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## The Computation of Loop Amplitudes in Gauge Theories

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### Abstract

We present a detailed derivation of a new and efficient technique based on the technology of four-dimensional heterotic strings, for computing one-loop amplitudes in gauge theories, along with expressions for the one-loop dimensionally regularized helicity amplitudes for the process with four external gluons. We also give a set of computational rules presupposing ignorance of string theory.



## 1. Introduction

Gauge theories form the backbone of the Standard Model of particle interactions. Perturbative calculations in these theories are one of the important tools in searching for new physics through confronting theories with experimental results. Because of the present and likely future importance of hadron collider experiments to our understanding of the Standard Model, and to our ability to uncover the new physics hiding part of the electroweak symmetries and generating the observed fermion mass hierarchies and weak mixing angles, perturbative calculations in quantum chromodynamics have special importance. They are important not only to precision measurements, but also to understanding from a theoretical point of view which QCD-associated predictions can be made reliably, and which cannot.

These perturbative calculations have traditionally been done with Feynman diagrams. The relatively large value of the QCD coupling constant makes it desirable to carry out such calculations to relatively high order (large number of external legs or large number of loops); but the proliferation of diagrams, and associated vertex algebra, makes such higher-order calculations increasingly difficult.

Recent years have seen substantial progress in improving the situation in tree-level calculations. Three main elements contributed to this progress: the use of a spinor helicity basis [1,2], such as that of Xu, Zhang, and Chang [3], for gluon polarization vectors; the color decomposition of the amplitudes [4,5,6,7]; and an improved technique for calculating the kinematical coefficients of the different color factors — the Berends-Giele recurrence relations for amplitudes [8]. The tree level color decomposition [5,9] and recurrence relations [10] emerge quite naturally from string theories.

In this paper, we present a new and efficient technology for calculating the kinematical coefficients at one-loop, along with the explicit calculation of the one-loop helicity amplitudes for the  $gg \rightarrow gg$  process. The technology is based on appropriately constructed heterotic string theories, although for practical applications one can rely on a set of rules presupposing ignorance of string theory. (The particular string theories we use are do *not* possess *space-time* supersymmetry; they contain a pure non-abelian gauge theory in the infinite string-tension or field-theory limit.)

There are a number of advantages in using a string-based formalism. In the Polyakov string formalism the loop momentum is already integrated out in the initial expression, and thereby all algebra associated with the non-abelian gauge vertices is bypassed. Furthermore, the string amplitude is well suited to use of the same spinor-helicity basis that has proved so useful in tree-level computations. The spinor helicity basis simplifies expressions with polarization vectors contracted

into external momenta or other polarization vectors; and these are the only types of terms to be found in the string loop amplitudes. It thereby reduces greatly the complexity of the expression for the amplitude, at the very starting point of the calculation. In contrast, in the conventional Feynman-diagram approach, the initial expression contains polarization vectors contracted with loop momenta; and a simple technique for evaluating conventional Feynman loop integrals within the framework of the spinor helicity method does not exist [2].

In a theory of closed strings, at each loop order there is only a single diagram. This allows one to write a systematic and compact expression for the  $n$ -point amplitudes at all loop orders. The string also provides a color decomposition of the loop amplitudes automatically, just as it does for the tree level amplitudes: the full amplitude appears as a sum of gauge-invariant *partial amplitudes* multiplied by traces over color matrices. We have discussed the color decomposition of one-loop amplitudes, and the structure of the color-summed corrections to matrix elements, in a previous paper [11]. The decomposition of the amplitude into smaller gauge-invariant partial amplitudes, each containing contributions from many Feynman diagrams, eliminates many of the large cancellations typical of Feynman diagram calculations in gauge theories. Another aspect of the string reorganization of the amplitude which exemplifies its advantage over the conventional Feynman-diagram approach is the absence of extra Faddeev-Popov ghost diagrams, even though the string formalism is completely covariant. For the string  $n$ -gluon amplitude at one loop, the ghost contributions are trivially accounted for in the string partition function.

Since the gluon amplitudes contain the usual infrared divergences found in relativistic massless amplitudes [12,13], we have developed string versions of several variants of ordinary dimensional regularization, including its most conventional form. This conventional scheme is distinguished by its conceptual simplicity, as all states are continued uniformly to  $4 - \epsilon$  dimensions, and by the ease with which one may compare to previous Feynman diagram calculations of Ellis and Sexton [14], which were done using this scheme. Other variants of dimensional regularization, in which all observed states remain in four-dimensions, are to be preferred in future calculations, because they reduce the amount of work required when using the spinor helicity formalism.

The new approach also lends itself to a richer set of consistency checks than the conventional one. Besides the usual checks on unitarity, gauge invariance and cancellation of the infrared divergences — which here may be applied to the separate pieces in the color decomposition — one also finds relationships between various partial amplitudes that emerge from the requirement of  $U(1)$  decoupling. We have discussed these decoupling identities at great length in ref. [11]. Alternatively, these decoupling identities could be used to reduce the number of partial amplitudes that must be computed directly.

We have performed a number of explicit checks on our computation of the four-point helicity amplitudes. The most direct check of our amplitudes is the comparison [15] of our result for the next-to-leading order correction to the *unpolarized* cross-section to the previous computation of Ellis and Sexton [14]. This provides the first complete check on their results and verifies in detail our understanding of the dimensional regularization issues in the new string-based formalism. We have checked the gauge invariance of the amplitude. Using the new techniques for performing soft and collinear phase space integrations developed by Giele and Glover [16], we have also checked that the infrared singularities have the correct form to cancel with the soft and collinear phase integrations for the corresponding five-point cross sections, in accordance with the Lee-Nauenberg [13] theorem regarding the complete cancellation of infrared divergences in physical processes. Finally, we shall demonstrate explicitly that the optical theorem, relating the absorptive (‘imaginary’) part of the amplitude to tree cross sections, is satisfied.

In deriving our string-based approach there were a number of seeming technical obstructions which had to be overcome. We have chosen to defer those details which are ultimately irrelevant to a practical use of the technology to a separate publication [17]. (One of these issues, that of the gluon wavefunction renormalization, has been analyzed elsewhere [18].)

In section 2 we review the framework of helicity amplitudes, which provide a convenient and compact basis for expressing the essential gauge-invariant content of gauge-theory scattering amplitudes. The formalism we introduce makes good use of the spinor helicity basis of Xu, Zhang and Chang [3], which we review in section 3. We then review the color decomposition for one-loop amplitudes in section 4, and the structure of the four-dimensional heterotic string amplitude in section 5. The ‘moduli’ which appear in the string amplitude correspond to Schwinger proper-time parameters in field theory; we discuss this correspondence in section 6. We discuss the construction of string versions of dimensional regularization in section 7. We then show how to simplify the string amplitude in the field-theory limit in section 8, and present a set of rules for performing practical calculations in section 9. The use of these rules does not require any understanding of the intricacies of string theory (only their derivation does). The partial amplitudes for the physical helicities of the four-gluon process in a pure glue theory are given in section 10 and 11. The former section, devoted to the finite helicity amplitudes, also presents several checks on the answers using the expected symmetry properties of the partial amplitudes and using the decoupling equations. In section 12 we discuss some of the details of the infrared pole structures in different versions of dimensional regularization. In section 13, we present the results for the remaining ‘ $\epsilon$ ’ helicities, and show the answer obtained for the *unpolarized* next-to-leading differential cross-section in the conventional dimensional regularization scheme agrees with the result previously obtained by Ellis

and Sexton [14]. In section 15, we check the optical theorem for the different helicity amplitudes; we conclude with a few general remarks in the final section.

We have also included a number of technical appendices. Appendix I contains a sample string model which contains a pure  $SU(N_c)$  factor in the field theory limit, and appendix II contains some additional technical details. The notation and normalizations for string quantities are given in appendix III. The field theory limit of string expressions are given in appendix IV; these expansions are necessary for the derivation of a set of rules for the field theory limit. We summarize the Feynman parameter integrals required for evaluating the loop amplitudes in appendix V, and the integrals required for checking the optical theorem in appendix VI.

## 2. Helicity Amplitudes

In a traditional Feynman diagram approach to the computation of the next-to-leading corrections to unpolarized multigluon scattering, one would write down all the contributing diagrams, in terms of the external polarization vectors and momenta represented by the symbols  $\epsilon_i$  and  $k_i$ ; perform all the algebra associated with the vertex factors; reduce all tensor integrals to scalar ones through a Passarino-Veltman [19] procedure; re-cast the propagator denominators in each diagram in terms of Feynman parameters; perform the momentum integrals; perform the Feynman parameter integrals; sum up the diagrams; construct the interference of the tree amplitude and the newly-calculated loop amplitude; and use on-shell identities of the form

$$\sum_j \epsilon_\mu^{(j)*} \epsilon_\nu^{(j)} = -g_{\mu\nu} \quad (2.1)$$

followed by a great deal of algebra, to sum over helicities.

In the four-point amplitude, for each independent color structure, there are forty-three formally independent terms arising from the number of ways in which four polarization vectors can be contracted into each other and into independent momenta. At tree level, there are fifteen terms, so that in forming the the interference of the loop with the tree we would find 645 terms none of which would be expected to vanish trivially.

However, the entire physical content of the amplitude resides in its values for physical helicities; the formal expression in terms of the polarization vectors and momenta contains a great deal of redundant information because of gauge invariance and crossing symmetry. The spinor helicity basis [1,3] provides a means of writing down helicity amplitudes for processes with external vectors in a compact and efficient form. In fact, the whole strategy of computing an unpolarized amplitude changes: it is often more efficient to compute each helicity *amplitude* numerically (using complex floating-point arithmetic), and then to square and sum over helicities numerically to arrive at a

numerical evaluation of a matrix element, rather than squaring analytically and then evaluating the resulting differential cross-section numerically.

We will find it convenient to work in a basis where all momenta are labelled as outgoing, so that for a  $2 \rightarrow n - 2$  scattering process,  $k_{1,2}^0 < 0$ , and momentum conservation takes the form  $\sum_{j=1}^n k_j = 0$ . (This also changes the sign of the incoming helicities.) We can order the amplitudes according to the net helicity,  $S = \left| \sum_{j=1}^n \sigma_j \right|$ , which ranges from  $S = n$  down by two units at a time, to  $S = 0$  (for  $n$  even) or  $S = 1$  (for  $n$  odd).

Each helicity amplitude has a perturbative expansion,

$$\begin{aligned}
\mathcal{A}_n(1^+, 2^+, 3^+, \dots, n^+) &= \mathcal{A}_n^{\text{tree}}(1^+, 2^+, 3^+, \dots, n^+) + \mathcal{A}_n^{1\text{-loop}}(1^+, 2^+, 3^+, \dots, n^+) + \mathcal{O}(g^{n+2}) \\
\mathcal{A}_n(1^-, 2^+, 3^+, \dots, n^+) &= \mathcal{A}_n^{\text{tree}}(1^-, 2^+, 3^+, \dots, n^+) + \mathcal{A}_n^{1\text{-loop}}(1^-, 2^+, 3^+, \dots, n^+) + \mathcal{O}(g^{n+2}) \\
\mathcal{A}_n(1^-, 2^-, 3^+, \dots, n^+) &= \mathcal{A}_n^{\text{tree}}(1^-, 2^-, 3^+, \dots, n^+) + \mathcal{A}_n^{1\text{-loop}}(1^-, 2^-, 3^+, \dots, n^+) + \mathcal{O}(g^{n+2}) \\
&\vdots
\end{aligned} \tag{2.2}$$

where the powers of the coupling constant and the renormalization scale dependence are implicit in the  $\mathcal{A}_n^{l\text{-loop}}$ . The first two helicity amplitudes, with  $S = n$  and  $S = n - 2$ , are special, because the Parke-Taylor equations [20] tell us that the tree-level amplitudes vanish; thus the expansions of these particular helicity amplitudes begins at  $\mathcal{O}(g^n)$  rather than  $\mathcal{O}(g^{n-2})$ :

$$\begin{aligned}
\mathcal{A}_n(1^+, 2^+, 3^+, \dots, n^+) &= \mathcal{A}_n^{1\text{-loop}}(1^+, 2^+, 3^+, \dots, n^+) + \mathcal{O}(g^{n+2}) \\
\mathcal{A}_n(1^-, 2^+, 3^+, \dots, n^+) &= \mathcal{A}_n^{1\text{-loop}}(1^-, 2^+, 3^+, \dots, n^+) + \mathcal{O}(g^{n+2})
\end{aligned} \tag{2.3}$$

This in turn implies that the one-loop contributions to these amplitudes are infrared finite; and irrelevant to the calculation of next-to-leading ( $\mathcal{O}(g^{2n-2})$ ) corrections to any cross section.

### 3. Review of the Spinor Helicity Basis

The spinor helicity basis provides an efficient way of evaluating helicity amplitudes of vector particles in gauge theories. We will follow the form introduced by Xu, Zhang, and Chang (XZC) [3], which we review in this section.

To every massless four-momentum  $k$ , one can associate a pair of Weyl spinors (with opposite helicities),  $|k_{\pm}\rangle$ . These are the basic objects used in this formalism. Amplitudes will eventually be expressed in terms of the complex-valued spinor products,

$$\langle k_{j-} | k_{l+} \rangle \quad \text{and} \quad \langle k_{j+} | k_{l-} \rangle. \tag{3.1}$$

It is convenient to define abbreviations for the various spinor products and the Lorentz product,

$$\begin{aligned}
\langle j l \rangle &= \langle k_j k_l \rangle = \langle k_{j-} | k_{l+} \rangle \\
[j l] &= [k_j k_l] = \langle k_{j+} | k_{l-} \rangle \\
(j l) &= \langle j l \rangle [l j] = 2k_j \cdot k_l
\end{aligned} \tag{3.2}$$

The spinor products are antisymmetric,

$$\langle j l \rangle = -\langle l j \rangle, \quad [j l] = -[l j] \quad (3.3)$$

and can be evaluated explicitly using [3],

$$\begin{aligned} \langle k_1 k_2 \rangle &= \sqrt{(k_1^t - k_1^z)(k_2^t + k_2^z)} \exp(i \operatorname{atan}(k_1^y/k_1^x)) - (1 \leftrightarrow 2) \\ &= \sqrt{\frac{k_2^t + k_2^z}{k_1^t + k_1^z}} (k_1^x + i k_1^y) - (1 \leftrightarrow 2) \\ [k_1 k_2] &= \operatorname{sign}(k_1^t k_2^t) (\langle k_2 k_1 \rangle)^*. \end{aligned} \quad (3.4)$$

For each on-shell gluon carrying momentum  $k$ , one must choose a *reference momentum*  $q$ , where  $q^2 = 0$ ,  $k \cdot q \neq 0$ . The reference momenta (which can be different for different gluons) can be chosen to eliminate many of different invariants involving the polarization vectors. The spinor helicity basis for the gluon polarization vectors is then

$$\varepsilon_\mu^{(+)}(k; q) = \frac{\langle q^- | \gamma_\mu | k^- \rangle}{\sqrt{2} \langle q k \rangle}, \quad \varepsilon_\mu^{(-)}(k, q) = \frac{\langle q^+ | \gamma_\mu | k^+ \rangle}{\sqrt{2} [k q]}, \quad (3.5)$$

Because of the antisymmetry of the spinor product, these automatically satisfy the on-shell condition  $k \cdot \varepsilon = 0$ . We use the convention that all momenta are outgoing; the effect of this is to flip helicity notation on an incoming line. Gauge-invariant quantities are independent of the choice of reference momentum  $q$ , because changing  $q$  just corresponds to a gauge transformation,

$$\varepsilon_\mu^{(+)}(k; q') = \varepsilon_\mu^{(+)}(k; q) + \frac{\sqrt{2} \langle q q' \rangle}{\langle q k \rangle \langle q' k \rangle} k_\mu \quad (3.6)$$

which follows from the rearrangement or Schouten identity

$$\langle 1 2 \rangle \langle 3 4 \rangle = \langle 1 4 \rangle \langle 3 2 \rangle + \langle 1 3 \rangle \langle 2 4 \rangle \quad (3.7)$$

The Fierz identity, in the form [3]

$$\langle 1^- | \gamma^\mu | 2^- \rangle \langle 3^+ | \gamma_\mu | 4^+ \rangle = 2 \langle 1 4 \rangle [3 2] \quad (3.8)$$

can be used to evaluate the dot products of polarization vectors (note also that  $\langle 1^- | \gamma^\mu | 2^- \rangle = \langle 2^+ | \gamma^\mu | 1^+ \rangle$ ).

Given the reference momenta, the various dot products are simply

$$\begin{aligned} \varepsilon_j^{(+)}(k_j; q_j) \cdot \varepsilon_l^{(-)}(k_l; q_l) &= \frac{\langle q_j l \rangle [q_l j]}{\langle q_j j \rangle [l q_l]} \\ k_j \cdot \varepsilon_l^{(+)}(k_l; q_l) &= \frac{\langle q_l j \rangle [j l]}{\sqrt{2} \langle q_l l \rangle} \\ k_j \cdot \varepsilon_l^{(-)}(k_l; q_l) &= \frac{[q_l j] \langle j l \rangle}{\sqrt{2} [l q_l]} \end{aligned} \quad (3.9)$$

The first step in the string-based calculation of gluon amplitudes is indeed the choice of reference momenta for the external gluons, and use of the spinor helicity basis. In making a choice of reference momenta, it is useful to keep the properties noted by Mangano et al. [5] in mind. With the first argument to a polarization vector denoting the momentum of the gluon, and the second its reference momentum, these properties are

$$\begin{aligned}
q \cdot \varepsilon^{(\pm)}(k; q) &= 0 \\
\varepsilon_j^{(\pm)}(k_j; q) \cdot \varepsilon_i^{(\pm)}(k_i; q) &= 0 \\
\varepsilon_j^{(\mp)}(k_j; q) \cdot \varepsilon_i^{(\pm)}(k_i; k_j) &= 0
\end{aligned} \tag{3.10}$$

so that it is desirable to choose the same reference momenta for all gluons of a given helicity, and to take this momentum to be the momentum of one of the opposite-helicity gluons. This will greatly reduce the number of non-vanishing  $\varepsilon_i \cdot \varepsilon_j$  invariants. Furthermore, within the set of choices suggested by these properties, it is preferable to choose a reference momentum that is cyclicly adjacent to the momentum of the gluon, since that does not introduce spurious poles (which must cancel) in momentum invariants.

In the computation of the four-point amplitude, examples of good choices of reference momenta are  $(k_4, k_1, k_1, k_1)$  for  $A(1^-, 2^+, 3^+, 4^+)$ ,  $(k_4, k_1, k_1, k_1)$  for  $A(1^+, 2^+, 3^+, 4^+)$ , and  $(k_4, k_4, k_1, k_1)$  for  $A(1^-, 2^-, 3^+, 4^+)$ . Good choices for the other partial amplitudes may be obtained by a relabelling of legs. These choices are not unique; other choices exist which are just as good. With these choices of reference momenta we find the following simplifications in computing the partial amplitude  $A(1^-, 2^-, 3^+, 4^+)$ ,

$$\begin{aligned}
\varepsilon_1 \cdot \varepsilon_2 &= \varepsilon_1 \cdot \varepsilon_3 = \varepsilon_1 \cdot \varepsilon_4 = \varepsilon_2 \cdot \varepsilon_4 = \varepsilon_3 \cdot \varepsilon_4 = 0, & k_4 \cdot \varepsilon_1 &= k_4 \cdot \varepsilon_2 = k_1 \cdot \varepsilon_3 = k_1 \cdot \varepsilon_4 = 0, \\
k_3 \cdot \varepsilon_1 &= -k_2 \cdot \varepsilon_1, & k_1 \cdot \varepsilon_2 &= -k_3 \cdot \varepsilon_2, & k_4 \cdot \varepsilon_3 &= -k_2 \cdot \varepsilon_3, & k_2 \cdot \varepsilon_4 &= -k_3 \cdot \varepsilon_4, \\
\varepsilon_2 \cdot \varepsilon_3 &= k_2 \cdot \varepsilon_3 k_3 \cdot \varepsilon_2 / k_2 \cdot k_3,
\end{aligned} \tag{3.11}$$

where the last line follows from an explicit computation making use of the Fierz identity (3.8). We have made use of momentum conservation in deriving some of these expressions. The reduction in the number of independent terms greatly reduces the labor in a calculation, from the point at which these expressions can be substituted into an emerging expression; as we shall see in section 5 these simplifications can be used in their entirety in the very first expression of a string based loop amplitude computation.

The ordinary  $(+, -)$  helicities suffice for next-to-leading computations in certain variants of dimensional regularization\*, but for computations in the more conventional variant, one must also

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\* In a scheme in which all helicities are treated as four-dimensional ones, such as the four-dimensional helicity



take into account the fact that vectors in  $4 - \epsilon$  dimensions have  $2 - \epsilon$  rather than 2 helicity states. To do this in the framework of the spinor helicity basis, one introduces [21] an additional  $[\epsilon]$ -helicity, with the following rules in  $4 - \epsilon$  dimensions,

$$\begin{aligned} k \cdot \varepsilon^{([\epsilon])}(k'; q) &= 0 \\ \varepsilon^{(\pm)}(k; q) \cdot \varepsilon^{([\epsilon])}(k'; q') &= 0 \\ \varepsilon_1^{([\epsilon])}(k; q) \cdot \varepsilon_2^{([\epsilon])}(k'; q) &= -\delta_{(-\epsilon)}^{i_1 i_2} \end{aligned} \quad (3.12)$$

In the last expression,  $i_1$  and  $i_2$  run over the  $-\epsilon$  additional dimensions; in squaring an amplitude (or forming an interference), one must sum over these additional indices:

$$\delta_{(-\epsilon)}^{i_1 i_2} \delta_{(-\epsilon)}^{i_2 i_3} = \delta_{(-\epsilon)}^{i_1 i_3}, \quad \delta_{(-\epsilon)}^{i_1 i_2} \delta_{(-\epsilon)}^{i_1 i_2} = -\epsilon. \quad (3.13)$$

It will be convenient to abbreviate  $\delta_{(-\epsilon)}^{i_1 i_2}$  to  $\delta_{(-\epsilon)}^{12}$ .

