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On the Viability of Lattice Perturbation Theory

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Abstract

In this paper we show that the apparent failure of lattice perturbation theory to account for Monte Carlo measurements of perturbative quantities results from choosing the bare lattice coupling constant as the expansion parameter. Using instead "renormalized" coupling constants defined in terms of physical quantities, like the heavy-quark potential, greatly enhances the predictive power of lattice perturbation theory. The quality of these predictions is further enhanced by a method for automatically determining the coupling-constant scale most appropriate to a particular quantity. We present a mean-field analysis that explains the large renormalizations relating lattice quantities, like the coupling constant, to their continuum analogues. This suggests a new prescription for designing lattice operators that are more continuum-like than conventional operators. Finally, we provide evidence that the scaling of physical quantities is asymptotic or perturbative already at β 's as low as 5.7, provided the evolution from scale to scale is analyzed using renormalized perturbation theory. This result indicates that reliable simulations of (quenched) QCD are possible at these same low β 's.



1 Introduction

In principle, nonperturbative lattice simulations allow the calculation of any quantity in QCD, without recourse to perturbation theory. In practice, however, perturbation theory is important to lattice QCD in several ways. Firstly, it provides the essential connection between lattice simulations, which are most effective for low-energy phenomena, and the high-energy arena of perturbative QCD phenomenology. This is accomplished through such constructs as the operator-product expansion. Secondly, perturbation theory can account for effects on low-energy phenomena due to the physics at distance scales shorter than the lattice spacing. Provided the lattice spacing a is small enough, systematic errors of order a and higher can be removed from the theory by perturbatively correcting the action and operators that define the lattice theory. This approach provides a cost-effective alternative to simply reducing the lattice spacing when systematic errors must be removed. Finally, lattice simulations and perturbation theory must agree for short distance quantities, where both approaches should be reliable, if we are to have confidence in simulation results for nonperturbative quantities.

It is disturbing therefore that Monte Carlo estimates for most short-distance quantities seem to agree poorly with perturbative calculations. An example is the vacuum expectation value of the lattice gluon operator U in Landau gauge. This is the lattice analogue of the expectation value $\langle A_\mu^2 \rangle$ of the square of the bare gauge field A_μ . Since $\langle A_\mu^2 \rangle$ is quadratically divergent, the loop integral in first-order perturbation theory is dominated by momenta of order the cutoff, and perturbation theory should be effective for cutoffs of order a couple of GeV or larger. However, the perturbative result, when expressed in terms of the bare coupling constant $\alpha_{\text{lat}} \equiv g_{\text{lat}}^2/4\pi$ of the lattice theory, is

$$\langle 1 - \frac{1}{3} \text{Tr}U \rangle_{\text{PT}} = 0.97 \alpha_{\text{lat}} = .078 \quad (1)$$

at $\beta \equiv 6/g_{\text{lat}}^2 = 6$.⁽¹⁾ This is almost a factor of two smaller than the nonperturbative result,

$$\langle 1 - \frac{1}{3} \text{Tr}U \rangle_{\text{MC}} = 0.139, \quad (2)$$

obtained from Monte Carlo simulations.[1] The coupling constant is quite small here ($\alpha_{\text{lat}} = 0.08$), and the loop momenta typically large ($q \approx \pi/a \approx$

⁽¹⁾ β is the parameter used to specify the bare coupling constant in the standard lattice action for QCD.

6 GeV). Perturbation theory ought to work; instead it seems to fail completely. Discouraging results such as this have arisen in a wide range of lattice calculations, leading to considerable pessimism about the viability of lattice perturbation theory at moderate β 's.

In this paper we show that although these facts are true they are misleading. We find that the key problem with previous calculations of this sort is in the choice of the expansion parameter for the perturbation series: α_{lat} is generally a very poor choice. There is no compelling reason in a field theory for using the bare coupling constant as the expansion parameter in weak-coupling perturbation theory. Standard practice is to express perturbation series in terms of some renormalized coupling constant, one usually defined in terms of a physical quantity. Indeed the renormalized coupling is usually a running coupling “constant” whose value in a particular expansion depends upon the length scales relevant in that process; there is no single expansion parameter for all series. The perturbative quantities important in lattice QCD generally involve lengths of order the lattice spacing a , and so one might expect little renormalization of the coupling from its bare value. However this argument, the usual rationale for using α_{lat} , ignores the possibility of a large scale-independent renormalization of the bare coupling. We find that just such a renormalization does occur in lattice QCD, making expansions in α_{lat} useless except at very large β 's.

Faced with large renormalizations, we must replace α_{lat} by a renormalized coupling. It is straightforward to reexpress lattice perturbation expansions in terms of any of the expansion parameters that have proven effective in continuum perturbation theory—for example, $\alpha_{\overline{\text{MS}}}(q)$ with some physically motivated momentum q . When this is done, we find that lattice perturbation theory becomes far more reliable. In fact, perturbation theory becomes about as effective for lattice quantities as it is for continuum quantities at comparable momenta.⁽²⁾

The large renormalization of α_{lat} is due to the structure of the link operators from which the theory is built. The nonlinear relation between the

⁽²⁾This situation in lattice theory parallels that for dimensional regularization, the other widely used regulator in QCD. Early calculations using dimensional regularization were expressed in terms of the minimal-subtraction coupling constant α_{MS} , the “natural” definition for that regulator. The results usually looked nonsensical, with large coefficients appearing in the higher-order terms of most expansions. Consequently a modified minimal-subtraction scheme, the $\overline{\text{MS}}$ scheme, was introduced for defining the coupling constant. This scheme, while somewhat arbitrary, did result in reasonable perturbation series, and has since become standard. An analogous shift, away from α_{lat} , is required in the study of lattice quantities.

link operator and the gauge field leads to large renormalizations of lattice operators relative to their continuum analogues, and these in turn result in large shifts of the coupling constants in the action. In this paper we present a simple nonperturbative procedure for removing the bulk of these large “tadpole” renormalizations from gluon and quark operators. This procedure elucidates the problems with α_{lat} . More importantly, perturbative expansions of the renormalization constants that relate quark currents and other composite operators on the lattice to their continuum counterparts become far more convergent once the tadpole contributions are removed.

In Section 2 of this paper we discuss the symptoms that result from a poor choice of expansion parameter in a perturbation series. We show how these symptoms afflict lattice expansions expressed in terms of α_{lat} , and we suggest a new, physically motivated procedure for renormalizing lattice perturbation theory.⁽³⁾ In Section 3, we compare predictions from our renormalized perturbation theory with nonperturbative results obtained from Monte Carlo simulations. We examine quark masses, $\langle \text{Tr}U \rangle$, and a variety of Wilson loops and Creutz ratios. We find impressive agreement for all quantities, with no tuning of the theory, even at β 's as low as 5.7. In Section 4 we discuss the origins of the large renormalizations that arise when comparing lattice quantities with their continuum analogues. We develop a new prescription for building lattice operators that are much closer in behavior to their continuum counterparts; in particular the large renormalizations disappear.⁽⁴⁾ The success of renormalized perturbation theory at low β 's suggests that the evolution of the coupling constant with lattice spacing is also perturbative and scaling asymptotic at these β 's. This important issue is discussed in Section 5. Finally, in Section 6, we summarize our conclusions, stressing their implications concerning the reliability of simulations on relatively coarse (and therefore much less costly) lattices.

2 Renormalized Lattice Perturbation Theory

2.1 A poor expansion parameter

If an expansion parameter α_{good} produces well behaved perturbation series for a variety of quantities, using an alternative expansion parameter $\alpha_{\text{bad}} \equiv \alpha_{\text{good}}(1 - 10,000 \alpha_{\text{good}})$ will lead to second-order corrections that

⁽³⁾For a discussion of these issues in the context of dimensionally regularized QCD perturbation theory, see [2]. A preliminary version of our lattice results is in [3].

⁽⁴⁾A preliminary version of this analysis is published in [4].

are uniformly large, each roughly equal to 10,000 α_{bad} times the first order contribution. Series expressed in terms of α_{bad} , although formally correct, are misleading if truncated and compared with data. The signal for a poor choice of expansion parameter is the presence in a variety of calculations of large second-order coefficients that are all roughly equal relative to first order.

A large coefficient appears in the first second-order calculation done on the lattice: the calculation of the gluonic three-point function used to relate the Λ parameter of the bare lattice coupling α_{lat} to the Λ 's of various continuum coupling constants.[5, 6] The coupling constant $\alpha(q)_{\text{mom}}$, defined in terms of this three point function at momentum q , has the expansion

$$\alpha(q)_{\text{mom}} = \alpha_{\text{lat}} \left\{ 1 + \alpha_{\text{lat}} (\beta_0 \ln(\pi/aq)^2 + 5.419) \right\}, \quad (3)$$

where $\beta_0 = 11/4\pi$. Naively, one expects that $\alpha(q = \pi/a)_{\text{mom}} \approx \alpha_{\text{lat}}$, since π/a is roughly the largest momentum on the lattice. The constant 5.419 spoils the equality; it results in very large ratios between continuum and lattice Λ 's.

Since continuum quantities are usually well behaved when expanded in terms of $\alpha(q)_{\text{mom}}$, it is immediately obvious that most other continuum quantities will have a similar constant term when expressed in terms of α_{lat} . For example, the heavy quark potential $V(q)$ at momentum transfer q has the expansion[7]

$$V(q) = -\frac{C_f 4\pi \alpha_{\text{lat}}}{q^2} \left\{ 1 + \alpha_{\text{lat}} \left(\beta_0 \ln \left(\frac{\pi}{aq} \right)^2 + 4.70 \right) \right\} \quad (4)$$

where $C_f = 4/3$ is the quark's color (Casimir) charge. Similar results hold for the e^+e^- hadronic cross section, derivatives of moments for deep inelastic ep scattering, etc.

