 of fig. 4 whose group $\pi_1(\mathbb{R}^3 - L_1)$ is presented in (4). Since the generators satisfy $x_1 x_2 = x_2 x_1$, the only allowed strings of type L_1 are those where the fluxes in the two components commute [7]. If H is nonabelian, this implies that there are less L_1-configurations than L_0-configurations.

A precise definition of essentially linked string configurations is somewhat more difficult than the analogous one for knots. The source of the difficulty is that changes in flux can occur even in some projections of a *trivial* link, as demonstrated above. The corresponding statement for knots does not hold. Intuitively, a configuration of type L is *essentially linked* if, for an *arbitrary* singular gauge choice, the flux in some component of L changes as it pierces the gauge surface coming from some other component. In other words, in all diagrams obtainable via a sequence of Reidemeister moves from a planar projection of this configuration, the flux in some component changes at a crossing with an arc belonging to a different component. This assures us that L_0-configurations can never be essentially linked—nor can configurations where all the changes in flux are due to self-crossings of individual components. (With this definition, there are also no theories possessing essentially linked strings of type L_1.) We now turn to examples which illustrate the existence of essentially knotted and essentially linked gauge-Higgs fields. These are the main results of this paper.

Let us first consider the case where the unbroken discrete subgroup H is abelian—for example $G = SU(n)$ breaking to its center, the cyclic group of order n, $H = \mathbb{Z}_n$. When H is abelian we can assign separately to each component of a link L an arbitrary element of H; any such assignment automatically satisfies all the Wirtinger relations. Moreover, the form of these relations implies that this exhausts all possibilities. That is, the flux in any given component *must* be constant, and hence there is no essential knotting or linking in these theories. Things may change when H is nonabelian. A commonly discussed theory with nonabelian strings is that of $G = SU(2)$ breaking to the 8-element quaternion group $H = Q_8$ [4]. The elements of $Q_8 \subset SU(2)$ are ± 1 and the Pauli matrices $\pm i\sigma_j$, $j = 1, 2, 3$. It is interesting to note that no essential knotting occurs in this model either. This can be seen by recognizing that in any essentially knotted string configuration, the images in H of at least two of the Wirtinger generators of $\pi_1(\mathbb{R}^3 - K)$ must not commute. Further, as mentioned earlier, the full set of these images must lie in a given conjugacy class of H. Now, there are five conjugacy classes of Q_8, namely, $\{1\}$, $\{-1\}$ and $\{i\sigma_j, -i\sigma_j\}$, $j = 1, 2, 3$. Since the members in any one of these classes clearly commute with each other, there is no essential knotting in the Q_8 model[3]. By contrast, there *is* essential linking in this theory. Consider the link L_2 of fig. 4, whose group $\pi_1(\mathbb{R}^3 - L_2)$ has the presentation

$$\pi_1(\mathbb{R}^3 - L_2) = \langle\, x_1, x_2, x_3, x_4 \mid x_1 = x_3^{-1}x_2x_3,\ x_2 = x_4^{-1}x_1x_4,\ x_3 = x_2^{-1}x_4x_2 \,\rangle, \qquad (6)$$

where we have deleted the redundant Wirtinger relation $x_4 = x_1^{-1}x_3x_1$. Six distinct essentially linked string configurations can be obtained from the homomorphisms $W_{jk} : \pi_1(\mathbb{R}^3 - L_2) \to Q_8$, $1 \le j \ne k \le 3$, defined by

$$W_{jk}(x_1) = i\sigma_j = -W_{jk}(x_2), \quad W_{jk}(x_3) = -i\sigma_k = -W_{jk}(x_4). \qquad (7)$$

[3] More generally, it can be shown that essential knotting never occurs in a model where H is a nilpotent group. This class of groups includes, among many others, the group Q_8 and all abelian groups.

The configurations associated with W_{jk} and W_{kj} are related by a 180^o rotation of L_2 about its vertical axis of symmetry in fig. 4. It is also straightforward to show that any other homomorphism leading to an essentially linked L_2-configuration is conjugate to one of these. For instance, consider the homomorphism $\tilde{W}_{jk}$ given by $\tilde{W}_{jk}(x_i) = -W_{jk}(x_i)$. We have $\tilde{W}_{jk} = h^{-1}W_{jk}h$, where $h = i\sigma_\ell$, $\ell \neq j, k$.