#### 4. The Color Decomposition

In analogy to the color decomposition of the tree-level amplitude [4,5,9],

$$\mathcal{A}_n(\{k_i, \sigma_i, a_i\}) = g^{n-2} \sum_{\rho \in S_n/Z_n} \text{Tr}(T^{a_{\rho(1)}} \dots T^{a_{\rho(n)}}) A_n(k_{\rho(1)}, \lambda_{\rho(1)}; \dots; k_{\rho(n)}, \lambda_{\rho(n)}) \quad (4.1)$$

there is also a color decomposition of loop amplitudes [22],

$$\mathcal{A}_n^{1\text{-loop}} = g^n \sum_{j=1}^{\lfloor n/2 \rfloor + 1} \sum_{\rho \in S_n/S_{n,j}} \text{Gr}_{n;j}(\rho(1), \dots, \rho(n)) A_{n;j}(k_{\rho(1)}, \lambda_{\rho(1)}; \dots; k_{\rho(n)}, \lambda_{\rho(n)}) \quad (4.2)$$

In these expressions,  $k_i$ ,  $\lambda_i$ , and  $a_i$  are respectively the momentum, helicity, and color index of the  $i$ -th external gluon. The  $T^a$  are the set of hermitian traceless  $N \times N$  matrices (normalized so that  $\text{Tr}(T^a T^b) = \delta^{ab}$ );  $S_n/Z_n$  is the set of non-cyclic permutations of  $\{1, \dots, n\}$ ;  $\text{Gr}_{n;j}$  denote the double-trace structures

$$\begin{aligned} \text{Gr}_{n;1}(1, \dots, n) &= \text{Tr}(1) \text{Tr}(T^{a_1} \dots T^{a_n}) \\ &= N_c \text{Tr}(T^{a_1} \dots T^{a_n}) \\ \text{Gr}_{n;j}(1, \dots, n) &= \text{Tr}(T^{a_1} \dots T^{a_{j-1}}) \text{Tr}(T^{a_j} \dots T^{a_n}), \end{aligned} \quad (4.3)$$

and  $S_{n;j}$  is the subset of the permutation group  $S_n$  that leaves the trace structure  $\text{Gr}_{n;j}$  invariant. ( $S_{n;1}$  is just the set of cyclic permutations of  $n$  objects,  $Z_n$ .) Four pure-gluon amplitudes in  $SU(N)$ ,

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scheme to be described in section 7, the ordinary helicities in fact suffice to all orders in perturbation theory. This was in fact the scheme that should have been implied in the discussion at the end of ref. [21], rather than Siegel's dimensional reduction; the latter effectively treats all helicities in four dimensions at tree-level, but is slightly different in loop calculations.

the partial amplitude  $A_{n;2}$  drops out since its coefficient includes a trace over a single  $SU(N)$  generator, which vanishes identically. In summing a squared amplitude over colors, however, the fact that the additional  $U(1)$  of  $U(N)$  decouples allows us to sum over  $U(N)$ . (That is easier because the  $U(N)$  Fierz identities are simpler than their  $SU(N)$  counterparts.) If we do that, we must retain  $A_{n;2}$  since its coefficient no longer vanishes trivially. In next-to-leading corrections to the cross section, it drops out anyway; but in higher-order corrections it does not. The decoupling equation for this partial amplitude could still be used to eliminate  $A_{n;2}$ , but that is not necessarily desirable, since formulæ which contain it may well be more compact than those which eliminate it in favor of a sum of  $A_{n;i}$ 's.

The four-point amplitude thus has the decomposition

$$\begin{aligned} \mathcal{A}_4^{1\text{-loop}} = g^4 \sum_{\sigma \in S_4/Z_4} N_c \text{Tr}(T^{\alpha_{\sigma(1)}} T^{\alpha_{\sigma(2)}} T^{\alpha_{\sigma(3)}} T^{\alpha_{\sigma(4)}}) A_{4;1}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) \\ + \sum_{\sigma \in S_4/Z_3} \text{Tr}(T^{\alpha_{\sigma(1)}}) \text{Tr}(T^{\alpha_{\sigma(2)}} T^{\alpha_{\sigma(3)}} T^{\alpha_{\sigma(4)}}) A_{4;2}(\sigma(1); \sigma(2), \sigma(3), \sigma(4)) \\ + \sum_{\sigma \in S_4/Z_2^2} \text{Tr}(T^{\alpha_{\sigma(1)}} T^{\alpha_{\sigma(2)}}) \text{Tr}(T^{\alpha_{\sigma(3)}} T^{\alpha_{\sigma(4)}}) A_{4;3}(\sigma(1), \sigma(2); \sigma(3), \sigma(4)) \end{aligned} \quad (4.4)$$

In the first term, the permutation  $\sigma$  lies in the set of all permutations  $S_4$  of four objects, with purely cyclic ones removed; in the second term, the set consists of all permutations which are inequivalent under a cycling of the last three indices; and in the last term,  $\sigma$  is again in the set of permutations of four objects but with two factors of  $Z_2$  removed, corresponding to exchanging the indices within each trace, as well as another  $Z_2$  removed corresponding to interchanging the two traces. (In this equation, we have abbreviated the dependence of the  $A_{n;j}$  on momentum  $k_l$  and helicity  $\lambda_l$  by writing the label  $l$  alone.)

The one-loop *partial amplitudes*  $A_{n;j}$  have properties analogous to those of their tree-level counterparts: they are gauge-invariant on-shell; satisfy a symmetry equation,

$$\forall \sigma \in S_{n;j}, \quad A_{n;j}(\sigma(1), \dots, \sigma(n)) = A_{n;j}(1, \dots, n) \quad (4.5)$$

and a reflection identity,

$$A_{n;j}(R_{n;j}(1, \dots, n)) = (-1)^n A_{n;j}(1, \dots, n) \quad (4.6)$$

where

$$R_{n;j}(i_1, \dots, i_n) = (i_{j-1}, \dots, i_1, i_n, \dots, i_j). \quad (4.7)$$

In the case of the four-point amplitude, these identities are

$$\begin{aligned}
A_{4;1}(1, 2, 3, 4) &= A_{4;1}(2, 3, 4, 1) = A_{4;1}(3, 4, 1, 2) = A_{4;1}(4, 1, 2, 3) \\
A_{4;2}(1; 2, 3, 4) &= A_{4;2}(1; 3, 4, 2) = A_{4;2}(1, 4, 2, 3) \\
A_{4;3}(1, 2; 3, 4) &= A_{4;3}(2, 1; 3, 4) = A_{4;3}(1, 2; 4, 3) = A_{4;3}(3, 4; 1, 2) \\
A_{4;1}(1, 2, 3, 4) &= A_{4;1}(4, 3, 2, 1) \\
A_{4;2}(1; 2, 3, 4) &= A_{4;2}(1; 4, 3, 2)
\end{aligned} \tag{4.8}$$

In addition the partial amplitudes satisfy a set of decoupling equations which we have discussed in detail in ref. [22]. In the case of the four-point function, these take the form

$$\begin{aligned}
A_{4;3}(1, 2, 3, 4) &= \sum_{\sigma \in S_4/Z_4} A_{4;1}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) \\
A_{4;2}(1, 2, 3, 4) &= - \sum_{\sigma \in Z_3\{2,3,4\}} A_{4;1}(1, \sigma(2), \sigma(3), \sigma(4)) \\
&= -\frac{1}{2} A_{4;3}(1, 2, 3, 4).
\end{aligned} \tag{4.9}$$

Using the decoupling equations, one can simplify the color-summed next-to-leading correction to the four-gluon process,

$$\sum_{\text{colors}} [\mathcal{A}_4^* \mathcal{A}_4]_{\text{NLO}} = 2g^6 N_c^3 (N_c^2 - 1) \text{Re} \sum_{\sigma \in S_4/Z_4} A_4^{\text{tree}^*}(\sigma) A_{4;1}(\sigma). \tag{4.10}$$

where we have abbreviated  $A_{n;j}(\sigma(1), \dots, \sigma(n))$  by  $A_{n;j}(\sigma)$ .

The leading-order result has the form

$$\sum_{\text{colors}} [\mathcal{A}_4^* \mathcal{A}_4]_{\text{LO}} = g^4 N_c^2 (N_c^2 - 1) \sum_{\sigma \in S_4/Z_4} |A_4^{\text{tree}^*}(\sigma)|^2. \tag{4.11}$$

The color decomposition emerges naturally from the heterotic string amplitude, which we review in the next section.

## 5. Heterotic String Amplitudes

If we wish to write an  $SU(N)$  gauge theory loop amplitude as the infinite-tension limit of a string amplitude, we must control the massless matter content of the string theory, because colored massless matter particles (if any) can run around the loops. In a string theory, control of the spectrum is particularly important, since even a single sector of a string theory typically contains different massless states whose contributions are summed together. (In contrast, in field theory, each different virtual particle has its own set of independent Feynman diagrams.) It is possible to

build heterotic string theories whose infinite-tension limit is a non-abelian gauge theory where one of the factors is an  $SU(N)$  with no matter fields. The technology needed for such a construction is precisely the one used to construct four-dimensional string models. We have discussed the construction of such models, using the fermionic formulation of Kawai, Lewellen, and Tye (KLT), in previous work [24]; a sample model is summarized in appendix I. We use a heterotic string because bosonic strings always contain unwanted massless scalars and tachyons, while consistent four-dimensional type II [25,26] and type I [27] superstrings do not have a rich enough variety of models for our purposes.

There are three basic kinds of four-dimensional heterotic string constructions: bosonic formulations [28], fermionic formulations [29,30], and direct superconformal field theory constructions [31]. (The appellations ‘bosonic’ and ‘fermionic’ refer solely to aspects of the string world-sheet and imply nothing about the spacetime nature of the models, which may be supersymmetric or nonsupersymmetric.) These formalisms have revealed a huge class of consistent four-dimensional superstring models; the freedom in constructing models is so large that string models with virtually any low energy gauge group with rank 22 or less can be built. In this paper, we again use the fermionic formulation in the notation of KLT [29]. We have found the fermionic formalism to be particularly straightforward to use for constructing models as well as computing scattering amplitudes, although one could use other formalisms as well.

It turns out that a fully consistent string is not needed for practical calculation, since in the field theory limit the details of the infinite tower of massive states is not relevant. (In particular, the modular transformation which relates the spectrum of massive states to the massless states is not important.) However, a consistent string does guarantee that no extraneous problems enter to affect the results.

We now review briefly the salient aspects of the KLT construction. The construction is based on an analysis of the one-loop string partition function or world-sheet torus: one imposes on it the constraints of world-sheet reparameterization invariance, world-sheet supersymmetry, freedom from the conformal anomaly, one-loop modular invariance [32]. One also demands that it contain a physically sensible spectrum, with an integral number of states appearing, each with the correct spin-statistics relation. (This last condition is actually equivalent to two- and higher-loop modular invariance [30,33].)

In the fermionic formulation all internal degrees of freedom (which carry the non-abelian gauge charges) are taken to be world-sheet fermions [34]. The degrees of freedom corresponding to uncompactified (genuine space-time) dimensions are left as world-sheet bosons. These satisfy the conventional world-sheet boundary conditions  $X(\sigma_1 + 2\pi n, \sigma_2 + 2\pi m) = X(\sigma_1, \sigma_2)$  on the one-loop

world sheet torus. Before imposition of the KLT conditions, each of the world-sheet fermions can have independent boundary conditions

$$\psi_l(\sigma_1 + 2\pi n, \sigma_2 + 2\pi m) = e^{2\pi i(\alpha_l n + \beta_l m)} \psi_l(\sigma_1, \sigma_2). \quad (5.1)$$

The points on the torus are parametrized by the complex quantity  $\sigma_1 + \tau\sigma_2$  where  $\tau$  is the modular parameter describing the ‘shape’ of the world-sheet torus. The modular transformations

$$\tau \rightarrow -1/\tau, \quad \tau \rightarrow \tau + 1, \quad (5.2)$$

are the last remnant of reparametrization invariance on the world-sheet after gauge fixing.

Corresponding to each set of boundary conditions of the  $n_\psi$  world-sheet fermions is a one-loop fermion partition function on the torus  $\mathcal{Z}_F[\vec{\alpha}, \vec{\beta}] = \prod_{l=1}^{n_\psi} \mathcal{Z}_1[\alpha_l, \beta_l]$ , where  $\mathcal{Z}_1$  is the partition functions for a single complex fermion. (The explicit value of these partition functions in terms of  $\vartheta$ -functions is given in Appendix III.)

The complete string partition function is an integral over all inequivalent tori of the separate world-sheet bosonic and fermionic contributions,

$$\int \frac{d^2\tau}{(\text{Im}\tau)^2} \mathcal{Z}_B \mathcal{Z}_F. \quad (5.3)$$

The KLT construction imposes modular invariance separately on the fermionic and bosonic contributions. Note again that ‘boson’ and ‘fermion’ do not refer to the space-time properties but only to world-sheet properties; in particular,  $\mathcal{Z}_F$  will contain contributions from states which are space-time bosons. At one loop, the ghost contributions enter trivially, and simply reduce the number of bosonic and fermionic degrees of freedom by two. In the KLT construction, the bosonic degrees of freedom are completely generic and require no special consideration. The choices of boundary conditions of the world sheet fermions, on the other hand, define the string model; different choices of boundary conditions on the world-sheet lead to different particle contents.

Under the modular transformations (5.2), the single-fermion partition functions transform as follows,

$$\begin{aligned} \tau \rightarrow -\frac{1}{\tau} : \quad \mathcal{Z}_1 \begin{bmatrix} \alpha_l \\ \beta_l \end{bmatrix} &\rightarrow e^{2\pi i(\beta_l - 1/2)(\alpha_l - 1/2)} \mathcal{Z}_1 \begin{bmatrix} \beta_l \\ -\alpha_l \end{bmatrix} \\ \tau \rightarrow \tau + 1 : \quad \mathcal{Z}_1 \begin{bmatrix} \alpha_l \\ \beta_l \end{bmatrix} &\rightarrow e^{2\pi i(\alpha_l^2 - \alpha_l + 1/6)} \mathcal{Z}_1 \begin{bmatrix} \alpha_l \\ \beta_l - \alpha_l \end{bmatrix}. \end{aligned} \quad (5.4)$$

Linear combinations of products of these single-fermion partition functions,

$$\mathcal{Z}_F = \sum_{\vec{\alpha}, \vec{\beta}} C_{\vec{\alpha}, \vec{\beta}} \mathcal{Z}_F[\vec{\alpha}, \vec{\beta}] = \sum_{\vec{\alpha}, \vec{\beta}} C_{\vec{\alpha}, \vec{\beta}} \prod_{l=1}^{n_\psi} \mathcal{Z}_1 \begin{bmatrix} \alpha_l \\ \beta_l \end{bmatrix}, \quad (5.5)$$

will be invariant under the modular transformations, for appropriate choices of coefficients  $C_{\beta}^{\alpha}$ . KLT give a set of rules of determining these coefficients so that the partition function is modular-invariant in addition to satisfying the other consistency requirements mentioned above.

Given the partition function for a given string model, the computation of scattering amplitudes is straightforward. In the Polyakov formalism the amplitude is given by the expectation value of the vertex operators [35,23] of the theory using the world-sheet action for free fermions and bosons,

$$\mathcal{A}_n^{\text{string}}(\{a_i, k_i, \varepsilon_i\}) \sim \int [DX][D\psi] \exp[-S] V^{a_1}(k_1, \varepsilon_1) \cdots V^{a_n}(k_n, \varepsilon_n) . \quad (5.6)$$

For a bosonic string, the world-sheet action  $S = (4\pi\alpha')^{-1} \int d^2\xi \eta_{\mu\nu} \partial_{\beta} X^{\mu} \partial^{\beta} X^{\nu}$  corresponds to free bosons  $X^{\mu}$ ; it is a bit more complicated for a four-dimensional heterotic string (see, for example, ref. [29]). The vertex operator for external gluon states is given in appendix III. Using Wick's theorem and expressions for the Green functions, these the expectation values can be computed explicitly. (The details of performing such computations in the operator formalism may be found in ref. [23].)

The one-loop dimensionally-regularized amplitude for  $n$  external gluons in a heterotic string

theory can be written as follows,\*

$$\begin{aligned}
\mathcal{A}_n = & i \frac{(4\pi)^{\epsilon/2}}{2(16\pi^2)} \lambda^{n/2-2+\epsilon/2} (\sqrt{2g\mu^{\epsilon/2}})^n T^{a_1}_{m_1} \widehat{m}_1 \dots T^{a_n}_{m_n} \widehat{m}_n \\
& \int \frac{d^2\tau}{(\text{Im}\tau)^2} \int \left( \prod_{i=1}^n d\theta_{i1} d\theta_{i2} d\theta_{i3} d\theta_{i4} \right) \int \left( \prod_{i=1}^n d^2\nu_i \right) \sum_{\vec{\alpha}, \vec{\beta}} C_{\vec{\beta}}^{\vec{\alpha}} \mathcal{Z}_{\vec{\beta}}^{\vec{\alpha}}(\tau) \\
& \prod_{i=1}^n \exp \left[ -\theta_{i1}\theta_{i2}\delta^{m_i}_{\widehat{m}_i} S_F \left[ \begin{matrix} \alpha_{m_i} \\ \beta_{\widehat{m}_i} \end{matrix} \right] \right] \\
& \prod_{i<j} \exp \left[ \lambda k_i \cdot k_j G_B(\nu_i - \nu_j) \right. \\
& \quad - \theta_{i1}\theta_{j2}\delta^{m_i}_{\widehat{m}_j} G_F \left[ \begin{matrix} \alpha_{m_i} \\ \beta_{\widehat{m}_j} \end{matrix} \right] (\nu_i - \nu_j) - \theta_{i2}\theta_{j1}\delta^{m_j}_{\widehat{m}_i} \hat{G}_F \left[ \begin{matrix} \alpha_{m_j} \\ \beta_{\widehat{m}_i} \end{matrix} \right] (\nu_i - \nu_j) \\
& \quad - \theta_{i3}\theta_{j3} \lambda k_i \cdot k_j G_F \left[ \begin{matrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{matrix} \right] (\bar{\nu}_i - \bar{\nu}_j) \\
& \quad + i\sqrt{\lambda}(\theta_{i3}\theta_{j4} k_i \cdot \epsilon_j + \theta_{i4}\theta_{j3} k_j \cdot \epsilon_i) G_F \left[ \begin{matrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{matrix} \right] (\bar{\nu}_i - \bar{\nu}_j) \\
& \quad - i\sqrt{\lambda}(\theta_{i3}\theta_{i4} k_j \cdot \epsilon_i - \theta_{j3}\theta_{j4} k_i \cdot \epsilon_j) \dot{G}_B(\bar{\nu}_i - \bar{\nu}_j) \\
& \quad + \theta_{i4}\theta_{j4} \epsilon_i \cdot \epsilon_j G_F \left[ \begin{matrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{matrix} \right] (\bar{\nu}_i - \bar{\nu}_j) \\
& \quad \left. + \theta_{i3}\theta_{i4}\theta_{j3}\theta_{j4} \epsilon_i \cdot \epsilon_j \ddot{G}_B(\bar{\nu}_i - \bar{\nu}_j) \right] \tag{5.7}
\end{aligned}$$

where  $\lambda = \pi\alpha'$  is the inverse string tension; the  $\theta_{i,j}$  are Grassmann integration variables, and the  $\nu_i$  are integrated over the torus specified by the modular parameter  $\tau$ ;  $\vec{\alpha}$  and  $\vec{\beta}$  are the various boundary conditions for the world-sheet fermions, over which one must sum in order to obtain a modular-invariant answer;  $\mathcal{Z}_{\vec{\beta}}^{\vec{\alpha}}$  is the partition function for a given set of boundary conditions;  $G_F \left[ \begin{matrix} \alpha_m \\ \beta_{\widehat{m}} \end{matrix} \right]$  are the left-mover fermionic Green functions, with  $\alpha_m$  and  $\beta_{\widehat{m}}$  the boundary conditions on the torus of world-sheet fermions associated with the gauge group of interest;  $G_F \left[ \begin{matrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{matrix} \right]$  are the right-mover fermionic Green functions, with  $\alpha_{\uparrow}$  and  $\beta_{\uparrow}$  the boundary conditions of the world-sheet fermions carrying the space-time index (which occupies the first right-mover position in the world-sheet boundary condition vector);  $G_B$  are the bosonic Green functions (dots indicate derivatives with respect to  $\bar{\nu}$ ); Our convention is that the left-movers form a bosonic string while the right-movers form a (world-sheet) superstring. Detailed expressions for all of these quantities are given in the appendices. The  $T^a$  are the usual charge matrices of  $U(N_c)$ . The overall factor of  $i$  ensures

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\* This form is valid in all string sectors which do not contain world-sheet zero-modes, such as the sectors of interest to us — those containing the gluons.

that the phase convention is the same as in a Feynman diagram computation. (The form of the amplitude given here differs slightly from the standard form, in that we have chosen to integrate over all  $\nu$ s, and have compensated by dividing by the volume of the torus, rather than fixing  $\nu_n$ .) The string amplitude may be diagrammatically represented in terms of a torus world sheet and punctures for the vertex operators, as depicted in fig. 1 for the four-point amplitude.

This expression yields a properly normalized amplitude in  $(4 - \epsilon)$  dimensions, with the parameter  $\mu$  being the usual scale parameter which absorbs the dimensions of the coupling constant. We shall discuss the details of the string versions of dimensional regularization in section 7.

A rather striking feature of the amplitude (5.7) is that it is valid for arbitrary numbers of gluon legs. In contrast, the usual Feynman rules do not yield a comparable concrete formula in any straightforward manner. A field theorist would also find the simplicity with which the ghost contributions enter into the amplitude rather remarkable. The contribution of the world sheet ghosts (which are not trivially related to the space-time ghosts) amounts to removing two degrees of freedom from both the bosonic and fermionic partition functions with no other change whatsoever. This simplicity is an indication that the string amplitude contains a non-trivial rearrangement of contributions to the amplitude that would emerge from different Feynman diagrams in the usual method of calculation.

In the string model we employ, the gauge group of interest is actually  $U(N_c)$  rather than  $SU(N_c)$ . The extra ‘photon’ is an integral part of the string theory, and amplitudes with an external ‘photon’ receive non-vanishing contributions from the self-contraction left-mover Green functions  $S_{\mathcal{F}}[\frac{\alpha}{\beta}]$ . The decoupling of this extra  $U(1)$  gauge boson in the infinite-tension limit can be used to derive various constraints on pieces of the gauge theory amplitude [11]. The self-contraction Green functions enter only into the calculation of the  $A_{n,2}$  partial amplitudes whose coefficients vanish trivially for  $SU(N_c)$  gluons, and for the most part we shall ignore them in the remainder of this paper.