A crucial point is that a similar constant term appears in the expansions for all short-distance lattice quantities that have been studied. For example, the corrections to the heavy-quark potential as a function of distance have the form[8]

$$V(R) = -\frac{C_f \alpha_{\text{lat}}}{R} \left\{ 1 + \alpha_{\text{lat}} \left(\beta_0 \ln \left(\frac{\pi R}{a} \right)^2 + C(R/a) \right) \right\}, \quad (5)$$

where $C(R/a)$ for various values of R is given in the following table:

R/a	2	4	6	∞
$C(R/a)$	5.5	5.5	5.6	5.711

(The constant for $R = \infty$ can be obtained by Fourier transforming the equation for $V(q)$ above.) Note that the constants $C(R/a)$ at finite R vary little from the one at $R = \infty$. This is expected since these corrections are dominated by quadratically UV divergent tadpole loops that are insensitive to the external momenta.

As we show later, similar terms are present in Wilson loops and Creutz ratios. Thus the pattern of second-order coefficients for lattice quantities strongly suggests that α_{lat} is a poor choice of expansion parameter.

2.2 A better expansion parameter

To define an improved (renormalized) expansion parameter, we must both choose a definition of the running coupling $\alpha_s(q)$ (“fix the scheme”) and specify how the scale q of the coupling is to be chosen (“set the scale”). It is natural and convenient in perturbation theory to tie the scale of the coupling to that of the loop momenta circulating in the Feynman diagrams. Thus, we want to define $\alpha_s(q)$ so that it approximates the coupling strength of a gluon with momentum q .⁽⁵⁾ It is also important that $\alpha_s(q)$ be defined in terms of a *physical* quantity, so as to avoid confusions, such as that between the MS and $\overline{\text{MS}}$ schemes, that are artifacts of arbitrary definitions.

2.2.1 Fixing the Scheme

Of the many physical quantities one might use to define an $\alpha_s(q)$, the heavy-quark potential $V(q)$ is among the most attractive.[2] Typically there is an integral over the momentum of the leading-order gluon, but the gluon in $V(q)$ has only momentum q . Thus it is particularly easy to tie the coupling constant’s argument to the gluon’s momentum for this quantity: we define $\alpha_V(q)$, the coupling strength of a gluon with momentum q , such that

$$V(q) = -\frac{C_f 4\pi \alpha_V(q)}{q^2} \quad (6)$$

⁽⁵⁾It is natural in a gauge theory to associate the scale of the coupling with the gluon’s momentum since every g in the theory is associated with a particular A_μ by gauge invariance. This association allows us to set the scale in a gauge invariant way.

with no higher-order corrections. We can easily relate α_V to the bare lattice coupling constant α_{lat} since $V(q)$ has been computed in terms of α_{lat} (Eq. 4):

$$\alpha_{\text{lat}} = \alpha_V(q) \left\{ 1 - \alpha_V \left(\beta_0 \ln(\pi/aq)^2 + 4.702 \right) \right\} + \mathcal{O}(\alpha_V^3). \quad (7)$$

for SU(3) color with no light-quark vacuum polarization. With this expression, any one-loop or two-loop lattice expansion can be reexpressed as a series in α_V . The q dependence of α_V is given by the usual formula,

$$\alpha_V^{-1}(q) = \beta_0 \ln(q/\Lambda_V)^2 + \beta_1/\beta_0 \ln \ln(q/\Lambda_V)^2 + \mathcal{O}(\alpha_V(q)), \quad (8)$$

where $\beta_0 = 11/4\pi$ (as before), $\beta_1 = 102/16\pi^2$, and

$$\Lambda_V = 46.08 \Lambda_{\text{lat}} \quad (9)$$

is the scale parameter for this scheme. The scale parameter Λ_{lat} for α_{lat} is defined implicitly by

$$\alpha_{\text{lat}}^{-1} = \beta_0 \ln(1/a\Lambda_{\text{lat}})^2 + \beta_1/\beta_0 \ln \ln(1/a\Lambda_{\text{lat}})^2 + \dots \quad (10)$$

Note that $\alpha_{\overline{\text{MS}}}(q)$ is numerically fairly close to $\alpha_V(q)$, and thus is another useful alternative to α_{lat} . In this case,

$$\alpha_{\text{lat}} = \alpha_{\overline{\text{MS}}}(q) \left\{ 1 - \alpha_{\overline{\text{MS}}} \left(\beta_0 \ln(\pi/aq)^2 + 3.880 \right) \right\} + \mathcal{O}(\alpha_{\overline{\text{MS}}}^3), \quad (11)$$

and the scale parameter is

$$\Lambda_{\overline{\text{MS}}} = 28.81 \Lambda_{\text{lat}}. \quad (12)$$

2.2.2 Setting the Scale

The coupling constant α_V is defined so that $\alpha_V(q^*)$ is the appropriate expansion parameter for a process in which the typical gluon momentum is q^* . For many processes it is possible to guess q^* fairly accurately. For example, power-law UV divergent quantities like $\langle \text{Tr}U \rangle$ are controlled by the lattice modes with the highest momenta, and so one expects $q^* \approx \pi/a$. Although such guesses are often sufficient, there is a simple automatic procedure that takes the guessing out of q^* . Such a procedure has proven invaluable in our systematic study of the reliability of perturbation theory.

Consider a one-loop perturbative contribution in our scheme:

$$I = \alpha_V(q^*) \int d^4q f(q) \quad (13)$$

where q is the gluon's momentum. The natural definition of q^* would be

$$\alpha_V(q^*) \int d^4q f(q) \equiv \int d^4q \alpha_V(q) f(q) \quad (14)$$

except that the second integral is singular. The singularity is due to the pole in the coupling constant at $q = \Lambda_V$. This pole is an artifact of the all orders summation of perturbative logarithms that is implicit in the formula for $\alpha_V(q)$ (Eq. 8). The singularity does not arise in any finite order of perturbation theory, as may be seen by replacing the running coupling constant $\alpha_V(q)$ in Eq. 14 by its expansion in terms of the coupling constant renormalized at some fixed scale μ :

$$\alpha_V(q) = \alpha_V(\mu) \{ 1 + \beta_0 \ln(q/\mu) \alpha_V(\mu) + (\beta_0 \ln(q/\mu) \alpha_V(\mu))^2 + \dots \} \quad (15)$$

None of these terms separately results in a singularity, but the sum of all terms diverges.

In fact it is incorrect to sum to all orders since the QCD perturbation series is an asymptotic series. The proper procedure is to retain only those terms consistent with the accuracy of the rest of the calculation. For our purposes we should retain only the first two terms in Eq. 15:

$$\begin{aligned} \alpha_V(q^*) \int d^4q f(q) &\equiv \alpha_V(\mu) \int d^4q f(q) + \\ &\beta_0 \alpha_V(\mu)^2 \int d^4q f(q) \ln(q/\mu)^2 + \dots \end{aligned} \quad (16)$$

Expanding $\alpha_V(q^*)$ in terms of $\alpha_V(\mu)$ in this equation, we obtain a simple definition for q^* (independent of μ):

$$\ln(q^{*2}) \equiv \frac{\int d^4q f(q) \ln(q^2)}{\int d^4q f(q)}. \quad (17)$$

2.2.3 Summary

To summarize, our general procedure for analyzing a perturbation series in lattice QCD involves replacing α_{lat} by $\alpha_V(q^*)$ using Eq. 7. The scale q^* is determined by probing the first-order calculation with factor $\ln q^2$, as in Eq. 17. In calculations that extend through two-loop order, we assume that the one-loop q^* , determined this way, is also appropriate for the two-loop contribution.

A special feature of expansions in $\alpha_V(q^*)$ is that they are unaffected through second order by quark vacuum-polarization insertions in the gluon

propagator. All such contributions are automatically absorbed into $\alpha_V(q^*)$, by virtue of its definition. As a consequence all of the perturbative expansions we use in this paper (second order as well as first) are identical in quenched and unquenched versions of QCD when they are expressed in terms of $\alpha_V(q^*)$. Only the evolution of $\alpha_V(q)$ changes: $\beta_0 \rightarrow (11 - \frac{2}{3}n_f)/4\pi$ and $\beta_1 \rightarrow (102 - 18n_f)/16\pi^2$ in Eq. 8 for n_f light-quark flavors. The n_f -independence of α_V expansions leads to an alternative procedure for determining q^* that is analyzed extensively, for continuum QCD, in [2].

3 Testing Renormalized Perturbation Theory

Our procedure for defining a renormalized coupling constant with a proper scale (Section 2.2) follows solely from known results in lattice perturbation theory, without regard to Monte Carlo data. Only now are we ready to consider the extent to which our renormalized perturbation theory agrees with Monte Carlo simulations of short-distance quantities.

