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MULTILOOP FEYNMAN DIAGRAMS AND DISTRIBUTION THEORY.

(IV) Asymptotic Expansions of Renormalized Diagrams.

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ABSTRACT

We complete formalization of the theory of Euclidean *As*-operation undertaken in the previous publications [1]–[3], by presenting formal regularization-independent proofs of general formulae for Euclidean asymptotic expansions of renormalized Feynman diagrams (inlcuding short-distance OPE, heavy mass expansions and mixed asymptotic regimes etc.) derived earlier in the context of dimensional regularization [8]–[11]. This result, together with the variant of the theory of UV renormalization developed in [3], demonstrates the power of the new techniques based on a systematic use of the theory of distributions and establishes the method of *As*-operation as a comprehensive full-fledged—and inherently more powerful—alternative to the BPHZ approach.

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1 Introduction

In the preceding three papers [1]–[3] we undertook a regularization-independent formalization of the heuristic reasoning behind a series of publications [4]–[11] in which efficient methods of perturbative calculations were found (for references to various 2-, 3-, 4- and 5-loop calculations performed using that techniques see [12]). The new techniques for studying multiloop Feynman diagrams is based on a systematic use of the ideas of the distribution theory, and the key notion is that of asymptotic expansion in the sense of distributions [4], [8], [1]. A very general context in which to construct such expansions is established by the *extension principle* [4], [1]—an abstract functional-analytic proposition analogous to the classical Hahn-Banach theorem. A specific realization of the recipe implied by the extension principle—and the key instrument of our techniques—is the so-called As-operation [8], [2]. The Euclidean version of As-operation constructed in [8], [2] is defined on a class of products of singular functions comprising integrands of Euclidean multiloop Feynman diagrams and returns their expansions in powers and logarithms of a small parameter (e.g., a mass) in the sense of distributions.

In the first paper [1] an analytical technique was developed for describing singularities of distributions, as well as a combinatorial formalism (*universum of graphs*) to work with hierarchies of graphs and their subgraphs—a formalism which makes it easy to utilize inherent recursive structures in problems involving multiloop diagrams. As a warm-up exercise, a very compact proof of a (localized) version of the familiar Bogoliubov-Parasiuk theorem in coordinate representation was presented with a purpose of illustrating in detail the typical ways of reasoning within the new techniques.

In [2] we began studying asymptotic expansions of Euclidean multiloop Feynman diagrams with respect to masses and momenta, which is done most naturally—cf. a detailed analysis of this problem from the point of view of apllications in [8]—in momentum representation. It was constructively proved that the As-operation exists for a large class of products of singular functions that includes integrands of Euclidean Feynman diagrams in momentum representation. The simplest example of such an expansion is with respect to a mass in a product of propagators, while the As-operation applied to such a product yields asymptotic expansions in powers and logs of the small mass, with coefficients given by explicit expressions.

In [3] the techniques of [1] and [2] was extended and applied to the problem of studying UV divergences in momentum representation. Thus, it was found [3] that a straightforward subtraction from the integrand of all those and only those terms of its asymptotic expansion "in the UV regime" that are responsible for UV divergences, is equivalent to the standard Bogoliubov R-operation. The UV finiteness of the R-operation in the new representation is ensured *by definition* and what had to be proved was equivalence to the standard formulation of the R-operation.[1] The class of subtraction schemes that naturally corresponds to the new definition (the so-called generalized minimal subtraction—or GMS—schemes) comprises massless schemes (including the MS scheme [14]) characterized by an extremely important property of polynomiality of the renormalization group functions in masses [16].

[1]which, strictly speaking, is not necessary, because one can prove correctness of the new representation directly, without reducing it to the standard construction. Such a proof would be easier but it requires to establish properties like existence of short-distance OPE considered in the present paper.

An extremely important result of [3] is that the coefficients of the As-operation constructed in [2] are exactly renormalized Feynman diagrams corresponding to subgraphs of the initial diagram. This fact has a dramatic technical impact on the problem of asymptotic expansions fof Feynman diagrams in momenta and masses.

The present paper addresses the problem of Euclidean asymptotic expansions of renormalized multiloop diagrams. Systematically using the techniques developed in [1]–[3], we present a compact and straightforward derivation of general Euclidean asymptotic expansions in the form of As-operation for integrated diagrams first introduced in [10]. Then the combinatorial techniques developed in [10] immediately allows one to obtain expansions for perturbative Green functions in OPE-like form.

The importance and feasibility of the general problem of Euclidean asymptotic expansions was realized in [8]–[11]. In those papers, a compact derivation of closed general formulae for such expansions was presented. The derivation of [8]–[11], however, aimed at obtaining the results in a shortest way and in a form immediately useful for phenomenological applications, so that a heavy use was made of the dimensional regularization and the MS scheme [14].[2] This left an open question of to what extent the results of [8]–[11] are independent of regularization. The question becomes even more interesting if one recalls the notorious difficulties that the dimensional regularization encounters when applied to models involving γ_5 or supersymmetry.

Another important realization of [8]–[11], [6] was that a proof of any asymptotic expansion—be it Wilson's OPE or a heavy-mass expansion or asymptotics of the quark formfactor in the Sudakov regime—is phenomenologically irrelevant unless the result exhibits *perfect factorization* of large and small parameters. At the technical level of diagram-by-diagram expansions, perfect factorization means that the expansions run in pure powers and logarithms of the expansion parameter. Such expansions possess the property of uniqueness (cf. the discussion in [2], section 1) which is tremendously useful from the technical point of view; for example, one immediately obtains that the As-operation must commute with multiplications by polynomials (see [9] and [2]). Another example is that one need not worry about properties like gauge invariance of the expansion in a given approximation: such properties are inherited by the expansion termwise from the initial amplitude, provided the expansion is "perfect" in the above sense.[3]

For the above reasons, we consider it our major task to clarify the issue of existence of "perfect" expansions in regularization-independent way.

It is a interesting fact that the derivation of OPE and, more generally, Euclidean asymptotic expansions presented in this paper—being more formalized than that of [8]–[11]—leads to a final formula which is much easier to deal with at the final stage of obtaining expansions for Green

[2]One can ponder on the tremendous heuristic potential of the dimensional regularization and the MS scheme. Although the understanding of the analytical aspects of the problem—including existence of the representation of the UV renormalization in the form described in [3]—was hardly lacking in [8]–[11], the presentation turned out feasible due to the property of dimensional regularization to nullify certain types of scaleless integrals.

[3]As was pointed out by J. Collins, such a property should be even more important for the problem of asymptotic expansions in Minkowskian regimes where both gauge invariance plays a greater role for phenomenological reasons, and the expansions one has to deal with are considerably more complicated. It may be said that the "relentless pursuit of perfection" (in the above sense) is one of the characteristic differences of the philosophy of the As-operation from the old BPHZ paradigm.

functions in a global "exponentiated" form. For example, unlike [11], we don't have to study inversion of the R-operation. However, as was stressed in [8], the derivation presented there was geared to the calculational needs of applied quantum field theory (primarily, applications to perturbative quantum chromodynamics) and, therefore, dealt explicitly with UV counterterms etc. From practical point of view, the formalism of the present paper offers, at least in its current form, no advantages as compared with the explicit recipes of [8]–[11].

Nevertheless, from theoretical point of view the formalization undertaken in the present series is more than just an exercise in rigour: there is the major unsolved problem of asymptotic expansions in non-Euclidean regimes, and it seems to be intrinsically intractable by the BPHZ method. On the other hand, extension of the As-operation to non-Euclidean regimes—taking into account the accumulated experience [19] which only needs to be properly organized within an adequate technical framework—seems to be a matter of near future. We hope, the experience gained in Euclidean problems will play a role in the more complicated cases.

One of the main points in any proof of OPE is to study the interaction of UV renormalization and the expansion proper. As we are going to show, within our formalism the problem reduces to double asymptotic expansions. Indeed, a renormalized Feynman diagram in the GMS formulation of [3] has a form of an integral of the remainder of the As-expansion of the integrand in the regime when all the dimensional parameters of the diagram are much less than the implicit UV cutoff. When one applies the second expansion with respect to some of the diagram's masses or external momenta, there emerges, essentially, a double As-expansion. All one has to prove is that the double expansion thus obtained factorizes into a composition of two commuting As-expansions and that the remainder of such a double expansion is bounded by a factorizable function of the small parameters.

The plan of the paper is as follows. In section 2 the problem of the asymptotic expansion of renormalized Feynman diagrams is reviewed, its heuristic analysis from the point of view of As-operation is given, and a recursive expansion formula is obtained. Section 3 is devoted to a detailed combinatorial analysis of that formula. In section 4 a convenient expression for expanded renormalized diagrams is obtained in a combinatorial form similar to UV R-operation, and its exponentiation on perturbative Green functions is considered. All analytical details are dealt with in section 5 where the key theorem on double asymptotic expansion is formulated and proved. Some less important technical results are relegated to two appendices.

The notations used in the present paper are largely the same as in [1]–[3].

2 Expansions of renormalized diagrams.

2.1 Formulation of the problem.

The feasibility and usefulness of considering asymptotic expansions of Feynman diagrams for general Euclidean asymptotic regimes was realized in [8]–[11]. The formulation of the Euclidean asymptotic expansion problem follows [9] except that we now use GMS-schemes instead of dimensional regularization and the MS-scheme (which is a special case of GMS-schemes) and consider only the simplified version of the expansion problem without contact terms.

Asymptotic regime. Let G be an arbitrary Euclidean multiloop Feynman diagram. Let $G(p,\ldots)$ be its unrenormalized momentum-space integrand, where p collectively denotes its integration (loop) momenta while dots stand for other dimensional parameters on which G also depends. Those parameters include masses (which enter the propagators of $G(p,\ldots)$) and external momenta, and will be referred to as *external parameters* of the diagram G. We wish to construct asymptotic expansion for the diagram G in the asymptotic regime when some of the external parameters of G are much larger than others.

Denote the large (or "heavy") external parameters of G collectively as M, and small (or "light") as m. Formally speaking, As-expansions that we study require presence of a scalar parameter with respect to which to expand. As an abuse of notation, we will use the same symbol m to represent such a parameter which goes to zero and to which all the light parameters are proportional. Thus, the asymptotic regime we wish to consider is described as $m \to 0$ and $M = O(1)$.

It makes sense to assume that neither the set M nor m are empty.

Perfect factorization. An extremely important requirement on expansions at the diagram level is that they should ran in powers and logs of the expansion parameter. At the level of Green functions, this means that the coefficient functions depending on large parameters should not contain non-analytic (logarithmic) dependences on light parameters. It has been realized that only expansion possessing the property of perfect factorization have phenomenological significance; in particular, only such expansions are useful in models with massless particles like QCD. Moreover, such expansions possess the property of uniqueness which turns out to be extremely useful; e.g. it simplifies study of gauge properties of expansions since they then inherit the gauge properties of non-expanded Green functions [20]. Detailed discussion of phenomenological and technical aspects of this requirement are discussed in detail in [9]. Here we only note that in the papers [13] where the fact of power-and-log nature of expansions was established at a formal level, no convenient explicit formulae were presented, while the standard BPHZ derivation of OPE resulted in expansions in which coefficient functions that contained all the dependence on the large momentum also depended on the light masses in a non-trivial way. OPE possessing the property of perfect factorization was obtained in [6].

Renormalization. We assume that the diagram G (i.e. the integral of $G(p,\ldots)$ over p in infinite bounds) is renormalized using the GMS prescription of [3]. (Following [3], we will denote GMS-renormalized diagram as $\mathcal{R}\circ G$.) The GMS prescription comprises all the schemes that possess the property of polynomial dependence of the corresponding renormalization group functions on dimensional parameters.[4] This is an important assumption both technically and conceptually.