We are now ready for an example in which essential knotting occurs. In this model, the group $SU(2)$ is broken down to the *binary dihedral group* D_{12}^* of order 12. This subgroup is the double cover of $D_6 \subset SO(3)$ which consists of the rotational symmetries of an equilateral triangle in 3-space; D_6 is isomorphic to the full permutation group on 3 objects. The group $D_{12}^* \subset SU(2)$ can be generated by two elements $a \equiv i\vec{\sigma}\cdot\vec{n_1}$ and $b \equiv i\vec{\sigma}\cdot\vec{n_2}$, where the fixed unit vectors $\vec{n_1}$ and $\vec{n_2}$ satisfy $\vec{n_1}\cdot\vec{n_2} = -1/2$. From the algebra of the Pauli matrices, we see that these generators obey $a^2 = (ab)^3 = b^2 = -1$. The conjugacy classes of D_{12}^* are given by $\{1\}$, $\{-1\}$, $\{a,\ b,\ -aba\}$, $\{-a,\ -b,\ aba\}$, $\{ab,\ ba\}$ and $\{-ab,\ -ba\}$. Now consider the trefoil knot T in fig. 1, whose group is presented in (2). There is a homomorphism $W : \pi_1(\mathbb{R}^3 - T) \to D_{12}^*$ given by $W(x_1) = a,\ W(x_2) = b$. The string configuration associated with W is essentially knotted since $W(x_1) \neq W(x_2)$. So, the D_{12}^* model has essentially knotted trefoils. Any other essentially knotted T-configuration in this theory is related to this one by a global H gauge transformation, followed by a continuous motion of the string in $\mathbb{R}^3$ (beginning and ending in the same position T). Essentially linked strings of type L_2 also occur here. For example, using the presentation in (6) we can define $W : \pi_1(\mathbb{R}^3 - L_2) \to D_{12}^*$ by $W(x_1) = a,\ W(x_2) = b,\ W(x_3) = ab$ and $W(x_4) = ba$.

At this point, it is worth remarking that there is a finite action path interpolating between *any* two closed string configurations. (Therefore, there is tunneling between the corresponding quantum string states.) In particular, one can deform a string of any type L into one of trivial link type. For an essentially knotted or essentially linked configuration, such a deformation will always require overlapping the cores of two arcs with *noncommuting* flux elements. In describing this process, it is useful to imagine a "bridge" of string forming between these two arcs as they pass through each other [5][9]. The flux through this bridge is the commutator of the two fluxes in the original arcs. An example is shown in fig. 5. Here, the initial configuration is of type L_2 and is associated with the homomorphism W_{12} in (7). As the two components are pulled apart, bridges appear which carry a flux $\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2 = (\sigma_1\sigma_2)^2 = -1$.

Some cautionary remarks should be made as well about the situation when the Lie group G is not simply connected. Here, $\pi_1(G/H)$ is no longer just given by H, and as a result the homomorphisms of $\pi_1(\mathbb{R}^3 - L)$ into H no longer label the gauge-Higgs strings of type L. In particular, knowing the H-flux assignments (Wilson loops in a faithful representation of G) for the arcs in a given planar projection of a string does not suffice to determine the configuration. The missing information, which cannot be obtained from Wilson loops, is provided by the abelian group $\pi_1(G)$. More precisely, we have the following short exact sequence (group extension)

$$\{e\} \to \pi_1(G) \xrightarrow{p_*} \pi_1(G/H) \xrightarrow{\delta} H \to \{e\}. \tag{8}$$

This can be obtained from the long exact homotopy sequence of the canonical projection map $p : G \to G/H$. (Note that if G is semisimple, then $\pi_1(G/H)$ is still finite. Otherwise, it is infinite. Further, it can be shown that the image of the 1-1 homomorphism p_* lies in the center of $\pi_1(G/H)$, that is, (8) is a *central* extension.) The arcs in a diagram for a link L now get assigned elements of this larger group $\pi_1(G/H)$, and the homomorphisms from $\pi_1(\mathbb{R}^3 - L)$ into $\pi_1(G/H)$ label the type L strings. Given such a string, with associated homomorphism $\phi : \pi_1(\mathbb{R}^3 - L) \to \pi_1(G/H)$, we can recover the H-flux of an arc with corresponding Wirtinger generator x_i as follows. First, compose the map ϕ with the *boundary homomorphism* δ in (8); this gives rise to a homomorphism $W_\phi : \pi_1(\mathbb{R}^3 - L) \to H$, that is, $W_\phi = \delta \circ \phi$. The H-flux of the above arc is then given by $W_\phi(x_i)$. For a knot K it is true that any homomorphism from $\pi_1(\mathbb{R}^3 - K)$ into H can be obtained by composing one of the above maps ϕ with δ. However this is not true in general for a multicomponent link L. As a consequence, flux assignments for a string of type L which seem perfectly reasonable from the point of view of the subgroup H, may not be allowed due to the presence of $\pi_1(G)$.