It will be helpful to distinguish three factors of the integrand: the left-mover contributions



(which are a function of the  $\nu_i$  alone),

$$\begin{aligned}
L(\{\nu_i\}) &= \int \left( \prod_{i=1}^n d\theta_{i1} d\theta_{i2} \right) \\
&\times \prod_{i=1}^n \exp \left[ -\theta_{i1}\theta_{i2}\delta^{m_i} \widehat{m}_i S_F \left[ \begin{matrix} \alpha_{m_i} \\ \beta_{\widehat{m}_i} \end{matrix} \right] \right] \\
&\times \prod_{i < j} \exp \left[ -\theta_{i1}\theta_{j2}\delta^{m_i} \widehat{m}_j G_F \left[ \begin{matrix} \alpha_{m_i} \\ \beta_{\widehat{m}_j} \end{matrix} \right] (\nu_i - \nu_j) - \theta_{i2}\theta_{j1}\delta^{m_j} \widehat{m}_i \hat{G}_F \left[ \begin{matrix} \alpha_{m_j} \\ \beta_{\widehat{m}_i} \end{matrix} \right] (\nu_i - \nu_j) \right] \\
&= \int \left( \prod_{i=1}^n d\theta_{i1} d\theta_{i2} \right) \\
&\times \prod_{i=1}^n \exp \left[ -\theta_{i1}\theta_{i2}\delta^{m_i} \widehat{m}_i S_F \left[ \begin{matrix} \alpha_{m_i} \\ \beta_{\widehat{m}_i} \end{matrix} \right] \right] \\
&\times \prod_{i \neq j} \exp \left[ -\frac{1}{2}\theta_{i1}\theta_{j2}\delta^{m_i} \widehat{m}_j G_F \left[ \begin{matrix} \alpha_{m_i} \\ \beta_{\widehat{m}_j} \end{matrix} \right] (\nu_i - \nu_j) - \frac{1}{2}\theta_{i2}\theta_{j1}\delta^{m_j} \widehat{m}_i \hat{G}_F \left[ \begin{matrix} \alpha_{m_j} \\ \beta_{\widehat{m}_i} \end{matrix} \right] (\nu_i - \nu_j) \right]
\end{aligned} \tag{5.8}$$

The form of the left-mover Green functions associated with any given trace structure is actually quite simple. Every left-mover Green function is uniquely associated with a Kronecker  $\delta$ -function which ties together two color charge matrices, and so the list of left-mover Green functions can be read off from the trace of color charge matrices. As examples, the following terms appear in the four-point string amplitude:

$$- \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (-\nu_{12}) G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (-\nu_{23}) G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (-\nu_{34}) G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (\nu_{14}) \tag{5.9a}$$

$$- \text{Tr}(T^{a_4} T^{a_3} T^{a_2} T^{a_1}) G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (\nu_{12}) G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (\nu_{23}) G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (\nu_{34}) G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (-\nu_{14}) \tag{5.9b}$$

$$- \text{Tr}(T^{a_1} T^{a_2}) \text{Tr}(T^{a_3} T^{a_4}) G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (-\nu_{12}) G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (\nu_{12}) G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (-\nu_{34}) G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (\nu_{34}) \tag{5.9c}$$

where we have used the relation  $\hat{G}_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (\nu) = -G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (-\nu)$  to rewrite the form appearing in equation (5.8).

The second factor of the integrand consists of the right-mover contributions,

$$\begin{aligned}
R(\{\bar{\nu}_i, k_i, \varepsilon_i\}) &= \int \left( \prod_{i=1}^n d\theta_{i3} d\theta_{i4} \right) \\
&\prod_{i<j}^n \exp \left[ -\theta_{i3}\theta_{j3}\lambda k_i \cdot k_j G_F \begin{bmatrix} \alpha_\uparrow \\ \beta_\uparrow \end{bmatrix} (\bar{\nu}_i - \bar{\nu}_j) \right. \\
&\quad + i\sqrt{\lambda}(\theta_{i3}\theta_{j4}k_i \cdot \varepsilon_j + \theta_{i4}\theta_{j3}k_j \cdot \varepsilon_i) G_F \begin{bmatrix} \alpha_\uparrow \\ \beta_\uparrow \end{bmatrix} (\bar{\nu}_i - \bar{\nu}_j) \\
&\quad - i\sqrt{\lambda}(\theta_{i3}\theta_{i4}k_j \cdot \varepsilon_i - \theta_{j3}\theta_{j4}k_i \cdot \varepsilon_j) \dot{G}_B(\bar{\nu}_i - \bar{\nu}_j) \\
&\quad + \theta_{i4}\theta_{j4}\varepsilon_i \cdot \varepsilon_j G_F \begin{bmatrix} \alpha_\uparrow \\ \beta_\uparrow \end{bmatrix} (\bar{\nu}_i - \bar{\nu}_j) \\
&\quad \left. + \theta_{i3}\theta_{i4}\theta_{j3}\theta_{j4}\varepsilon_i \cdot \varepsilon_j \ddot{G}_B(\bar{\nu}_i - \bar{\nu}_j) \right]
\end{aligned} \tag{5.10}$$

This is a function of the  $\bar{\nu}_i$  alone, except for the zero-mode piece in the bosonic Green functions, which also depends on  $\nu_i$ . In this part, one may use the spinor helicity simplifications, then perform the Grassmann integrals, and then integrate the  $\bar{\nu}_i$  integrals by parts to remove all double-derivatives of the bosonic Green function,  $\ddot{G}_B(\bar{\nu})$ . As we shall discuss in section 8, integrating by parts is important for deriving a simple set of rules in the field theory limit. Before performing the integration by parts, there is a simple correspondence between the kinematic tensors and products of Green functions arising from the structure of the right-mover contributions (5.10); the integration by parts, however, destroys this simple correspondence.

The third and last factor of the integrand consists of the partition function and parts common to both left- and right-movers,

$$E(\{\nu_i, \bar{\nu}_i, k_i, \varepsilon_i\}) = \sum_{\bar{\alpha}, \bar{\beta}} C_{\bar{\beta}}^{\bar{\alpha}} \mathcal{Z}_{\bar{\beta}}^{\bar{\alpha}}(\tau) \prod_{i<j}^n \exp[\lambda k_i \cdot k_j G_B(\nu_i - \nu_j)] . \tag{5.11}$$

The overall normalization will be denoted by  $\mathcal{N}$ . The partition function is composed of a product of zero-mode, left-, and right-mover pieces

$$\mathcal{Z}_{\bar{\beta}}^{\bar{\alpha}}(\tau) = (\text{Im } \tau)^{-2+\epsilon/2} \mathcal{Z}_R^{\bar{\alpha}_R} \mathcal{Z}_L^{\bar{\alpha}_L} \tag{5.12}$$

Before summing over the world sheet boundary conditions, in the Neveu-Schwarz sector, the left-mover bosonic string partition function has a leading behavior of  $q^{-1}$  (where  $q = e^{2\pi i\tau}$ ), corresponding to a mass-squared level of  $-4\pi/\lambda$ , while the right-mover superstring partition function has a leading behavior of  $\bar{q}^{-1/2}$ , corresponding to a mass-squared level of  $-2\pi/\lambda$ . The appearance of the dimensional regularization parameter  $\epsilon$  will be discussed in section 7.

As an example of the structure of a right-mover contribution and of the required integration by parts, consider the right-mover terms associated with the factor  $\varepsilon_1 \cdot \varepsilon_2$  in the three-point amplitude ( $n = 3$ )

$$\begin{aligned}
& i\sqrt{\lambda}\varepsilon_1 \cdot \varepsilon_2 \left( \tilde{G}_B(\bar{\nu}_{12})(k_1 \cdot \varepsilon_3 \dot{G}_B(\bar{\nu}_{13}) + k_2 \cdot \varepsilon_3 \dot{G}_B(\bar{\nu}_{23})) \right) \\
& + \lambda k_1 \cdot k_2 (\dot{G}_B(\bar{\nu}_{23})\varepsilon_3 \cdot k_2 + \dot{G}_B(\bar{\nu}_{13})\varepsilon_3 \cdot k_1) G_F \left[ \begin{matrix} \alpha_\uparrow \\ \beta_\uparrow \end{matrix} \right] (\bar{\nu}_{12})^2 \\
& + \lambda (\varepsilon_3 \cdot k_1 k_2 \cdot k_3 - \varepsilon_3 \cdot k_2 k_1 \cdot k_3) G_F \left[ \begin{matrix} \alpha_\uparrow \\ \beta_\uparrow \end{matrix} \right] (\bar{\nu}_{12}) G_F \left[ \begin{matrix} \alpha_\uparrow \\ \beta_\uparrow \end{matrix} \right] (\bar{\nu}_{13}) G_F \left[ \begin{matrix} \alpha_\uparrow \\ \beta_\uparrow \end{matrix} \right] (\bar{\nu}_{23}) .
\end{aligned} \tag{5.13}$$

These right-mover Green functions contain a  $\tilde{G}_B(\bar{\nu}_{12})$ , and we must perform an integration by parts to remove it: multiply the offending right-mover terms by the common factors (5.11), integrate by parts with respect to one of the  $\bar{\nu}_i$ , and then divide out the common contributions to obtain the transformed form of the right-mover contributions. For the  $\tilde{G}_B(\bar{\nu}_{12})\dot{G}_B(\bar{\nu}_{13})$  term, the appropriate variable is  $\bar{\nu}_2$ ; the transformation takes

$$\tilde{G}_B(\bar{\nu}_{12})\dot{G}_B(\bar{\nu}_{13}) \longrightarrow \lambda \dot{G}_B(\bar{\nu}_{12})\dot{G}_B(\bar{\nu}_{13}) \left( k_2 \cdot k_1 \dot{G}_B(\bar{\nu}_{21}) + k_2 \cdot k_3 \dot{G}_B(\bar{\nu}_{23}) + k_2 \cdot k_4 \dot{G}_B(\bar{\nu}_{24}) \right) \tag{5.14}$$

where the terms in parentheses have been pulled down from the exponentiated Green functions in equation (5.11). There are no surface terms since the torus has no boundaries; at the locations of the vertex operators an analytic continuation in the external momenta removes the potential singularities which might have obstructed the integration by parts [11]. For the  $\tilde{G}_B(\bar{\nu}_{12})\dot{G}_B(\bar{\nu}_{23})$  term, the appropriate variable is  $\bar{\nu}_1$ , and this transformation also removes the  $\tilde{G}_B(\bar{\nu}_{12})$ .

In general, when we perform the  $\theta_{i1}$  and  $\theta_{i2}$  integrals, the  $\delta^m_n$  will contract the indices of the charge matrices, and we will obtain an expression for the string amplitude as a sum of kinematic coefficients,

$$\begin{aligned}
\mathcal{A}_n^{\text{string}} &= \sum_i \text{Tr}(T^{a_1} \dots T^{a_i}) \dots \text{Tr}(T^{a_1} \dots T^{a_m}) \\
&\times \mathcal{N} \int \frac{d^2\tau}{(\text{Im}\tau)^2} \int d^2\nu_i L(\{\nu\}) E(\{\nu, k\}) R(\{\bar{\nu}, k, \varepsilon\})
\end{aligned} \tag{5.15}$$

where the sum runs over all trace structures with up to  $n$  traces, and all inequivalent ways of ordering the charge matrices in any given trace structure. In this way the string induces a natural color decomposition on the amplitude; in the field theory limit the coefficient of each color trace structure is a partial amplitude discussed in the last section. In order to evaluate a particular partial amplitude we need only focus on a particular trace structure in the full amplitude, which in turn implies that we need focus only on a particular left-mover Green function structure. Furthermore, the factorization of the amplitude into separate left- and right-mover factors implies that the manipulations discussed above — substitution of the spinor helicity basis, the right-mover Grassmann

integrations, and the integrations by parts — can be performed on the right-movers independently of the structure of the left-movers.

As an example of the expressions one obtains after such manipulations, consider the right-mover contributions to the helicity amplitude  $\mathcal{A}_4(1^-, 2^-, 3^+, 4^+)$ . If we use the reference momenta discussed in section 3, substitute the simplifications (3.11) in the right-mover terms, and integrate by parts to remove the remaining  $\tilde{G}_B$ 's, the right-mover expression becomes

$$\begin{aligned}
R(1^-, 2^-, 3^+, 4^+) = \lambda^2 \frac{st}{4} \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} & \left[ \left( \dot{G}_B^2(\bar{\nu}_{43}) - G_F^2(\bar{\nu}_{43}) \right) \left( \dot{G}_B(\bar{\nu}_{31}) \dot{G}_B(\bar{\nu}_{32}) \right. \right. \\
& \left. \left. - \dot{G}_B(\bar{\nu}_{21}) \dot{G}_B(\bar{\nu}_{32}) + \dot{G}_B(\bar{\nu}_{21}) \dot{G}_B(\bar{\nu}_{31}) + G_F^2(\bar{\nu}_{21}) - \dot{G}_B^2(\bar{\nu}_{21}) \right) \right. \\
& + \left( \dot{G}_B^2(\bar{\nu}_{21}) - G_F^2(\bar{\nu}_{21}) \right) \left( \dot{G}_B(\bar{\nu}_{42}) \dot{G}_B(\bar{\nu}_{43}) - \dot{G}_B(\bar{\nu}_{32}) \dot{G}_B(\bar{\nu}_{43}) + \dot{G}_B(\bar{\nu}_{32}) \dot{G}_B(\bar{\nu}_{42}) \right) \\
& + G_F(\bar{\nu}_{21}) G_F(\bar{\nu}_{31}) G_F(\bar{\nu}_{42}) G_F(\bar{\nu}_{43}) - G_F(\bar{\nu}_{21}) G_F(\bar{\nu}_{32}) G_F(\bar{\nu}_{41}) G_F(\bar{\nu}_{43}) \\
& + G_F(\bar{\nu}_{31}) G_F(\bar{\nu}_{32}) G_F(\bar{\nu}_{41}) G_F(\bar{\nu}_{42}) - \dot{G}_B(\bar{\nu}_{31}) \dot{G}_B(\bar{\nu}_{32}) \dot{G}_B(\bar{\nu}_{41}) \dot{G}_B(\bar{\nu}_{42}) \\
& \left. - \dot{G}_B(\bar{\nu}_{21}) \dot{G}_B(\bar{\nu}_{31}) \dot{G}_B(\bar{\nu}_{42}) \dot{G}_B(\bar{\nu}_{43}) + \dot{G}_B(\bar{\nu}_{21}) \dot{G}_B(\bar{\nu}_{32}) \dot{G}_B(\bar{\nu}_{41}) \dot{G}_B(\bar{\nu}_{43}) \right] \tag{5.16}
\end{aligned}$$

This is an example of a ‘first-line’ simplification obtained using the spinor helicity basis; comparable simplifications emerge for the other helicities.

Before proceeding to extract the infinite-tension or field-theory limit, there are a number of technical points which must be addressed. In the corners of moduli space where the loop is isolated on an external leg, the standard Polyakov amplitude is in fact ill-defined, because of ‘0/0’ ambiguities which must be resolved. These can be resolved using a prescription due to Minahan [36] which preserves modular invariance. We have performed a detailed analysis of this question in previous work [18], and have also shown a connection between the Minahan prescription and possible off-shell continuations of the amplitude [37], but for calculations in the field theory limit using a dimensional regularization scheme it turns out that the *coefficient* of the ‘0/0’ vanishes because of a complete cancellation of the UV and IR divergences, just as in field theory when loops are isolated on external on-shell legs [38]. We also note at this point that with the use of the Minahan prescription, the wavefunction renormalization in the string theory vanishes (independent of the renormalization scheme), and so the amplitude (5.7) in fact *is* the  $S$ -matrix element as well. The vanishing of the coefficient of the ‘0/0’ ambiguity means that this technical issue is in fact irrelevant to the derivation of the field-theory limit, and so we shall not discuss it any further. We will perform a closer examination of the vanishing of the coefficient in another paper [17].

Since the string model of interest does not possess space-time supersymmetry, the dilaton tadpole does not vanish. One might fear that this would lead to difficulties with gauge invariance or decoupling, but a more detailed analysis [17] reveals that for our purposes this is not the case.

For practical computations, one may simply note that the would-be dilaton tadpole contributions come from an isolated region of parameter space, where the locations of all the external vertex operators come together, and therefore they can simply be dropped.

The one technical issue that *is* important for practical calculations is that of regulating infrared divergences; we shall discuss this question in section 7.

Before turning to that issue, we may note that although the moduli of the string amplitude,  $\tau$  and the  $\nu_i$ , may appear strange to a field theorist, they have a simple interpretation in as Schwinger proper-time variables; we discuss this interpretation in the next section.

## 6. A Connection to Field Theory: Schwinger Proper Time

At first sight, the string amplitude (5.7) does not look much like an ordinary quantum field theory amplitude; in particular, the integration parameters or string ‘moduli’ may appear rather exotic in nature. These parameters (or more precisely their imaginary parts) are in fact just ordinary Schwinger proper-time parameters of the same sort that one would encounter in field theory.

This interpretation can be understood most easily in the operator formalism of string theory. In this formalism the one-loop string amplitude is written as a trace of vertex operators separated by string propagators

$$\mathcal{A}_n = \text{Tr} \left( V_1 \frac{1}{L_0 + \tilde{L}_0 - 2} V_2 \frac{1}{L_0 + \tilde{L}_0 - 2} \cdots V_n \frac{1}{L_0 + \tilde{L}_0 - 2} \right) \quad (6.1)$$

where the  $V_i$  are vertex operators, one for each external state, and  $1/(L_0 + \tilde{L}_0 - 2)$  is the string propagator.  $L_0 - 1$  is the world-sheet hamiltonian for the left-mover modes of a string while  $\tilde{L}_0$  is hamiltonian for the right-movers. The string propagator is a space-time scalar which propagates an infinite tower of states with increasing space-time spin; besides the usual vibrational string modes there is an overall translation mode which can be identified with the momentum variable of a point particle theory. (For simplicity’s sake we restrict our attention to bosonic string theory, but the essential features are identical in all conventional string theories.)

The general form of the string theory loop amplitude (6.1) is analogous to that of a field theory loop consisting of a trace of vertices separated by propagators. There are, however, some general properties of a string theory, central to its computational simplicity, that are quite different from the properties of a field theory; for example, a single string propagator, devoid of Lorentz indices, propagates an infinite tower of space-time bosons, with ever increasing spins. For computational purposes, string theory takes vector and higher-spin states, and turns them into scalar-like states. One should contrast this with the field theory, in which the Lorentz structure of the Feynman

propagators and vertices becomes significantly more complicated as one increases the spin of the propagating states, leading to laborious tensor-loop computations.

Another remarkable feature of oriented closed strings is that only one diagrammatic topology contributes at each order of perturbation theory. It is therefore possible to (effectively) integrate out the loop momenta at a given order in perturbation theory in one fell swoop and thus to write closed-form expressions like the one-loop amplitude (5.7). In field theory each loop amplitude would be associated with a number of different diagrammatic topologies, each in turn leading to a different loop integral.

In the operator formalism of string theory one may evaluate the loop integral (or trace) by introducing proper-time variables for each propagator:

$$\frac{1}{L_0 + \tilde{L}_0 - 2} = \int_0^\infty dt \exp(-t(L_0 + \tilde{L}_0 - 2)) = \int_0^\infty dt \exp(-t(p^2/2 + \text{string oscillators})) \quad (6.2)$$

where  $p$  is the zero-mode or string momentum.

At this stage, the introduction of a proper-time parameter is identical to the field theory case (6.2), so the  $(4 - \epsilon)$ -dimensional zero-mode integrals produce a result identical to the corresponding field theory loop integrals (7.1). The string amplitude also contains non-trivial contributions from the string oscillators, of course.

In string theory, the left- and right-mover parts of the theory are completely decoupled except for a ‘level-matching’ condition [23]. This condition demands that for any physical state, the left- and right-mover world-sheet Hamiltonians must take on identical values, so that  $(L_0 - \tilde{L}_0)|\psi\rangle = 0$ . One can include this condition in the propagator by introducing a phase in the proper time integral

$$\int_0^\infty dt \int_0^{2\pi} e^{-t(L_0 + \tilde{L}_0 - 2)} e^{i\phi(L_0 - \tilde{L}_0)} = \begin{cases} 1/(L_0 + \tilde{L}_0 - 2), & (L_0 - \tilde{L}_0)|\psi\rangle = 0 \\ 0, & (L_0 - \tilde{L}_0)|\psi\rangle \neq 0 \end{cases} \quad (6.3)$$

since the eigenvalues of  $L_0 - \tilde{L}_0$  are integers. Combining  $t$  and  $\phi$  gives a complex proper-time parameter; by convention in string theory  $t$  is put in the imaginary part and  $\phi$  in the real part.

The sum of all complex proper time parameters around the loop

$$\tau \equiv \frac{i}{2\lambda} \sum_{i=1}^n (t_i + i\phi_i) \quad (6.4)$$

is the ‘modular parameter’. In the geometrical picture of string theory this parameter specifies the inequivalent tori over which one must integrate.

Those parameters appearing in the conventional string amplitude which describe the locations of the vertex operators on the world-sheet torus are partial sums over the complex proper time parameters,

$$\nu_j \equiv \frac{i}{2\lambda} \sum_{l=1}^j (t_l + i\phi_l). \quad (6.5)$$

This simple interpretation of the string amplitude integration parameters in terms of proper-time variables makes possible the identification of string versions of dimensional regularization with their field-theory counterparts. We discuss the construction of string versions of dimensional regularization in the next section.

## 7. Dimensional Regularizations in String Theory

Bare loop diagrams in a massless quantum field theory possess both ultraviolet and infrared divergences. A variety of regularization schemes could be used to regulate the ultraviolet divergences, though in non-abelian gauge theories various forms of dimensional regularization are by far the most popular. In dealing with infrared divergences, one must find a scheme in which one can regulate both the divergent phase space integrals over soft and collinear momenta in the higher-point tree diagrams, as well as the corresponding divergences in the loop diagram. For practical calculations, there is simply no choice other than a form of dimensional regularization. We should therefore construct a form of dimensional regularization for the new technology presented in this paper; to do so, we will construct a dimensional regularization for the string amplitudes. (We note in passing that although the string amplitudes do not require an ultraviolet regulator, all relativistic quantum-mechanical theories in four dimensions, including string theories, with massless particles display infrared divergences in on-shell amplitudes with fixed particle number [12,13], and thus four-dimensional string amplitudes require an infrared regulator as well.)

In discussing dimensional regularization, it is helpful to keep in mind several different classes of particles: internal, or virtual, ones; external particles which are soft, or collinear with other external particles, and are thus unobserved; and external particles which are observed. The former two classes we shall lump together as *unobserved* particles, while the last class we shall term simply *observed*. Unitarity demands that the treatment of unobserved particles be uniform, independent of whether they are internal or external. It is also helpful to keep in mind the possibility of differing treatment of a gluon's momentum and polarization vector.

All dimensional regularization schemes entail continuing the momentum integrals (both the loop integrals and the integrals over soft and collinear phase space) to  $4 - \epsilon$  dimensions in order to render them finite. There are, however, at least three versions of dimensional regularization, which differ in their treatment of the polarization vectors (or helicities) of the observed and unobserved particles:

- (a) The original 't Hooft and Veltman scheme (HV) [39] in which all unobserved polarization vectors are also continued to  $4 - \epsilon$  dimensions (so that unobserved gluons have  $2 - \epsilon$  helicity

- states), but observed gluons are kept in four dimensions (so that observed gluons have 2 helicity states);
- (b) the ‘conventional’ dimensional regularization\* (CDR) used, for example, by Ellis and Sexton [14] in which both observed and unobserved polarization vectors are continued to  $4 - \epsilon$  dimensions (so that all gluons have  $2 - \epsilon$  helicity states); and
- (c) a four-dimensional helicity scheme (FDH) which naturally arises when using the spinor helicity formalism. In this scheme all helicities (of both observed and unobserved particles) are treated in four dimensions (so that all gluons have 2 helicity states).