Having converted all of our perturbative expansions from α_{lat} to α_V , we need some way of determining values of α_V that are appropriate to particular simulations. The most straightforward procedure is to measure α_V in the simulations.⁶ This can be done, for example, by measuring the heavy-quark potential, or, more simply, by measuring the trace of the plaquette operator U_{plaq} (the 1×1 Wilson loop). The improved perturbative expansion for the logarithm of $\text{Tr}U_{\text{plaq}}$ is

$$-\ln\langle \frac{1}{3}\text{Tr}U_{\text{plaq}} \rangle = 4.18879 \alpha_V(3.41/a) \left\{ 1 - 1.19 \alpha_V + \mathcal{O}(\alpha_V^2) \right\}. \quad (18)$$

Given data for this quantity, one can easily solve for $\alpha_V(3.41/a)$. The coupling $\alpha_V(q)$ for other q 's can then be obtained using standard two-loop evolution (Eq. 8). We have extracted $\alpha_V(3.41/a)$ in this way from data for quenched QCD at several β 's. The results, evolved down to $q = 1/a$, are given in Table 1. We also give values for α_{lat} and for $\alpha_{\overline{\text{MS}}}(1/a)$, the latter being obtained from the measured α_V using the relation $\Lambda_{\overline{\text{MS}}} = 0.6252 \Lambda_V$.

Our choice of $-\ln\langle \frac{1}{3}\text{Tr}U_{\text{plaq}} \rangle$ for determining α_V is for convenience; we have not attempted to optimize this choice. One could use any other short-distance quantity whose α_V -expansion is known through second order.

⁶As we discuss in Section 4, α_V can also be computed directly from α_{lat} without using Monte Carlo data. However this procedure is probably less accurate than measuring α_V , particularly at lower β 's.

β	$-\ln W_{11}$	α_{lat}	$\alpha_{\overline{\text{MS}}}(1/a)$	$\alpha_V(1/a)$
5.7	0.5995	0.0838	0.2579	0.3552
6	0.5214	0.0796	0.1981	0.2467
6.1	0.5025	0.0783	0.1860	0.2275
6.2	0.4884	0.0770	0.1774	0.2144
6.3	0.4740	0.0758	0.1690	0.2020
6.4	0.4610	0.0746	0.1617	0.1913
9	0.2795	0.0531	0.0815	0.0878
12	0.1954	0.0398	0.0532	0.0558
18	0.1227	0.0265	0.0317	0.0326

Table 1: Monte Carlo data for logarithm of the plaquette, together with the coupling constant values used in this study

Other alternatives might be Creutz ratios of small loops, whose perturbative expansions might be more convergent, or a combination of Wilson loops chosen so that potential nonperturbative area-law contributions cancel (eg, $\ln \text{Tr} U_{2 \times 2} - 4 \ln \text{Tr} U_{1 \times 1}$).

Note that, as discussed earlier, the formula used in measuring α_V (Eq. 18) is valid also for unquenched QCD, as are all of the α_V expansions that follow. Thus precisely the same techniques and tests we use here can be applied to the unquenched case. We have not yet done this, but we expect similar results.

3.1 $\langle A_\mu^2 \rangle$

The lattice equivalent of $\langle A_\mu^2 \rangle$ is $\langle 1 - \frac{1}{3} \text{Tr} U \rangle$, which is given in perturbation theory by $0.97 \alpha_s$. The one-loop contribution comes from a quadratically divergent tadpole graph, and we therefore expect that it is dominated by momenta of order the lattice cutoff π/a . Using the procedure of Section 2.2 we find $q^* = 2.80/a$. In Fig. 1 we compare perturbative results for $\langle 1 - \frac{1}{3} \text{Tr} U \rangle$ with Monte Carlo data[1] at several values of β . We present results from perturbation expansions in α_{lat} , in our favorite coupling constant $\alpha_V(q^*)$, and in $\alpha_{\overline{\text{MS}}}(q^*)$. The data agree with perturbation theory to within 10–15% for all $\beta \geq 5.7$ when α_V or $\alpha_{\overline{\text{MS}}}$ is used. Uncalculated terms of order α_V^2 or higher in the perturbation theory could easily account for the remaining differences; the differences between the α_V and $\alpha_{\overline{\text{MS}}}$ predictions give an indication of how important such terms might be. Of course part of the difference between perturbation theory and Monte Carlo might be

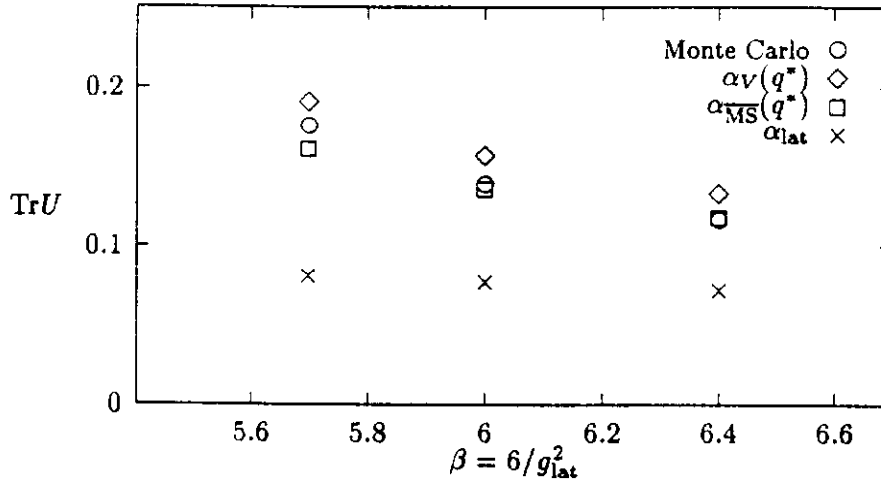


Figure 1: The expectation value of the trace of a link in Landau gauge, calculated by Monte Carlo (circles), and in first order perturbation theory using for the expansion parameter $\alpha_V(q^*)$ (diamonds), $\alpha_{\overline{\text{MS}}}(q^*)$ (boxes), and α_{lat} (crosses).

nonperturbative, particularly at the lowest β . The data disagree with the α_{lat} expansion by almost a factor of two.

3.2 Mass renormalization for Wilson quarks

A famous example of the “failure” of lattice perturbation theory is the calculation of the additive mass renormalization for Wilson quarks. The bare mass in Wilson’s formulation of the lattice quark action is renormalized by an additive power-law divergent term. The critical quark mass, for which this term is canceled (leaving the quark massless), is given in perturbation theory by $m_c a \equiv 1/2\kappa_c - 4 = -5.457\alpha_s$. [9] (Here, κ is the “hopping parameter” used to parameterize the quark mass in lattice gauge theory.) The linear divergence in this result suggests that the important momenta here are of order π/a . We find $q^* = 2.58/a$ using our procedure (Eq. 17). In Fig. 2 we compare perturbative results for m_c with Monte Carlo data [10] at several values of β . Again we see that the data agree with our renor-

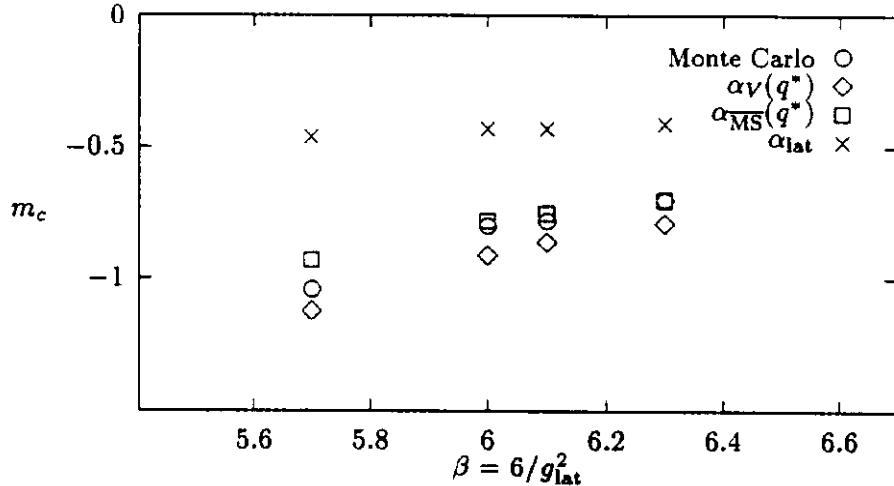


Figure 2: The critical quark mass m_c for Wilson quarks, calculated by Monte Carlo (circles), and in first order perturbation theory using for the expansion parameter $\alpha_V(q^*)$ (diamonds), $\alpha_{\overline{\text{MS}}}(q^*)$ (boxes), and α_{lat} (crosses).

malized perturbation theory to within 10–15% for all β 's, but disagree with perturbation theory using α_{lat} by almost a factor of two.

3.3 Wilson loops and Creutz ratios

Aside from the heavy-quark potential and the coupling constant, Wilson loops are the only lattice quantities for which two-loop perturbation theory has been calculated. Consequently they provide the most stringent tests of perturbation theory. Large Wilson loops have badly behaved perturbative expansions for a trivial reason: they contain a self-energy contribution proportional to the length of the loop. For large loops, contributions to this self-energy approximately exponentiate, so we expect that the logarithm of a Wilson loop is better behaved in perturbation theory than the loop itself.⁽⁷⁾

⁽⁷⁾Our data confirms that perturbation theory works better for logarithms of the W_{mn} than for the W_{mn} themselves, the expansions for the latter failing completely for even modestly large loops. Curiously the pathologies in the W_{mn} expansions seem to cancel

Taking Creutz ratios[11] of Wilson loops further improves perturbation theory by reducing the effects of both the divergent contributions associated with the perimeter of the loop and those coming from the corners of the loop.