Conceptually, the asymptotic expansions in masses and momenta obtained within such schemes possess, as we will see, the above mentioned property of perfect factorization.[5]

Technically, the GMS prescription amounts [3] to subtraction from the integrand of those and only those terms in the asymptotic expansion in the UV regime that generate UV divergences, while the necessary modification of the logarithmic terms at zero momentum (the operation $\tilde{\mathbf{r}}$) does not affect the dependences on the dimensional parameters. The net effect, as we will see, is to trivialize the problem of expansion of renormalized diagrams by reducing it to a study of double As-expansions (see subsec.2.3 below).

Renormalization introduces an additional external parameter for G besides its masses and external momenta. Such parameter is usually denoted as μ. The dependence on μ is known explicitly: $\mathcal{R}{\circ}G$ is a polynomial in $\log\mu$. Therefore, it is of no practical consequence whether μ is considered as a heavy or light parameter. For definiteness, we will treat it as a heavy parameter.

External momenta as fixed parameters. The simplest version of the expansion problem emerges if one fixes them at some values and then treats them on an equal footing with masses. This is what we will assume for now. It should be emphasized that in principle it is not necessary to fix the momenta at generic non-zero or otherwise non-exceptional values as long as the initial expression one wishes to expand is well defined. We will discuss this point in more detail below. Here we only wish to note that the precise conditions of when a diagram is well-defined depend on the details of the structure of specific Feynman diagrams in a specific model, and it is not our aim to discuss such conditions. The only important thing is that our techniques is insensitive to such details.

A somewhat more complicated (and more general) version of the problem would be to consider the diagram as a distribution in the external momenta. Here one expects additional terms to appear in the expansion; such "contact" terms should be proportional to δ-functions of linear combinations of the external momenta (cf. below the discussion of IR singularities of the non-expanded diagrams). This case will be considered in a separate publication. Note that within the framework of dimensional regularization and the MS-scheme explicit expressions for contact terms were obtained in [10], [11].

Linear restrictions on momenta. Another aspect of the problem is how one divides the external momenta into heavy and light. The point is that certain sums of heavy external momenta should be allowed to be light, i.e. $O(m)$, in some physically meaningful situations. This amounts to imposing linear restrictions on the heavy external momenta. Such restrictions were analyzed in detail in [10], [11] where an important class of *natural restrictions* was identified.

[4] The first and most important example is the MS-scheme [14], for which the polynomiality property was established in [16].

[5] The fact that the property of perfect factorization is a necessary condition for existence of OPE in the MS-scheme was observed in [15]. For non-GMS-schemes perfectly factorized expansions have a more complicated form.

The natural restrictions can be described as follows. One divides the external lines of G through which heavy external momenta flow, into several "bunches". Then for each bunch one imposes a single restriction that the (algebraic) sum of the corresponding momenta is $O(m)$ (i.e. it is equal to a combination of the light momenta only). Otherwise the heavy momenta are assumed to take generic values. Note that there always is at least one such restriction due to overall momentum conservation: the sum of all heavy momenta must be equal to the sum of all light momenta. As a rule our analysis at the level of individual diagrams is quite general, but the interpretation of the results at the level of Green functions turns out to be more transparent for asymptotic regimes with natural restrictions.

Infrared divergences. It should be pointed out that Feynman integrands in models with massless particles possess singularities at finite values of p due to massless propagators. Such singularities can formally be even non-integrable but, nevertheless, spurious in the sense that they cancel out after performing integrations and taking into account specific algebraic properties of the integrands like gauge invariance (cf. the Kinoshita-Poggio-Quinn theorem [17]), or they may require special treatment whose exact form is determined by additional considerations. Such considerations are "orthogonal" to the expansion problem proper (in the sense explained below) and their discussion goes beyond the scope of the present paper. However, the following remarks can be made here.

One can remove such singularities from the initial integrand using a version of the special subtraction operation $\tilde{\mathbf{R}}$ introduced in [2, 3]. Then one can perform all the reasoning of the present paper taking into account that our techniques is essentially insensitive to presence of such $\tilde{\mathbf{R}}$ in the integrand (see section 7 in [2] and our subsect.3.4).

Alternatively, one can regularize such singularities by introducing a mass, m_0, for the massless particles, and after the expansion in m_0 is done, to consider the limit $m_0 \to 0$. This would be technically equivalent to considering double expansion in the regime $m_0 \ll m \ll M$. Such a problem can be studied by straightforward application of the results of the present paper. Indeed, our main concern here is exactly the extension of the results on simple As-expansions to the case of double As-expansions (see below subsec.2.3). Needless to say, further extension to three-fold expansions etc. is completely straightforward. The net effect is that the multiple As-expansions of the above type can be obtained by performing simple expansions sequentially, in any order. Thus, one can first expand in the regime when m_0 and m are much less than M, and after that, perform termwise expansion of the result for $m_0 \ll m$. How many terms in the expansion in m_0 one should retain, and what one should do with the singularities in m_0 , is to be decided from the specifics of the problem. This way to proceed is essentially equivalent to the first one based on the use of the operation $\tilde{\mathbf{R}}$, because the As-expansion in m_0—as any As-expansion—can be expressed as an $\tilde{\mathbf{R}}$ with suitably chosen finite counterterms. However, introduction of a non-zero mass m_0 has an advantage of not requiring new notations.

Either way, the factors in the final expansion that depend on the heavy parameters of the problem (e.g. the coefficient functions of the OPE which depend on the heavy momentum Q) are independent of the IR structure of the initial diagram.[6] Only the factors in which

[6] Another way to put it is to say that the OPE coefficient functions are analytical in the light mass parameters, including the regulator mass m_0. Therefore, whatever one does with m_0 afterwards (e.g. taking it to 0) will not, essentially, affect the coefficient functions.

the dependence on the light parameters is concentrated are sensitive to IR structure. This is essentially the property of perfect factorization.

To reiterate: what happens when m_0 is taken to zero depends on details like gauge invariance of the model and UV-renormalization procedure adopted, and affects only those factors in the expansion which contain the non-trivial dependence on the light parameters. On the other hand, what we wish to concentrate on is the analytical aspects of the expansion problem proper, avoiding inessential details. All that can be said in this respect is that our techniques offers efficient ways to deal with such singularities. For the above reasons, in what follows we will simply ignore such singularities, in order to avoid unnecessary notational complications. This will allow us to concentrate on the non-trivial aspects of our techniques.

Lastly, it is convenient to assume that the set of heavy parameters M is non-empty, because otherwise the expansions in κ and m coincide and the problem degenerates into a trivial one. (Triviality means that the renormalized diagram itself is a power of m times a polynomial in $\log m$, so that its As-expansion coincides with the diagram itself.)

2.2 Expansion of "regularized" diagram.

The GMS-renormalized diagram can be represented as follows [3]:

$$\mathcal{R}\circ G(M,m) \equiv \mathcal{R}\circ \int dp\, G(p,M,m) \tag{2.1}$$

$$\equiv \lim_{\Lambda\to\infty} \int dp\, H^{\Lambda}(p) \left[G(p,M,m) - \tilde{\mathbf{r}}\circ \mathbf{As}'_{\kappa}\circ G(p,\kappa M,\kappa m)\right]_{\kappa=1}.$$

We have introduced an auxiliary parameter κ on the r.h.s. of (2.1) in order to formally describe the asymptotic expansion of the integrand in the UV regime in which all dimensional parameters of G are small as compared to the cutoff Λ.

What we ultimately wish to do is to determine the explicit expression for the As-expansion of (2.1) in $m \ll M$ which we denote by the same symbol **As** as we use for the As-operation on non-integrated products:

$$\mathcal{R}\circ G(M,m) \underset{m\to 0}{\simeq} \mathbf{As}_m\circ\mathcal{R}\circ G(M,m) = ? \tag{2.2}$$

Such a notation is very natural both in view of the general definitions of section 2 of [2] and because the new version of the operation **As** (2.2), as we will see, is closely related to the operation **As** already defined on non-integrated products. Note that the As-operation was first introduced for *integrated* diagrams, in [10].

It is natural to try to find the expression for (2.2) by applying the operation $\mathbf{As}_m$, which has already been defined on products of singular functions, to the expression in square brackets on the r.h.s. of (2.1). As we will see, this is indeed possible. Moreover, the corresponding calculation exhibits a recursive pattern: in order to derive (2.2) for G itself one has to assume validity of an expansion of the type (2.2) for integrated renormalized graphs with a lesser number of loops than G.

Studying (2.1) at fixed Λ. First of all, let us fix $\Lambda < \infty$ and study the expansion at $m \to 0$ of the resulting "regularized" integral. One can immediately write down the expansion for the first term on the r.h.s. of (2.1), $\int dp\, H^\Lambda(p)\, G(p,M,m)$, by directly using the techniques of [2], i.e. by applying the As-operation to $G(p,M,m)$:

$$\int dp\, H^\Lambda(p)\, G(p,M,m) \underset{m\to 0}{\simeq} \int dp\, H^\Lambda(p)\, \mathbf{As}_m \circ G(p,M,m).$$

In order to expand the remaining contributions to (2.1), one should consider the structure of the expression $\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ G$ in more detail. From eqs.(2.22)–(2.24) in [3] it follows that:

$$\begin{aligned} &\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ G(p,\kappa M,\kappa m)|_{\kappa=1} \qquad (2.3)\\ &= \sum_{\gamma<G}\sum_\alpha \tilde{\mathbf{R}}\circ\left\{\delta_\alpha(p_\gamma)\,\mathbf{T}_\kappa\circ G\backslash\gamma(p,\kappa M,\kappa m)\right\}_{\kappa=1} \times \mathcal{R}\circ \int dp'_\gamma\, \mathcal{P}_\alpha(p'_\gamma)\,\gamma(p'_\gamma,M,m). \end{aligned}$$

(Note that the summation here runs over subgraphs corresponding to the singularities of the formal expansion in κ.[7])

As a convenient abuse of notations we will often omit κ in expressions like the r.h.s. of (2.3) everywhere except in $\mathbf{T}_\kappa$:

$$\mathbf{T}_\kappa\circ G\backslash\gamma(p,\kappa M,\kappa m)_{\kappa=1} \to \mathbf{T}_\kappa\circ G\backslash\gamma(p,M,m). \qquad (2.4)$$

This should cause no confusion.

The r.h.s. of (2.3) is a sum of terms in which the dependences on the integration momenta p and on the external parameters are completely factorized. Indeed, the distribution $\tilde{\mathbf{R}}\circ\{\delta_\alpha(p_\gamma)\,\mathbf{T}_\kappa\circ G\backslash\gamma(p,\ldots)\}$ is a polynomial of the external parameters M and m (this is due to the action of $\mathbf{T}_\kappa$ and the fact that $\tilde{\mathbf{R}}$ does not affect M- and m-dependences). On the other hand, the factor $\mathcal{R}\circ\int dp'_\gamma\,\mathcal{P}_\alpha(p'_\gamma)\,\gamma(p'_\gamma,M,m)$ contains a non-trivial dependence on M and m but is independent of p. Moreover, the latter factor has the same form as the initial expression (2.1) up to a replacement $p \to p_\gamma$ and $G(p) \to \mathcal{P}_\alpha(p_\gamma)\times\gamma(p_\gamma)$. Therefore, it is natural to make an inductive assumption that the operation **As** has been defined on the products $\mathcal{R}\circ\int dp'_\gamma\,\mathcal{P}_\alpha(p'_\gamma)\,\gamma(p'_\gamma,M,m)$ for all $\gamma < G$. Then the As-expansion of (2.3) is given by applying such **As** to the last factor in (2.3):

$$\begin{aligned} \tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ G(p,M,m) &\underset{m\to 0}{\simeq} \mathbf{As}_m\circ\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ G(p,M,m) \qquad (2.5)\\ &= \sum_{\gamma<G}\sum_\alpha \tilde{\mathbf{R}}\circ\left\{\delta_\alpha(p_\gamma)\,\mathbf{T}_\kappa\circ G\backslash\gamma(p,M,m)\right\} \times \mathbf{As}_m\circ\mathcal{R}\circ\int dp'_\gamma\,\mathcal{P}_\alpha(p'_\gamma)\,\gamma(p'_\gamma,M,m). \end{aligned}$$

It is important to note that there has emerged a compact recursive pattern which is characteristic of our techniques: expansion of the GMS-renormalized diagram G (2.1) is reduced to an essentially similar problem but with a lesser number of integration momenta.