We summarize the defining properties of the various regularization schemes in Table 1.

		Conventional	't Hooft-Veltman	Four-dimensional helicity
Momentum components	Unobserved particles	$4 - \epsilon$	$4 - \epsilon$	$4 - \epsilon$
	Observed particles	$4 - \epsilon$	4	4
Helicities	Unobserved particles	$2 - \epsilon$	$2 - \epsilon$	2
	Observed particles	$2 - \epsilon$	2	2

**Table 1:** Defining properties of the various dimensional regularization schemes.

A satisfactory regulator must respect gauge invariance and unitarity. The conventional scheme is conceptually simpler, as all quantities are continued uniformly to  $4 - \epsilon$  dimensions, and its consistency is widely accepted [40]. We will not address the complete consistency of the 't Hooft-Veltman and FDH schemes in all situations, although the desired properties are not hard to prove at one loop. We will however show explicitly the consistency of the latter schemes in the four-point calculation. The FDH is similar to Siegel’s regularization by dimension reduction [41] since the number of helicity states is fixed at 2; however, in dimensional reduction the two physical states are split between a  $(4 - \epsilon)$ -dimensional vector (which has  $2 - \epsilon$  states) and  $\epsilon$  scalars which must be treated separately.

(The reader might worry at this point about the use of helicity amplitudes in the context of dimensional regularization, given the difficulties encountered with  $\gamma_5$  and chirality in such schemes.

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\* For historical reasons, this scheme (with the use of specific pole subtraction prescription) is sometimes referred to as the ‘ $\overline{\text{MS}}$  scheme’. That nomenclature is misleading and confusing, since it is possible to use an  $\overline{\text{MS}}$  prescription in any of the schemes we are discussing. We will therefore reserve the term ‘ $\overline{\text{MS}}$ ’ for the name of the ultraviolet subtraction prescription.



The definition of  $\gamma_5$  is however irrelevant to the calculations considered in this paper, and indeed irrelevant to any calculation not involving parity violation; the easiest way to understand this is to imagine performing the calculation without resorting to the spinor helicity basis. In that case, one would obtain an answer expressed in terms of the polarization vectors as formal objects, and no  $\gamma_5$ 's would be encountered; at the very end (in principle even after construction of differential cross-sections and cancellation of infrared divergences), one could then use the spinor helicity basis. In the string-based calculation, we are effectively at the very end in the very beginning, since the starting expression (5.7) is already expressed in terms of dot products of external quantities.)

While the conventional scheme is conceptually simpler, the other two schemes have practical advantages when using the spinor helicity formalism, which is most naturally used in four dimensions. Use of the conventional scheme would require the computation of additional  $\epsilon$ -helicity amplitudes, using the conventions of section 3. While the formalism we present in this paper can of course be used for such a computation as well, it does entail extra (unnecessary) work. We have introduced the FDH scheme because it is in a sense the most natural within the context of the string formalism when using the spinor helicity basis. From a practical point of view, however, for the calculation of the one-loop corrections, the amount of work required in the FDH scheme or in the 't Hooft-Veltman scheme is the same; indeed, one may trivially include a parameter multiplying the difference between the two schemes.

We turn now to the construction of string versions of these dimensional regularization schemes. Although the meaning of these regularization prescriptions in terms of the underlying conformal field theory on the string world-sheet remains obscure, it is also completely irrelevant to practical calculations. The important question is the equivalence of our string versions of these regularization prescriptions to their field theory counterparts.

As we do not know how to map field-theory Feynman diagrams into the QCD amplitudes generated by the string, we rely on an indirect verification of the equivalence. We shall show that all the string versions preserve on-shell gauge invariance, and are thus at least consistent regulators; we shall argue that the field content of the string versions of the regulators reduce to the desired field content in the field-theory limit; we shall argue that the relations between the different schemes are precisely those expected between their field theory counterparts; and in section 13, we shall compare our results for the unpolarized corrections to gluon-gluon scattering, using the string version of the CDR scheme, to the result of the previous Feynman diagram computation of Ellis and Sexton [14]. We shall find complete agreement [15], verifying that our understanding of dimensional regularization in string theory is sound.

The string versions of the various forms of dimensional regularization that we have constructed

are based on the work of Green, Schwarz and Brink (GSB) [42]. These authors started with a ten-dimensional superstring which was regulated by assuming that  $10 - D$  of the spatial dimensions are circular with radius  $R$ . This compactification is analogous to the usual Kaluza-Klein compactifications of field theory, except that a non-integer number of dimensions are compactified. In field theory, a compactification to non-integer dimensions was used by Siegel [41] to construct his dimensional reduction scheme. This scheme entails a compactification of the theory from four dimensions to  $4 - \epsilon$  dimensions. In string theory, the GSB regularization scheme automatically leads to modular invariant string amplitudes, since it is nothing more than a simple toroidal compactification of the string. The compactification would result in the insertion of an additional factor  $(\text{Im } \tau)^{5-D/2} F_c^{10-D}(\tau)$  (defined in Appendix III) into the ten-dimensional superstring amplitude. Green, Schwarz, and Brink did not discuss the treatment of the external polarization vectors explicitly.

For four-dimensional strings, the simplest regularization scheme to construct is the FDH scheme. The only modification required to the unregulated amplitude is a change in the dimensionality of the string zero-mode (or loop momentum) integral. The operator formalism of string theory is convenient for understanding the effect of this modification to the string amplitude, as it contains the loop momentum explicitly.

One can perform the string momentum integral in  $4 - \epsilon$  dimensions as follows. In string theory, the loop momentum integration consists of two types of contributions. There is a conventional  $4 - \epsilon$  dimensional momentum integral

$$\int_0^\infty dt_1 \int_0^\infty dt_2 \cdots \int_0^\infty dt_{n-1} \int_0^\infty dt_n \int \frac{d^{4-\epsilon} p}{(2\pi)^{4-\epsilon}} \exp\left(-\sum_{i=1}^n t_i p_i^2 / 2 + \sqrt{\lambda} \epsilon_i \cdot p_i + \text{oscillators}\right) \quad (7.1)$$

where

$$p_i = p - k_1 - k_2 - \cdots - k_{i-1} = p + k_i + \cdots + k_n \quad (7.2)$$

and the proper time parameters  $t_i$  are related to the conventional string parameters as described in the previous section. (The real parts of the string parameters are not relevant for the discussion of the string momentum integral.) The  $p_i^2$  terms arise from the string propagators, while the  $\epsilon_i \cdot p_i$  terms arise from the zero-mode momentum dependence of the string vertex operators. For convenience we have exponentiated the  $\epsilon_i \cdot p_i$  terms; at the end, one must of course extract those terms linear in each and every polarization vector. In addition to the integral over momenta, in string theory one also has a discrete sum over the compactified zero-mode momenta and winding number modes. These will yield a factor of  $F_c^\epsilon(\tau)$ , whose definition is given in Appendix III.

After completing the square (see also chapter 8 of ref. [23]), the string amplitude is then of the

form

$$\begin{aligned}
\mathcal{A}_n \sim & \int d\text{Im } \tau \int_0^{\text{Im } \tau} d\text{Im } \nu_{n-1} \int_0^{\text{Im } \nu_{n-2}} d\text{Im } \nu_{n-2} \cdots \int_0^{\text{Im } \nu_2} d\text{Im } \nu_1 (\text{Im } \tau)^{-2+\epsilon/2} F_c^\epsilon \\
& \times \exp \left[ \sum_{i < j}^n \left( \lambda k_i \cdot k_j \text{Im } \nu_{ij} (1 + \text{Im } \nu_{ij} / \text{Im } \tau) \right) + \sqrt{\lambda} (\epsilon_i \cdot k_j - \epsilon_j \cdot k_i) (\text{Im } \nu_{ji} / \text{Im } \tau - 1/2) \right. \\
& \left. - \epsilon_i \cdot \epsilon_j / 2 \text{Im } \tau \right] \left( \text{oscillator contributions} \right)
\end{aligned} \tag{7.3}$$

where we have changed variables to the imaginary parts of the conventional string parameters using the relations (6.4) and (6.5). Thus, the only explicit dimension dependence arising from the string zero-mode integration is an additional factor of  $(\text{Im } \tau)^{\epsilon/2} F_c^\epsilon(\tau)$ , as in the work of GSB.

The factor  $F_c^\epsilon(\tau)$  maintains the modular invariance of the amplitude, but reduces to unity in the field theory limit. Thus the only modification in the field-theory limit is an additional factor of the Schwinger proper time  $(\text{Im } \tau)^{\epsilon/2}$ . Except for those terms arising from contractions of the polarization vectors induced by the string loop momentum integral (these are precisely terms containing  $\vec{G}_B$ 's), in the fermionic formulation, the number of gluon helicity states circulating around the loop is determined by the right-mover fermion carrying the 'spin' index. This number is not altered by the GSB-type compactification procedure for the string loop momentum. (For consistency, when evaluating the contractions of polarization vectors with other polarization vectors induced by the string loop-momentum integral we assume that the loop momentum is continued to  $D > 4$  so that the polarization vectors (and hence the helicities) remain uniformly in four dimensions. Further details of the string loop momentum integration will be provided in ref. [17].) By construction we leave all observable particles in four dimensions.

We can now modify this scheme in order to match the two more standard forms of dimensional regularization. Let us consider first the modification necessary to obtain the 't Hooft-Veltman scheme. From Table 1, the difference between the 't Hooft-Veltman scheme and the FDH scheme is that in the FDH the number of unobserved — in our case virtual or internal — gluon helicity states is 2 while in the 't Hooft-Veltman scheme the number of helicity states is  $2 - \epsilon$ . Thus, in order to obtain a string version of the 't Hooft-Veltman scheme we must remove a set of  $\epsilon$  massless bosonic degrees of freedom from the string loop. The generalized Gliozzi-Scherk-Olive (GSO) projector [43] which controls the string spectrum gives us the means to accomplish this; subtracting a set of  $\epsilon$  adjoint massless states from the loop turns out to be relatively straightforward. The contribution of an additional massless bosonic state (with its associated tower of massive states) transforming as an adjoint under the gauge group can be written as a constant times the difference of amplitudes in two different models, one with no matter content, and the other with additional scalars. Since each of the two amplitudes is modular-invariant, the difference is modular-invariant, and the contribution we

subtract from our original amplitude will thus be modular-invariant. This will provide a consistent string version of the 't Hooft-Veltman scheme.

More precisely, the relation between amplitudes in the two schemes is

$$\mathcal{A}_n^{\text{HV}}(\text{pure glue}) = \mathcal{A}_n^{\text{FDH}}(\text{pure glue}) - \frac{\epsilon}{N_s} \left( \mathcal{A}_n^{\text{FDH}}(\text{pure glue} + N_s \text{ real scalars}) - \mathcal{A}_n^{\text{FDH}}(\text{pure glue}) \right) \quad (7.4)$$

We begin by considering the addition of a discrete set of  $N_s$  adjoint real scalar states to the Neveu-Schwarz sector of the string loop. This increases the number of propagating states in the loop from 2 (for the two gluon helicities) to  $2 + N_s$ . Given the model presented in Appendix I, we could, for example, replace the basis vector  $W_4$  with

$$W'_4 = \left( 0^9 \frac{1}{2} \ 0 \frac{1}{2} \ 00 \left| \frac{1}{2} \begin{pmatrix} 00 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} 00 & 1 \\ 2 \end{pmatrix} \right) . \quad (7.5)$$

This would add one complex (or two real) scalars to the massless content of the theory arising from the Neveu-Schwarz sector (corresponding to the fifth right-mover oscillator); these additional states will also transform as adjoints under the gauge group of interest, since their left-mover structure will be identical to that of the gauge bosons. In general, changing the sets of boundary conditions modifies the string amplitude in both the sum over boundary conditions and in the set of coefficients  $C_{\beta}^{\alpha}$  in a rather complicated way. When we restrict our attention to the Neveu-Schwarz sector, however, the modifications are quite simple, and for this reason we do not really need a specific choice of model; the fact that such choices exist suffices.

It turns out that again only the partition function is modified,

$$\mathcal{Z}^{\text{HV}}(\text{pure glue}) = \mathcal{Z}^{\text{FDH}}(\text{pure glue}) - \frac{\epsilon}{N_s} \left( \mathcal{Z}^{\text{FDH}}(\text{pure glue} + N_s \text{ real scalars}) - \mathcal{Z}^{\text{FDH}}(\text{pure glue}) \right) . \quad (7.6)$$

As we shall discuss in the following section, the partition function for the right-movers in a multigluon amplitude has the following expansion in powers of  $\hat{q}^{1/2} = e^{-\pi \text{Im} \tau}$ ,

$$\text{phase} \times \hat{q}^{-1/2} \left( 1 - 2\hat{q}^{1/2} \times \text{phase} + \dots \right) \quad (7.7)$$

where the omitted terms are irrelevant in the field theory limit. Changing the number of scalars changes only the coefficient of the  $\hat{q}^{1/2}$ , from 2 to  $2 + N_s$ , so that calculations are transformed to the 't Hooft-Veltman scheme by replacing the above expansion with

$$\text{phase} \times \hat{q}^{-1/2} \left( 1 - 2 \left( 1 - \frac{\epsilon}{2} \right) \hat{q}^{1/2} \times \text{phase} + \dots \right) . \quad (7.8)$$

In field-theory, one moves from the 't Hooft-Veltman form of dimensional regularization to the conventional one by removing a set of 'epsilon' helicities for the external states [21]. The same is

true in string theory; in addition to the usual  $(+, -)$  helicities, one now has ‘ $[\epsilon]$ ’ helicities (or more properly ‘ $[-\epsilon]$ ’ helicities) that must be summed over in constructing differential cross sections,

$$\begin{aligned} \left[ \sum_{\text{helicities}} |\mathcal{A}_n|^2 \right]_{\text{HV}} &= \sum_{\lambda_1, \dots, \lambda_n = (+, -)} |\mathcal{A}_n(1^{\lambda_1}, \dots, n^{\lambda_n})|^2 \\ \left[ \sum_{\text{helicities}} |\mathcal{A}_n|^2 \right]_{\text{CDR}} &= \sum_{\lambda_1, \dots, \lambda_n = (+, -, [\epsilon])} |\mathcal{A}_n(1^{\lambda_1}, \dots, n^{\lambda_n})|^2 . \end{aligned} \tag{7.9}$$

Using the rules described in section 3, one can perform such calculations in the framework of the spinor helicity basis.

In traditional Feynman diagram calculations, one squares the amplitude and sums over helicities, and thus only scalar quantities appear visibly as the results of calculations. Quantities with indices — such as helicity amplitudes — do not appear explicitly. In the helicity approach, in contrast, the results of calculations *do* involve tensorial quantities. It is for this reason that the distinction between the ‘t Hooft-Veltman and conventional dimensional regularization schemes are sharper, and more apparent, than a mere difference in renormalization prescriptions. In particular, the use of either the ‘t Hooft-Veltman or FDH regulators are preferable to the use of the conventional scheme, since the latter requires the calculation of amplitudes with ‘epsilon’-helicities, which increases the amount of work involved.

We summarize those calculational differences between the three schemes that remain in the infinite-tension limit in table 2.

	FDH	‘t Hooft-Veltman	Conventional
Factor of $(\text{Im } \tau)^{-\epsilon/2}$	Yes	Yes	Yes
Remove $\epsilon$ bosonic states	No	Yes	Yes
$\epsilon_i^{(4)} \rightarrow \epsilon_i^{(4-\epsilon)}$	No	No	Yes

**Table 2:** Modifications needed to construct various string versions of dimensional regularization from the unregularized string amplitude.

The main motivation for choosing a dimensional regularization scheme is the issue of infrared divergences. Having chosen such a scheme, however, we must also handle the ultraviolet divergences in a conventional manner. Field theory computations are performed in terms of ‘bare’ quantities,

that is with an infinite renormalization scale. In order to re-express Green functions in terms of couplings and fields renormalized at physical scales, one must multiply by the appropriate (infinite) renormalization constant. In one-loop perturbation theory with a dimensional regulator, this corresponds simply to subtracting the  $1/\epsilon$  ultra-violet pole; but of course there are finite ambiguities. Within any of three regulators, we can perform renormalization according to the  $\overline{\text{MS}}$  prescription — subtracting only the ultra-violet pole — or according to the  $\overline{\text{MS}}$  prescription [44], where one subtracts

$$(n-2)\beta_0 g^2 \frac{1}{\epsilon} \frac{\Gamma^2(1-\epsilon/2)\Gamma(1+\epsilon/2)}{\Gamma(1-\epsilon)} (4\pi)^{\epsilon/2} (\mu^2)^{\epsilon/2} \mathcal{A}_n^{\text{tree}}(1, \dots, n) \quad (7.10)$$

from the  $n$ -point amplitude ( $\beta_0 = -11N_c/(3 \cdot 16\pi^2)$ ). In the string formalism, the amplitudes are ultraviolet-finite so long as the inverse string tension  $\lambda$  is not actually zero; but in the limit, we of course recover the usual ultraviolet divergences of field theory, and we perform the subtraction in the usual manner.

We turn next to the discussion of the gauge invariance of these various schemes. We want to show that the regulated amplitudes are invariant under the shift  $\epsilon_i \rightarrow \epsilon_i + k_i$ .

In string theory, the substitution of the external momentum  $k_i$  for the corresponding polarization vector  $\epsilon_i$  formally leads to the vanishing of the unregulated amplitude, because one obtains the integral of a total derivative in punctured moduli space. We shall begin with a review of this formal proof of gauge invariance in the string theory. Start with the gluon vertex operator, as given in appendix III, and set  $\epsilon = k$ ; the vertex operator becomes

$$V(\nu, \bar{\nu})|_{\epsilon=k} = -\sqrt{2}g\sqrt{\lambda}T_i^{\alpha j} : \Psi^{\dagger}(\nu)\Psi_j(\nu)\partial_{\bar{\nu}}e^{ik \cdot X(\nu, \bar{\nu})} : . \quad (7.11)$$

If we now compute expectation values using this vertex operator instead of the usual one for the first external gluon, we obtain a result whose right-mover and common exponential part is a total derivative in  $\bar{\nu}_1$ ; that is with  $\epsilon_1$  replaced by  $k_1$ , the integrand of the amplitude (5.7) is a total derivative in  $\bar{\nu}_1$ . As discussed above, the various dimensional regularization schemes modify only the partition function and possibly the external polarization vectors; the important point is that none of these changes alter the fact that the integrand is a total derivative in  $\nu_1$ , because they do not affect the structure of the Green functions. As a result, the dimensional regularization schemes do not alter the formal argument.

Since the torus has no boundaries one might then conclude that both the amplitude with a longitudinal external gluon vanishes since the integrand is a total derivative. This is almost right, but there are subtleties that must be addressed. The left-mover contains poles, and thus is not necessarily analytic everywhere in  $\nu_1$ . As a result, when we attempt to integrate the right-mover factors by parts, in order to prove the vanishing of the resulting longitudinal amplitude, the

derivative in  $\bar{\nu}_1$  can hit a left-mover Green function, and this in principle gives a non-vanishing contribution: for example,  $\partial_{\rho_1} \nu_1^{-1} = 2\pi\delta^{(2)}(\nu_1)$ . The typical form of the  $\nu_1$  dependence in the amplitude near a left-mover pole is

$$(\nu_1 - \nu_j)^{-1+\lambda s_{ij}/\pi} \quad (7.12)$$

which is not necessarily analytic as  $\nu_1 \rightarrow \nu_j$  when the momentum invariants are in the physical region. However, analytically continuing to sufficiently positive values of the  $s_{ij}$  renders the expression completely analytic in  $\nu_1$  so that the  $\bar{\nu}_1$  derivative vanishes.

There are further subtleties in the regions of the integral where the loop is isolated on the first external leg, and in the region where the loop is isolated at the end of a (dilaton) tadpole. The latter contribution can be eliminated in the string theory using Green-Seiberg [45] type contact terms, and in any event may be shown to drop out [17] in the gauge theory (infinite-tension) limit. In the former case, the momentum invariant in which we want to continue is  $k_1^2$ , which vanishes identically on shell. Because of 0/0 ambiguities mentioned in the previous section, contributions from this region of moduli space must anyway be defined by an appropriate ‘offsheets’ prescription during which the momentum invariant does *not* vanish. The appropriate analytic continuation can then be performed. The limit of vanishing momentum invariants must be taken at the end. A further twist occurs in this limit: we obtain terms proportional to  $(K^2)^\epsilon$  where  $K^2$  is a momentum invariant which vanishes in the on-shell limit. While an appropriate prescription for such terms may seem at first elusive, an analogous set of terms arises when working with ordinary Feynman diagrams [38]. In field theory calculations, the prescription that has been used is to take the limit  $K^2 \rightarrow 0$  first, so that one sets such terms to zero; this is interpreted as a complete cancellation of infrared and ultraviolet contributions. The present formalism makes it possible to prove that this is the prescription consistent with gauge invariance, and we shall do so elsewhere [17]. It is amusing that the two regions of moduli space which contain subtleties in the proof of gauge invariance ultimately do not contribute to the regularized QCD amplitudes.

## 8. Derivation of Rules for Field Theory Limit

In order to calculate the gauge-theory scattering amplitude for  $n$  external gluons, we must compute the partial amplitudes  $A_{n;j}(1, 2, \dots, n)$  for  $1 \leq j \leq \lfloor n/2 \rfloor + 1$ . The full amplitude can then be obtained using the color decomposition (4.2), and the next-to-leading correction to the differential cross-section can be expressed in terms of these partial amplitudes using equation (9.16) of ref. [11].

These partial amplitudes appear in the dimensionally-regulated amplitude (5.7) as the coefficients of the trace structures  $\text{Tr}(T^{a_1} \dots T^{a_{j-1}}) \text{Tr}(T^{a_j} \dots T^{a_n})$ ; after integrating over the various

Grassmann parameters, we should pick the coefficient of each of these trace structures, and compute it in the infinite tension limit  $\lambda \rightarrow 0$ . As explained in section 5, the left-mover terms are in a direct correspondence with the trace structures, so we need not start with the full expression (5.8) and perform the Grassmann integrations explicitly; we can simply write down the left-mover Green functions associated with the given trace structure. Because of the factorization of right- and left-mover pieces (other than the zero-mode contributions in equation (5.11)), we will get the same initial right-mover integrand for each partial amplitude. The contributions to the different partial amplitudes will differ because of differences in pole structure and integration region determined by the different left-mover Green functions.

The first step, as explained in section 3, is to choose a set of reference momenta for the external gluons, and substitute into equation (5.10) the expressions resulting from use of the spinor helicity basis for the various dot products of polarization vectors and external momenta.