For these reasons, we concentrate in this study on the logarithms of small Wilson loops and on Creutz ratios χ_{mn} defined by

$$\chi_{mn} \equiv -\ln \left(\frac{W_{mn} W_{m-1, n-1}}{W_{m, n-1} W_{m-1, n}} \right). \quad (19)$$

where W_{mn} is one third the expectation value of the trace of the $m \times n$ planar Wilson loop:

$$W_{mn} \equiv \frac{1}{3} \langle \text{Tr} U_{m \times n} \rangle. \quad (20)$$

We compare perturbative predictions with new data generated on a 16^4 lattice at β 's ranging from 5.7 to 18.[12] We use one-loop and two-loop perturbation-theory coefficients computed for a 16^4 lattice[8], and include the leading-order contribution from the zero mode.[13] Thus our perturbation theory is accurate up to uncalculated terms of order α_V^3 , and of order α_V^2/V , due to the zero mode, where $V = 16^4$ is the volume of the lattice. The finite-volume errors becomes significant for larger loops and so we limit ourselves to 5×5 loops and smaller.

In Fig. 3 we show results for χ_{22} , calculated through first order in α_s , and also through second order. The pattern at first order is similar to that in our previous examples: expansions in $\alpha_V(q^*)$ and $\alpha_{\text{lat}}(q^*)$ give reliable results at all β 's; the expansion in α_{lat} is off by almost a factor of four at $\beta = 5.7$, and still by almost 30% at $\beta = 12$. The second-order corrections significantly improve agreement between the data and the α_V and $\alpha_{\overline{\text{MS}}}$ expansions, with errors ranging from a few percent at $\beta = 5.7$ to a few tenths of a percent at $\beta = 12$. The remaining discrepancy could easily be accounted for by uncalculated corrections of order α_s^3 , although again nonperturbative effects may play a role at the lowest β 's. The second-order expansion with α_{lat} gives results that are at least an order of magnitude worse than those from the other two expansions (at all β 's). By comparison with the others, the convergence of this expansion is very sluggish—a unambiguous symptom of a bad expansion parameter.⁽⁸⁾

the pathologies in α_{lat} when m and n are small, making the α_{lat} expansion more accurate than the α_V expansion for these loops. Neither expansion is as accurate as expanding — in W_{mn} in powers of α_V and then exponentiating. This last procedure gives good results

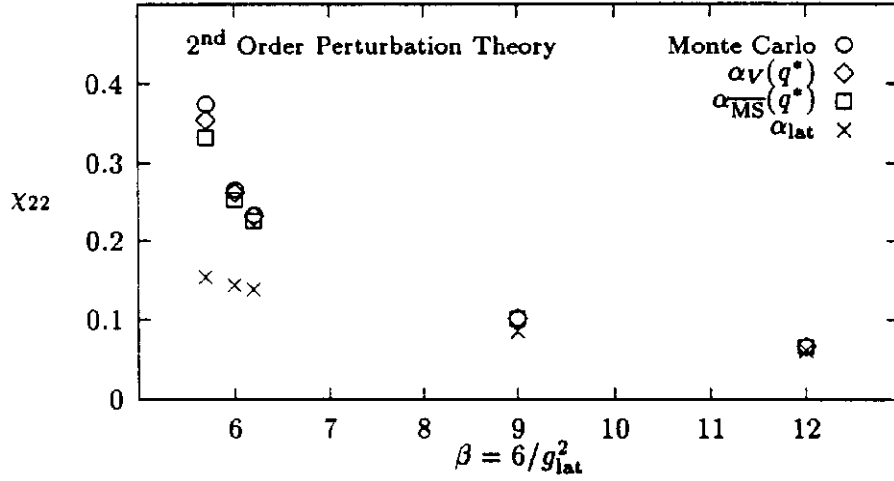
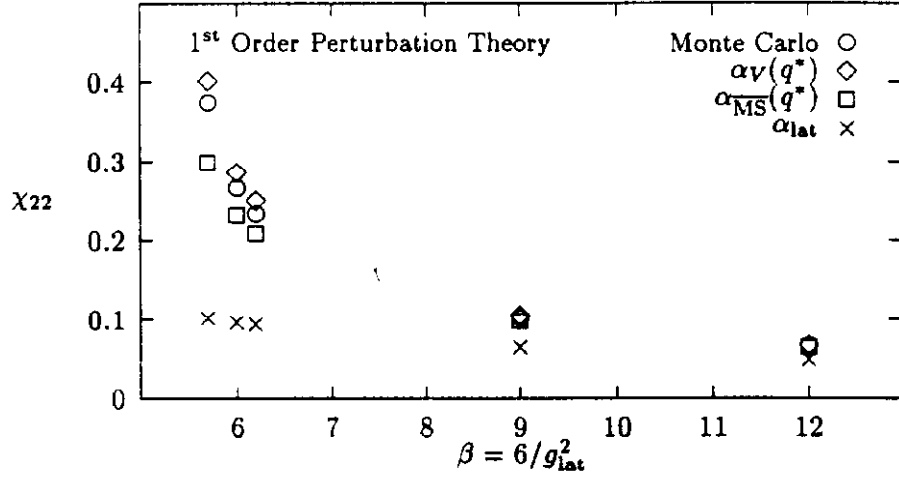


Figure 3: Results for Creutz ratio χ_{22} at different couplings β from Monte Carlo simulations (circles), and from perturbation theory (using $\alpha_V(q^*)$ (diamonds), $\alpha_{\overline{MS}}(q^*)$ (boxes), and α_{lat} (crosses)). The first plot shows perturbation theory through one-loop order, the second through two-loop order.

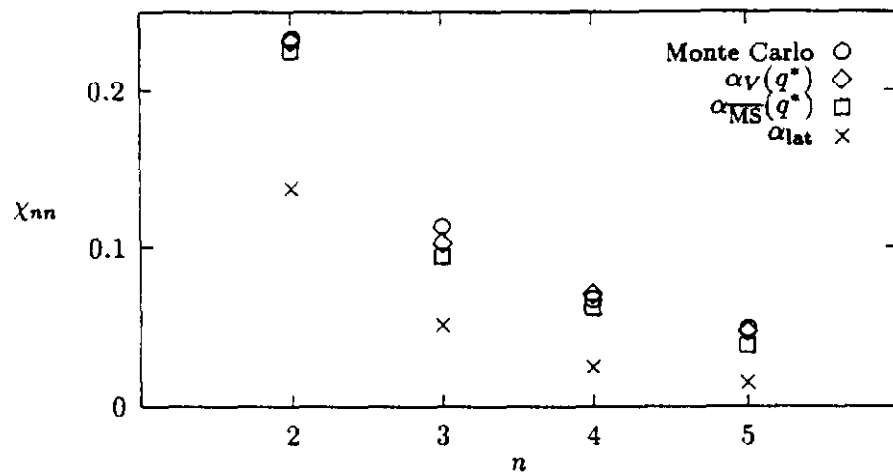


Figure 4: Results from perturbation theory (with $\alpha_V(q^*)$ (diamonds), $\alpha_{\overline{MS}}(q^*)$ (boxes), and α_{lat} (crosses)) and Monte Carlo simulations (circles) for diagonal Creutz ratios $\chi_{n,n}$ at $\beta = 6.2$.

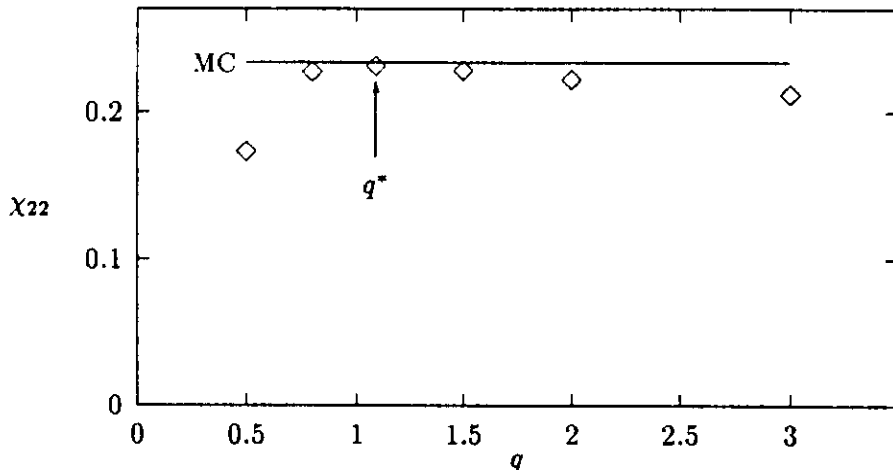


Figure 5: Results for the Creutz ratio χ_{22} from Monte Carlo simulations (line) and from the perturbation expansion in $\alpha_V(q)$ (diamonds) versus the scale q .

In Fig. 4 we show two-loop results with each of the coupling constants for a variety of different Creutz ratios at $\beta = 6.2$. The α_V and $\alpha_{\overline{\text{MS}}}$ expansions are again far superior for all of the ratios.

We expect smaller momentum scales for Creutz ratios than for the loops themselves since many of the divergent contributions to loop expectation values cancel in the ratios. Our scale setting procedure indicates that q^* is $1.09/a$ for χ_{22} , and smaller for ratios involving larger loops.

The importance of choosing a proper q^* is illustrated in Fig. 5, where the two-loop prediction for χ_{22} has been reexpressed in terms of $\alpha_V(q)$ and plotted versus q . Taking $q = \pi/a$, for example, rather than $q = q^* \equiv 1.09/a$ results in a 10% error rather than a 1% error. This situation should be con-

(when β is large) for all loops out to 8×8 , the largest we checked.

⁸Note that some of our results have been anticipated in the literature. The fact that perturbative results for Creutz ratios are better behaved when expanded in terms of $\alpha_{\overline{\text{MS}}}$ than when expanded in terms of α_{lat} was pointed out in [14]. The fact that perturbative results for Creutz ratios are better behaved when expanded in terms of an α , defined from any given ratio than when expanded in terms of α_{lat} was pointed out in [15].