[7] We will also have to deal with singularities of, and the corresponding operation $\tilde{\mathbf{R}}^m$ for, the expansion in m. The resulting notational complications will be dealt with in subsec.3.1 below.

In the case of a single loop momentum in the initial diagram G (or one-dimensional p if one does not limit the discussion to Feynman diagrams proper), our expansion problem degenerates into a trivial one. Because of this and owing to the explicit recursive pattern in (2.5) we can assume that the problem has been solved for all GMS-renormalized diagrams with lesser number of loop momenta, i.e. that the operation **As** on the r.h.s. of (2.5) is well defined. Using this inductive assumption completes the expansion of the "regularized" integral.

Explicit expressions for $\mathbf{As}_m \circ \mathcal{R} \circ G$ which provide solution to the above recursive procedure, will be presented below in section 3.

2.3 The limit $\Lambda \to \infty$.

The only important question of analytical nature that needs to be answered is whether the asymptotic expansion constructed above for fixed Λ remains such after taking the limit $\Lambda \to \infty$, i.e. whether the As-operation defined by the expression[8]:

$$\mathbf{As}_m \circ \mathcal{R} \circ \int dp\, G(p, M, m) \stackrel{\text{def}}{=} \lim_{\Lambda \to \infty} \int dp \circ H^{\Lambda}(p) \mathbf{As}_m \circ [1 - \tilde{\mathbf{r}} \circ \mathbf{As}'_{\kappa}] \circ G(p, M, m) \qquad (2.6)$$

delivers a true As-expansion for the integral (2.1). The answer is yes, and it can be justified in two steps.

(i) As a first step, it is natural to consider existence of the limit on the r.h.s. of (2.6). To this end we split the integration region in (2.6) into two parts by introducing an intermediate cut-off at the radius μ in order to explicitly separate the non-trivial asymptotic region $p \to \infty$ from the point $p = 0$ where the expression is complicated by the operator $\tilde{\mathbf{r}}$ but the expansion is essentially straightforward:

$$\mathbf{As}_m \circ \mathcal{R} \circ \int dp\, G(p, M, m) \equiv \int dp\, H^{\mu}(p)\, \mathbf{As}_m \circ [1 - \tilde{\mathbf{r}} \circ \mathbf{As}'_{\kappa}] \circ G(p, M, m) \qquad (2.7)$$

$$+ \lim_{\Lambda \to \infty} \int dp\, H^{\Lambda}_{\mu}(p) \mathbf{As}_m \circ [1 - \mathbf{As}'_{\kappa}] \circ G(p, M, m).$$

(For definition of the functions H see subsec.8.5 of [1].) The second term in this expression will be finite if the two As-operations commute:

$$\mathbf{As}_m \circ \mathbf{As}'_{\kappa} = \mathbf{As}'_{\kappa} \circ \mathbf{As}_m. \qquad (2.8)$$

Then the operator $1 - \mathbf{As}'_{\kappa}$ can be taken out to the left of $\mathbf{As}_m$, so that existence of the limit $\Lambda \to \infty$ will follow automatically (recall in this respect the motivations and construction of the operation $\mathcal{R}$ in section 1 of [3].)

The commutativity (2.8) (see also (3.6)) is one of the central results of the present paper. Its nature is essentially algebraic. Here we only note that explicit formulae for $\mathbf{As} \circ \mathcal{R} \circ G$ follow from (3.6) (see section 3).

[8]As should be clear from the above construction, the composition of the two As-operations here is purely algebraic: the second operation is applied termwise to the series generated by the first one irrespective of approximation properties of the resulting expression.

(*ii*) The second step is analytical in nature: one verifies that the remainder of the expansion (2.7) vanishes at the required rate as $m \to 0$. This is formally expressed as

$$[1-\mathbf{As}^n_m]\circ\mathcal{R}\circ\int dp\, G(p,M,m) \tag{2.9}$$

$$\stackrel{\text{def}}{=} \lim_{\Lambda\to\infty}\int dp\, H^\Lambda(p)[1-\mathbf{As}^n_m]\circ[1-\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa]\circ G(p,M,m) = o(m^n).$$

To check (2.9), one splits the integration region as in (2.7):

$$[1-\mathbf{As}^n_m]\circ\mathcal{R}\circ\int dp\, G(p,M,m)$$

$$\equiv \int dp\, H^\mu(p)[1-\mathbf{As}^n_m]\circ[1-\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa]\circ G(p,M,m)$$

$$+\lim_{\Lambda\to\infty}\int dp\, H^\Lambda_\mu(p)[1-\mathbf{As}^n_m]\circ[1-\mathbf{As}'_\kappa]\circ G(p,M,m).$$

For the first term on the r.h.s. the estimate (2.9) is true by definition of **As** and because $H^\mu(p)$ is an ordinary test function. For the second term, one represents $H^\Lambda_\mu(p)$ as an integral over spherical layers of radius λ (see subsec.8.5 of [1]):

$$H^\Lambda_\mu(p) = \int_\mu^\Lambda \frac{d\lambda}{\lambda}\,\eta_\lambda(p). \tag{2.10}$$

Then one rescales the integration variable $p \to \lambda p$ and uses the uniformity properties of G to arrive at the following expression:

$$\int_\mu^\Lambda \frac{d\lambda}{\lambda}\int dp\,\eta_1(p)\,[1-\mathbf{As}^n_m]\circ[1-\mathbf{As}_{\kappa/\lambda}]\circ G(p,M\kappa/\lambda,m\kappa/\lambda). \tag{2.11}$$

Recall that one can retain only those terms in $\mathbf{As}'_\kappa$ that are responsible for UV divergences (see the text immediately after (1.4) of [3]). Since λ always divides κ in the above expression, one can see that the $o(m^n)$ estimate for (2.11) follows from an estimate of the type

$$\left|\int dp\,\eta_\lambda(p)\,[1-\mathbf{As}^n_m]\circ[1-\mathbf{As}^l_\kappa]\circ G(p,\kappa M,\kappa m)\right| < o(m^n)\times o(\kappa^l). \tag{2.12}$$

This can be adopted as (part of) an exact analytical interpretation of the informal statement that the algebraic composition of the two (commuting) *As*-operations yields a true double asymptotic expansion in the sense of distributions for the integrand $G(p,\kappa M,\kappa m)$.

The inequality (2.12) is easy to understand at a heuristic level. Indeed, the remainder of an asymptotic expansion is often estimated (at least in the cases when the expansion has a relatively simple analytical nature—in our case one deals with expansions of integrals of rational functions, however cumbersome) by the last discarded term, which in our case is $O(m^{n+1})\times O(\kappa^{l+1})$ (up to inessential logarithms), which immediately explains (2.12).

Actually, the inequalities that we prove in Theorem 1 (see below section 5) are more stringent than (2.12); in particular, they also describe dependence of the bounds on the support of the test function.

2.4 Summary

We have exhibited the recursive structure of the *As*-operation for GMS-renormalized diagrams and identified the inductive assumptions as well as the propositions that have to be proved. To proceed, we first have to study the structure of our expressions in more detail and derive an explicit expression for $\mathbf{As}_m \circ \mathcal{R} \circ \int dp\, G$. This will allow us to check the formal commutativity of the two *As*-operations $\mathbf{As}_m$ and $\mathbf{As}_\kappa$. Second, we have to present and prove the inequalities that guarantee validity of (2.12).

3 Explicit expressions for $\mathbf{As}_m \circ \mathcal{R} \circ G$.

We are now going to derive explicit formulae for (2.2). We will do this assuming (by induction) that the two *As*-operations commute on subgraphs of G. After that it will not be difficult to check the commutativity on Γ itself. Note that the recursion is correct because whatever property one wishes to prove for a graph, one only has to make assumptions about its subgraphs.

3.1 Some notations.

Operation $\mathcal{R}$ on integrands. So far we have been using the operation $\mathcal{R}$ defined on integrated diagrams. But since now we will have to work with integrands, it is convenient to use the same symbol $\mathcal{R}$ to denote the operation of UV subtractions *prior* to integrations over p:

$$\mathcal{R} \stackrel{\text{def}}{=} 1 - \tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa. \tag{3.1}$$

This operation is defined on graphs G as well as on the products of the form $\mathcal{P}_a(p_\gamma) \times \gamma(p_\gamma)$ where $\mathcal{P}_a(p_\gamma)$ is a polynomial while γ is any (κ- or m-) subgraph of G.

$\mathcal{R}$ associated with $\tilde{\mathbf{R}}$. Although the entire arbitrariness in the definition of $\mathcal{R}$ for an individual diagram G is in the operator $\tilde{\mathbf{r}}$ (more precisely, $\tilde{\mathbf{r}}_{(G)}$), it is more convenient to think about $\tilde{\mathbf{r}}$ in terms of the operation $\tilde{\mathbf{R}}$ (which also involves operators $\tilde{\mathbf{r}}_{(\gamma)}$ for $\gamma < G$). This is because all the explicit expressions for $\mathcal{R}$ (cf. eq.(2.22) of [3]) involve $\tilde{\mathbf{R}}$. On the other hand, the two points of view are equivalent if one recalls that in the problems of perturbative quantum field theory one deals with the entire universum of graphs: specifying the family of operations $\tilde{\mathbf{R}}_{(G)}$ on the entire universum of graphs G is equivalent to specifying the family of operators $\tilde{\mathbf{r}}_{(G)}$. Then, to fix an operation $\mathcal{R}$ (or, equivalently, $\tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa$) on a hierarchy of graphs, it is sufficient to fix an operation $\tilde{\mathbf{R}}$ on it.

For the above reasons we will say that an operation $\mathcal{R}$ (and $\tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa$) is *associated with* some operation $\tilde{\mathbf{R}}$, whenever it is necessary to indicate this kind of relationship between the two operations.

κ- and m-subgraphs. Since there are two expansions—in κ and m—in our problem that one has to deal with simultaneously, there are two systems of singular planes, complete subgraphs etc. in the same graph G.[9] In order to distinguish the objects from the two systems, we will call them κ- and m-subgraphs etc.

We will retain the symbols $\tilde{\mathbf{R}}$ and $\mathcal{R}$ for the operations associated with κ-singularities, and use the symbols $\tilde{\mathbf{R}}^m$ and $\mathcal{R}^m$ for the operations associated with m-singularities.

By default, a subgraph in a formula is κ-subgraph. Presence of m-subgraphs will be explicitly indicated.

The relation between the two systems of subgraphs is based on a simple principle: any factor $g \in G$ which is "m-singular" (i.e. develops singularities after expansion in m) is automatically "κ-singular"—because the set of momenta and masses with respect to which the expansion is done in the latter case comprises all such parameters in the former case. Thus, to every m-subgraph Γ there corresponds a unique κ-subgraph H obtained from Γ by "κ-completion".