If we then examine these right-mover terms, we will see that every factor of the Grassmann parameter  $\theta_{i3}$  carries along a factor of  $\sqrt{\lambda}$ , excepting the  $\theta_{i3}$  multiplying the double derivatives of the bosonic Green function ( $\vec{G}_B$ ). Thus after performing all the  $\theta_{i3}$  integrals, each term not containing double derivatives will carry an overall factor of  $\lambda^{n-2+\epsilon/2}$  (after combining with the factors of  $\lambda$  in the over-all normalization). It is however possible (and desirable) to integrate the right-movers by parts and remove all double derivatives of the bosonic Green functions. (We showed in appendix II of ref. [11] that it is always possible to do so. The procedure required is completely mechanical.) After such integrations by parts, the right-mover contribution contains only fermionic Green functions and single derivatives of bosonic Green functions; and all terms in the amplitude have a uniform factor of  $\lambda^{n-2+\epsilon/2}$  in front. In the following discussion, we shall assume throughout that such an integration by parts has been carried out. (This form is also preferable in that it makes the world-sheet supersymmetry of the right-movers manifest, in the sense that disappearance of the poles associated with the fictitious  $F_1$ -formalism tachyon is manifest. The manifest supersymmetry can be used as a check on the algebra — or on the computer program performing the algebra; substituting  $-G_F[\alpha_\tau^\dagger]_{\beta_\tau}(\bar{\nu})$  for each  $\dot{G}_B(\bar{\nu})$  should yield zero in this form of the amplitude.) These steps are common to all partial amplitudes, and thus need be done only once for each helicity configuration.

The presence of these explicit powers of  $\lambda$  in front of the amplitude means that only those regions of the integral which yield an appropriate number of powers of  $\lambda^{-1}$  will survive in the gauge-theory limit of vanishing  $\lambda$ . There are two sources of such powers: the large  $\text{Im } \tau$  region of the modular parameter integral, and poles in the differences of the locations of the massless-vector vertex operators,  $\nu_{ij} = \nu_i - \nu_j$ . Even after extracting as many powers of  $\lambda^{-1}$  as possible from poles



in the  $\nu_{ij}$ , it turns out that surviving contributions come only from the large- $\text{Im } \tau$  region. The  $\nu_{ij}$  pinch contributions yield trees of massless vectors sewn onto a loop; and large  $\text{Im } \tau$  means that only the massless particles survive to run around the loop, as depicted in fig. 2.

In extracting surviving contributions to the amplitude, we must therefore ‘pinch’ together various sets of  $\nu$ s at a set of locations on the world-sheet torus, and then extract the large- $\text{Im } \tau$  contributions to the modular-parameter integral. In this limit, the theta functions that comprise the various world-sheet Green functions have simple expansions in terms of ordinary transcendental functions, which makes it straightforward to compute the integrals explicitly. The types of terms which arise in the field theory limit are similar to the types of terms one would expect from a Schwinger proper-time formulation of gauge theory after the momentum integrations have been performed. (The terms themselves are more compact than their Feynman diagram counterparts.)

In the field theory limit we end up with a loop integral of the form

$$\int^{\infty} d\text{Im } \tau (\text{Im } \tau)^{n_{\ell}-3+\epsilon/2} e^{-\lambda K \text{Im } \tau} = \frac{\Gamma(n_{\ell} - 2 + \epsilon/2)}{(\lambda K)^{n_{\ell}-2+\epsilon/2}} \quad (\lambda \rightarrow 0) \quad (8.1)$$

where  $n_{\ell}$  is the number of surviving  $\nu$  variables which remain after integrating out the pinched variables, and where  $K$  is a function of the momentum invariants and Feynman parameters. (Each surviving  $\nu$  variable is integrated over the torus, producing a factor of  $\text{Im } \tau$ ; there are two powers of  $(\text{Im } \tau)^{-1}$  from the modular measure, and a factor of  $(\text{Im } \tau)^{-1+\epsilon/2}$  from the partition function in  $(4 - \epsilon)$  dimensions.) Combining with the over-all uniform factor of  $\lambda^{n-2+\epsilon/2}$ , we have a factor of  $\lambda^{n-n_{\ell}}$  left over. The integration of any pinched variable, as we shall see below, can contribute at most one power of  $\lambda^{-1}$ ; and since we have only  $n - n_{\ell}$  pinched variables, we see that each integral over a pinched variable *must* contribute a factor of  $\lambda^{-1}$  in order to cancel the overall powers of  $\lambda$  and thereby obtain a non-vanishing contribution in the infinite-tension limit.

For the purposes of this section, it will be convenient to use the surviving remnant of conformal invariance on the torus to fix the coordinate  $\nu_n$  of the last leg to be the modular parameter  $\tau$ . This does not change any of the counting arguments in the previous paragraph, since an explicit power of  $\text{Im } \tau$  will appear in the integrand to compensate for the missing integral.

The remainder of this section is divided into five parts. In the first, we discuss the pinch integrations, followed by three parts in which we discuss in turn the large- $\text{Im } \tau$  expansions of the left-mover, the right-movers, and the common exponentials. In the last part, we discuss the integration regions.

## 8.1 Pinch Integrations

We begin with a general discussion of the structure of pinch contributions. In a pinch region,

$\nu_{ij} = \nu_i - \nu_j \rightarrow 0$ , the Green functions have simple expansions,

$$\begin{aligned}
\exp(G_B(\nu_{ij})) &\longrightarrow |\nu_{ij}|^{-1/\pi} \times \text{constant} \\
\dot{G}_B(\bar{\nu}_{ij}) &\longrightarrow -\frac{1}{2\pi\bar{\nu}_{ij}} + \mathcal{O}(\bar{\nu}_{ij}) \\
G_F\left[\begin{smallmatrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{smallmatrix}\right](\bar{\nu}_{ij}) &\longrightarrow \frac{1}{2\pi\bar{\nu}_{ij}} + \mathcal{O}(\bar{\nu}_{ij}) \\
G_F\left[\begin{smallmatrix} 1/2 \\ \beta \end{smallmatrix}\right](\nu_{ij}) &\longrightarrow \frac{1}{2\pi\nu_{ij}} + \mathcal{O}(1)
\end{aligned} \tag{8.2}$$

In general we may pinch together different sets of vertex locations  $\nu_i$  at several different locations on the string world-sheet. Each separate set of pinched variables may be integrated independently. The various pinch contributions may be classified by drawing ordinary one-loop  $\phi^3$  diagrams with  $n$  external legs. The diagrams classifying the pinch contributions in the four-point amplitude are shown in fig. 3. Diagram 3a corresponds to a contribution with no pinched variables; diagram 3b to contributions with a single pair of variables pinched together; diagram 3c to contributions with two separate pairs of variables pinched at separate locations on the string world-sheet; and diagram 3d to contributions where three variables are pinched together. These diagrams may seem a bit odd from a field theory point of view, since there are neither four-point vector interaction vertices nor ghost lines. The string theory includes those aspects of gauge theory interactions automatically. Momentum factors in the numerator can cancel momentum poles leaving contributions equivalent to four-point contact terms; and as discussed in section 5, the ghost contributions affect only the string partition function, and there they are already included in a strikingly simple manner.

Consider first pinching only two variables together,  $\nu_i \rightarrow \nu_j$ . In this limit, we can expand the integrand. The highest pole that can emerge from the left-movers is  $\nu^{-2}$ , corresponding to the left-mover vacuum level of  $-4\pi/\lambda$ . Because of the manifest world-sheet supersymmetry of the right-movers after the integration by parts, the same degree pole does not exist (it would correspond to the propagation of a fictitious  $F_1$  formalism tachyon), so the highest pole that can emerge from the right-movers is  $\bar{\nu}^{-1}$ , corresponding to the propagation of massless states. Thus the integrals we must consider are

$$P_{m,n}^{i,j} = \int d^2\nu_i |\nu_{ij}|^{-\lambda k_i \cdot k_j / \pi} \nu_{ij}^m \bar{\nu}_{ij}^n \tag{8.3}$$

where  $m \geq -2$ ,  $n \geq -1$ , and the factor of  $|\nu_{ij}|^{-\lambda k_i \cdot k_j / \pi}$  comes from the pinch limit of the exponentiated Green functions (5.11). If the orders of the left- and right-mover poles are not identical, the phase integration makes the integral vanish,

$$P_{m,n}^{i,j} = \delta_{m,n} P_n^{i,j} . \tag{8.4}$$

Thus we need consider only the case  $m, n \geq -1$ . For the lower bound, we find

$$\int d^2\nu_i \frac{1}{|\nu_{ij}|^{2+\lambda k_i \cdot k_j / \pi}} = -\frac{2\pi^2}{\lambda k_i \cdot k_j} . \tag{8.5}$$

so that we would obtain the factor of  $\lambda^{-1}$  necessary for a contribution to survive in the infinite-tension limit. Other integer powers of  $|\nu_{ij}|$  correspond to the propagation of a massive string state; these yield expressions of the form  $(\lambda k_i \cdot k_j - \text{integer})^{-1}$  which fail to produce an overall factor of  $\lambda^{-1}$ , and thus are not relevant in the field theory limit. In the infinite-tension limit, we can therefore summarize the pinch integrals in the following form,

$$P_{m,n}^{i,j} = \delta_{m,-1} \delta_{n,-1} \times \left( -\frac{2\pi^2}{\lambda k_i \cdot k_j} \right). \quad (8.6)$$

Now, each left-mover and each right-mover Green function, whether fermionic or bosonic, contributes a single pole of its argument in the pinch limit of its argument. Thus the only terms which survive are those which have exactly one left-mover and exactly one right-mover fermion with an argument of the pinched variable. Other terms vanish. The contribution of each surviving term is given by the following rule: remove the pair of Green functions that produce the pole, replace it with the value from equation (8.6) times the appropriate signs from equation (8.2), and continue with the analysis of the other factors in the term. In the other factors, one must set  $\nu_i = \nu_j$ , which substitution we perform by replacing  $\nu_{\min(i,j)}$  with  $\nu_{\max(i,j)}$ . This procedure generalizes trivially to pinch structures such as that depicted in diagram 3b, with pinches at different locations on the whole, since these pinches are independent, and may be treated independently.

In the particular case of the four-point amplitude, we need not worry about pinching together more than two variables to one point, as depicted for example in diagram 3d, because in these contributions the loop is isolated on an external leg, and as discussed in section 7, the contribution therefore vanishes in dimensional regularization because of a cancellation of ultraviolet and infrared regions.

In general, of course, there will be nontrivial contributions from pinching together some subset of more than two variables  $\{\nu_p\}_{j=1}^p$ , leaving the remaining ones  $(\{\nu_u\}_{j=1}^{n-p})$  unpinched. In this case, we will again find that only terms which lead to single left- or right-mover poles in the various pinch variables survive; and it is possible to perform the integrals in a manner analogous to that above. (See equations (7.6–7.8) of ref. [37] for an example.) However, it is possible to derive a simpler set of rules for the results of such integrations. We defer the discussion of a set of simpler rules regarding the multi-pinch structure [46] since these are not needed for the four-point amplitude discussed in this paper.

## 8.2 Large Proper Time Expansions

After integrating out all the pinched variables, we must take the large  $\text{Im } \tau$  limit in order to reduce the string loop into a field theory loop. To understand the structure of the surviving contributions as  $\text{Im } \tau \rightarrow \infty$  we should therefore must consider the expansion of the partition function

and the Green functions in powers of  $\hat{q}^{1/2} = e^{-\pi \text{Im } \tau}$ . The integer powers of  $\hat{q}^{1/2}$  correspond to mass[-squared] levels of the string in units of  $4\pi/\lambda$ . As we shall discuss later in this section, the correspondence can be understood simply through the interpretation of  $\text{Im } \tau$  as a Schwinger proper-time parameter; in ordinary field theory, contributions from massive states would also be of the form  $e^{-t m^2}$  where  $t$  is a proper-time parameter. Terms with negative powers of  $\hat{q}^{1/2}$  would correspond to the propagation of tachyonic states in the loop, but these are removed by the generalized GSO projector; terms with positive integer powers of  $\hat{q}^{1/2}$  would disappear in the infinite-tension limit, because they would fail to produce inverse powers of  $\lambda$  in front of the integral. In this fashion, the massive tower of states in the string disappears, as expected, in the infinite-tension limit. Only terms with no surviving integer powers of  $\hat{q}^{1/2}$  will give rise to massless-particle contributions in the gauge theory; and such contributions can only arise in certain sectors of the models. In the case of the particular model we are using, there is only one sector with massless states — the Neveu-Schwarz sector or  $W_0$  sector in the notation of KLT [29].

In any term in the expansion of the integrand, the powers of  $\hat{q}^{1/2}$  coming from the left-movers are accompanied by powers of  $e^{i\pi \text{Re } \tau}$ , while the powers of  $\hat{q}^{1/2}$  coming from the right movers are accompanied by powers of  $e^{-i\pi \text{Re } \tau}$ . Thus unless the left-movers supply the same number of powers of  $\hat{q}^{1/2}$  in any given term as do the right-movers, there will be a surviving factor of  $e^{\pm i\pi \text{Re } \tau \times \text{integer}}$ , and the  $\text{Re } \tau$  integral will kill the term. That is, the  $\text{Re } \tau$  integral (which in the field theory limit varies between  $-1/2$  and  $+1/2$ ) enforces the level-matching condition of the string, and allows us to consider the expansions of the left- and right-movers separately. (The origin of level matching conditions in the integration over the real parts of the parameters should come as no surprise, since that is precisely the role the real parts play in the operator formalism.)

In this limit, the Green functions behave as follows,

$$\begin{aligned}
\exp(G_B(\nu)) &\rightarrow \exp\left(\text{Im } \tau(\hat{\nu}^2 - |\hat{\nu}|)\right) \times \text{constant} \\
\dot{G}_B(\bar{\nu}) &\rightarrow \frac{i}{2}(-\text{sign}(\hat{\nu}) + 2\hat{\nu}) \\
G_F\left[\frac{1/2}{\beta}\right](\nu) &\rightarrow -i \text{sign}(\hat{\nu}) \left( e^{\pi i \text{sign}(\hat{\nu}) \text{Re } \nu} e^{-\pi \text{Im } \tau |\hat{\nu}|} - \hat{q}^{1/2} e^{\pi i \text{Re } \tau} e^{-\pi i \text{sign}(\hat{\nu}) \text{Re } \nu} e^{\pi \text{Im } \tau |\hat{\nu}|} e^{2\pi i \text{sign}(\hat{\nu}) \beta} \right. \\
&\quad \left. - \hat{q} e^{2\pi i \text{Re } \tau} e^{-\pi i \text{sign}(\hat{\nu}) \text{Re } \nu} e^{\pi \text{Im } \tau |\hat{\nu}|} e^{4\pi i \text{sign}(\hat{\nu}) \beta} \right) \\
G_F\left[\frac{1/2}{\beta}\right](\bar{\nu}) &\rightarrow +i \text{sign}(\hat{\nu}) \left( e^{-\pi i \text{sign}(\hat{\nu}) \text{Re } \bar{\nu}} e^{-\pi \text{Im } \tau |\hat{\nu}|} - \hat{q}^{1/2} e^{-\pi i \text{Re } \tau} e^{+\pi i \text{sign}(\hat{\nu}) \text{Re } \bar{\nu}} e^{\pi \text{Im } \tau |\hat{\nu}|} e^{-2\pi i \text{sign}(\hat{\nu}) \beta} \right)
\end{aligned} \tag{8.7}$$

where  $\hat{\nu} = \text{Im } \nu / \text{Im } \tau = -\text{Im } \bar{\nu} / \text{Im } \tau$ . The left-mover partition function in the Neveu-Schwarz

sector, which contains the gauge bosons, goes as follows in the large  $\text{Im } \tau$  limit,

$$\mathcal{Z}_L \left[ \begin{array}{c} \vec{\alpha}_{0L} \\ \vec{\beta}_L \end{array} \right] \longrightarrow \hat{q}^{-1} e^{-2\pi i \text{Re } \tau} (1 + 24\hat{q} e^{2\pi i \text{Re } \tau}) \\ \times \left( 1 - 2\hat{q}^{1/2} e^{\pi i \text{Re } \tau} \sum_{i=1}^{\text{len } \alpha_{0L}} \cos 2\pi \beta_{Li} + 4\hat{q} e^{2\pi i \text{Re } \tau} \sum_{i<j=1}^{22} \cos 2\pi \beta_{Li} \cos 2\pi \beta_{Lj} \right) \quad (8.8)$$

In the following discussion, we will find it convenient to distinguish between the different types of terms in the left-mover partition function. For the sample model of appendix I, the  $\mathcal{O}(\hat{q}^{-1/2})$  left-mover contributions to the partition function can be grouped into three types: those associated with the  $SU(N_c)$  gauge group  $G$  of interest (where  $N_c = 9$  for the model in appendix I), those associated with the secondary  $SU(N'_c)$  gauge group  $G'$  (where  $N'_c = 9$  for the model in appendix I), and those associated with any remaining string gauge groups. Labeling the the world-sheet boundary conditions associated with gauge group of interest  $\beta_G$  and those associated with the secondary gauge group as  $\beta_{G'}$ , the  $\mathcal{O}(\hat{q}^{-1/2})$  terms in the partition function can be rewritten as follows,

$$-2\hat{q}^{-1/2} e^{\pi i \text{Re } \tau} \sum_{j=1}^{22} \cos 2\pi \beta_{Lj} = -2\hat{q}^{-1/2} e^{\pi i \text{Re } \tau} \left( N_c \cos(2\pi \beta_G) + N'_c \cos(2\pi \beta_{G'}) + \sum_{j=N_c+N'_c+1}^{22} \cos 2\pi \beta_{Lj} \right) \quad (8.9)$$

These expansions might seem complicated at first sight, but as we shall see, combinations of Green functions and partition functions always yield simple expressions. The simplifications result from the constraint that only terms not exponentially suppressed in  $\text{Im } \tau$  survive. Furthermore, the generalized GSO projector (which arises from the summation over string world-sheet boundary conditions) removes all contributions in the field-theory limit from states which decouple in that limit from the  $SU(N_c)$  gauge group of interest.

### 8.3 Expansion of the Left-Movers

Let us consider first the structure of the left-movers. We must extract the coefficient of the  $\hat{q}^0$  term from the product of the Green functions and the partition function. In expanding this product, there are in principle four possible ways of obtaining the two powers of  $\hat{q}^{1/2}$  needed to cancel off the leading  $\hat{q}^{-1}$  from the left-mover partition function: (a) a power of  $\hat{q}$  from the partition function, combined with the leading ( $\hat{q}^0$ ) power from the Green functions (b) a power of  $\hat{q}$  from one of the Green functions (c) one power of  $\hat{q}^{1/2}$  from the Green functions, and one from the expansion of the partition function, or (d) two powers of  $\hat{q}^{1/2}$  from two different Green functions.

From equation (8.7), we see that the leading term in the expansion of the left-mover Green function  $G_F \left[ \begin{array}{c} 1 \\ \beta \end{array} \right]$  contains a decaying exponential in  $|\hat{\nu}_i| \text{Im } \tau$ . A product of such terms alone would

fail to produce any powers of  $\lambda^{-1}$  after performing the  $\text{Im } \tau$  integral, and would therefore vanish in the field theory limit. (In the region where the  $\hat{\nu}_i$  vanish, one fails to obtain the required powers of  $\lambda^{-1}$ , except for the pinch limits discussed above.) On the other hand, the coefficients of the  $\hat{q}^{1/2}$  and  $\hat{q}$  terms in the expansion of the Green functions contain growing exponentials in the  $|\hat{\nu}_i| \text{Im } \tau$  that can cancel the decaying ones present in the leading-order term. Thus the only terms that will survive are those with an appropriate combination of leading-order and higher-order terms from the Green functions. In particular, at least one of the powers of  $\hat{q}^{1/2}$  *must* come from the Green functions, and so alternative (a) is not viable.

For the left-movers of the model in Appendix I, we may group the world sheet ‘time’ boundary conditions, which control the generalized GSO projection into triplets  $(W + 0 \cdot W_1, W + 1 \cdot W_1, W + 2 \cdot W_1)$ . Each of the time-boundary conditions in any given triplet shares the same coefficient  $C_{\vec{\beta}}^{W_0}$ , since  $W_1$  has a zero in the first right-mover position or “spin-component” (which denotes the world-sheet fermion carrying the space-time index) and the coefficient  $C_{\vec{\beta}}^{W_0} = -\cos(2\pi\beta_{\uparrow})/\mathcal{M} = (-1)^{n_0+n_2+n_3+n_4+1}/\mathcal{M}$  where  $\vec{\beta} \equiv \sum_{i=0}^4 n_i W_i$  and  $\mathcal{M} = 48$  for the model in Appendix I. In our model, the complex exponentials  $e^{-2\pi i\beta_{\sigma}}$  and  $e^{-4\pi i\beta_{\sigma}}$  are simply the cube roots of either 1 or  $-1$ , and so will vanish when summed over all world-sheet time-boundary conditions. Only terms where the factors of  $e^{\pm 2\pi i\beta_{\sigma}}$  completely cancel can survive. This tells us that terms where the  $\hat{q}$  comes from the third term in the expansion of a single Green function cannot contribute, because these would not give rise to an appropriate ‘interference’, so option (b) is not viable either.

We are thus left with the two alternatives (c) and (d). In general, when we extract terms proportional to  $\hat{q}^{1/2}$  from a product of Green functions, we will end up with a factor of the form

$$\exp \left[ \left( |\hat{\nu}_{kl}| - \sum |\hat{\nu}_{ij}| \right) \text{Im } \tau \right] . \quad (8.10)$$

As mentioned previously, in order to avoid an eventual exponential suppression in  $\text{Im } \tau$ , the sum must add up to cancel the leading term within the exponential exactly. This will happen only if each  $\hat{\nu}_i$  appears once with a positive and once with a negative sign after expressing the absolute values in terms of the  $\hat{\nu}_i$ s directly. After fixing  $\nu_n = \text{Im } \tau$ , that is  $\hat{\nu}_n = 1$ , we may divide the integration over the  $\hat{\nu}_i$  remaining after the pinch integrations into different regions, where in each region these variables have a definite ordering, for example  $\hat{\nu}_1 \leq \hat{\nu}_2 \leq \dots \hat{\nu}_{n-1} \leq \hat{\nu}_n$ . We will call each of these orderings an *integration ordering*. This ordering can be given a diagrammatic interpretation in terms of the labels of the legs attached to a loop as depicted in fig. 4a.

In this particular ordering, we will avoid an exponential decay only if the first term inside the exponential is  $\hat{\nu}_{j_{\min}, j_{\max}}$ , and if the terms inside the sum have the form  $\hat{\nu}_{j, j+1}$ . Such terms are an example of a *cyclic set*: given a sequence of  $n$  variables  $\{\nu_1, \dots, \nu_n\}$ , the corresponding cyclic set is the set of differences  $\nu_{ij}$  of the form  $\nu_{j, j+1}$  (where the indices are taken mod  $n$ ). We shall

term a cyclic set *indivisible* if the underlying sequence of variables — the *base set* — cannot be partitioned so that the cyclic set is the union of the cyclic sets generated by the different partitions. For example, given the sequence of four variables  $\nu_1, \dots, \nu_4$ , then  $\{\nu_{12}, \nu_{23}, \nu_{34}, \nu_{41}\}$  is an indivisible cyclic set;  $\{\nu_{12}, \nu_{34}\}$  is a divisible cyclic set; and  $\{\nu_{12}, \nu_{13}, \nu_{14}\}$  is not a cyclic set at all. A  $m$ -times divisible set is one that can be partitioned into  $m$  indivisible cyclic sets.