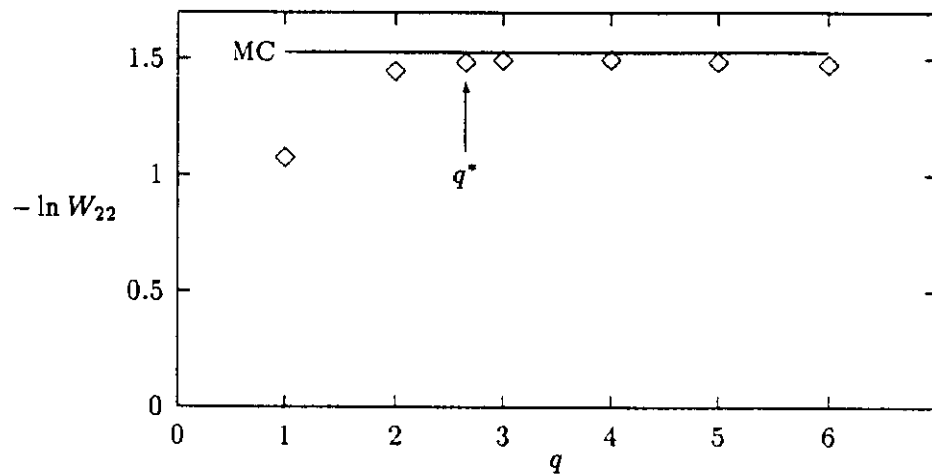


Figure 6: Results for $-\ln W_{22}$ from Monte Carlo simulations (line) and from the perturbation expansion in $\alpha_V(q)$ (diamonds) versus the scale q .

trusted with that for $-\ln W_{22}$ (Fig. 6). This quantity is significantly more ultraviolet than χ_{22} , having $q^* = 2.65/a$. Here our perturbative estimate degrades significantly if we use, say, $q = 1/a$ rather than q^* to set the scale of our expansion parameter. These examples illustrate the importance of our scale-setting procedure when high precision is required. Small departures from q^* are unimportant, at least for reasonably convergent series; but deviations by factors of two or more can affect the reliability of a perturbative estimate.

4 Mean field theory

We have shown that perturbation theory works well when a proper coupling constant is used, but it is still important to understand the origins of the large mismatch between the lattice coupling and the continuum couplings. This mismatch is one of many examples where a large renormalization is required to relate a lattice quantity to its continuum analogue. In this section we explore the connection between operators on the lattice and in the continuum.

4.1 Tadpole improvement

We usually design lattice operators by mapping them onto analogous operators in the continuum theory. For gauge fields, this mapping is based upon the expansion

$$U_\mu(x) \equiv e^{iagA_\mu(x)} \rightarrow 1 + iagA_\mu(x). \quad (21)$$

This expansion seems plausible when the lattice spacing a is small, but it is misleading since further corrections do *not* vanish as powers of a in the quantum theory. Higher-order terms in the expansion of U_μ contain additional factors of gaA_μ , and the A_μ 's, if contracted with each other, generate ultraviolet divergences that precisely cancel the additional powers of a . Consequently these terms are suppressed only by powers of g^2 (not a), and turn out to be uncomfortably large. These are the QCD tadpole contributions.

The tadpoles spoil our intuition about the connection between lattice operators and the continuum, and so we should not be surprised if the lattice theory is not quite what we expected (because of large renormalizations). In order to regain this intuition we must refine the naive formula that connects the lattice operator to the continuum operator (Eq. 21). Consider the

vacuum expectation values of these operators. In the continuum, the expectation value of $1 + iagA_\mu(x)$ is 1. In the lattice theory, tadpole corrections renormalize the link operator so that its vacuum expectation value (in, say, Landau gauge) is considerably smaller than 1 (see Fig. 1). This suggests that the appropriate connection with continuum fields is more like

$$U_\mu(x) \rightarrow u_0(1 + iagA_\mu(x)), \quad (22)$$

where u_0 , a number less than one, represents the mean value of the link. Gauge invariance requires that parameter u_0 enter as an overall constant.⁹

The mean-field parameter u_0 depends upon the parameters of the theory. It can be measured easily in a simulation. Simply measuring the link expectation value gives zero since the link operators are gauge dependent. However relations such as Eq. 22 only make sense in smooth gauges, like Landau gauge. Thus one might define u_0 to be the expectation value of the link operator in Landau gauge. A simpler, gauge-invariant definition uses the measured value of the plaquette:

$$u_0 \equiv \langle \frac{1}{3} \text{Tr} U_{\text{plaq}} \rangle^{1/4}. \quad (23)$$

Several other definitions are possible; all give similar results. At $\beta = 6$, for example, u_0 is 0.86 from the Landau-gauge link and 0.88 from the plaquette.

Our improved relation, Eq. 22, between lattice and continuum gauge-field operators suggests that all links U_μ that appear in lattice operators should be replaced by U_μ/u_0 , where u_0 is measured in the simulation. The operators U_μ/u_0 are much closer in their behavior to their continuum analogues; large tadpole renormalizations are largely canceled out by the u_0 (and the cancellation is nonperturbative since u_0 is measured rather than calculated). This is the key ingredient in our tadpole-improvement procedure for lattice operators. Several illustrations follow in succeeding sections.

⁹This formula follows simply from a renormalization-group argument. The tadpole contributions come mainly from the gauge-field modes with the highest momenta. Consequently the tadpoles can be removed by splitting the gauge field into ultraviolet (UV) and infrared (IR) parts (in a smooth gauge), and integrating out the UV parts. Averaging over the UV modes, the link operator is replaced by its IR part:

$$U_\mu \rightarrow u_0 e^{iagA_\mu^{\text{IR}}} \approx u_0(1 + iagA_\mu^{\text{IR}}),$$

where now the Taylor expansion of the exponential is quite convergent. Parameter u_0 contains the averaged UV contribution. It enters only as an overall constant since the link operator functions as a gauge connection both before and after averaging.

4.2 α_V from α_{lat}

Our new prescription for building continuum-like operators suggests that

$$\tilde{S}_{\text{gluon}} = \sum \frac{1}{2\tilde{g}^2 u_0^4} \text{Tr}(U_{\text{plaq}} + \text{h.c.}). \quad (24)$$

is a better gluon action for lattice QCD. In particular, perturbation theory in \tilde{g}^2 is much more like continuum perturbation theory (ie, no tadpoles). Of course this tadpole-improved action becomes the normal gluon action if we identify

$$\begin{aligned} \tilde{g}^2 &= g_{\text{lat}}^2 / u_0^4 \\ &= g_{\text{lat}}^2 / \langle \frac{1}{3} \text{Tr}(U_{\text{plaq}}) \rangle. \end{aligned} \quad (25)$$

This is a very important relationship; it tells us that the correct expansion parameter for the *usual* theory is \tilde{g}^2 rather than g_{lat}^2 . The difference is significant: for example, $\tilde{g}^2 \approx 1.7g_{\text{lat}}^2$ at $\beta = 6$ (using the measured value of the plaquette to relate the couplings). It is a big mistake to expand in powers of α_{lat} rather than $\tilde{\alpha}_{\text{lat}} \equiv \tilde{g}^2/4\pi$.⁽¹⁰⁾

If our mean-field analysis is correct, $\tilde{\alpha}_{\text{lat}}$ should be roughly equal to $\alpha_V(\pi/a)$. This is confirmed by perturbation theory which implies that

$$\alpha_V(\pi/a) = \frac{\alpha_{\text{lat}}}{\langle \frac{1}{3} \text{Tr} U_{\text{plaq}} \rangle} (1 + 0.513 \alpha_V + \mathcal{O}(\alpha_V^2)); \quad (26)$$

the difference between the two coupling constants is only a few percent at $\beta = 6$. This formula provides a nonperturbative relationship between the bare lattice coupling α_{lat} and α_V when measured values for the plaquette are used.

Note that since the renormalization is multiplicative, its main effect is to rescale the argument of the running coupling constant. This suggests that we define α_V by

$$\alpha_V(46.08/a) = \alpha_{\text{lat}} \left(1 + \mathcal{O}(\alpha_V^2) \right) \quad (27)$$

⁽¹⁰⁾Our analysis of the gauge-field action was anticipated by Parisi [16] who gives a simple analysis for the compact abelian theory. To see what effect the (UV-divergent) tadpoles have on infrared modes, we can split the gauge field in UV and IR components, and average over the UV part. Then the abelian gauge action becomes

$$\langle g^{-2} \cos(gF_{\mu\nu}^{\text{UV}} + gF_{\mu\nu}^{\text{IR}}) \rangle^{\text{UV}} = g^{-2} \langle \cos(gF_{\mu\nu}^{\text{UV}}) \rangle \cos(gF_{\mu\nu}^{\text{IR}})$$

and the effective coupling for the IR modes is g^2 divided by the UV part of the plaquette expectation value.