3.2 General formula for the expansion $\mathbf{As}_m \circ \mathcal{R} \circ G$.

Let us first explain the structure of (2.8). The dependence on p, m and M in (2.8) can be described as follows:

$$\mathbf{As}'_\kappa \circ G(p, M, m) = \sum D(p) \times C(M, m), \tag{3.2}$$

which correponds to the expansion (2.3). It is clear that $D(p)$ are distributions well-defined everywhere except for the point $p = 0$. The action of $\mathbf{As}_m$ in (2.5) can be described as

$$\mathbf{As}_m \circ \mathbf{As}'_\kappa \circ G(p, M, m) \equiv \sum D(p)\ \mathbf{As}_m \circ C(M, m) = \sum D(p)\ A(M)\ B(m) \tag{3.3}$$

(the explicit fromulae for $\mathbf{As}_m \circ C$ are yet to be determined). On the other hand,

$$\mathbf{As}_m \circ G(p, M, m) = \sum C'(p, M)\ B'(m) \tag{3.4}$$

(cf. the explicit expression below in (3.7)), and

$$\mathbf{As}'_\kappa \circ \mathbf{As}_m \circ G(p, M, m) \equiv \sum \mathbf{As}'_\kappa \circ C'(p, M)\ B'(m) = \sum D'(p)\ A'(M)\ B'(m). \tag{3.5}$$

So far we don't know the form of C' and how $\mathbf{As}'_\kappa$ does its job on C' (which will be explained below). Nevertheless, the commutativity (2.8) implies that $D = D'$ etc.[10] It follows immediately that the commutativity is preserved if one replaces $\mathbf{As}'_\kappa$ by $\tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa$:

$$\mathbf{As}_m \circ \tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa = \tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa \circ \mathbf{As}_m. \tag{3.6}$$

Indeed, on both sides of (3.6) the operator $\tilde{\mathbf{r}}$ acts—as demonstrated above—on exactly the same p-dependent distributions $D(p)$.

[9] In general, one should also consider the third system of subgraphs—the one corresponding to singularities of unrenormalized non-expanded integrand $G(p, \ldots)$ that were discussed in subsect.2.1. In order to keep notations simple, we agreed not to indicate explicitly the possible presence of such singularities, the corresponding operation $\tilde{\mathbf{R}}$ etc.

[10] Strictly speaking D etc. depend on a summation index so that one may need to take linear combinations to establish the equality.

The explicit version of (3.4) is analogous to (2.3) (cf. eqs.(2.22)-(2.24) of [3]):

$$\mathbf{As}_m \circ G(p,M,m) = \sum_{\substack{\emptyset \leq \Gamma < G \\ \Gamma \text{ is } m\text{-subgraph}}} \sum_a \tilde{\mathbf{R}}^m \circ [\delta_a(p_\Gamma)\, \mathbf{T}_m \circ G \backslash \Gamma(p,M,m)] \times \mathcal{R}^m \circ \langle \mathcal{P}_{a,\Gamma} * \Gamma \rangle, \quad (3.7)$$

where

$$\mathcal{R}^m \circ \langle \mathcal{P}_{a,\Gamma} * \Gamma \rangle \equiv \langle \mathcal{P}_{a,\Gamma} * \mathcal{R}^m \circ \Gamma \rangle \equiv \lim_{\Lambda \to 0} \int dp'_\Gamma\, H^\Lambda(p)\, \mathcal{P}_{a,\Gamma}(p'_\Gamma)\, \mathcal{R}^m \circ \Gamma(p'_\Gamma, m). \quad (3.8)$$

Note that since Γ is an m-subgraph of G, the object (3.8) is an almost-uniform (in the sense of [2], subsec.1.2) function of m and is independent of M and p. On the other hand, all the dependence on p is concentrated in the square-bracketed factor on the r.h.s. whose m-dependence is trivial while the M-dependence is not.

We now wish to apply the operation $\mathcal{R} = 1 - \tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa$ termwise to the above expansion (similarly to (3.5)). Owing to (3.6) this is equivalent to $\mathbf{As}_m \circ \mathcal{R} \circ G$. Since As-expansion of a product is a product of As-expansions [2] and since the only non-trivial dependence on κ is via M in $G \backslash \Gamma(p,M,m)$, one has:

$$\mathbf{As}_m \circ \mathcal{R} \circ G(p,M,m) \quad (3.9)$$

$$= \sum_{\substack{\emptyset \leq \Gamma < G \\ \Gamma \text{ is } m\text{-subgraph}}} \sum_a \mathcal{R} \circ \tilde{\mathbf{R}}^m \circ [\delta_{a,\Gamma}(p_\Gamma)\, \mathbf{T}_m \circ G \backslash \Gamma(p,M,m)] \times \mathcal{R}^m \circ \langle \mathcal{P}_{a,\Gamma} * \Gamma \rangle.$$

The action of $\mathcal{R}$ on the somewhat unusual expression in square brackets will be explained in the next subsection.

The above eq.(3.9) constitutes the final result of the analysis of the problem of Euclidean symptotic expansions of Feynman diagrams as seen from the point of view of the abstract theory of As-expansions of products of singular functions. Below in section 4 we will transform it—using specific properties of Feynman diagrams proper—to a more convenient form similar to the As-operation as presented in [11]. The immediate remarks to be made here are as follows.

(*i*) The non-analytic dependences on the heavy parameters M and the light parameters m are clearly factorized in (3.9). Indeed, the expression in angle brackets is independent of M by construction. On the other hand, the M-dependent distributions over p in square brackets are pure power series in m (due to the action of $\mathbf{T}_m$ and the fact that neither $\tilde{\mathbf{R}}^m$ nor $\mathcal{R}$ affect the resulting powers of m). Therefore, it is clear—in the context of ordinary short-distance OPE—that the angle-bracketed expressions correspond to matrix elements of OPE (the polynomials $\mathcal{P}$ then correspond to vertices with composite operators) while the square-bracketed expressions, to coefficient functions.

(*ii*) All the quantities in (3.9) are finite by construction: the angular-bracketed "matrix element" has its UV divergences removed by $\mathcal{R}^m$, while the IR and UV singularities of the square-bracketed "coefficient functions" are eliminated by $\tilde{\mathbf{R}}^m$ and $\mathcal{R}$, respectively.

(*iii*) The expression (3.9) as a whole is independent of the specific choice of the operation $\tilde{\mathbf{R}}^m$ and the associated operation $\mathcal{R}^m$. This is a usual feature of representation of an As-expansion in terms of an intermediate $\tilde{R}$-operation (recall that an As-expansion is unique [9], [2]).

3.3 Defining $\tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa \circ \tilde{\mathbf{R}}^m \circ [\ldots]$ in (3.9).

Consider the following object from (3.9) (p'_Γ is the integration momentum implied in $\langle \ldots \rangle$; it is isomorphic to p_Γ):

$$\sum_a \delta_a(p_\Gamma)\, \mathbf{T}_m \circ G\backslash\Gamma(p, M, m) \times \mathcal{P}_{a,\Gamma}(p'_\Gamma) \qquad (\Gamma \text{ is } m\text{-subgraph}). \tag{3.10}$$

When δ_a is integrated out, its derivatives also affect $G\backslash\Gamma$. Let us exhibit this explicitly using the fact that the above pattern of the polynomials $\mathcal{P}_a$ and the δ-functions δ_a is characteristic of the operation of Taylor expansion (cf. eq.(9.8) in [1]). We use the following elementary identity:

$$\sum_a \delta_a(p_\Gamma) F(p_\Gamma) \times \mathcal{P}_a(p'_\Gamma) = \sum_b \delta_b(p_\Gamma) \times \mathbf{T}_{p'_\Gamma} \circ F(p'_\Gamma)\, \mathcal{P}_b(p'_\Gamma). \tag{3.11}$$

Now split the variable p as

$$p = (p_\Gamma, p_{G\backslash\Gamma}),$$

and define

$$m_\Gamma = (m, p_\Gamma), \qquad m'_\Gamma = (m, p'_\Gamma). \tag{3.12}$$

Then we can rewrite (3.10) as

$$\text{eq.}(3.10) = \sum_b \delta_b(p_\Gamma) \times \left[\mathbf{T}_{m'} \circ G\backslash\Gamma(p_{G\backslash\Gamma}, M, m')\right] \mathcal{P}_b(p'_\Gamma). \tag{3.13}$$

The variables $p_{G\backslash\Gamma}$ on the r.h.s. parameterize the singular plane π_Γ on which the entire expression is localized. Note that $G\backslash\Gamma$ is being expanded in p'_Γ, so that all the singularities are with respect to $p_{G\backslash\Gamma}$.

The above formula allows one to define $\tilde{\mathbf{R}}^m$ on such expressions (cf. the reasoning in subsect.2.4 in [3]) as follows:

$$\sum_a \tilde{\mathbf{R}}^m \circ \left[\delta_a(p_\Gamma)\, \mathbf{T}_m \circ G\backslash\Gamma(p, M, m)\right] \times \mathcal{P}_{a,\Gamma}(p'_\Gamma) \qquad (\Gamma \text{ is } m\text{-subgraph}) \tag{3.14}$$

$$\stackrel{\text{def}}{=} \sum_b \delta_b(p_\Gamma) \times \tilde{\mathbf{R}}^m \circ \mathbf{T}_{m'} \circ G\backslash\Gamma(p_{G\backslash\Gamma}, M, m')\, \mathcal{P}_b(p'_\Gamma)$$

$$= \delta(p_\Gamma) \times \tilde{\mathbf{R}}^m \circ \mathbf{T}_{m'} \circ G\backslash\Gamma(p_{G\backslash\Gamma}, M, m') + \ldots,$$

where on the r.h.s. we have not shown the terms proportional to derivatives of the δ-function—such derivatives vanish after integration over p which eventually has to be done. Note that there need not be any correlation between the definitions of the operation $\tilde{\mathbf{R}}^m$ on $\mathbf{T}_m G$ and on $\mathbf{T}_{m'} \circ G\backslash\Gamma$ (cf. the reasoning in section 2 of [3]).

In (3.9), the expression (3.14) is being acted on by $\mathcal{R} = 1 - \tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa$. Here one only has to deal with the non-trivial dependence on M on the r.h.s., and it is easy to understand that $\mathbf{As}'_\kappa$ should be applied termwise to the r.h.s., effectively getting combined with $\tilde{\mathbf{R}}^m$.[11]

[11] In [2] the formulae for the *As*-operation were presented only for a class of singular functions without non-integrable singularities prior to expansion. In the present case, we for a first time encounter a situation where an *As*-operation—$\mathbf{As}'_\kappa$—is being applied to an expression (3.14) which involves an $\tilde{R}$-operation. As was noted in [2], extension the formula for *As*-operation to the case of singular initial expression is straightforward (see also below subsect.3.4).

It remains to note that, as usual, the operator $\tilde{\mathbf{r}}$ on $\mathbf{As}'_\kappa{\circ}\tilde{\mathbf{R}}^m{\circ}\mathbf{T}_{m'}{\circ}G\backslash\Gamma$ (or, equivalently, $\tilde{\mathbf{R}}$ on the $\mathbf{T}_\kappa{\circ}\mathbf{T}_{m'}{\circ}G\backslash\Gamma \equiv \mathbf{T}_{\kappa'}G\backslash\Gamma$, where $\kappa' = (\kappa, p'_\Gamma)$) can be chosen independently from $\tilde{\mathbf{r}}$ on $\mathbf{As}'_\kappa{\circ}\tilde{\mathbf{R}}^m{\circ}\mathbf{T}_m{\circ}G$ (cf. section 2 of [3]).

Finally, eq.(3.9) takes the form:

$$\mathbf{As}_m{\circ}\mathcal{R}{\circ}G(p, M, m) \tag{3.15}$$

$$= \sum_{\substack{\emptyset \le \Gamma < G \\ \Gamma \text{ is } m\text{-subgraph}}} \delta(p_\Gamma) \times \mathcal{R}{\circ} \int dp'_\Gamma \tilde{\mathbf{R}}^m{\circ}\mathbf{T}_{m'}{\circ}G\backslash\Gamma(p_{G\backslash\Gamma}, M, m')\, \mathcal{R}^m{\circ}\Gamma(p'_\Gamma, M, m) + \ldots.$$

Note that $G\backslash\Gamma$ depends on p'_Γ through m'. It should also be remembered that the integration over p'_Γ should be understood in the sense of the principal value (operation $*$).