As noted in section 5, the left-mover Green function are in one-to-one correspondence to the trace structures appearing in the amplitude. For example, the left-mover Green functions associated with  $\text{Tr}(T^{\alpha_1} T^{\alpha_2} \dots T^{\alpha_n})$ , and thus contributing to  $A_{n;1}(1, 2, \dots, n)$  are

$$(-1)^{n-1} G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{1,n}) G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (-\nu_{12}) G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (-\nu_{23}) \dots G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (-\nu_{n-1,n}) \quad (8.11)$$

where the sign comes from the Grassmann integrations and from replacing  $\hat{G}_F[\frac{\alpha_G}{\beta_G}]$ 's with  $G_F[\frac{\alpha_G}{\beta_G}]$ 's,

$$\begin{aligned} & \int \left( \prod_{i=1}^n d\theta_{i1} d\theta_{i2} \right) [-\theta_{1,1}\theta_{n,2}] [-\theta_{1,2}\theta_{2,1}] [-\theta_{2,2}\theta_{3,1}] \dots [-\theta_{n-1,2}\theta_{n,1}] \\ & \quad \times G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{1,n}) \hat{G}_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{12}) \hat{G}_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{23}) \dots \hat{G}_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{n-1,n}) \\ & = \int \left( \prod_{i=1}^n d\theta_{i1} d\theta_{i2} [-\theta_{i1}\theta_{i2}] \right) \\ & \quad \times (-1)^{n-1} G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{1,n}) G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (-\nu_{12}) G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (-\nu_{23}) \dots G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (-\nu_{n-1,n}) \end{aligned} \quad (8.12)$$

which reduces to equation (8.11), since the Grassmann integrations yield unity. The arguments of the Green functions in these equations are a cyclic set (or more precisely the negatives of the arguments are a cyclic set generated from  $\nu_1, \dots, \nu_n$ ), and we will thus call the product of Green functions itself a cycle. Similarly, we will refer to a product of Green functions whose arguments are a divisible cyclic set as a divisible cycle of Green functions.

The expansion of the cycle of Green functions in equation (8.11) will yield a single power of  $\hat{q}^{1/2}$  with no accompanying exponential suppression only for the integration ordering  $\hat{\nu}_1 \leq \hat{\nu}_2 \leq \dots \hat{\nu}_{n-1} \leq \hat{\nu}_n = 1$  depicted in 4a and the reversed ordering  $\hat{\nu}_{n-1} \leq \hat{\nu}_{n-2} \leq \dots \hat{\nu}_1 \leq \hat{\nu}_n = 1$  depicted in 4b, since only for these orderings will the factor multiplying the  $\text{Im}\tau$  inside the exponential of equation (8.10) vanish. (Variables that have been eliminated through the pinch integrations are of course omitted from the ordering.) In the first ordering the factor of  $\hat{q}^{1/2}$  emerges from  $G_F[\frac{\alpha_G}{\beta_G}](\nu_{1,n})$ , while in the reversed ordering, the factor emerges from  $G_F[\frac{\alpha_G}{\beta_G}](-\nu_{n-1,n})$ .

The left-mover fermionic Green functions always come in (divisible) cyclic sets, where each indivisible cyclic set is associated with a single trace. In the full string theory, there is no limit (up to the number of Green functions) to the number of indivisible cyclic sets that can appear in

any term; but in the infinite-tension limit, as pointed out above, each indivisible cyclic set also carries a power of  $\hat{q}^{1/2}$ , and we can have no more than two if we want to obtain a non-vanishing contribution. The only surviving trace structures in the gauge-theory limit are those with one or two non-trivial traces corresponding to cases (c) and (d), respectively. This argument is unaffected by the pinch integrations, since a cyclic set cannot be pinched off [11].

In case (c), which arises in the computation of  $A_{n;1}$ , the other power of  $\hat{q}^{1/2}$  must come from the expansion of the partition function. In the partition function, the  $\hat{q}^{1/2}$  is multiplied by a sum over oscillators. The sum over all boundary conditions with coefficients  $C_{\beta}^{\vec{\alpha}}$  which builds up the generalized GSO projection operator will eventually leave only the sum over those oscillators that correspond to the gauge group of interest; there are  $N_c$  of these for an  $SU(N_c)$  model. This will yield an explicit factor of  $N_c$ , which we have anticipated in the definition of the trace structure  $\text{Gr}_{n;1}$ . Let us consider first the contribution of two left-mover Green functions to  $A_{n;1}$ :

$$\begin{aligned}
& \frac{1}{N_c} \sum_{\{\vec{\beta}\}} C_{\vec{\beta}}^{\vec{\alpha}_0} \mathcal{Z}_L(-1)^{n-1} G_F \begin{bmatrix} \alpha_G \\ \beta_G \end{bmatrix} (-\nu_{12}) G_F \begin{bmatrix} \alpha_G \\ \beta_G \end{bmatrix} (\nu_{12}) \\
& \rightarrow \frac{(-1)^{n-1}}{N_c} \sum C_{\vec{\beta}}^{\vec{\alpha}_0} \left[ \hat{q}^{-1} e^{-2\pi i \text{Re } \tau} \left( 1 - \hat{q}^{1/2} e^{\pi i \text{Re } \tau} \sum_{i=1}^{22} \cos 2\pi \beta_{L_i} \right) \right] \\
& \quad \times (-i) \left( e^{\pi i \text{Re } \nu_{21}} e^{-\pi |\nu_{21}| \text{Im } \tau} - \hat{q}^{1/2} e^{\pi i \text{Re } \tau} e^{-\pi i \text{Re } \nu_{21}} e^{\pi |\nu_{21}| \text{Im } \tau} e^{2\pi i \beta_G} \right) \\
& \quad \times (+i) \left( e^{\pi i \text{Re } \nu_{21}} e^{-\pi |\nu_{21}| \text{Im } \tau} - \hat{q}^{1/2} e^{\pi i \text{Re } \tau} e^{-\pi i \text{Re } \nu_{21}} e^{\pi |\nu_{21}| \text{Im } \tau} e^{-2\pi i \beta_G} \right) \quad (8.13) \\
& \rightarrow \frac{(-1)^{n-1}}{N_c} \sum C_{\vec{\beta}}^{\vec{\alpha}_0} \left[ N_c \cos(2\pi i \beta_G) + N'_c \cos(2\pi i \beta_{G'}) \right] \\
& \quad + \sum_{i=N_c+N'_c+1}^{22} \cos(2\pi i \beta_{L_i}) \Big] \times \left[ e^{2\pi i \beta_G} + e^{-2\pi i \beta_G} \right]
\end{aligned}$$

where the explicit factor of  $1/N_c$  compensates for the  $N_c$  in the definition of  $\text{Gr}_{4;1}$ , and where we have discarded all terms exponentially suppressed in  $\text{Im } \tau$ .

This can be simplified further through the same consideration (of that part of the sum over all boundary conditions which corresponds to summing over multiples of the boundary condition basis vector  $W_1$ ) that was used to eliminate alternative (b). In addition to terms with a dependence on  $\beta_G$ , the partition function also includes terms with a dependence on  $\beta_{G'}$ , so in this case, the sum over triplets ( $W + 0 \cdot W_1, W + 1 \cdot W_1, W + 2 \cdot W_1$ ) eliminates all terms lacking an interference except those of the form  $\exp[\pm 2\pi i (\beta_G - \beta_{G'})]$ . These terms are however eliminated by the sum over other boundary conditions, as discussed in appendix II. This then leaves us with

$$-\frac{1}{N_c} \sum C_{\vec{\beta}}^{\vec{\alpha}_0} \left( 2N_c + N'_c (e^{2\pi i (\beta_G - \beta_{G'})} + e^{-2\pi i (\beta_G - \beta_{G'})}) \right) \quad (8.14)$$



as potential contributions. It turns out that the interference terms between the  $G$  and  $G'$  gauge groups  $e^{\pm 2\pi i(\beta_G - \beta'_G)}$  are removed by the sum over the other boundary conditions, whose discussion we defer to appendix II. Thus, the only remaining terms are from the  $N_c$  gauge oscillators, as one might have expected, since those are the oscillators corresponding to the gauge charges carried by the external legs. This will leave a factor of

$$(-1)^{n-1} 2, \quad (8.15)$$

for the left-mover contribution (after performing the remaining sums over world-sheet boundary conditions) in this two-Green function case.

Terms with more than two left-mover Green functions arranged in a cycle of the form in equation (8.11) can be analyzed in a similar fashion. In the large  $\text{Im } \tau$  limit the surviving contribution for such a term (for the ordering  $\hat{\nu}_1 \leq \hat{\nu}_2 \leq \dots \leq \hat{\nu}_n$ ) is

$$\begin{aligned} & \frac{1}{N_c} \sum_{\{\beta\}} C_{\beta}^{\bar{\alpha}_0} \mathcal{Z}_L (-1)^{n-1} G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (\nu_{1,n}) G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (-\nu_{12}) G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (-\nu_{23}) \cdots G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (-\nu_{n-1,n}) \\ & \rightarrow (-1)^{n-1} \frac{1}{N_c} \sum C_{\beta}^{\bar{\alpha}_0} \mathcal{Z}_L \left[ \hat{q}^{-1} e^{-2\pi i \text{Re } \tau} \left( 1 - \hat{q}^{1/2} e^{\pi i \text{Re } \tau} + \sum_{i=1}^{\text{len } W_{0L}} \cos 2\pi \beta_{L_i} \right) \right] \\ & \quad \times (+i) \left( e^{\pi i \text{Re } \nu_{n,1}} e^{-\pi |\hat{\nu}_{n,1}| \text{Im } \tau} - \hat{q}^{1/2} e^{\pi i \text{Re } \tau} e^{-\pi i \text{Re } \nu_{n1}} e^{\pi |\hat{\nu}_{n,1}| \text{Im } \tau} e^{-2\pi i \beta_G} \right) \\ & \quad \times (-i) \left( e^{\pi i \text{Re } \nu_{21}} e^{-\pi |\hat{\nu}_{21}| \text{Im } \tau} - \hat{q}^{1/2} e^{\pi i \text{Re } \tau} e^{-\pi i \text{Re } \nu_{21}} e^{\pi |\hat{\nu}_{21}| \text{Im } \tau} e^{2\pi i \beta_G} \right) \\ & \quad \times (-i) \left( e^{\pi i \text{Re } \nu_{32}} e^{-\pi |\hat{\nu}_{32}| \text{Im } \tau} - \hat{q}^{1/2} e^{\pi i \text{Re } \tau} e^{-\pi i \text{Re } \nu_{32}} e^{\pi |\hat{\nu}_{32}| \text{Im } \tau} e^{2\pi i \beta_G} \right) \\ & \quad \vdots \\ & \quad \times (-i) \left( e^{\pi i \text{Re } \nu_{n,n-1}} e^{-\pi |\hat{\nu}_{n,n-1}| \text{Im } \tau} - \hat{q}^{1/2} e^{\pi i \text{Re } \tau} e^{-\pi i \text{Re } \nu_{n,n-1}} e^{\pi |\hat{\nu}_{n,n-1}| \text{Im } \tau} e^{2\pi i \beta_G} \right) \\ & \rightarrow (-1)^{n+n_L} i^{n_L} \frac{1}{N_c} \sum C_{\beta}^{\bar{\alpha}_0} \frac{1}{2} \left[ N_c (e^{2\pi i \beta_G} + e^{-2\pi i \beta_G}) + N'_c (e^{2\pi i \beta_{G'}} + e^{-2\pi i \beta_{G'}}) \right. \\ & \quad \left. + \sum_{i=N_c+N'_c+1}^{22} (e^{2\pi i \beta_{L_i}} + e^{-2\pi i \beta_{L_i}}) \right] \times \left[ -e^{-2\pi i \beta_G} \right] \end{aligned} \quad (8.16)$$

where any variables and Green functions removed during the pinch integrations are understood to be omitted, and where we have again discarded all terms exponentially suppressed in  $\text{Im } \tau$  on the last line. Following the same reasoning concerning the cancellation of the phase factors of  $e^{2\pi i \beta_G}$  employed above for the two-Green function case, we find a factor of

$$(-1)^{n+n_L} i^{n_L} \quad (8.17)$$

We remind the reader that  $n$  is the total number of external legs for the process at hand, while  $n_l$  is the number of legs attached to the loop after performing the pinch integrals. (Each pinch reduces the number of legs by one.) The factor of 2 difference between this case and the earlier case of two left-mover Green functions is due to the fact that in the former case either Green function can supply an  $\mathcal{O}(\hat{q}^{1/2})$  term, while in the latter case only  $G_F[\frac{\alpha_G}{\beta_G}](\nu_{1,n})$  can supply the  $\mathcal{O}(\hat{q}^{1/2})$  term. However, this factor is (eventually) compensated by the fact that there is another ordering of the  $\hat{\nu}$ 's which gives a contribution for this cycle of Green functions, that with  $\hat{\nu}_{n-1} \leq \hat{\nu}_{n-2} \leq \dots \leq \hat{\nu}_1 \leq \hat{\nu}_n$ . This reversed integration ordering gives a contribution with a factor of

$$(-1)^n i^{n_l} \quad (8.18)$$

for the left-movers. All other integration orderings give no contributions for this cycle of Green functions.

In case (d), which arises in the computation of  $A_{n;j}$  ( $3 \leq j \leq \lfloor n/2 \rfloor + 1$ ; the computation of  $A_{n;2}$  is essentially the same, except that the first cycle of Green functions is replaced by a self-contraction Green function), we start with two indivisible cycles of Green functions,

$$\begin{aligned} & (-1)^n \left( G_F \left[ \frac{\alpha_G}{\beta_G} \right] (\nu_{1,j-1}) G_F \left[ \frac{\alpha_G}{\beta_G} \right] (-\nu_{12}) G_F \left[ \frac{\alpha_G}{\beta_G} \right] (-\nu_{23}) \cdots G_F \left[ \frac{\alpha_G}{\beta_G} \right] (-\nu_{j-2,j-1}) \right) \\ & \times \left( G_F \left[ \frac{\alpha_G}{\beta_G} \right] (\nu_{j,n}) G_F \left[ \frac{\alpha_G}{\beta_G} \right] (-\nu_{j,j+1}) G_F \left[ \frac{\alpha_G}{\beta_G} \right] (-\nu_{j+1,j+2}) \cdots G_F \left[ \frac{\alpha_G}{\beta_G} \right] (-\nu_{n-1,n}) \right) \end{aligned} \quad (8.19)$$

each contributing a power of  $\hat{q}^{1/2}$  in the infinite-tension limit. (The sign  $(-1)^n = (-1)^{(n-j+1)-1} (-1)^{(j-1)-1}$  again comes from replacing  $\hat{G}_F[\frac{\alpha_G}{\beta_G}]$ 's with  $G_F[\frac{\alpha_G}{\beta_G}]$ 's.) Throughout the following discussion, we will assume again that variables and Green functions that have been removed by the pinch integrations are omitted.

Following the same reasoning used in the discussion of case (c) above, we can see that there are four classes of integration orderings without an exponential suppression of the form (8.10),

$$\hat{\nu}_1 \leq \hat{\nu}_2 \leq \dots \leq \hat{\nu}_{j-1}, \quad \hat{\nu}_j \leq \hat{\nu}_{j+2} \cdots \leq \hat{\nu}_n = 1 \quad (8.20a)$$

$$\hat{\nu}_{j-1} \leq \hat{\nu}_{j-2} \leq \dots \leq \hat{\nu}_1, \quad \hat{\nu}_{n-1} \leq \hat{\nu}_{n-2} \cdots \leq \hat{\nu}_j \leq \hat{\nu}_n = 1 \quad (8.20b)$$

$$\hat{\nu}_{j-1} \leq \hat{\nu}_{j-2} \leq \dots \leq \hat{\nu}_1, \quad \hat{\nu}_j \leq \hat{\nu}_{j+2} \cdots \leq \hat{\nu}_{n-1} \leq \hat{\nu}_n = 1 \quad (8.20c)$$

$$\hat{\nu}_1 \leq \hat{\nu}_2 \leq \dots \leq \hat{\nu}_{j-1}, \quad \hat{\nu}_j \leq \hat{\nu}_{j+2} \cdots \leq \hat{\nu}_{n-1} \leq \hat{\nu}_n = 1 \quad (8.20d)$$

In addition to contributing a power of  $\hat{q}^{1/2}$ , each cycle also contributes a factor of  $\exp[\pm 2\pi i \beta_G]$ . In order to obtain a non-vanishing result after the sum over string world-sheet boundary conditions one cycle must contribute a factor of  $e^{2\pi i \beta_G}$ , while the other contributes its inverse, a factor of  $e^{-2\pi i \beta_G}$ . This imposes a restriction on the relative ordering of the variables in the two cyclic

sets: if they are ordered in the same direction, as in equations (8.20a) and (8.20b), we would obtain a vanishing result in the field theory limit because then the phases would add to give  $e^{\pm 4\pi i \beta_G}$ . This would then vanish after summing over the triplet of world-sheet boundary conditions generated by  $W_1$ . For example, in the field theory limit with ordering (8.20a) the Green functions in equation (8.19) expand to

$$\begin{aligned}
& \left[ (+i) \left( e^{\pi i \operatorname{Re} \nu_{j-1,1}} e^{-\pi |\dot{\nu}_{j-1,1}| \operatorname{Im} \tau} - \hat{q}^{1/2} e^{\pi i \operatorname{Re} \tau} e^{-\pi i \operatorname{Re} \nu_{j-1,1}} e^{\pi |\dot{\nu}_{j-1,1}| \operatorname{Im} \tau} e^{-2\pi i \beta_G} \right) \right. \\
& \quad \left. (-i) \left( e^{\pi i \operatorname{Re} \nu_{21}} e^{-\pi |\dot{\nu}_{21}| \operatorname{Im} \tau} - \hat{q}^{1/2} e^{\pi i \operatorname{Re} \tau} e^{-\pi i \operatorname{Re} \nu_{21}} e^{\pi |\dot{\nu}_{21}| \operatorname{Im} \tau} e^{2\pi i \beta_G} \right) \right. \\
& \quad \quad \quad \vdots \\
& \times (-i) \left( e^{\pi i \operatorname{Re} \nu_{j-1,j-2}} e^{-\pi |\dot{\nu}_{j-1,j-2}| \operatorname{Im} \tau} - \hat{q}^{1/2} e^{\pi i \operatorname{Re} \tau} e^{-\pi i \operatorname{Re} \nu_{j-1,j-2}} e^{\pi |\dot{\nu}_{j-1,j-2}| \operatorname{Im} \tau} e^{2\pi i \beta_G} \right) \left. \right] \\
& \times \left[ (+i) \left( e^{\pi i \operatorname{Re} \nu_{nj}} e^{-\pi |\dot{\nu}_{nj}| \operatorname{Im} \tau} - \hat{q}^{1/2} e^{\pi i \operatorname{Re} \tau} e^{-\pi i \operatorname{Re} \nu_{nj}} e^{\pi |\dot{\nu}_{nj}| \operatorname{Im} \tau} e^{-2\pi i \beta_G} \right) \right. \\
& \quad \times (-i) \left( e^{\pi i \operatorname{Re} \nu_{j+1,j}} e^{-\pi |\dot{\nu}_{j+1,j}| \operatorname{Im} \tau} - \hat{q}^{1/2} e^{\pi i \operatorname{Re} \tau} e^{-\pi i \operatorname{Re} \nu_{j+1,j}} e^{\pi |\dot{\nu}_{j+1,j}| \operatorname{Im} \tau} e^{2\pi i \beta_G} \right) \\
& \quad \quad \quad \vdots \\
& \quad \times (-i) \left( e^{\pi i \operatorname{Re} \nu_{n,n-1}} e^{-\pi |\dot{\nu}_{n,n-1}| \operatorname{Im} \tau} - \hat{q}^{1/2} e^{\pi i \operatorname{Re} \tau} e^{-\pi i \operatorname{Re} \nu_{n,n-1}} e^{\pi |\dot{\nu}_{n,n-1}| \operatorname{Im} \tau} e^{2\pi i \beta_G} \right) \left. \right] \\
& \longrightarrow \hat{q} e^{2\pi i \operatorname{Re} \tau} (-1)^{n+n_\ell} i^{n_\ell} \left[ e^{-2\pi i \beta_G} \right] \left[ e^{-2\pi i \beta_G} \right] \\
& \longrightarrow 0
\end{aligned} \tag{8.21}$$

where we have obtained a vanishing result because the phase  $e^{2\pi i \beta_G}$  did not cancel. The result is the same for the ordering (8.20b). Only in those cases where the two cycles of Green functions have the opposite ordering, as in equations (8.20c) and (8.20d), do the phases cancel. For these orderings, the left-movers simplify in the field theory limit, to

$$(-1)^{n+n_\ell-j_\ell+1} i^{n_\ell} \tag{8.22}$$

(for the ordering (8.20c)) where  $j_\ell - 1$  of the first  $j - 1$  variables remain after the pinch integrations and

$$(-1)^{n+j_\ell-1} i^{n_\ell} \tag{8.23}$$

(for the ordering (8.20d)).