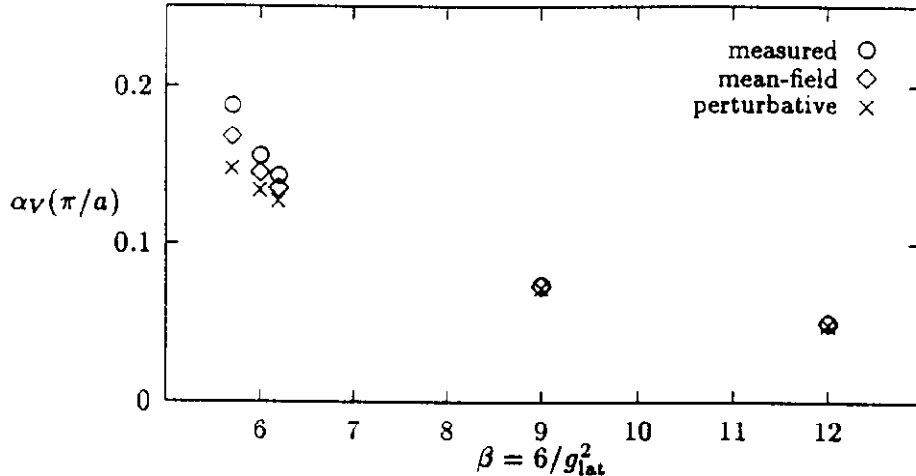


Figure 7: Values of $\alpha_V(\pi/a)$ as determined by measuring $-\ln(\frac{1}{3}\text{Tr}U_{\text{plaq}})$ (circles), by using a nonperturbative mean-field formula to relate it to the bare coupling (diamonds), and by using perturbation theory to relate it to the bare coupling (crosses).

(since $\Lambda_V = 46.08\Lambda_{\text{lat}}$), and then use two-loop evolution to determine $\alpha_V(q)$ for any other scale q . This provides a purely perturbative relation between α_V and α_{lat} .

In Fig. 7 we compare the measured values of $\alpha_V(\pi/a)$ from Section 3 with values obtained from the mean-field formula, Eq. 26, and from the perturbative formula, Eq. 27. Large coupling-constant renormalizations are automatically incorporated when α_V is measured, and so the validity of our mean-field analysis is tested by the extent to which the mean-field values agree with the measured values. All three methods produce results consistent up to corrections of order α_V^3 . The first two methods of determining α_V are probably preferable to the perturbative formula at low β 's since they incorporate some higher-order and nonperturbative effects.

Our prescription for defining tadpole-improved lattice operators is crucial in other, related contexts. One example is in defining operators to represent the chromoelectric and magnetic fields. These are needed for the operators

that remove $\mathcal{O}(a, a^2)$ errors from quark actions. The standard cloverleaf definitions, \mathbf{E}_{cl} and \mathbf{B}_{cl} , involve a products of four link operators, just like the plaquette [17]. Thus the tadpole-improved operators,

$$\tilde{\mathbf{E}}_{\text{cl}} = \mathbf{E}_{\text{cl}} / \langle \frac{1}{3} \text{Tr}(U_{\text{plaq}}) \rangle \quad (28)$$

$$\tilde{\mathbf{B}}_{\text{cl}} = \mathbf{B}_{\text{cl}} / \langle \frac{1}{3} \text{Tr}(U_{\text{plaq}}) \rangle, \quad (29)$$

are almost twice as large at $\beta = 6$. The plaquette factors account for the bulk of the very large renormalizations found in perturbation theory for operators containing cloverleaf fields. Such operators play an important role in all formulations of heavy-quark dynamics; omitting the tadpole renormalization leads to severe underestimates of their effects.

4.3 Improved Wilson fermions

Our tadpole-improvement scheme provides valuable insights into the pattern of large renormalizations in lattice QCD, and it is generally trivial to implement. As another example consider the tadpole-improved action for Wilson quarks:

$$S_q = \sum_x \bar{\psi} \psi + \tilde{\kappa} \sum_{x,\mu} \bar{\psi} \left((1 + \gamma_\mu) \frac{U_\mu}{u_0} \right) \psi + \dots \quad (30)$$

Again, this action is identical to the usual one if we relate the modified parameters, here the hopping parameter $\tilde{\kappa}$, to the usual ones by rescaling with u_0 :

$$\tilde{\kappa} = \kappa u_0. \quad (31)$$

The modified hopping parameter should be more continuum-like; for example, the tree-level value that gives massless quarks, $\tilde{\kappa}_c = 1/8$, should be roughly correct for interacting quarks as well, at least at high β 's. Thus an approximate nonperturbative formula for the critical value of the usual hopping parameter is

$$\kappa_c \approx 1/8 u_0. \quad (32)$$

This formula accounts for about 75% of the renormalization of the hopping parameter when β is large, as is evident if we compare the perturbative expansions for the two sides. By combining these perturbative expansions, we obtain a tadpole-improved perturbation theory for the critical bare mass m_c (where $m_c a \equiv 1/2\kappa_c - 4$, as in Section 3.2):

$$m_c a = -4 \left(1 - \langle \frac{1}{3} \text{Tr} U_{\text{plaq}} \rangle^{1/4} \right) - 1.268 \alpha_V (1.03/a) + \mathcal{O}(\alpha_V^2). \quad (33)$$

β	$\delta m_c a$	perturbative	tadpole-improved	measured
5.7	-0.44	-1.11	-1.00	-1.04(2)
6.0	-0.31	-0.91	-0.80	-0.80(2)
6.1	-0.29	-0.86	-0.76	-0.78(2)
6.3	-0.22	-0.79	-0.70	-0.70(2)

Table 2: Mass renormalization for Wilson fermions at different couplings β as computed using ordinary α_V perturbation theory, tadpole-improved perturbation theory, and Monte Carlo simulation. Also listed is the first-order perturbative correction to the mean-field estimate of m_c , $\delta m_c a = -1.268\alpha_V(1.03/a)$.

Using the measured value of the plaquette operator, this formula should be more accurate than the purely perturbative formula used in Section 3.2 since large tadpole renormalizations are being summed to all orders. Higher-order perturbative corrections should be smaller for the improved formula, as should nonperturbative effects. This seems to be the case as we show in Table 2, where the two predictions are compared with Monte Carlo data.

In the continuum limit, the tadpole-improved lagrangian for massless quarks becomes

$$2\bar{\kappa}\bar{\psi}\gamma_\mu\partial^\mu\psi + \mathcal{O}(a). \quad (34)$$

This indicates that $\sqrt{2\bar{\kappa}}\psi$ is the lattice quantity that corresponds to the continuum quark field. Since $\bar{\kappa} \approx 1/8$ when the quarks are massless, a tadpole-improved operator for massless quarks on the lattice is⁽¹¹⁾

$$\tilde{\psi} \equiv \psi/2. \quad (35)$$

This lattice operator has roughly the same normalization as the continuum field; in particular, there are no large tadpole contributions to the renormalization constant relating them. This is important in designing new lattice operators involving quark fields. For example, if one wants matrix elements of the continuum current $\bar{\psi}\gamma^\mu\gamma^5\psi$, then one should simulate with the lattice operator

$$\bar{\tilde{\psi}}\gamma^\mu\gamma^5\tilde{\psi} = \frac{1}{4}\bar{\psi}\gamma^\mu\gamma^5\psi. \quad (36)$$

⁽¹¹⁾This result is only valid for $\kappa \approx \kappa_c$. Away from κ_c , $\tilde{\psi}$ becomes [18, 4]

$$\tilde{\psi} \equiv (1 - 3\kappa/4\kappa_c)^{1/2} \psi.$$

Note that the κ dependence here is quite different from that of the commonly used (but incorrect) $\sqrt{2\kappa}\psi$.

Our procedure differs somewhat from common practice. Frequently the quark renormalization factor is taken to be $\sqrt{2\kappa_c}$ for massless quarks, rather than the factor $\sqrt{2\bar{\kappa}_c} = 1/2$ that we use. The former factor differs significantly from $1/2$ unless β is quite large. In our mean-field analysis, the conventional factor $\sqrt{2\kappa_c}$ is in effect divided by $\sqrt{8\kappa_c}$. This additional factor removes the bulk of the large tadpole corrections usually found in calculations of renormalization constants for quark operators. We have verified this for a variety of two, three and four-quark operators. Our results are presented in Table 3.⁽¹²⁾ There we present the renormalization factors for each operator (continuum divided by lattice) as computed in perturbation theory, and in perturbation theory but with the (nonperturbative) mean-field factors removed. Perturbative expansions are much improved by extracting the mean-field factors, particularly for operators with lots of fields (or cloverleaf fields, as in the last table entry).⁽¹³⁾

The results here are all for massless or nearly massless quarks. Tadpole-improved operators and actions for heavy quarks are discussed in [18, 4], for Wilson quarks, and in [22], for nonrelativistic quarks (NRQCD).

5 Implications for Scaling

A key issue in QCD concerns the onset of asymptotic or perturbative scaling: how small must the lattice spacing be before the variation of the coupling constant is perturbative. The variation of the coupling with changing lattice spacing is determined by the beta function, which, at short distances, is a perturbative quantity like any other. If perturbation theory successfully predicts a range of short-distance quantities, it is likely that it also correctly predicts the beta function. Thus our results in Section 3 provide indirect evidence in support of perturbative scaling. Our results also test the scaling properties of the coupling constant directly. This is because at

⁽¹²⁾These results are for Wilson parameter $r = 1$; very similar results arise for $r = 1/2$.