Eq.(3.15) eliminates the last unknown in (3.9) and represents a convenient starting point for studying exponentiation of As-operation on collections of Feynman diagrams corresponding to Green functions (section 4 below).

3.4 As-operation on products involving $\tilde{\mathbf{R}}$ and δ-functions. Proof of commutativity of the two As-operations.

Both in (3.9) and in (3.15) one has to consider As-expansions of products involving and $\tilde{R}$-operation and/or δ-functions. Therefore, it is worthwhile to consider this point from a general point of view. Moreover, it turns out that the object

$$\tilde{\mathbf{R}}^m{\circ}\left[\delta_{a,\Gamma}(p_\Gamma)\, \mathbf{T}_m{\circ}G\backslash\Gamma(p, M, m)\right] \tag{3.16}$$

from (3.9) can be analyzed without performing projections onto the plane singled out by the $\delta_{a,\Gamma}(p_\Gamma)$—in complete analogy with expressions of the form $\tilde{\mathbf{R}}^m{\circ}G$. The simplicity and straightforward character of the resulting formal proof of commutativity (3.6) is another example of how the meticulous attention to the formalism and notations in the preceding papers [1]-[3] pays off.

Indeed, with some experience with the formalism and understanding of the mechanism of the As-operation, an explicit expression for $\mathcal{R}$ on the distribution in square brackets in (3.9) can be written down offhand. In view of (3.1), it is sufficient to present expressions for the operation $\tilde{\mathbf{r}}{\circ}\mathbf{As}'_\kappa$.

To start, recall the expression for $\tilde{\mathbf{r}}{\circ}\mathbf{As}'_\kappa{\circ}G$ (2.3), which we here rewrite in slightly different notations similar to those used above:

$$\tilde{\mathbf{r}}{\circ}\mathbf{As}'_\kappa{\circ}G(p, M, m) = \sum_{\substack{\emptyset \le \gamma < G \\ \gamma \text{ is } \kappa\text{-subgraph}}} \sum_b \tilde{\mathbf{R}}{\circ}\left\{\delta_b(p_\gamma)\, \mathbf{T}_\kappa{\circ}G(p, M, m)\right\} \times \mathcal{R}{\circ}\langle \mathcal{P}_b * \gamma \rangle. \tag{3.17}$$

First we wish to write down a similar expression for the product G replaced as

$$G(p, M, m) \to G'(p, M, m) = \tilde{\mathbf{R}}^m{\circ}\mathbf{T}_m{\circ}G(p, M, m). \tag{3.18}$$

To put it simply, some of the factors have been replaced (owing to the action of $\mathbf{T}_m$) by arbitrarily singular factors; on top of everything, the singularities of the resulting expression have been subtracted using $\tilde{\mathbf{R}}^m$. Expressions of this form appear in (3.15).

Second, we also wish to consider expressions obtained from (3.18) by replacing the group of factors corresponding to the m-subgraph Γ by one factor—the δ-function δ_a:

$$G'(p,M,m) \to G''(p,M,m) = \tilde{\mathbf{R}}^m\circ[\delta_a(p_\Gamma)\,\mathbf{T}_m\circ G\backslash\Gamma(p,M,m)]\,. \tag{3.19}$$

The crucial thing to realize is that neither the philosophy nor the reasoning of the theory of As-operation developed in [2] need to be changed in order to deal with such products. Indeed, as to the singularities and the additional R-operation (the operation $\tilde{\mathbf{R}}^m$ in our case), one only has to bear in mind the following. As a "formal expansion" one should take the original product *without* the additional R-operation but with the factors that can be expanded, expanded. One then proceeds to constricting the counterterms (introducing an intermediate $\tilde{R}$-operation etc.) treating on an equal footing both the "old" singularities (i.e. corresponding to the factors that are singular prior to expansion) and those generated by formal expansion. However, in constructing the counterterms via consistency conditions, one uses the initial expression which contains *both* non-expanded factors *and* the additional R-operation. The entire procedure becomes perfectly obvious if one recalls the philosophy of constructing the expansion by considering it first in an open region in the space of p where all the factors are regular and then expanding the domain of definition of the expansion by adding counterterms porportional to δ-functions. By analogy with (3.17), one immediately obtains a similar expression for G':

$$\begin{aligned}&\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ\tilde{\mathbf{R}}^m\circ\mathbf{T}_m\circ G(p,M,m) &(3.20)\\ &= \sum_{\substack{\emptyset\le\gamma<G\\ \gamma\text{ are }\kappa\text{-subgraphs}}}\sum_b \tilde{\mathbf{R}}\circ\{\delta_b(p_\gamma)\mathbf{T}_\kappa\circ G\backslash\gamma(p,M,m)\}\times\mathcal{R}\circ\langle\mathcal{P}_b * \tilde{\mathbf{R}}^m\circ\mathbf{T}_m\circ\gamma(p,M,m)\rangle.\end{aligned}$$

Note that we have replaced $\mathbf{T}_\kappa\circ\mathbf{T}_m$ acting on $G\backslash\gamma$ by $\mathbf{T}_\kappa$.

Turning to the expression (3.19), it is not difficult to realize that the δ-function is as good as any other factor from the point of view of the formalism—provided one considers it as a singular factor. Its only effect is that now every κ-subgraph h must contain the δ-function plus, perhaps, some other κ-singular factors. A simple combinatorial observation is that the subgraphs h are in one-to-one correspondence with κ-subgraphs γ of G such that $\gamma\ge\Gamma$. Without more ado we get:

$$\begin{aligned}&\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ\tilde{\mathbf{R}}^m\circ[\delta_a(p_\Gamma)\,\mathbf{T}_m\circ G\backslash\Gamma(p,M,m)] &(3.21)\\ &= \sum_{\Gamma\le\gamma<G}\sum_b \tilde{\mathbf{R}}\circ\{\delta_b(p_\gamma)\,\mathbf{T}_\kappa\circ G\backslash\gamma(p,M,m)\}\times\mathcal{R}\circ\langle\mathcal{P}_b * \tilde{\mathbf{R}}^m\circ[\delta_a(p_\Gamma)\,\mathbf{T}_m\circ\gamma\backslash\Gamma(p,M,m)]\rangle.\end{aligned}$$

One should remember that in this expression γ are κ-subgraphs in G, while Γ are m-subgraphs. Summation runs over γ in accordance with the above observation about one-to-one correspondence between κ-subgraphs in G and in the product G'' given by (3.19).

It is a good excercise to verify equivalence of (3.21) and the expressions that can be obtained along the lines of the preceding subsection.

It remains to note that now one can easily check the formal commutativity of the two As-operations (3.6) by substituting (3.20) into the r.h.s. of (3.6) and using the expression for its l.h.s. that has already been discussed—provided $\mathbf{As}_m$ on the l.h.s. can be put under integration. The final justification for the latter comes from inequalities of the type (2.12) that will be obtained in section 5.

4 "Diagrammatic" form of $\mathbf{As}_m \circ \mathcal{R} \circ G$.

The formulae derived in the preceding section are not immediately useful for deriving As-expansions for Green functions in OPE-like form. To this end one should recast eq.(3.15) into a form that would take into account specific properties of Feynman diagrams that were irrelevant at the analytical stage.

The diagrammatic analysis of Euclidean As-expansions was performed in much detail in [10], [11]. The combinatorial aspects which we are going to discuss in this section are not very sensitive to whether one deals with expansions in dimensionally regularized form, as in [11], or in formalism without regularizations, as in the present paper. Therefore, we will give only an outline of the reasoning and consider just two key examples: the ordinary short-distance OPE and the expansion in heavy masses (which extends the decoupling theorem of Appelquist and Carrazzone—for a review see [16]. An interested reader can find further examples and a detailed description of the combinatorial techniques in [11].

4.1 Using factorization properties of $G\backslash\Gamma$.

At this point we may suppose that G is an ordinary Feynman diagram and use the factorization properties of the expression $G\backslash\Gamma(p, M, m)$.[12]

An m-subgraph Γ is any set of lines and vertices of G such that every line from Γ is singular after expansion in m (irrespective of whether or not the line is singular before expansion); Γ must also satisfy the completeness condition (see subsect.4.2 in [1]). In the present case completeness of Γ means the following:

(i) When one nullifies all light external momenta from the set m as well as all the momenta flowing through the lines of Γ, no other m-singular line of G will have its momentum nullified owing to momentum conservation at vertices;

(ii) Γ contains all those and only those vertices of G whose all incident lines belong to Γ; no heavy external momentum from the set M is allowed to enter into such a vertex.

Consider the complement of Γ in G, denoted as $G\backslash\Gamma$. The graphical image for $G\backslash\Gamma$ is obtained by deleting the lines and vertices belonging to Γ from the diagram G. We have already encountered a similar situation in [3], where $G\backslash\Gamma$ decomposed into a set of 1PI UV-subgraphs.

[12] The reasoning below follows section 3 of [3] which in turn is reminiscent of [11].

Consider the connected components of $G \backslash \Gamma$, denoting them generically as ξ:

$$G \backslash \Gamma = \prod_i \xi_i.$$

Denote the set of the loop momenta that are internal with respect to ξ as p_ξ (ξ may have no loops at all in which case p_ξ is empty; this has no effect on our formalism). Then the variable p can be decomposed as follows:

$$p = (p_\Gamma, p_1, \ldots, p_i, \ldots).$$

Then

$$G \backslash \Gamma (p_{G \backslash \Gamma}, M, m, p_\Gamma) = \prod_i \xi_i(p_i, M, m_i),$$

where we have introduced the notation m_i for the collection of parameters which contains those light parameters from m, as well as those components of p_Γ, on which ξ_i depends.

The As-expansion we are dealing with is independent of the choice of the operation $\tilde{\mathbf{R}}^m$. in particular, we may fix $\tilde{\mathbf{R}}^m$ to be factorized in the sense of section 11 of [1]. Then the associated operation $\mathcal{R}$ will be also factorized (see Appendix A). Therefore, performing integration over p, we obtain instead of (3.9) the following expression:

$$\mathbf{As}_m \circ \mathcal{R} \circ \int dp\, G(p, M, m) = \sum_{\substack{\emptyset \leq \Gamma < G \\ \Gamma \text{ is } m\text{-subgraph}}} \mathcal{R}^m \circ \int dp_\Gamma \left(\prod_i \Delta^{\mathrm{as}} \circ \xi_i(M, m_i) \right) \Gamma(p_\Gamma, m), \tag{4.1}$$

where:

$$\Delta^{\mathrm{as}} \circ \xi_i(M, m_i) = \mathcal{R} \circ \int dp_i\, \tilde{\mathbf{R}}^m \circ \mathbf{T}_{m_i} \circ \xi_i(p_i, M, m_i). \tag{4.2}$$

(Note that only the "counterterms" Δ are sensitive to the operation $\mathcal{R}$ in the initial diagram, while the operation $\mathcal{R}^m$ used to perform UV subtractions on the r.h.s. of (4.1) is associated (in the sense of subsect.3.1) with the operation $\tilde{\mathbf{R}}^m$ used to subtract IR singularities from the formal expansion on the r.h.s. of (4.2). Recall that p_i are loop momenta of ξ_i.)

A "fool-proof" recipe for enumeration of subgraphs in (4.1). It is interesting to note, following [11], that the condition of m-completness of Γ admits a universal and very convenient "fool-proof" reformulation. The above formulae will remain correct if, instead of summing over m-subgraphs Γ, one performs summations over all collections of pairwise non-intersecting and otherwise arbitrary subgraphs ξ. Then in order to nullify superfluos terms it is sufficient to demand that

(i) whenever the operation $\mathbf{T}_m$ generates meaningless expressions of the type $1/0$ (due to a propagator carrying only a combination of light external momenta) Δ_i should be equated to zero;

(ii) if setting $M = \infty$ in Γ produces a factor $1/\infty$ in denominator (due to a heavy line that happened to remain outside all ξ_i's) then such terms should be put to zero in the sum in (4.1).