We can now summarize the factors produced by the left-mover Green function in the field theory limit,

Two Green functions:

$$\frac{(-1)^{n-1}}{N_c} \sum CZ_L G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (\nu_{i_1 i_2}) G_F \left[ \begin{matrix} \alpha_G \\ \beta_G \end{matrix} \right] (\nu_{i_2 i_1}) \longrightarrow (-1)^{n-1} 2 \tag{8.24a}$$

Three or more Green functions in a single cycle:

$$\frac{1}{N_c} \sum CZ_L (-1)^{n-1} G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{i_1 i_2}) G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{i_2 i_3}) \cdots G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{i_{n_\ell} i_1}) \rightarrow \begin{cases} (-1)^{n+n_\ell} i^{n_\ell}, & \hat{\nu}_{i_1} \leq \hat{\nu}_{i_2} \leq \cdots \leq \hat{\nu}_{i_{n_\ell-1}} \\ (-1)^{n_\ell} i^{n_\ell}, & \hat{\nu}_{i_1} \geq \hat{\nu}_{i_2} \geq \cdots \geq \hat{\nu}_{i_{n_\ell-1}} \\ 0 & \text{otherwise} \end{cases} \quad (n_\ell \geq 3) \quad (8.24b)$$

Four Green functions in two cycles:

$$\sum CZ_L (-1)^n G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{i_1 i_2}) G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{i_2 i_1}) G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{i_3 i_4}) G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{i_4 i_3}) \rightarrow (-1)^n 2 \quad (8.24c)$$

Five or more Green functions in two cycles:

$$\begin{aligned} & \sum CZ_L (-1)^n G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{i_1 i_2}) G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{i_2 i_3}) \cdots G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{i_{j_\ell-1} i_1}) \\ & \times G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{i_{j_\ell} i_{j_\ell+1}}) G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{i_{j_\ell+1} i_{j_\ell+2}}) \cdots G_F \left[ \begin{smallmatrix} \alpha_G \\ \beta_G \end{smallmatrix} \right] (\nu_{i_{n_\ell} i_{j_\ell}}) \\ & \rightarrow \begin{cases} (-1)^{n+j_\ell-1} i^{n_\ell}, & \hat{\nu}_{i_1} \leq \hat{\nu}_{i_2} \leq \cdots \leq \hat{\nu}_{i_j}, \quad \hat{\nu}_{i_{j+1}} \geq \hat{\nu}_{i_{j+2}} \geq \cdots \geq \hat{\nu}_{i_{m-1}} \\ (-1)^{n+n_\ell+j_\ell-1} i^{n_\ell}, & \hat{\nu}_{i_1} \geq \hat{\nu}_{i_2} \geq \cdots \geq \hat{\nu}_{i_j}, \quad \hat{\nu}_{i_{j+1}} \leq \hat{\nu}_{i_{j+2}} \leq \cdots \leq \hat{\nu}_{i_{m-1}} \\ 0, & \text{otherwise} \end{cases} \quad (8.24d) \end{aligned}$$

where  $\hat{\nu}_{i_m} = 1$  has been fixed. The schematic notation  $\sum C$  indicates that these simplification rules include all effects from summing over the world sheet boundary conditions. If there are more than two cycles of left-mover Green functions there will be no field theory contribution. In the case that  $n_\ell = 2$ , there is only one integration ordering (since one of the legs is at a fixed location), whereas for  $n_\ell \geq 3$  we see that the left-mover factors contributed by a given integration ordering and the reversed integration ordering always differ by a factor of  $(-1)^{n_\ell}$ .

#### 8.4 Expansion of the Right-Movers

We turn next to the consideration of the large  $\text{Im } \tau$  limit of the right-mover contributions. The analysis of the right-movers is similar to that performed above for the left-movers. There are several differences, most importantly the change in the vacuum energy of the right-mover superstring, which is at  $m^2 = -1/2$  (in units of  $4\pi/\lambda$ ). The leading term (before GSO projection) is therefore  $\hat{q}^{-1/2}$ . Thus, in the field theory limit we can obtain at most a single additional power of  $\hat{q}^{1/2}$  from either the Green functions or from the partition function.

After performing the  $\theta_{i_3}$  and  $\theta_{i_4}$  integrations in equation (5.10), and integrating by parts to remove all double-derivatives of the bosonic Green functions  $\vec{G}_B$ , a general term in the right-mover

factor is a product of bosonic Green functions and cycles of fermionic Green functions. There are two types of terms that survive in the field theory limit, those with (a) no fermionic Green functions or (b) a single cycle of fermionic Green functions. Terms with more than a single cycle of fermionic Green functions will not contribute because, just as in the left-mover case, each cycle of fermionic Green functions contributes at least a factor of  $\hat{q}^{1/2}$ , and here more than one power would yield an over-all positive power.

Let us consider first the surviving contributions for the FDH scheme, where no modifications to the string amplitude of equation (5.7) are necessary; we will consider the modifications (7.4) necessary for the HV (and CDR) schemes later. We begin with case (a), where only bosonic Green functions are present. Since the next-to-leading order term in the bosonic Green functions is of  $\mathcal{O}(\hat{q})$ , the infinite-tension limit will leave only the leading contributions; the next-to-leading term will result in an overall positive power of  $\hat{q}^{1/2}$ . For the bosonic Green functions, there is no exponential suppression of the form (8.10) in the leading terms (unlike the fermionic Green functions) so the product of all the leading order terms from the bosonic Green functions survives in the field theory limit. Since the bosonic Green functions cannot supply the required factor of  $\hat{q}^{1/2}$ , this factor must come from the right-mover partition function. Thus a factor of this type in the amplitude becomes

$$\sum_{\vec{\beta}} C_{\vec{\beta}}^{\vec{\alpha}_0} \mathcal{Z}_R \prod_{l=1}^m \dot{G}_B(\vec{\nu}_{i_l, j_l}) \longrightarrow \sum C_{\vec{\beta}}^{\vec{\alpha}_0} \hat{q}^{-1/2} e^{\pi i \text{Re } \tau} \left( 1 - 2\hat{q}^{1/2} e^{-\pi i \text{Re } \tau} \cos 2\pi\beta_\tau - 2\hat{q}^{1/2} e^{-\pi i \text{Re } \tau} \sum_{i=2}^{10} \cos 2\pi\beta_{Ri} \right) \left( \frac{i}{2} \right)^m \prod_{l=1}^m \left( -\text{sign}(\hat{\nu}_{i_l, j_l}) + 2\hat{\nu}_{i_l, j_l} \right) \quad (8.25)$$

in the field theory limit. In the Neveu-Schwarz sector, which contains the gluons, the KLT coefficients  $C_{\vec{\beta}}^{\vec{\alpha}}$  are

$$\begin{aligned} C_{\vec{\beta}}^{\vec{\alpha}_0} &= -\cos(2\pi\beta_\tau)/\mathcal{M} \\ &= -(-1)^{n_0+n_2+n_3+n_4}/48 \end{aligned} \quad (8.26)$$

where the first form is true for the  $W_0$  sector in any model, while the second form is for the specific model detailed in appendix I. The  $n_i$  are the coefficient of the basis vector  $W_i$  in the given boundary condition vector  $\vec{\beta} = \sum n_i W_i$ . We may note the basis vector  $W_1$ , which was used to simplify the left-mover structure, does not enter into the considerations of the right-movers. We can also rewrite

the various  $\cos 2\pi\beta_{R_i}$  in terms of these  $n_i$ ; we obtain

$$\begin{aligned}
C_{\beta}^{\vec{\alpha}_0} \cos 2\pi\beta_{R_1} &= -1 & C_{\beta}^{\vec{\alpha}_0} \cos 2\pi\beta_{R_6} &= -(-1)^{(n_2+n_4)} \\
C_{\beta}^{\vec{\alpha}_0} \cos 2\pi\beta_{R_2} &= -(-1)^{(n_3+n_4)} & C_{\beta}^{\vec{\alpha}_0} \cos 2\pi\beta_{R_7} &= -(-1)^{(n_2)} \\
C_{\beta}^{\vec{\alpha}_0} \cos 2\pi\beta_{R_3} &= -(-1)^{(n_4)} & C_{\beta}^{\vec{\alpha}_0} \cos 2\pi\beta_{R_8} &= -(-1)^{(n_3)} \\
C_{\beta}^{\vec{\alpha}_0} \cos 2\pi\beta_{R_4} &= -(-1)^{(n_3)} & C_{\beta}^{\vec{\alpha}_0} \cos 2\pi\beta_{R_9} &= -(-1)^{(n_2)} \\
C_{\beta}^{\vec{\alpha}_0} \cos 2\pi\beta_{R_5} &= -(-1)^{(n_4)} & C_{\beta}^{\vec{\alpha}_0} \cos 2\pi\beta_{R_{10}} &= -(-1)^{(n_2+n_3)}.
\end{aligned} \tag{8.27}$$

Because each coefficient  $n_i$  in the boundary condition vector is summed over the values 0 and 1, only those terms independent of all the  $n_i$  can survive; here, only the terms involving the ‘spin’ component survive, leaving

$$\frac{1}{\mathcal{M}} \sum_{\beta} 2 \cos^2(2\pi\beta_{\uparrow}) = 2 \tag{8.28}$$

for the coefficient of the  $\hat{q}^{1/2}$  term in the partition function. While the precise form of the coefficients is model-dependent, this result is not; any model with only gluons transforming under the gauge group will yield the same result.

Combining this with the large- $\text{Im}\tau$  behavior of the bosonic Green function given in equation (8.7), the contribution of the right-mover Green functions in this case is simply

$$2 (-i)^{n_t} \prod_{i=1}^{n_t} \left( \frac{\text{sign}(\hat{\nu}_{i_j i})}{2} - \hat{\nu}_{i_j i} \right) \tag{8.29}$$

Next, consider case (b), where the term does contain fermionic Green functions. The product of the leading contributions (in  $\hat{q}$ ) from the fermionic Green functions is exponentially suppressed, just as for the left-mover Green functions. In order to get a non-vanishing result the exponentials of the form (8.10) must again cancel completely between the next-to-leading order term in one fermionic Green function and the leading terms in the remaining Green functions. Unlike the case of the left-movers, however, the expansion of the right-mover partition function begins with  $\hat{q}^{-1/2}$  rather than  $\hat{q}^{-1}$ , so we can have at most one cycle of fermionic Green functions; since each cycle contributes a factor of  $\hat{q}^{1/2}$ , terms with more than one cycle will end up with a positive integer power of  $\hat{q}^{1/2}$ , and so will disappear in the infinite-tension limit.

If we choose the ordering  $\hat{\nu}_1 \leq \hat{\nu}_2 \leq \dots \leq \hat{\nu}_n$  (such as arises in the calculation of  $A_{n;1}$ ), the only cyclic sets of Green functions that will not vanish in the field theory limit are those of the form

$$G_F \left[ \begin{smallmatrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{smallmatrix} \right] (\bar{\nu}_{j,m}) G_F \left[ \begin{smallmatrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{smallmatrix} \right] (\bar{\nu}_{j,j+k}) G_F \left[ \begin{smallmatrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{smallmatrix} \right] (\bar{\nu}_{j+k,j+k+k'}) \dots G_F \left[ \begin{smallmatrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{smallmatrix} \right] (\bar{\nu}_{m-k'',m}) \tag{8.30}$$

where  $j < m$  and  $k, k', k'' > 0$ . That is, only cycles built upon a subset of the variables ordered in the base set in the same way as in the ordering determining the integration region, can survive.

In the infinite-tension limit, such cycles simplify as follows ( $n_f$  denotes the number of fermionic Green functions),

$$\begin{aligned}
& \sum_{\vec{\beta}} C_{\vec{\beta}}^{\vec{\alpha}_0} \mathcal{Z}_R G_F \left[ \begin{matrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{matrix} \right] (\bar{\nu}_{j,m}) G_F \left[ \begin{matrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{matrix} \right] (\bar{\nu}_{j,j+k}) G_F \left[ \begin{matrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{matrix} \right] (\bar{\nu}_{j+k,j+k+k'}) \cdots G_F \left[ \begin{matrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{matrix} \right] (\bar{\nu}_{m-k'',m}) \\
& \rightarrow \sum_{\vec{\beta}} C_{\vec{\beta}}^{\vec{\alpha}_0} (-i) \left( e^{-\pi i \operatorname{Re} \nu_{m,j}} e^{-\pi |\hat{\nu}_{m,j}| \operatorname{Im} \tau} - \hat{q}^{1/2} e^{-\pi i \operatorname{Re} \tau} e^{\pi i \operatorname{Re} \nu_{m,j}} e^{\pi |\hat{\nu}_{m,j}| \operatorname{Im} \tau} \cos(2\pi \beta_{\uparrow}) \right) \\
& \quad \times (-i) \left( e^{-\pi i \operatorname{Re} \nu_{j+k,j}} e^{-\pi |\hat{\nu}_{j+k,j}| \operatorname{Im} \tau} - \hat{q}^{1/2} e^{-\pi i \operatorname{Re} \tau} e^{\pi i \operatorname{Re} \nu_{j+k,j}} e^{\pi |\hat{\nu}_{j+k,j}| \operatorname{Im} \tau} \cos(2\pi \beta_{\uparrow}) \right) \\
& \quad \times (-i) \left( e^{-\pi i \operatorname{Re} \nu_{j+k+k',j+k}} e^{-\pi |\hat{\nu}_{j+k+k',j+k}| \operatorname{Im} \tau} \right. \\
& \quad \quad \left. - \hat{q}^{1/2} e^{-\pi i \operatorname{Re} \tau} e^{\pi i \operatorname{Re} \nu_{j+k+k',j+k}} e^{\pi |\hat{\nu}_{j+k+k',j+k}| \operatorname{Im} \tau} \cos(2\pi \beta_{\uparrow}) \right) \\
& \quad \quad \quad \vdots \\
& \quad \times (-i) \left( e^{-\pi i \operatorname{Re} \nu_{m,m-k''}} e^{-\pi |\hat{\nu}_{m,m-k''}| \operatorname{Im} \tau} \right. \\
& \quad \quad \left. - \hat{q}^{1/2} e^{-\pi i \operatorname{Re} \tau} e^{\pi i \operatorname{Re} \nu_{m,m-k''}} e^{\pi |\hat{\nu}_{m,m-k''}| \operatorname{Im} \tau} \cos(2\pi \beta_{\uparrow}) \right) \\
& \rightarrow \frac{1}{\mathcal{M}} \sum_{\vec{\beta}} \cos^2(2\pi \beta_{\uparrow}) \times \begin{cases} -2, & (n_f = 2) \\ (-i)^{n_f}, & (n_f > 2) \end{cases} \\
& \rightarrow \begin{cases} -2, & (n_f = 2) \\ (-i)^{n_f}, & (n_f > 2) \end{cases}
\end{aligned} \tag{8.31}$$

where we have dropped all exponentially suppressed terms. Any bosonic Green functions multiplying a cycle of fermionic Green functions should be expanded according to equation (8.7), yielding a factor

$$(-i)^{n_{\ell} - n_f} \prod_{l=1}^{n_{\ell} - n_f} \left( \frac{\operatorname{sign}(\hat{\nu}_{i_l j_l})}{2} - \hat{\nu}_{i_l j_l} \right) \tag{8.32}$$

with  $n_{\ell}$  replaced by  $n_{\ell} - n_f$ .

A cycle of Green functions built upon a set ordered differently than the integration ordering, for example,

$$G_F \left[ \begin{matrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{matrix} \right] (\bar{\nu}_{j,m}) G_F \left[ \begin{matrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{matrix} \right] (\bar{\nu}_{j,j+k}) G_F \left[ \begin{matrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{matrix} \right] (\bar{\nu}_{j+k-k',j+k}) \cdots G_F \left[ \begin{matrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{matrix} \right] (\bar{\nu}_{m-k'',m}), \tag{8.33}$$

with the same ordering of the  $\hat{\nu}$  used above, will vanish in the field-theory limit. Conversely, the cycle of equation (8.30) will yield a non-vanishing factor only for those integration orderings where either the order of the variables in the base set of the cycle  $\{\bar{\nu}_j, \bar{\nu}_{j+k}, \bar{\nu}_{j+k+k'}, \dots, \bar{\nu}_{m-k''}, \bar{\nu}_m\}$  is consistent with the integration ordering, or where the reversed ordering  $\{\bar{\nu}_m, \bar{\nu}_{m-k''}, \dots, \bar{\nu}_{j+k+k'}, \bar{\nu}_{j+k}, \bar{\nu}_j\}$  is consistent with the integration ordering. For every integration ordering with which the base set

is consistent, the cycle of right-mover fermionic Green functions (8.30) will contribute an identical factor, the one given in equation (8.31). For the reversed integration ordering (i.e. those integration orderings with which the reversed ordering of the base set is consistent), the contributed factor will be

$$\begin{cases} -2, & (n_f = 2) \\ i^{n_f}, & (n_f > 2) \end{cases} \quad (8.34)$$

That is, the contributions of a cycle of fermionic Green functions in a given integration ordering and in the reversed ordering differ by a factor of  $(-1)^{n_f}$ .

So much for the expansions in the FDH regularization scheme. What are the modifications necessary to obtain the expansion for the HV regularization scheme? If we introduce scalars in the Neveu-Schwarz sector, for example by replacing  $W_4$  from the set of basis vectors for the model we are using with  $W'_4$  as given in equation (7.5), we will alter only terms which depend on  $n_4$ , the coefficient of  $W_4$  in the boundary condition vector  $\vec{\beta}$ . In the modified model, only terms which depend on  $\beta_{R2}, \beta_{R3}, \beta_{R8}$ , and  $\beta_{R9}$  will end up with a dependence on  $n_4$ . In particular, the fermionic Green functions depend only on  $\beta_\uparrow$ , and are thus the infinite-tension limits of terms containing any fermionic Green functions are left unaltered by the addition of scalars. The only terms which are affected are those which contain only bosonic Green functions, case (a) above.

For these terms, again only terms independent of the remaining  $n_i$  can survive; these now include not only the  $\cos 2\pi\beta_\uparrow$  term in the partition function, but also the  $\cos 2\pi\beta_{R5}$  term. Equation (8.28) is thus replaced by

$$4 \quad (8.35)$$

as the coefficient of  $\hat{q}^{1/2}$ . As promised in section 7, the addition of scalars effectively modifies only the partition function in the field-theory limit.

The model modified by replacing the  $W_4$  vector with  $W'_4$  contains one complex, or two real adjoint scalars in the Neveu-Schwarz sector; thus, the prescription for obtaining the HV scheme amplitude (and the ordinary helicity contributions in the CDR scheme),

$$\mathcal{Z}^{\text{HV}}(\text{pure glue}) = \mathcal{Z}^{\text{FDH}}(\text{pure glue}) - \frac{\epsilon}{N_s} (\mathcal{Z}^{\text{FDH}}(\text{pure glue} + N_s \text{ real scalars}) - \mathcal{Z}^{\text{FDH}}(\text{pure glue})), \quad (8.36)$$

is simply to replace the coefficient of the  $\hat{q}^{1/2}$  in the partition function (8.28) with

$$2 - \epsilon \quad (8.37)$$

(thus this coefficient simply counts the number of independent helicity states circulating in the loop) so that equation (8.29) becomes

$$2 \left(1 - \delta_R \frac{\epsilon}{2}\right) (-i)^{n_\ell} \prod_{l=1}^{n_\ell} \left( \frac{\text{sign}(\hat{v}_{i_l j_l})}{2} - \hat{v}_{i_l j_l} \right) \quad (8.38)$$



where  $\delta_R = 0$  for the FDH scheme, and  $\delta_R = 1$  for the HV and CDR schemes.

A simple check on that the modification of the partition function includes the effect of adjoint scalars correctly comes from the computation of the gauge theory  $\beta$ -function. Using this string-based technology, we computed the  $\beta$ -function in previous work [47], obtaining the standard result,

$$\beta_{\text{YM}} = \frac{g^2 N_c}{16\pi^2} \left( \frac{1}{3} - 4 \right). \quad (8.39)$$

In this equation, the  $1/3$  arises from terms which contain only bosonic Green functions (after all pinch integrals have been performed), while the second term,  $-4$ , arises from terms which contain fermionic Green functions. Following the above discussion, to include the effect of adjoint scalars we should simply multiply the contributions from the pure-bosonic Green function terms by  $(1 + N_s/2)$ . For the  $\beta$ -function of a theory with  $N_s$  real adjoint scalars, we would obtain

$$\beta_{\text{YM+scalars}} = \frac{g^2 N_c}{16\pi^2} \left( \frac{1 + N_s/2}{3} - 4 \right) \quad (8.40)$$

in agreement with the usual field theory result [48].

### 8.5 Expansion of the Exponentiated Green Functions

The third factor in the integrand of the string amplitude are the contributions common to the left- and right-movers, equation (5.11). The partition function and its expansion have already been taken into account in the above discussions of the expansions of the left- and right-movers; this leaves us with the zero-mode pieces — the exponentiated bosonic Green functions.

These produce the standard Feynman-parameter denominator of a loop integrand in the infinite-tension limit (after the  $\text{Im } \tau$  integration). In the large  $\text{Im } \tau$  limit, these exponentiated Green functions behave as follows,

$$\exp(G_B(\nu_i - \nu_j)) \longrightarrow \exp\left(\text{Im } \tau (\hat{\nu}_{ij}^2 - |\hat{\nu}_{ij}|)\right) \quad (8.41)$$

The kinematic factor  $K$  in equation (8.1) thus becomes

$$- \sum_{i < j}^{n_L} k_i \cdot k_j \hat{\nu}_{ij} (1 - |\hat{\nu}_{ij}|) \quad (8.42)$$

and the denominator of the parametric integral is

$$\left[ \sum_{i < j}^{n_L} k_i \cdot k_j |\hat{\nu}_{ij}| (1 - |\hat{\nu}_{ij}|) \right]^{n_L - 2 + \epsilon/2} \quad (8.43)$$

We can put the denominator in a more standard form by changing variables from the  $\hat{\nu}$  to a set of standard Feynman parameters  $a_j$ . In general, we may have to sum a given contribution over

several orderings of the parameters  $\hat{\nu}_i$  that survive after the pinch integrations have been performed; consider the integration ordering  $\hat{\nu}_{\sigma(1)} \leq \hat{\nu}_{\sigma(2)} \leq \dots \leq \hat{\nu}_{\sigma(n_\ell)} = 1$  where  $\sigma \in S_{n_\ell}$ . The required change of variables is just

$$\hat{\nu}_{i_k} = \sum_{j=1}^k a_j \quad (8.44)$$

where the  $n_\ell$  Feynman parameters satisfy the constraint

$$\sum_{j=1}^{n_\ell} a_j = 1 \quad (8.45)$$

and are positive. The denominator of the parametric integral then becomes

$$\begin{aligned} & \left[ \sum_{j<l}^{n_\ell} k_{\sigma(j)} \cdot k_{\sigma(l)} \left( \sum_{m=j+1}^l a_m \right) \left( 1 - \sum_{m=j+1}^l a_m \right) \right]^{n_\ell-2+\epsilon/2} = \\ & \left[ \sum_{j<l}^{n_\ell} k_{\sigma(j)} \cdot k_{\sigma(l)} \left( \sum_{m=j+1}^l a_m \right) \left( \sum_{m=1}^j a_m + \sum_{m=l+1}^{n_\ell} a_m \right) \right]^{n_\ell-2+\epsilon/2} \end{aligned} \quad (8.46)$$

For each independent pinch configuration, the string integral thus becomes a standard Feynman parameter integral

$$\int_0^1 \prod_{j=1}^{n_\ell} da_j \delta \left( 1 - \sum_{i=1}^{n_\ell} a_i \right) P_L(\{a_i\}) \left[ \sum_{j<l}^{n_\ell} k_{\sigma(j)} \cdot k_{\sigma(l)} \left( \sum_{m=j+1}^l a_m \right) \left( \sum_{m=1}^j a_m + \sum_{m=l+1}^{n_\ell} a_m \right) \right]^{-n_\ell+2-\epsilon/2} \quad (8.47)$$

where  $P_L$  is a polynomial given by the earlier reduction of left- and right-mover Green functions. Note that no Passarino-Veltman reduction of vector integrals is required; as promised, the string has taken care of that.