⁽¹³⁾The continuum operators used in this comparison were defined using the $\overline{\text{MS}}$ scheme. Our choice of normalization scale, $\mu = 1/a$, was somewhat empirical; a more systematic determination of the appropriate scale is possible using a variation of the techniques discussed in Section 2. Also, there is another obvious nonperturbative procedure for normalizing the operator for lattice quarks. The quark field's normalization should be roughly the square root of the normalization of either the vector or axial-vector current since these currents are conserved (or partially conserved). Inspection of our table indicates that using the average normalization of the two currents to define the quark normalization gives even better results than those shown in the table.

operator		perturbative	tadpole-improved
$\overline{\psi}\psi$	[19]	$(1 - 1.39 \alpha_V) 2\kappa_c$	$(1 - 0.03 \alpha_V) / 4$
$\overline{\psi}\gamma^5\psi$	[19]	$(1 - 2.40 \alpha_V) 2\kappa_c$	$(1 - 1.03 \alpha_V) / 4$
$\overline{\psi}\gamma^\mu\psi$	[19]	$(1 - 2.19 \alpha_V) 2\kappa_c$	$(1 - 0.82 \alpha_V) / 4$
$\overline{\psi}\gamma^\mu\gamma^5\psi$	[19]	$(1 - 1.68 \alpha_V) 2\kappa_c$	$(1 - 0.31 \alpha_V) / 4$
$\overline{\psi}\sigma^{\mu\nu}\psi$	[19]	$(1 - 1.80 \alpha_V) 2\kappa_c$	$(1 - 0.44 \alpha_V) / 4$
$C_1^L[\psi\psi\psi]$	[20]	$(1 - 2.78 \alpha_V)(2\kappa_c)^{1.5}$	$(1 - 0.73 \alpha_V) / 8$
$d_1^L[\psi\psi\psi]$	[20]	$(1 - 2.89 \alpha_V)(2\kappa_c)^{1.5}$	$(1 - 0.84 \alpha_V) / 8$
$d_4^L[\psi\psi\psi]$	[20]	$(1 - 2.76 \alpha_V)(2\kappa_c)^{1.5}$	$(1 - 0.71 \alpha_V) / 8$
$\overline{\psi}\psi O_+\psi\psi$	[21]	$(1 - 3.81 \alpha_V)(2\kappa_c)^2$	$(1 - 1.08 \alpha_V) / 16$
$\overline{\psi}\psi O_-\psi\psi$	[21]	$(1 - 3.66 \alpha_V)(2\kappa_c)^2$	$(1 - 0.93 \alpha_V) / 16$
$\overline{\psi}\psi O_1\psi\psi$	[21]	$(1 - 3.78 \alpha_V)(2\kappa_c)^2$	$(1 - 1.05 \alpha_V) / 16$
$\overline{\psi}\psi O_2\psi\psi$	[21]	$(1 - 3.55 \alpha_V)(2\kappa_c)^2$	$(1 - 0.83 \alpha_V) / 16$

Table 3: Renormalization constants relating continuum ($\overline{\text{MS}}$ at $\mu = 1/a$) and lattice operators for massless quarks. Results are for the ratio of continuum to lattice matrix elements, both without and with tadpole factors removed.

each β we measure the coupling α_V at $q = 3.41/a$, using data for the very ultraviolet-divergent plaquette, and then we perturbatively evolve α_V down to scales ranging from $0.4/a$ to $2.8/a$ to compute estimates for a variety of less ultraviolet quantities. (Note that α_V at $\beta = 6$ increases by more than 50% when evolving from $q = 3.41/a$ down to $q = 1.09/a$, the scale for Creutz ratio χ_{22} .) The success of our many perturbative estimates is compelling evidence that coupling-constant evolution is mostly perturbative for all β 's down to 5.7, and possibly even for lower ones.⁽¹⁴⁾ Of course, this discussion only applies to the α_V and $\alpha_{\overline{\text{MS}}}$ definitions of the coupling; α_{lat} is poorly behaved, but also largely irrelevant given our new perturbative techniques.

Our conclusion, that scaling is asymptotic even at $\beta = 5.7$, contradicts standard lore. This lore derives from studies of scale invariant ratios of Λ_{lat} with physical quantities like the deconfining temperature T_c or the string tension σ . Such ratios, which should become independent of β at the onset of asymptotic scaling, show considerable variation with β for β 's less than

⁽¹⁴⁾This result is supported by recent calculations of the beta function itself for a range of scales using a finite-size scaling technique in SU2 gauge theory[23]. This study probes different scales by examining a single quantity on a series of lattices with different lattice spacings. Our study probes different scales on single lattices by examining a variety of quantities, some more ultraviolet than others. In both cases the evolution of the coupling constant is tracked over a large range of scales.

6.2. This is illustrated by the upper plots in Fig. 8. These show the $1P$ - $1S$ mass splitting ΔM for the ψ [24] and Υ [25] meson families divided by Λ_{lat} , as well as $\sqrt{\sigma}/\Lambda_{\text{lat}}$ [26], for a range of β 's. Scaling violations of order 30–40% are readily apparent between $\beta = 5.7$ to $\beta = 6.1$. However, the situation changes completely if we replace the Λ_{lat} 's in these ratios by Λ_V 's determined (perturbatively) from the α_V 's we used in Section 3. When compared with Λ_V , the data are consistent with asymptotic scaling to within a few percent.

Exact scaling is not expected for any physical quantity. There are certainly finite-lattice-spacing errors in each of the measurements we use here. These errors have been analyzed carefully for the Υ data [4]; they result in roughly 10% scaling violation over the range shown. Errors for the other two quantities are probably smaller since Υ 's are the smallest mesons. Note that the $1P$ - $1S$ splitting in quarkonium mesons is one of few hadronic measurements that is suitable for studying scaling. This is because the splitting is almost completely insensitive to the heavy-quark's mass, and so depends only upon the coupling constant. (This is also why the ψ results shown are nearly indistinguishable from the Υ results.)

The evidence suggests that physically interesting quantities like mass splittings or the string tension scale perturbatively even at low β 's. The problem with the standard lore is that α_{lat} does not scale perturbatively (at least through two-loop order). This is clear from our studies of perturbation theory. These studies also indicate that renormalized couplings like α_V are perturbative, and this is why the ratios with Λ_V scale so well. Of course, ratios of physical quantities should scale properly as well, and they do (see Fig. 9).

It has been apparent for some time that the deconfining temperature scales better when analyzed in terms of a modified coupling constant similar to ours.[27] Now it is apparent that the modified coupling constant is just a continuum coupling constant like α_V . Furthermore it is clear that the failure of scaling was intimately related to the lack of convergence of perturbation theory for short-distance quantities like κ_c or χ_{22} . Both problems are resolved by replacing α_{lat} with α_V .

6 Summary

The use of lattice perturbation theory in conjunction with simulations has been hampered by two problems:

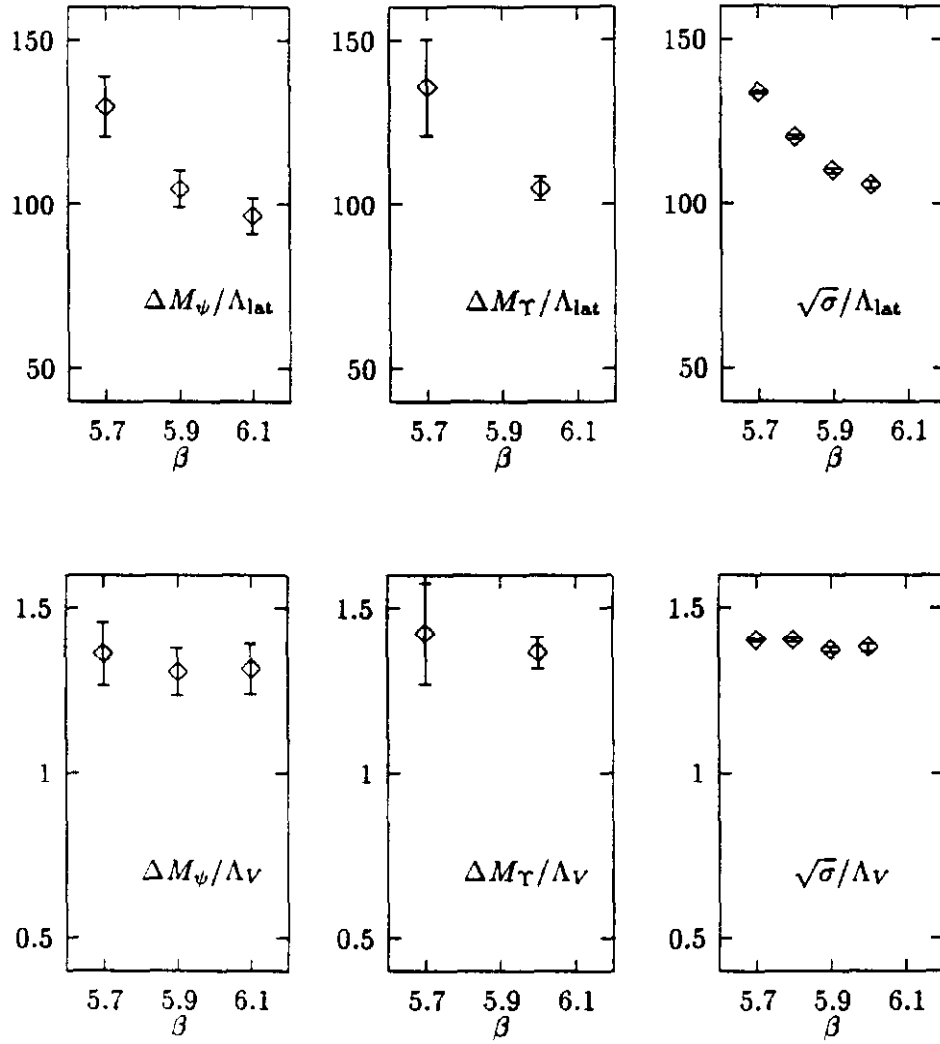


Figure 8: Ratios of the square root of the string tension σ and the $1P$ - $1S$ mass splittings ΔM for ψ 's and Υ 's with Λ_{lat} (top row) and with Λ_V (bottom row).