When G is not simply connected, the only change in the definitions of essentially knotted and linked configurations is to replace H by $\pi_1(G/H)$. More specifically, the configuration associated with $\phi : \pi_1(\mathbb{R}^3 - K) \to \pi_1(G/H)$ is essentially knotted if $\phi(x_i) \neq \phi(x_j)$ for some pair of Wirtinger generators x_i and x_j labelling distinct arcs in a diagram for the knot K. An L-configuration is essentially linked if, in all diagrams obtainable via a sequence of Reidemeister moves from a projection of this multicomponent string, the $\pi_1(G/H)$ assignment in some component changes at a crossing with an arc from a different component. Using the fact that (8) is a central extension, one can prove for knots that $\phi(x_i) \neq \phi(x_j)$ for some i and j if and only if $W_\phi(x_k) \neq W_\phi(x_\ell)$ for some (possibly distinct) pair k and ℓ. Hence, simply checking if the H-flux changes as you traverse the diagram for K still suffices to determine if the string associated with ϕ is essentially knotted[4]. The corresponding statement for links is not true—the H-flux in each component can remain unchanged at every undercrossing with another component, yet the string may still be essentially linked. To illustrate this, let us consider a model which breaks $SO(3)$ down to the subgroup $H = \mathbb{Z}_2 \times \mathbb{Z}_2$ consisting of all diagonal matrices. Since $\pi_1(SO(3)) = \mathbb{Z}_2$, the above analysis is relevant. All of the results concerning knotting and linking in this theory are identical to those in the Q_8 model discussed earlier since the coset spaces $SU(2)/Q_8$ and $SO(3)/\mathbb{Z}_2 \times \mathbb{Z}_2$ are homeomorphic. (In particular $\pi_1(SO(3)/\mathbb{Z}_2 \times \mathbb{Z}_2)$, which is a central extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$, is isomorphic to $\pi_1(SU(2)/Q_8) = Q_8$.) Thus, we know by our treatment of the Q_8 model that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ theory possesses essentially linked configurations. However in this latter theory, the essential linking can never be detected by watching the H-fluxes since here $H = \mathbb{Z}_2 \times \mathbb{Z}_2$ is abelian. This example demonstrates only part of the danger in considering only H and neglecting $\pi_1(G)$ in a discussion of essential linking. The comments at the end of the preceding paragraph imply that from the structure of H alone, it may seem as though a theory possesses an essentially linked configuration with certain H-flux labels, while the additional constraints imposed on the possible H-flux assignments by the presence of $\pi_1(G)$ actually rule it out.

Our analysis can also be extended to include cases where the unbroken subgroup H is not discrete. (We will again assume below that the full gauge group G is simply connected, though it is easy to extend the results to the general case.) Here we have $\pi_1(G/H) = \pi_0(H)$, where $\pi_0(H)$ is the group of disconnected components[5] of H. It is most natural to define essential knotting and linking in these theories using homomorphisms $\phi : \pi_1(\mathbb{R}^3 - L) \to \pi_0(H)$. Such a homomorphism ϕ assigns to each arc in a diagram for the link L the *path component* of H to which the flux running through it belongs. ϕ does not care where the flux actually lies within this component. Because of this property, the

[4] This implies that essential knotting never occurs in theories where H is abelian, even if $\pi_1(G/H)$ is nonabelian. More generally, if H is nilpotent then so is $\pi_1(G/H)$, and no essential knotting occurs.

[5] More precisely, $\pi_0(H)$ is the quotient group H/H_c, where H_c is the connected component of the identity in H. When H is discrete, H_c consists of the identity element alone and we recover $\pi_1(G/H) = H$.

reader may wonder whether it would be better to use maps from $\pi_1(\mathbb{R}^3 - L)$ into the complete subgroup H for our definitions. This approach would have, however, some rather undesirable consequences. For example, consider an *unbroken* $SU(2)$ gauge theory. In the above notation this means $G = H = SU(2)$. There are plenty of homomorphisms of, say, the trefoil group $\pi_1(\mathbb{R}^3 - T)$ into $SU(2)$ which have nonabelian images (for instance, the map into the D^*_{12} subgroup discussed earlier). Thus, there are $SU(2)$ field configurations whose energy density is localized on an embedded trefoil in $\mathbb{R}^3$, and which are essentially knotted by this alternative definition. But, of course, this unbroken gauge theory contains no topological strings at all; $\pi_1(G/H) = \pi_0(SU(2))$ is trivial. Hence the flux will not want to stay confined to the trefoil in this configuration, and will eventually decay away. It seems a minimal requirement that an essentially knotted configuration should tend to retain its flux. Therefore, we favor the original definition using $\pi_0(H)$ which avoids the unwanted scenario.