Such a reformulation is very convenient for studying exponentiation of expansions of Green functions.

Interpreted graphically, eq.(4.1) corresponds to shrinking the subgraphs ξ_i to vertices to which there correspond the factors (4.2) which are polynomials of the momenta entering the new vertices.

The two formulae (4.1) and (4.2) represent a fundamental explicit expression for the *As*-operation on a renormalized Feynman diagram.

4.2 Exponentiation of the *As*-operation into OPE-like form.

The two expressions (4.1) and (4.2) have exactly the same combinatorial structure as that of the *As*-expansions in the dimensionally regularized form studied in [11]. As was pointed out there, similarity of their structure to that of the R-operation in the MS scheme allows one to easily obtain expansions for entire collections of diagrams corresponding to Green functions in the global OPE-like form. In fact, the situation here is even simpler than in [11] because now all the terms in the expansion (4.1) which is a starting point for studying exponentiation, are finite. Therefore, one need not perform the step of inversion of the R-operation, which was the most cumbersome part of [11] (the role of inverted R-operation is played by the operation $\tilde{\mathbf{R}}^m$ in (4.2)). Repeating the reasoning of [11] *mutatis mutandis* one can immediately write down exponentiated forms for expansions of Green functions.

Recall that for each asymptotic regime one only has to find, starting from the basic definitions of the m-subgraph Γ, diagrammatic characterization of the connected components ξ_i of its complement $G\backslash\Gamma$—wherein the above "fool-proof" enumeration recipe is very convenient.

(i) Consider the case corresponding to the short-distance OPE. Then one has only two (one independent—after taking into account momentum conservation) heavy external momenta, while all the masses are considered as light parameters. One finds:

$$\mathcal{R}\circ < T\left\{\int dx e^{iqx} A(x)B(0)\, \exp i[\mathcal{L}+\varphi J]\right\} >_0 \tag{4.3}$$

$$\simeq_{q^2\to-\infty} \sum_i C_i(q)\,\mathcal{R}^m\circ < T\{O_i(0)\, \exp i[\mathcal{L}+\varphi J]\} >_0,$$

with the coefficient functions C_i specified by the following expression:

$$\sum_i C_i(q)\,O_i(0) = \mathcal{R}\circ\tilde{\mathbf{R}}^m\circ\mathbf{T}_m\circ\int dx e^{iqx}\, T\{A(x)B(0)\, \exp i\mathcal{L}\}^{conn}. \tag{4.4}$$

To correctly interpret these expressions one should keep in mind that the standard perturbative formalism of interaction representation is used here. Thus, A, B and O_i are local monomials of free fields (*without* normal ordering) while radiative corrections are generated by the chronological (T-) exponents of the interaction Lagrangian $\mathcal{L}$ (integration over the space time is included into $\mathcal{L}$). UV renormalization is performed by the operations $\mathcal{R}^m$ and $\mathcal{R}$. The operation $\mathbf{T}_m$ acts as follows: one expands the T-product on the r.h.s. of (4.4) in Wick normal products of the free fields, retains only connected diagrams that cannot be divided into disconnected parts by cutting any one of propagators corresponding to the light fields (cf. the above "fool-proof" recipe) and the operation $\mathbf{T}_m$ expands the resulting loop diagrams both in masses and the momenta corresponding to the free fields in normal products.

Individual coefficient functions C_i can be extracted by taking corresponding matrix elements of both sides of (4.4) (only tree-level diagrams will be present on the l.h.s. since there is no T-exponentiated Lagrangian there to generate radiative corrections). Such a procedure is analogous to the algorithm of calculating coefficient functions of OPE in the MS scheme described in [7].

We see that our formulae are in direct correspondence with the formulae and algorithms developed at an informal level in [6]-[11]. This should be no wonder because the methods we used were developed from the very beginning as a straightforward formalization of the reasoning of those works.

It is also worth stressing that our formalism contains nothing similar to the oversubtraction techniques of [21].

(*ii*) As a second example, consider the case when the set of heavy parameters M consists of only heavy masses. Consider the generating functional of Green functions of light particles:

$$\mathcal{R}\circ < T \exp[i\mathcal{L}(\varphi, \Phi) + \varphi J] >_0, \tag{4.5}$$

where $\mathcal{L}(\varphi, \Phi)$ is the total (integrated over space-time) Lagrangian of the system which depends on heavy and light particles. Supposing that typical momenta of φ and the masses of the light fields m are of the same order of magnitude and much less than the masses M of the heavy fields, one obtains:

$$\text{eq.(4.5)} = \mathcal{R}^m\circ < T \exp[i\mathcal{L}_{\text{eff}}(\varphi) + \varphi J] >_0, \tag{4.6}$$

where the effective low-energy Lagrangian whose expression is similar to (4.4):

$$\begin{aligned} i\mathcal{L}_{\text{eff}}(\varphi) &= \mathcal{R}\circ\tilde{\mathbf{R}}^m\circ\mathbf{T}_m\circ \{T \exp i\mathcal{L} - 1\}^{\text{light}-1\text{PI}} \\ &\equiv \sum_n g_{n,\text{eff}}(M) \int dx O_n(x), \end{aligned} \tag{4.7}$$

where $g_{n,\text{eff}}(M)$ are the couplings of the effective Lagrangian. Note (cf. [11]) that $\mathcal{L}_{\text{eff}}$ can contain contributions that are quadratic in the light fields. This corresponds to the finite M-dependent field renormalization in the usual formulation of the decoupling theorem (for a review see [16]. Also note that only analytic dependence on the light masses is allowed in $\mathcal{L}_{\text{eff}}$. "light-1PI" means (cf. the above "fool-proof" recipe) that one has to take into account only diagrams that have no heavy external fields and such that they cannot be divided into two disconnected pieces by cutting a line corresponding to a light particle.

This completes our discussion of the structure of Euclidean asymptotic expansions of Feynman diagram within the formalism of the *As*-operation.

5 Double As-expansions: existence and properties.

The aim of the present section is to prove a theorem on double As-expansions which summarizes all analytical facts that are necessary for derivation of Euclidean asymptotic expansions of renormalized Feynman diagrams. Explicit formulae have already been presented in section 3.

We will perform the reasoning in an abstract manner of [1]–[2], without explicit mentioning of Feynman diagrams proper. All the analytical formulae here—however cumbersome in appearance—are based on a primordially primitive power counting. The apparent abstruseness is due to presence of two expansion parameters and another one used to describe singularities—each of the three accompanied by an integer-valued index etc. Nevertheless, the powerful formalism of [1]–[2] allows one to use the recursive structures inherent in the problem and cut through all the complexities of the proof in an explicit algebraic fashion.

An abstract mathematical character of the following text makes it necessary to recycle some of the physics-inspired notations used in the preceding sections: the symbols m and M—alongside of n and N—will be used for interger-valued indices while the two expansion parameters will be denoted as κ and σ. We will not need M in its old meaning. Other notations follow [1]–[2].

5.1 Double As-expansions.

We start with a formal definition of double As-expansion and present a simple but rather interesting lemma, which can be considered as a generalization of the uniqueness property to the case of double As-expansions.

Let $G(\kappa,\sigma)$ be a distribution which depends parametrically on two real parameters κ and σ from a rectangle $(0,\kappa_0)\times(0,\sigma_0)$.

Suppose there exist asymptotic expansions of G in powers and logs of the parameters κ and σ. In the notations of [1], [2] the sum of terms of order κ^n is denoted as $\mathbf{as}^n_\kappa\circ G$ and the partial sum of the terms through the power n, as $\mathbf{As}^n_\kappa\circ G$ (and similarly for σ). By the definition of As-expansion the following asymptotic estimate must be fulfilled for all σ:

$$(1-\mathbf{As}^n_\kappa)\circ G = o(\kappa^n).$$

Each term of the expansion is a distribution parametrically depending on σ. Assume that there exist As-expansions of those distributions in σ. Denote the double series thus obtained as $\mathbf{As}_\sigma\circ\mathbf{As}_\kappa\circ G$. We can reverse the order of expansions and ask a natural question, whether the two resulting double series $\mathbf{As}_\sigma\circ\mathbf{As}_\kappa\circ G$ and $\mathbf{As}_\kappa\circ\mathbf{As}_\sigma\circ G$ coincide. Generally speaking, they don't (the simplest example is the numeric function $1/(\kappa+\sigma)$). However, it is possible to formulate a necessary condition for the commutativity of the two As-operations based on the notion of *double As-expansion.*

Consider the *double remainder* $\langle(1-\mathbf{As}^n_\kappa)\circ(1-\mathbf{As}^m_\sigma\circ)\circ G,\varphi\rangle$. By definition, it is $o(\kappa^n)$ for all σ, but its behaviour as $\sigma\to 0$ is a priori unpredictable. It is natural to introduce the following definition:

Definition. A double series in powers and logs of κ and σ—its partial sum of terms through $O(\kappa^n) \times O(\sigma^m)$ is denoted as $\mathbf{As}^{n,m}_{\kappa,\sigma}\circ G$—is called *double As-expansion* if:

$$\left|\langle(1 - \mathbf{As}^n_\kappa - \mathbf{As}^m_\sigma\circ + \mathbf{As}^{n,m}_{\kappa,\sigma})\circ G, \varphi\rangle\right| < o(\kappa^n)\, o(\sigma^m),$$

and there exist integers n_0, m_0 such that $\mathbf{as}^{n,m}_{\kappa,\sigma}\circ G = 0$, provided $n \leq n_0$ or $m \leq m_0$.

One can see that the double As-expansion is unique. Moreover, its existence implies nice properties of the double series obtained by termwise composition of the two one-parameter As-expansions like $\mathbf{As}_\kappa\circ\mathbf{As}_\sigma\circ G$, which can be summarized in the following elementary lemma:

Lemma 1. If there exist a double As-expansion of the graph G then there exist series $\mathbf{As}_\kappa\circ\mathbf{As}_\sigma\circ G$ and $\mathbf{As}_\sigma\circ\mathbf{As}_\kappa\circ G$, and, moreover:

$$\mathbf{As}_\sigma\circ\mathbf{As}_\kappa\circ G = \mathbf{As}_\kappa\circ\mathbf{As}_\sigma\circ G = \mathbf{As}_{\kappa,\sigma}\circ G.$$

Therefore, to prove the commutativity of the two As-operations it is sufficient to construct the double As-expansion—which is the purpose of the rest of this section.

5.2 Object of expansion.

The objects we are working with the so-called graphs (formally defined in section 1 of [1] and section 7 of [2]). A graph in this sense is an abstraction to describe products of singular functions encountered on a regular basis in studies of multiloop diagrams (e.g. integrands of multiloop diagrams in momentum-representation). As we now wish to study expansions in two parameters, the notations of [1], [2] should be extended. Namely, the linear functions $l_g(p)$ (which describe the way the integration (loop) momenta p are combined in the argument of the g-th factor) is now required to have the form:

$$l_g(p) = l'_g(p) + \kappa\sigma l''_g + \kappa l'''_g, \tag{5.1}$$

where l''_g and l'''_g represent linear combinations of small and large external momenta, respectively, and are independent of p. Some of the functions F_g which used to depend on the expansion parameter κ, now acquire dependence on the second expansion parameter σ of the form:

$$F_g(q,\kappa) \to F_g(q,\kappa\sigma), \tag{5.2}$$

i.e. instead of κ we now have the product $\kappa\sigma$. Otherwise, the properties of the functions F remain the same.

The assumptions (5.1)–(5.2) are crucial for existence of double As-expansion.

To simplify formulae, we assume that the formal expansions of F's start from $\kappa^0\sigma^0$, which can always be achieved by multiplying F by the corresponding powers of κ and σ.