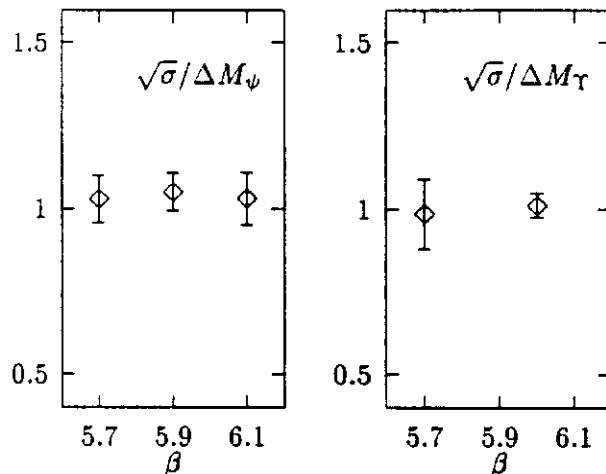


Figure 9: Ratios of the square root of the string tension σ with the $1P$ - $1S$ mass splittings ΔM for ψ 's and Υ 's.

- expansions in powers of the bare lattice coupling α_{lat} consistently underestimate perturbative effects, sometimes by factors of 2 or 4;
- expansions for many quantities (and particularly renormalization constants for lattice operators) have large coefficients due to tadpole diagrams and consequently converge poorly, if at all.

We have addressed both problems in this paper. We have shown here that lattice perturbation theory works well when a proper coupling constant is used; and it can be made about as convergent as the continuum theory by systematically removing tadpole contributions.

The first problem is remedied by replacing α_{lat} with a renormalized coupling constant like $\alpha_V(q^*)$, where scale q^* is customized (in a predetermined way) to the quantity under study. The coupling constant α_V is defined in terms a physical quantity, the heavy-quark potential, and it can either be measured (Section 3) or it can be determined from the bare lattice coupling α_{lat} using formulas from mean-field theory (Section 4.2). Perturbation theory, when expressed in terms of $\alpha_V(q^*)$, is remarkably effective even at $\beta = 5.7$.

The second problem, large tadpole-induced renormalizations, is remedied by simple redefinitions of the basic operators used to define the lattice theory. Every U_μ in a naive lattice operator is replaced with U_μ/u_0 , where u_0 is a measured constant representing the mean value of the link (Section 4); and every renormalized low-mass (Wilson) quark field $\sqrt{2\kappa_c}\psi$ is replaced by $\psi/2$. The new operators obtained this way are rescaled versions of the naive operators. Their normalizations are very close to those of their continuum analogues; renormalization constants for composite operators built from these tadpole-improved operators have perturbative expansions that are far more convergent. Tadpole improvement is essential for operators, like the cloverleaf operators for $F_{\mu\nu}$, that involve many links; without it normalizations are wrong by as much as a factor of 2, and perturbation theory becomes useless.

Our examples suggest that lowest-order perturbation theory in α_V gives results for short-distance quantities that are typically correct to within 10–20% at $\beta = 6$. Expansions in α_{lat} can be off by factors 2 or 3 at the same β . Adding in higher-order corrections usually reduces errors by factors of 2–5 for α_V expansions, and by very little for α_{lat} expansions. In many situations, α_V expansions can be made still more accurate through tadpole-improvement, where powers of the mean-field parameter u_0 are factored out of the expansion leaving behind a more convergent series. While our procedure for determining the proper scale q^* for the coupling consistently leads to excellent expansions, expansions in $\alpha_V(\pi/a)$ for quantities defined over one or two lattice spacings usually give errors that are within a factor of 2–3 of those obtained with $\alpha_V(q^*)$.

The fact that perturbation theory seems to be working at $\beta = 5.7$ implies that asymptotic or perturbative scaling should also work. We verified this here, for a number of physical quantities, by comparing their dependence on β with that of the scale parameter Λ_V for the renormalized coupling α_V ; scale invariant ratios of these quantities with Λ_V (as opposed to Λ_{lat}) showed little variation all the way down to $\beta = 5.7$. These results suggests that the lattice spacings used in current simulations are small enough for reliable studies of QCD. Indeed, if anything, they are unnecessarily small. It is probably much more cost effective to simulate QCD at $\beta = 5.7$, while removing the $\mathcal{O}(a, a^2)$ errors that are important by correcting the action. Previous efforts at improving lattice actions have not been too successful; but these relied upon the use of expansions in α_{lat} , and naive lattice operators. It is likely that the use of α_V perturbation theory, and tadpole-improved operators to correct the action will be much more successful. The poten-

tial savings in computer resources make it imperative that this option be thoroughly investigated.

Appendix

This appendix presents data for some of the figures in tabular form.

β	Perturbation Theory			M. C. Data
	α_{lat}	$\alpha_{\overline{\text{MS}}}(q^*)$	$\alpha_V(q^*)$	
5.7	0.081	0.161	0.191	0.176
6	0.077	0.136	0.157	0.139
6.4	0.072	0.118	0.133	0.117

Table 4: $\langle 1 - \frac{1}{3}\text{Tr}U \rangle$ (Landau gauge)—the expectation value of the trace of a link in Landau gauge, calculated in first order perturbation theory and by Monte Carlo simulation for various β 's. Statistical errors in all data presented are of order one in the last digit quoted or smaller.

β	Perturbation Theory			M. C. Data
	α_{lat}	$\alpha_{\overline{\text{MS}}}(q^*)$	$\alpha_V(q^*)$	
5.7	-0.46	-0.93	-1.12	-1.04
6	-0.43	-0.78	-0.91	-0.80
6.1	-0.43	-0.75	-0.86	-0.78
6.3	-0.41	-0.70	-0.79	-0.70

Table 5: $a m_c$ —the critical quark mass m_c for Wilson quarks with $r=1.0$, calculated in first order perturbation theory and by Monte Carlo simulation for various β 's. Statistical errors in the simulation data are of order 2 in the last digit quoted.

β	First Order			Second Order			M. C. Data
	α_{lat}	$\alpha_{\overline{\text{MS}}}$	α_V	α_{lat}	$\alpha_{\overline{\text{MS}}}$	α_V	
5.7	0.10160	0.29790	0.40100	0.15429	0.33181	0.35347	0.37343
6	0.09652	0.23170	0.28560	0.14407	0.25225	0.26150	0.26558
6.2	0.09341	0.20840	0.24990	0.13794	0.22497	0.23146	0.23317
9	0.06435	0.09749	0.10490	0.08548	0.10112	0.10167	0.10173
12	0.04826	0.06399	0.06703	0.06015	0.06556	0.06570	0.06574
18	0.03217	0.03826	0.03930	0.03746	0.03882	0.03885	0.03885

Table 6: χ_{22} —the expectation value of the Creutz ratio χ_{22} calculated in first and second order perturbation theory and by Monte Carlo.

n	Perturbation Theory			M. C. Data
	α_{lat}	$\alpha_{\overline{\text{MS}}}(q^*)$	$\alpha_V(q^*)$	
2	0.13794	0.22497	0.23146	0.23317
3	0.05207	0.09467	0.10312	0.11348
4	0.02525	0.06283	0.07121	0.06793
5	0.01512	0.03881	0.04780	0.04949

Table 7: $\chi_{n,n}$ —diagonal Creutz ratios $\chi_{n,n}$ as computed in second-order perturbation theory and by Monte Carlo simulation at $\beta = 6.2$.

$a q$	0.5	0.8	$a q^* = 1.09$	1.5	2.0	3.0
χ_{22}	0.174	0.227	0.231	0.228	0.222	0.212

Table 8: Perturbative predictions for Creutz ratio χ_{22} using expansion parameter $\alpha_V(q)$ with various q 's at $\beta = 6.2$. Monte Carlo simulation gives $\chi_{22} = 0.233$.

$a q$	1	2	$a q^* = 2.65$	3	4	6
$-\ln W_{22}$	1.074	1.446	1.485	1.492	1.494	1.474

Table 9: Perturbative predictions for $-\ln W_{22}$ using expansion parameter $\alpha_V(q)$ with various q 's at $\beta = 6.2$. Monte Carlo simulation gives $W_{22} = 1.527$.

β	measured	mean field	perturbative
5.7	0.188	0.168	0.148
6	0.156	0.145	0.134
6.2	0.143	0.135	0.127
9	0.074	0.073	0.072
12	0.050	0.050	0.049
18	0.031	0.030	0.030

Table 10: $\alpha_V(\pi/a)$ as measured (from $-\ln W_{11}$), as computed from the bare lattice coupling using nonperturbative mean-field theory, and as computed in perturbation theory.

β	$\Delta M_\psi/\Lambda_{\text{lat}}$	$\Delta M_\Upsilon/\Lambda_{\text{lat}}$	$\sqrt{\sigma}/\Lambda_{\text{lat}}$	$\Delta M_\psi/\Lambda_V$	$\Delta M_\Upsilon/\Lambda_V$	$\sqrt{\sigma}/\Lambda_V$
5.7	130(9)	136(15)	134(1)	1.36(9)	1.42(15)	1.40(1)
5.8			120(1)			1.40(1)
5.9	105(6)		110(1)	1.31(7)		1.37(1)
6.0		105(4)	105(1)		1.36(5)	1.38(1)
6.1	96(6)			1.31(7)		

Table 11: Scale invariant ratios for the $1P$ - $1S$ mass differences ΔM for ψ 's and Υ 's and the square root of the string tension σ .

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