We close with a few brief comments. First, we can also consider infinite length "open knots" in the framework of this paper. These can be thought of as ordinary knots which have been cut open at some point, and the two resulting ends then stretched off to spatial infinity in different directions. No modification of our techniques is needed to study these new configurations. (Similar statements hold for open links.) There are also configurations which possess vertices of three or more strings; for example, the three-string vertex created when two noncommuting fluxes pass through each other, as in fig. 5. Our classification applies here as well, but additional techniques must be used to calculate the group $\pi_1(\mathbb{R}^3 - \Gamma)$ for such an embedded *graph* Γ. More specifically, a relation must be added for each vertex V which states that any single flux incident on V must equal an appropriately ordered product of the remaining fluxes. We can further consider theories in which the three-dimensional space manifold M is different than $\mathbb{R}^3$. For an embedded link L in M, the group $\pi_1(M - L)$ replaces $\pi_1(\mathbb{R}^3 - L)$ in our analysis. Again, the procedure given earlier for finding a presentation of $\pi_1(\mathbb{R}^3 - L)$ must be generalized for $\pi_1(M - L)$.

Finally, all of our results have analogues in models containing ungauged or "global" strings, such as those used to describe ordered media in condensed matter physics. Indeed some related work, although with different emphasis, has been done in this area [9]. As an example, the global counterpart of the Q_8 model discussed here has been used to describe biaxial nematic liquid crystals. Such correspondences raise the interesting possibility that some of the exotic knotted and linked string configurations presented above may be detected in the laboratory.

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Figure Captions

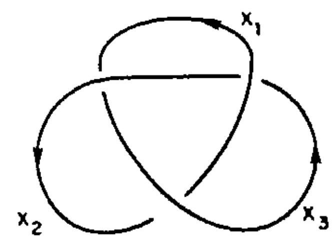
Fig. 1. A projection of the (right-handed) trefoil knot T. The arcs in the diagram have been labelled by the Wirtinger generators x_1, x_2 and x_3.

Fig. 2. An isolated crossing in a link diagram and its corresponding Wirtinger relation.

Fig. 3. Two diagrams, $P^{(1)}_{L_0}$ (left) and $P^{(2)}_{L_0}$, for the trivial link L_0.

Fig. 4. The Hopf link L_1 (left) and the link L_2.

Fig. 5. A deformation of an essentially linked string of type L_2 into a trivial link.



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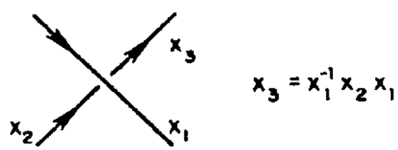

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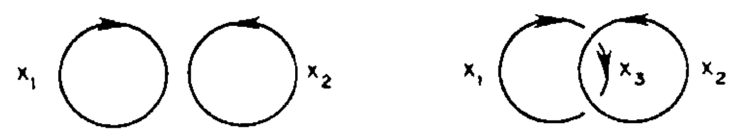

Fig. 3. Two diagrams, $P^{(1)}_{L_0}$ (left) and $P^{(2)}_{L_0}$, for the trivial link L_0.

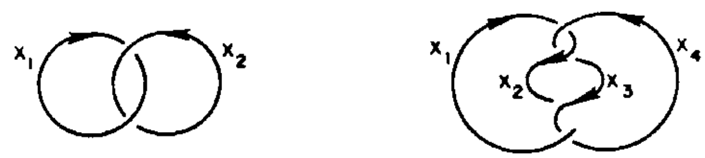

Fig. 4. The Hopf link L_1 (left) and the link L_2.

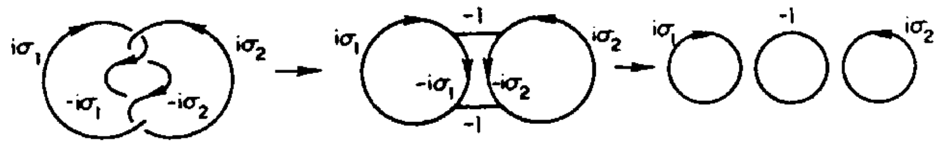

Fig. 5. A deformation of an essentially linked string of type L_2 into a trivial link.