We are going to show that the graph $G(p,\kappa,\sigma)$ has a double asymptotic expansion in powers and logarithms of κ and σ with the remainder bounded by an expression in which the dependencies on κ and σ are factorized, i.e. that there exist double As-expansion of G.

The theorem on double asymptotic expansions presented below is, essentially, a logical outcome of the conditions (5.1)–(5.2). Heuristically, it is clear why this is so: a numeric function of the form $1/(1+\kappa+\kappa\sigma)$ with a structure analogous to (5.1), can be expanded into a double As-expansion in κ and σ. The latter property is naturally inherited by any products of such functions. The pathological cases of the sort mentioned in subsec.5.1 are prohibited by the imposed restrictions.

Theorem 1. Under the above conditions, there exists a double asymptotic expansion of the graph $G(p,\kappa,\sigma)$; it is given by a termwise composition of the two As-operations as described in the preceding sections (see also below eqs.(5.5)–(5.6); and it has the following properties:

(a) $\mathbf{As}_\kappa \circ \mathbf{As}_\sigma \circ G = \mathbf{As}_\sigma \circ \mathbf{As}_\kappa \circ G = \mathbf{As}_{\kappa,\sigma} \circ G$.

(b) $\mathbf{As}^{n,m}_{\kappa,\sigma} \circ G = 0$ for $n < A^{\kappa G}$ or $m < A^{\sigma G}$, where

$$A^{\kappa G} = \max_{\substack{\Gamma \leq G \\ \Gamma \text{ is } \kappa\text{-subgraph}}} (0, \omega_\Gamma), \qquad A^{\sigma G} = \max_{\substack{\Gamma \leq G \\ \Gamma \text{ is } \sigma\text{-subgraph}}} (0, \omega_\Gamma).$$

(c) The operation $\mathbf{As}_{\kappa,\sigma}$ is local in the sense of [2].

(d) For the terms of the expansion the following estimate holds: for all $\varphi \in \mathcal{D}(P)$ such that $\operatorname{rad\,supp} \varphi \leq d$,

$$\left|\langle \mathbf{as}^{n,m}_{\kappa,\sigma} \circ G, \varphi \rangle\right| \leq \kappa^n \sigma^m d^{-\omega_G - n} \sum_{k \geq 0} d^k \mathcal{S}^k(\varphi) \Lambda(d,\kappa,\sigma).$$

(Here and below we do not indicate the upper limits of summations since their exact expressions are cumbersome and of no practical use. They can, however, be restored from the proofs in a straightforward manner.)

(e) The remainder of the double expansion, defined as

$$\Delta_{n,m} \stackrel{\text{def}}{=} 1 - \mathbf{As}^n_\kappa - \mathbf{As}^m_\sigma + \mathbf{As}^{n,m}_{\kappa,\sigma},$$

satisfies the following "factorizable" estimate: one can fix a constant $C > 0$ such that for all $d > C\kappa$ and $\varphi \in D(P)$ with $\operatorname{rad\,supp} \varphi \leq d$:

$$|\langle \Delta_{n,m} \circ G, \varphi \rangle| \leq \kappa^{n+1} \sigma^{m+1} d^{-\omega_G - n - 1} \sum_{k \geq 0} d^k \mathcal{S}^k(\varphi) \Lambda(d,\kappa,\sigma). \tag{5.3}$$

(f) The expansion possesses the following *minimality property*:

$$\langle \mathbf{as}^{n,m}_{\kappa,\sigma} \circ G * \mathcal{P}_{a,G} \rangle = \langle \Delta_{n,m} \circ G * \mathcal{P}_{a,G} \rangle = 0, \quad \text{for} \quad |a| \leq \omega_G + n.$$

5.3 Proof of theorem 1

The proof of the theorem will be carried out by induction over the hierarchy of m-subgraphs $\Gamma < G$. It is convenient to include into the induction the following lemma containing useful auxiliary estimates:

Lemma 2.

(i) $\forall\varphi \in \mathcal{D}(\mathcal{P})$, $\operatorname{rad\,supp}\varphi \leq d$:

$$|\langle \mathbf{as}^n_\kappa \circ (1 - \mathbf{As}^m_\sigma)\circ G\rangle| \leq \kappa^n \sigma^{m+1} d^{-\omega_G - n} \sum_{k\geq 0} d^k \mathcal{S}^k(\varphi)\Lambda(d,\kappa,\sigma),$$

(ii) $\exists C > 0\ \forall d > C\kappa\ \forall\varphi \in \mathcal{D}(P)$, $\operatorname{rad\,supp}\varphi \leq d$:

$$|\langle \mathbf{as}^m_\sigma \circ (1 - \mathbf{As}^n_\kappa)\circ G\rangle| \leq \kappa^{n+1} \sigma^{m} d^{-\omega_G - n-1} \sum_{k\geq 0} d^k \mathcal{S}^k(\varphi)\Lambda(d,\kappa,\sigma).$$

The statements of the theorem and the lemma are trivial for the empty graph $G = 1$. Let us suppose that they hold for any subgraph of G. The proof can be divided into three logical steps. First, one defines As-expansion as a distribution on $\mathcal{D}(P\backslash\{0\})$ using decompositions of unit (cf. section 10 of [2]) and verify the conditions of the theorem for it. Second, one performs a natural extension of the distributions obtained at the first step onto the space of functions from $\mathcal{D}(P)$ with zero of the order $B_n = \omega_G + n + 1$ (such space is denoted as $\mathcal{D}_{B_n}(P)$). Third, one continues the As-expansion onto the entire $\mathcal{D}(P)$ and determines a finite renormalization to ensure asymptotic estimates.

It is convenient to carry out the first and second steps simultaneously.

Steps 1–2. To begin with, take a function $\varphi \in \mathcal{D}_{B_n}(P)$ and a cutoff $\eta_\lambda \in \mathcal{D}(P\backslash\{0\})$. Using the sector decomposition of unit we define $\mathbf{as}_{\kappa,\sigma}\circ G$ on $\varphi\eta_\lambda$:

$$\langle \mathbf{as}^{N,M}_{\kappa,\sigma}\circ G\,,\,\varphi\eta_\lambda\rangle = \sum_{\Gamma \triangleleft G}\ \sum_{\substack{n\leq N\\ m\leq M}} \langle \mathbf{as}^{n,m}_{\kappa,\sigma}\circ\Gamma\,,\,\mathbf{t}^{N-n}_\kappa \circ \mathbf{t}^{M-m}_\sigma \circ G\backslash\Gamma\,\theta_\Gamma\,\varphi\eta_\lambda\rangle.$$

Using the estimate (d) for $\Gamma < G$ (which holds by inductive assumptions) and the auxiliary estimate (B.1) we conclude that:

$$\left|\langle \mathbf{as}^{N,M}_{\kappa,\sigma}\circ G\,,\,\varphi\eta_\lambda\rangle\right| \leq \kappa^n\sigma^m \sum_{k\geq B_n} \lambda^{k-\omega_G-n}\mathcal{S}^k(\varphi)\Lambda(\lambda,\kappa,\sigma).$$

Integration over λ (cf. [2]) completes steps 1–2 for the estimate (d).

The estimate (i) is proved similarly using the auxiliary estimate (B.2).

The estimates (ii) and (e) are of a somewhat different kind, which should be clear from their look. First of all, we prove the following lemma:

Lemma 3. $\forall\varphi$, $\operatorname{rad\,supp}\varphi \leq C\kappa$:

$$|\langle \mathbf{as}^m_\sigma \circ G\,,\,\varphi\rangle| \leq \sigma^m \sum_{k\geq 0} \kappa^{k-\omega_G}\mathcal{S}^k(\varphi)\Lambda(\sigma,\kappa),$$

$$|\langle (1 - \mathbf{As}^m_\sigma)\circ G\,,\,\varphi\rangle| \leq \sigma^{m+1} \sum_{k\geq 0} \kappa^{k-\omega_G}\mathcal{S}^k(\varphi)\Lambda(\sigma,\kappa).$$

The unusual way of how κ enters the r.h.s. is due to two reasons. First, now the test function is localized in a neighbourhood of radius $O(\kappa)$ so that κ plays the role normally reserved for d. Second, the formal expansion of G in σ prior to expansion in κ results in a situation with several maximal subgraphs (in the context of this lemma we are dealing only with m-subgraphs). This would normally prevent one from obtaining estimates describing singular behaviour of subgraphs (recall that in the proofs of [1]–[2] one normally deals with one maximal subgraph whose singular plane is the point $p = 0$ so that behaviour near $p = 0$ can be described by dependence on the radius of support of test functions). In the present case, however, the singular product expanded in σ depends parametrically on κ—in an interesting way (here the reader should review the pattern of how the factors G depend on κ and σ—see subsect.5.2). Consider the factors that develop singularities after expansion in σ but prior to expansion in κ. The only dependence on κ that remains in such factors is in their momentum arguments. This means that the eventual expansion in κ will result in an $O(\kappa)$ shift[13] of their singular planes, while after expansion in κ there will remain only one maximal subgraph (G itself) whose singular plane is $p = 0$. Therefore, our standard estimates are still meaningful provided the test functions satisfy the condition in the lemma.

The proof proceds as follows. In section 5 of [1] there was constructed a decomposition of unit isolating the singular planes of maximal (m-) subgraphs of G, whereby a smooth function η_Γ is assigned to each $\Gamma \in S_{\max}[G]$, so that

$$\sum_{\Gamma\in S_{\max}[G]} \eta_\Gamma(p) \equiv 1. \tag{5.4}$$

It is convenient (in fact, natural) to choose η_Γ to have the form $\eta_\Gamma(p/\kappa)$, so that the decomposition of unit works for all κ. Using (5.4), one can write down the following identity:

$$\langle \mathbf{as}^m_\sigma \circ G\,,\,\varphi\rangle = \sum_{\Gamma\in S_{\max}[G]}\sum_{m\leq M}\langle \mathbf{as}^m_\sigma\circ\Gamma\,,\,\mathbf{t}^{M-m}_\sigma\circ G\backslash\Gamma\,\eta_\Gamma\,\varphi\rangle.$$

Since every Γ is maximal in itself, the parameter σ appears in its expansion only in combination $\kappa\sigma$. Therefore, the estimate (9.10) from [2] may be applied to $\mathbf{as}^m_\sigma\circ\Gamma$ after replacing $d \to \kappa$ and $\kappa \to \kappa\sigma$:

$$|\langle \mathbf{as}^m_\sigma\circ G\,,\,\varphi\rangle| \leq \max_{\Gamma\in S_{\max}[G]}\max_{m\leq M}(\kappa\sigma)^m\sum_{k\geq 0}\kappa^{k-m-\omega_\Gamma}\mathcal{S}^k(\mathbf{t}^{M-m}_\sigma\circ G\backslash\Gamma\,\eta_\Gamma\varphi).$$

(We assume that $\sigma \leq 1$ and, hence, $\sigma\kappa \leq \kappa$ and $\kappa > \operatorname{rad}\operatorname{supp}\varphi$.) Noticing that:

$$\mathcal{S}^k_{\operatorname{supp}(\eta_\Gamma\varphi)}(\mathbf{t}^{M-m}_\sigma\circ G\backslash\Gamma) \leq \sigma^{M-m}\kappa^{-d_{G\backslash\Gamma}-k}\Lambda(\sigma,\kappa),$$

we obtain the desired estimate. The second statement of the lemma is proved similarly.

Now let us return to the theorem. We now have to deal with singularities of the expansion in κ and, therefore, with κ-subgraphs. To prove the estimates (ii) and (e) we choose the decomposition of unit $1 = H^\kappa(p) + H_\kappa(p)$ (H^κ non-zero in a neighbourhood of $p = 0$), with H^κ fixed so that $\forall\Gamma \triangleleft G$ and $\forall g \in G\backslash\Gamma$:

$$\mathbf{supp}(H_\kappa\theta_\Gamma)\bigcap\mathcal{O}^\kappa_{g(G)} = \emptyset,$$

[13] and/or rotation in the more general situation of the expansion problem with contact terms.

where $\mathcal{O}^{\kappa}_{g(G)}$ is the κ-vicinity of g defined as in [2] but with the following modifications. The singular plane π^{σ}_{g} of an element g at $\sigma = 0$ can be displaced from the one at $\kappa = 0$, π^{κ}_{g}, by const $\times\, \kappa$. Therefore we can take a neighbourhood of the plane π^{κ}_{g} with a radius const $\times\, \kappa$ and containing the singular plane π^{σ}_{g} for all $\kappa \to 0$.

Since rad supp $H^{\kappa} = C\kappa$, we use our usual representation of H_{κ} as an integral over spherical layers of radius λ, $H_{\kappa} = \int_{C\kappa}^{d} d\lambda/\lambda\, \eta_{\lambda}$, and get for (ii) with φ replaced by $\varphi\eta_{\lambda}$ in analogy with the proof of (d),(i) for $\lambda > C\kappa$ (using (B.3)):

$$|\langle \mathbf{as}^{m}_{\sigma}\circ(1 - \mathbf{As}^{n}_{\kappa})\circ G\,,\, \varphi\eta_{\lambda}\rangle| \leq \kappa^{n+1}\sigma^{m} \sum_{k\geq B_n} \lambda^{k-\omega_G-n-1}\mathcal{S}^{k}(\varphi)\Lambda(\lambda,\kappa,\sigma).$$

Integrating over λ from $C\kappa$ to d we immediately obtain a "half" of (ii) (i.e. for φH_{κ} instead of φ). The second half,

$$\langle \mathbf{as}^{m}_{\sigma}\circ(1 - \mathbf{As}^{n}_{\kappa})\circ G\,,\, \varphi H^{\kappa}\rangle = \langle \mathbf{as}^{m}_{\sigma}\circ G\,,\, \varphi H^{\kappa}\rangle - \sum_{l\leq m}\langle \mathbf{as}^{l}_{\sigma}\circ\mathbf{as}^{n}_{\kappa}\circ G\,,\, \varphi H^{\kappa}\rangle,$$

is estimated with the help of lemma 3 (the first term) and (d) which has been already proved. The estimate (e) may be obtained in the same manner. This completes steps 1–2 of the proof.

Step 3. The extension procedure have already been performed for the series $\mathbf{As}_{\kappa}\circ G$ and $\mathbf{As}_{\sigma}\circ G$ in [2]. Our purpose is to extend $\mathbf{As}_{\kappa,\sigma}\circ G$ to the distribution on $\mathcal{D}(P)$ maintaining all the estimates. It can be done in two strokes. The first stroke is to apply the special subtraction operator $\tilde{\mathbf{r}}$ to $\mathbf{as}^{n,m}_{\kappa,\sigma}\circ G$ so that it obey the estimates (d). This procedure is performed in the manner of [2]. The second (and the last) stroke is the finite renormalization of $\mathbf{as}^{n,m}_{\kappa,\sigma}\circ G$ to satisfy the estimate (d) (and, of course, (i) and (ii)). Namely, let:

$$\mathbf{as}^{n,m}_{\kappa,\sigma}\circ G = \tilde{\mathbf{r}}\circ\mathbf{as}^{n,m}_{\kappa,\sigma}\circ G + \sum_{|a|=\max(\omega_G+n,0)} \delta_a(p)\, E^{\kappa,\sigma}_{a,m}. \tag{5.5}$$

It is straightforward to check that the choice

$$E^{\kappa,\sigma}_{a,m} = \langle \mathcal{P}_{a,G} * \left[\mathbf{as}^{m}_{\sigma}\circ G - \tilde{\mathbf{r}}\circ\tilde{\mathbf{r}}\circ\mathbf{As}'^{\,n}_{\kappa}\circ\mathbf{as}^{m}_{\sigma}\circ G\right]\rangle \tag{5.6}$$

satisfies all the requirements, which completes the proof of the theorem 1.

A few remarks are in order.

(a) The equations (5.5) and (5.6) give explicit recursive formulae for the double As-expansion. They can be resolved along lines of [3] which has already been done in the precedent section.

(b) The theorem can be readily generalized to the case of N-fold expansions. For example, if one wished to study a two-scale expansion of a renormalized diagram, one would have to use a three-fold As-expansion etc.

(c) In our main inequality (5.3) the graph G is compared to a rather weird expression $\mathbf{As}_{\kappa}\circ G + \mathbf{As}_{\sigma}\circ G - \mathbf{As}_{\kappa,\sigma}\circ G$. However, this is only necessary for obtaining factorized bounds. Should one need just an approximation for G irrespective of whether it should be achieved due to smallness of κ or σ or both—as is often the case in applications—it is sufficient to use a suitably truncated series $\mathbf{As}_{\kappa,\sigma}\circ G$.

Conclusions.

A theory that pretends to be a comprehensive alternative to the BPHZ method should be able to address at the formal level, as a minimum, the problem of UV renormalization and that of operator product expansions. The theory of As-operation, which has already enjoyed success in applications, has now fulfilled this criterion.

As was observed in [12] the key difference between the two paradigms—BPHZ and our techniques based on the As-operation—is how the basic dilemma of the theory of multiloop Feynman diagrams is resolved. The dilemma consists in the conflict between the inherently recursive nature of the problems of perturbative quantum field theory involving hierarchies of Feynman diagrams, and the singular nature of the objects participating in such recursions. The BPHZ approach consists in systematically getting rid of the singularities by explicitly resolving the corresponding recursive patterns and thus reducing the problem to absolutely convergent integrals. However, those recursive structures are inherently natural, and to ignore them—as the BPHZ approach does—means to lose the heuristic advantage of dealing with complicated objects in a manner respectful of their true nature.

The techniques of the As-operation, on the contrary, allows one to preserve and make efficient use of the recursive structures by offering means to directly work with singular expressions. As a result, formal proofs become algebraically explicit and compact, while the final calculational formulae, powerful.

To put it shortly: the BPHZ formalism is only an instrument of proof; the techniques of the As-operation is also an instrument of discovery.

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Appendices.

A Factorizability of the operation $\mathcal{R}$.

Let us prove factorizability of the operation $\mathcal{R}$. It is always possible to fix the associated $\tilde{R}$-operation to be factorized whence the factorization of $\mathcal{R}$ follows. The corresponding proof was not given in [3] and we carry it out here.

For clarity, we consider the case of just two factors and use the general notation consistent with the reasoning of [3].

Let $G'(p',\kappa)$ and $G''(p'',\kappa)$ be two graphs, with p' and p'' independent. We assume that the operation $\mathbf{As}_\kappa$ is well-defined on both of them. We wish to prove that, provided the operation $\tilde{\mathbf{r}}$ is chosen to be factorized, the operation $\mathcal{R} = 1 - \tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa$ factorizes as follows:

$$\mathcal{R}\circ\int dp'\,dp''\,G'(p',\kappa)\,G''(p'',\kappa) = \mathcal{R}\circ\int dp'\,G'(p',\kappa) \times \mathcal{R}\circ\int dp''\,G''(p'',\kappa). \tag{A.1}$$

We will present simple arguments which use only factorizability of the operation $\mathbf{As}_\kappa$ (which follows from uniqueness of As-expansions—see section 2 of [2]) and the fact that the expression $\mathcal{R}\circ\int dp\,G(p)$ (where $p = (p',p'')$) is exactly the coefficient of $\delta(p)$ in the As-expansion of $G(p,\kappa)$ in κ in the sense of distributions:

$$\mathbf{As}_\kappa\circ G(p,\kappa) = \tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ G(p,\kappa) + \delta(p)\,\mathcal{R}\circ\int d\bar{p}\,G(\bar{p},\kappa) + \text{higher derivatives of } \delta(p). \tag{A.2}$$

The proof runs as follows. First one writes:

$$\mathbf{As}_\kappa\circ[G'(p',\kappa)\times G''(p'',\kappa)] \tag{A.3}$$

$$= \tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ[G'(p',\kappa)\times G''(p'',\kappa)] + \delta(p')\delta(p'')\,\mathcal{R}\circ\int d\bar{p}'d\bar{p}''\,G'(\bar{p}',\kappa)G''(\bar{p}'',\kappa) + \dots.$$

Then for each factor one has a similar expression; e.g. for G':

$$\mathbf{As}_k\circ G'(p',\kappa) = \tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ G'(p',\kappa) + \delta(p')\,\mathcal{R}\circ\int d\bar{p}'\,G'(\bar{p}',\kappa) \tag{A.4}$$

Recall that $\mathbf{As}_k$ factorizes (see section 2 of [2]):

$$\mathbf{As}_\kappa\circ[G'(p',\kappa)G''(p'',\kappa)] = \mathbf{As}_\kappa\circ G'(p',\kappa)\times\mathbf{As}_\kappa\circ G''(p'',\kappa). \tag{A.5}$$

Substituting (A.3) and (A.4) into (A.5) and comparing terms (taking into account that we always choose $\tilde{\mathbf{r}}$ to be factorizable—see section 5 of [2]) one finds, first, that

$$\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ[G'(p',\kappa)\times G''(p'',\kappa)] = \tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ G'(p',\kappa)\times\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ G''G''(p'',\kappa) \tag{A.6}$$

$$+\mathbf{As}_\kappa\circ\Gamma'(p',\kappa)\times\delta(p'')\,\mathcal{R}\circ\int d\bar{p}''\,\Gamma''(\bar{p}'',\kappa) + \mathbf{As}_\kappa\circ\Gamma''(p'',\kappa)\times\delta(p')\,\mathcal{R}\circ\int d\bar{p}'\,\Gamma'(\bar{p}',\kappa),$$

whence follows (A.1).

It remains to note that the factorization conditions that we always impose on $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{R}}$ ensure factorizability of $\mathcal{R}$.

B Auxiliary estimates.

Let H be a subproduct of G, $H \subset \Gamma$, and let K be a compact region not intersecting any of the singular planes of H, which is formally expressed as follows:

$$K \subset P_{(G)} \setminus \bigcup_{g \in H} \pi_g.$$

Then

$$\mathcal{S}^k_{p\in\lambda K}(\mathbf{t}^n_\kappa \circ \mathbf{t}^m_\sigma \circ H) < \left(\frac{\kappa}{\lambda}\right)^n \sigma^m \frac{\Lambda(\lambda,\kappa)}{\lambda^{d_H+k}}, \tag{B.1}$$

$$\mathcal{S}^k_{p\in\lambda K}(\mathbf{t}^n_\kappa \circ (1-\mathbf{T}^m_\sigma) \circ H) < \left(\frac{\kappa}{\lambda}\right)^n \sigma^{m+1} \frac{\Lambda(\lambda,\kappa)}{\lambda^{d_H+k}}. \tag{B.2}$$

Moreover, there exists a constant C (depending on H and K) such that for $\lambda > C\kappa$:

$$\mathcal{S}^k_{p\in\lambda K}((1-\mathbf{T}^n_\kappa) \circ \mathbf{t}^m_\sigma \circ H) < \left(\frac{\kappa}{\lambda}\right)^{n+1} \sigma^m \frac{\Lambda(\lambda,\kappa)}{\lambda^{d_H+k}}, \tag{B.3}$$

$$\mathcal{S}^k_{p\in\lambda K}((1-\mathbf{T}^n_\kappa) \circ (1-\mathbf{T}^m_\sigma) \circ H) < \left(\frac{\kappa}{\lambda}\right)^{n+1} \sigma^{m+1} \frac{\Lambda(\lambda,\kappa)}{\lambda^{d_H+k}}. \tag{B.4}$$

All the above estimates formalize elementary power counting with respect to each of the three parameters—κ, σ and λ.

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