## 8.6 Integration Orderings

Over what orderings of the  $\hat{\nu}$  variables must we integrate? As can be seen from eq. (8.24), in computing  $A_{n;j}(1, 2, \dots, n)$ , only a limited set of orderings contributes. To express these sets in a compact form, it is convenient to introduce the notion of the set of *mergings*  $M(\{a_i\}; \{b_j\})$  of two sets  $\{a_i\}$  and  $\{b_j\}$ , which is simply the set of all permutations of the set  $\{a_i, b_j\}$  set of mergings is simply which preserve the order within each of the constituent subsets separately. For our purposes, we want a notation for the set of all mergings of the two subsets  $\{a_i\}_{i=j-1}^1$  and  $\{a_i\}_{i=j}^{n-1}$  of a single set  $\{a_i\}_{i=1}^{n-1}$ :

$$M_{n;j}(\{a_i\}_{i=1}^n) = M(\{a_i\}_{i=j-1}^1; \{a_i\}_{i=j}^{n-1}) \quad (8.48)$$

(Recall that we have fixed  $\hat{\nu}_n = 1$ , so it is always ordered after the remaining parameters; this is the reason for the seeming discrepancy between  $n$  and  $n - 1$  in this notation. Note also that the first  $j - 1$  labels appear in *reverse* order.)

For the contribution without pinches, the analysis of the left-mover contributions would tell us that in calculating  $A_{n;j}(1, 2, \dots, n)$  we must integrate over the orderings  $M_{n;j}(\{\hat{\nu}_i\}_{i=1}^n)$  and the reversed orderings  $M_{n;n-j+1}(\{\hat{\nu}_i\}_{i=n-1}^1, \hat{\nu}_n)$ , which we shall abbreviate by the notation  $M_{n;j}(1, 2, \dots, n)$  and  $M_{n;n-j+1}(n-1, n-2, \dots, 1, n)$  respectively. If we have eliminated some variables through pinching, leaving  $n_\ell$  of which  $j_\ell - 1$  are amongst the first  $j - 1$ , we must integrate over the orderings  $M_{n_\ell;j_\ell}(1, \dots, n)$  and  $M_{n_\ell;n_\ell-j_\ell+1}(n, \dots, 1, n)$ , where the variables removed by the pinch integrations are understood to be omitted from the argument lists.

This is not quite the whole story, because the contribution from a given ordering and its reversed ordering are related in a simple manner. This is easiest to see after changing to Feynman parameters. First, we may note that the left-mover and right-mover fermionic Green functions factors change by at most a sign, while the expansion of the bosonic Green functions in the ordering  $\hat{\nu}_1 \leq \hat{\nu}_2 \leq \dots \leq \hat{\nu}_{n-1} \leq \hat{\nu}_n = 1$  is ( $j < l$ )

$$\begin{aligned} G_B(\nu_{jl}) &\rightarrow \text{Im } \tau (\hat{\nu}_{lj}^2 - \hat{\nu}_{lj}) \\ \dot{G}_B(\bar{\nu}_{jl}) &\rightarrow \frac{i}{2} (1 + 2\hat{\nu}_{jl}) \end{aligned} \quad (8.49)$$

Using the same change of variables (8.44) employed above, these become

$$\begin{aligned} G_B(\nu_{jl}) &\rightarrow \text{Im } \tau \left( \sum_{m=j+1}^l a_m \right) \left( 1 - \sum_{m=j+1}^l a_m \right) \\ &= \text{Im } \tau \left( \sum_{m=j+1}^l a_m \right) \left( \sum_{m=1}^j a_m + \sum_{m=l+1}^n a_m \right) \\ \dot{G}_B(\bar{\nu}_{jl}) &\rightarrow \frac{i}{2} \left( 1 - 2 \sum_{m=j+1}^l a_m \right) \end{aligned} \quad (8.50)$$

For the reversed ordering,  $\hat{\nu}_{n-1} \leq \hat{\nu}_{n-2} \leq \dots \leq \hat{\nu}_1 \leq \hat{\nu}_n = 1$  we have

$$\begin{aligned} G_B(\nu_{jl}) &\rightarrow \text{Im } \tau (\hat{\nu}_{lj}^2 + \hat{\nu}_{lj}) \\ \dot{G}_B(\bar{\nu}_{jl}) &\rightarrow \frac{i}{2} (-1 + 2\hat{\nu}_{jl}) \end{aligned} \quad (8.51)$$

Here, we employ a slightly differently mapping to Feynman parameters, which corresponds to a relabelling of the variables introduced in equation (8.44),

$$\hat{\nu}_j = \sum_{m=j+1}^m a_m \quad (8.52)$$

With this mapping, the latter expansions are simply

$$\begin{aligned}
G_B(\nu_{jl}) &\rightarrow \text{Im} \tau \left( - \sum_{m=j+1}^l a_m \right) \left( - \sum_{m=j+1}^l a_m + 1 \right) \\
\dot{G}_B(\bar{\nu}_{jl}) &\rightarrow \frac{i}{2} \left( -1 + 2 \sum_{m=j+1}^l a_m \right)
\end{aligned}
\tag{8.53}$$

The expression for  $G_B$  is equal to that for the original ordering, equation (8.50), while the expression has changed by a sign. The measure in both cases is identical, so that we may conclude that the contribution of a given ordering and the reversed ordering can differ by at most a sign.

Let us consider this possible sign more carefully. After all pinch integrations have been performed, each right-mover term contains  $n_f$  fermionic and  $n_b - n_f$  bosonic Green functions. We saw earlier that the contribution of the fermionic Green functions changes sign when we reverse the ordering of the integration variables; since the factor contributed by each bosonic Green function also changes sign, the net change in each right-mover term is simply  $(-1)^{n_f}$ . As we saw in the discussion of the left-mover expansions, the factors contributed by the left-movers also change by the same sign between a given integration ordering and the reversed one. We thus conclude that the contribution to any partial amplitude of a given integration ordering and of the reversed integration ordering are identical. We may thus pick one of each pair, and multiply that contribution by 2. The integration orderings we must sum over are then reduced to those contained in the set  $M_{n_\ell, j_\ell}(1, \dots, n)$ . This can be given a diagrammatic interpretation, namely that the contributions of the two diagrams 4a and 4b are identical. For all diagrams with three or more legs attached to the loop, there are two orderings giving identical contributions.

### 8.7 Collecting the Pieces

In the field-theory limit, the string integral simplifies considerably, and the various Green functions, which in the full string theory consist of products of theta functions, simplify to numbers or polynomials in the Feynman parameters. The above expansions allow one to write down a simple set of rules for calculating each partial amplitude. Each partial amplitude will receive contributions from various pinch diagrams with certain orderings of the integration variables. These contributions have the schematic form

$$\frac{1}{\text{Pinch factors}} \int d(\text{Feynman parameters}) \frac{\text{Polynomial}(\text{Feynman parameters})}{\text{Feynman denominator}}
\tag{8.54}$$

wherein the ‘Pinch factors’ are simply products of certain external momentum invariants  $k_i \cdot k_j$ , times constants, and the Feynman denominator is the standard one encountered in a  $\phi^3$  scalar field theory in  $4 - \epsilon$  dimensions.

In the pinch factors, we must keep track of the signs; for a generic neighboring pair of indices  $j$  and  $l$ , a pinched left-mover Green function will appear in the form  $G_F[\frac{\alpha_G}{\beta_G}](-\nu_{jl})$ ; the right-mover Green functions will appear in the form  $G_F[\frac{\alpha_l}{\beta_l}](\bar{\nu}_{jl})$  or  $\dot{G}_B(\bar{\nu}_{jl})$ , leading respectively to pinch factors of  $1/2k_j \cdot k_l$  and  $-1/2k_j \cdot k_l$  after performing the pinch integrals. If the indices are the first and last indices within a trace, however, the left-mover Green function will appear in the form  $G_F[\frac{\alpha_G}{\beta_G}](\nu_{jl})$ , giving rise to an additional sign.

As noted in the previous subsection, we can pick one out of each pair of equivalent integration orderings, in which case we get an over-all factor of 2, and we can ignore the left-mover structure, *except* in the case that we have four legs attached to the loop after the pinch integrations, corresponding to two independent traces (i.e.  $A_{n;j>l}$ ), in which case we get an additional factor of 2. The factors of  $i^{n_l}$  emerging from the left-movers combine with the factors of  $(-i)^{n_l}$  emerging from the right-movers and disappear. Only a factor of  $(-1)^{n+l-1}$  is left over.

Amongst the right-movers, contributing cycles of fermionic Green functions will then yield a factor of 2 if they contain only two Green functions, 1 otherwise; terms with no fermionic Green functions should be multiplied by  $2(1 - \delta_{R\epsilon}/2)$ , and every bosonic Green function  $\dot{G}_B(\bar{\nu}_{jl})$  should be replaced by  $\text{sign}(\dot{\nu}_{jl})/2 - \dot{\nu}_{jl}$ .

It is possible to decouple these expansions from their string theory origins, as we discuss in the next section.

## 9. Rules for the Field Theory Calculation

The simplifications derived in the preceding section enable us to write down a set of rules to be used in performing calculations with the new technology. This set of rules does not require any knowledge of string theory for its application.

In the calculation of the  $n$ -gluon amplitude, one begins by picking a set of reference momenta for the external gluons, along the lines suggested in section 3, and substituting the resulting expressions for the various dot products of external momenta and polarization vectors ( $k_i \cdot \epsilon_j$  and  $\epsilon_i \cdot \epsilon_j$ ) into the following expression,

$$\begin{aligned} \mathcal{K}(\{x_i, k_i, \epsilon_i\}) = & \int \left( \prod_{l=1}^n d\theta_l d\bar{\theta}_l \right) \\ & \prod_{j<l}^n \exp \left[ \begin{aligned} & -\theta_j \theta_l k_j \cdot k_l G_F(x_{jl}) + i(\theta_j \bar{\theta}_l k_j \cdot \epsilon_l + \bar{\theta}_j \theta_l k_l \cdot \epsilon_j) G_F(x_{jl}) \\ & - i(\theta_j \bar{\theta}_j k_l \cdot \epsilon_j - \theta_l \bar{\theta}_l k_j \cdot \epsilon_l) \dot{G}_B(x_{jl}) + \bar{\theta}_j \bar{\theta}_l \epsilon_i \cdot \epsilon_j G_F(x_{jl}) \\ & + \theta_j \bar{\theta}_j \theta_l \bar{\theta}_l \epsilon_j \cdot \epsilon_l \dot{G}_B(x_{jl}) \end{aligned} \right] \end{aligned} \quad (9.1)$$

which we shall call the kinematic tensor. In this expression, the  $G_F$ s and  $\dot{G}_B$ s, to which we shall refer as the ‘ $F$ ’ and ‘ $B$ ’ Feynman parameter functions, may be treated as ‘black boxes’; the rules will eventually turn products of these objects into a polynomial in the (ordinary) Feynman parameters. The other parameter function appearing in this expression,  $\bar{G}_B$ , is the derivative of  $B$ -parameter function with respect to its argument. The  $F$ - and  $B$ -parameter functions are antisymmetric functions of their arguments, which are differences of variables labelled by the indices of external gluons:  $x_{ji} = x_j - x_i$ . (These  $x$  variables are in fact sums of Feynman parameters, but the rules take this into account implicitly, and we shall not have to make use of this fact explicitly.) The  $\theta_j$  and  $\bar{\theta}_j$  are Grassmann parameters; after substituting the spinor-helicity basis values for the dot products, one must integrate with respect to these parameters. (We remind the reader that integration with respect to such Grassmann parameters is equivalent, up to a sign, to expansion of the exponential in equation (9.1), followed by extraction of the linear term in each parameter.)

Next, one must remove all derivatives of the  $B$ -parameter function, by integrating by parts the following (formal) integral over the  $x$ -variables,

$$\int \left( \prod_j^n dx_j \right) \mathcal{E}(\{x_l, k_l\}) \mathcal{K}(\{x_l, k_l\}) \quad (9.2)$$

as appropriate. (There are no boundary terms.) In this expression,

$$\mathcal{E}(\{x_l, k_l\}) = \exp \left[ \sum_{j < l}^n k_j \cdot k_l G_B(x_{jl}) \right], \quad (9.3)$$

where  $G_B$  is the antiderivative of the  $B$ -parameter function. It is possible to write down a systematic algorithm for this procedure, but one isn’t really needed for calculations done manually. For certain helicity amplitudes, for example  $(- + \cdots +)$ , with an appropriate choice of reference momenta, the derivatives of the  $B$ -parameter function disappear on the first line, without any need to perform this step of integration by parts. After performing the integrations by parts, one should remove the integrals and the  $\mathcal{E}$  function, and retain only the transformed form of the kinematic tensor. The latter now is a sum of terms, where each term is a product of spinor products, Lorentz products, and a combined total of  $n$   $F$ - and  $B$ -parameter functions. (The number of each type of parameter function in a given term will vary from term to term.) One may perform the following check on the integration by parts procedure: the resulting kinematic tensor should vanish identically under the substitution  $\dot{G}_B(x) \rightarrow -G_F(x)$ .

Knowledge of the functional form of the Feynman parameter functions is in fact not required after this step; only the dependence on the *labels* of the external gluons is important for subsequent

steps. It is therefore convenient to replace the notation used above with a notation tracking only the external labels,

$$\begin{aligned}
G_F(x_{jl}) &\rightarrow G_F^{jl} \\
\dot{G}_B(x_{jl}) &\rightarrow \dot{G}_B^{jl}, \quad j < l \\
\dot{G}_B(x_{jl}) &\rightarrow -\dot{G}_B^{lj}, \quad j > l
\end{aligned} \tag{9.4}$$

After this replacement, the first index of the Feynman parameter functions will always be smaller than the second one. The steps described below may occasionally generate a Feynman parameter function  $G_X^{ij}$  (denoting collectively the  $G_F^{a,b}$  and  $\dot{G}_B^{a,b}$ ), where the first index is greater than the second one; in such a case, one should replace it with  $-G_X^{ji}$ .

At this point, one should write down all planar  $\phi^3$  diagrams with  $n$  external legs, excluding diagrams with tadpoles, subject to the requirement that the legs be numbered in a cyclicly consecutive fashion around the loop. (Cyclicly consecutive is determined by the trace structure  $\text{Gr}_{n;j}$ ; for the trace structure  $\text{Gr}_{n;1}$ , cyclicly consecutive means consecutive mod  $n$ .) These diagrams will be used as a guide in the evaluation of the amplitude. Each diagram consists of some number of trees attached to the loop. We will denote the number of legs attached directly to the loop by  $n_L$ . Diagrams with only two legs attached to the loop where one of the two legs is an external one give a vanishing contribution in dimensional regularization, and should be omitted from the following discussions. It is helpful to label the internal vertices of the trees according to the following rule, working from the outside inwards: each vertex should be labelled with the larger index of the two lines feeding into it from an outermore vertex. In addition, it is convenient to think of each external momentum as flowing into the line labelled by the given index. Momentum conservation is enforced at each vertex, so the momentum attached to a given line is the same as for the corresponding field theory diagram. It is *not* necessary to track the momentum beyond the trees, and one should *not* introduce a momentum variable for the loop. Each line in a tree attached to the loop branches as we go outwards from the loop; we will refer to the set external legs reached via this branching as the *leaves* of a given line. Because of the cyclicly consecutive labelling of the external legs, the leaves of any tree line in the diagram are consecutively numbered either mod  $j - 1$  or mod  $n - j + 1$ .

What one calculates directly in the new technology is not the amplitude, but its irreducible gauge-invariant pieces, the partial amplitudes; the full amplitude then appears as a sum over partial amplitudes with appropriate color traces, as in equation (4.2). Each partial amplitude will eventually be written as a sum over these diagrams of a standard integral over Feynman parameters of a certain polynomial in the Feynman parameters divided by a standard Feynman denominator

for a loop with  $n_l$  legs,

$$A \sim \sum_{\text{diagrams}} \int d \left( \begin{array}{c} \text{Feynman} \\ \text{parameters} \end{array} \right) \frac{\text{Polynomial} \left( \begin{array}{c} \text{Feynman} \\ \text{parameters} \end{array} \right)}{\text{Feynman loop denominator}}. \quad (9.5)$$

The rules we shall describe below transform the kinematic tensor into the polynomial in the Feynman parameters times additional poles in the external momentum invariants. The diagrams guide this transformation; in general, of course, the resulting polynomials (including the number of Feynman parameters!) are different for different diagrams.

The first step in the transformation is to account for the contribution of the trees attached to the loops. For any given diagram, each propagator in the tree (including the propagator attaching the tree to the loop) contributes a factor of

$$\frac{1}{K^2} \quad (9.6)$$

where  $K$  is the momentum flowing through that line. The momentum flowing through a given line is of course just the sum of the momenta of its leaves. These factors should be carried along with the kinematic factor as one analyzes it follows the steps described below.

The contribution of any term to a given diagram is determined by the indices of the Feynman parameters functions in that term. In general, each tree with  $l$  leaves attached to the loop will remove  $l - 1$  Feynman parameter functions from the term, replacing them with either 1, 0, or  $-1$ . One may then proceed to analyze the other trees. Here, we give only the rule for analyzing single three-point vertices attached to the loop. Similar rules can be formulated for more complex trees attached to the loop, but as we have deferred the derivation of such rules, so we shall defer the statement of them. In the particular case of the four-point amplitude, the rules for three-point vertices alone suffice, because the only diagram with a more complex tree attached to the loop (one containing two three-point vertices and three external legs) has only two legs attached to the loop, and one must necessarily be an external one; it therefore vanishes, as stated above.

The contributions of the trees depend on the particular partial amplitude we are calculating, and so it is convenient to introduce a bit of nomenclature to refer to the arguments of the various partial amplitudes. Recall that we must calculate  $\lfloor n/2 \rfloor + 1$  partial amplitudes in general (though one may rely on the  $U(1)$  decoupling equations to reduce this number by two). Each partial amplitude  $A_{n;j}$  is associated with a color structure with two traces, the first containing  $j - 1$  color matrices corresponding to the first  $j - 1$  arguments, the second containing the remainder. (If  $j = 1$ , the first trace is then trivial; it is replaced in the associated color factor with the number of colors, and all the arguments are associated with the second and only non-trivial trace.) By extension, we



will refer to the first  $j - 1$  indices or arguments of  $A_{n;j}(1, \dots, n)$  as ‘belonging’ to the first trace, and the set of indices  $\{1, \dots, j - 1\}$  as ‘constituting’ the first trace, and so on.

With this nomenclature, we can phrase the rules for the attached simple trees as follows. For each such simple tree, consisting of a three-point vertex with external legs  $a$  and  $b$  attached to the loop (note that  $b = (a + 1) \bmod n$  for  $A_{n;1}$ , and either  $b = (a + 1) \bmod j - 1$  or  $b = (a + 1) \bmod n - j + 1$  for the other  $A_{n;j}$ ), examine the Feynman parameter functions in a given term. The term will contribute only if there is either a lone  $G_F^{a,b}$  or a lone  $G_B^{a,b}$  amongst the parameter functions. If neither is present, or more than one of each is present, or both are present, the term does not contribute, and one should throw it away. Furthermore, the term will only contribute to the given partial amplitude  $A_{n;j}$  if both indices belong to the same trace, but do not constitute the entire trace. (The requirement that the indices not constitute the entire trace eliminates contributions to  $A_{n;3}$  with legs 1 and 2 attached to a tree, and also any contributions with attached trees to  $A_{4;3}$ .) Otherwise, the  $G_X^{a,b}$  should be replaced according to the following rule,

$$\begin{aligned} G_F^{a,b} &\rightarrow 1, & \dot{G}_B^{a,b} &\rightarrow -1 & (a,b) \neq (1, j-1) \text{ and } (a,b) \neq (j, n) \\ G_F^{a,b} &\rightarrow -1, & \dot{G}_B^{a,b} &\rightarrow 1 & (a,b) = (1, j-1) \text{ or } (a,b) = (j, n) \end{aligned} \quad (9.7)$$

(The sign change for indices  $(1, j - 1)$  and  $(j, n)$  is essentially due to the difference of the cyclic ordering of the labels of the diagram compared to the indices of the  $G_X^{a,b}$ .) We can summarize this equation as follows,

$$G_X^{a,b} \rightarrow (-1)^{\delta_{B,X}} (-1)^{\delta_{(a,b)}^{(1,j-1)} + \delta_{(a,b)}^{(j,n)}} \quad (9.8)$$

After this replacement of the Feynman parameter functions, one replaces all appearances of the index  $\min(a, b)$  in the other Feynman parameter functions with  $\max(a, b)$ , and then proceeds examine the indices attached to the other trees.

At the end of this stage of the computation, one has reduced each term to a product of spinor and Lorentz products (including the poles brought in by the attached trees) times  $n_\ell$  Feynman parameter functions, that is one for each leg attached directly to the loop. Of the legs attached to the loop,  $j_\ell - 1$  belong to the first trace, and  $n_\ell - j_\ell + 1$  belong to the second. Note that  $1 \leq j_\ell \leq j$ . We will call the first set of indices  $T_1$ , and the second set  $T_2$ ; and the diagram depicted in fig. 5 with the trees amputated we will term the *reduced* diagram.

For most of the partial amplitudes  $A_{n;j}$ , each original diagram represents in fact the sum of several terms with different Feynman denominators corresponding to the different orderings of the external legs that respect the order within each trace. (The preceding steps in the calculation are independent of the ordering, but following steps depend on it.) Each different Feynman denominator corresponds to a different labelling of the reduced diagram. At this point, one should write down all inequivalent labellings of the reduced diagram, with the surviving labels, where the cyclic ordering

of the labels respects the ordering within the second trace, and where the cyclic ordering of the labels belonging to the first trace is the *reverse* of the ordering within the trace. Thus each reduced diagram will spawn a set of *ordered reduced* diagrams, each of which should be evaluated independently below. For the particular case of  $A_{n;1}$ , the partial amplitude giving rise to the leading- $\mathcal{N}_c$  behavior of the next-to-leading corrections, there is in fact only one possible labelling of the reduced diagram.

The orderings with which one must decorate a reduced diagram in order to obtain the class of diagrams to evaluate can be expressed in terms of the *mergings* of the indices associated with the two traces. (Recall that the set of mergings  $M(\{a_i\}; \{b_j\})$  of two sets  $\{a_i\}$  and  $\{b_j\}$  is simply the set of all permutations of the set  $\{a_i, b_j\}$  which preserve the order within each of the constituent subsets separately.) The set of orderings relevant to the calculation of  $A_{n;j}$  for the given diagram we shall call  $M_{n_i;j_i}(T_1, T_2)$ , where

$$M_{n_i;j_i}(\{a_i\}_{i=1}^n) = \left\{ \sigma \in S_n \left| \begin{array}{l} \sigma(a_{j'-1}) < \sigma(a_{j'-2}) < \dots < \sigma(a_1) < \sigma(a_{n'}) \\ \text{and} \\ \sigma(a_{j'}) < \sigma(a_{j'+1}) < \dots < \sigma(a_{n'-1}) < \sigma(a_{n'}) \end{array} \right. \right\} \quad (9.9)$$

Note that the first  $j' - 1$  labels appear in *reverse* order, and that the last label —  $n'$  — always appears last. In the special case that  $j' = 1$  (this is always the case when calculating  $A_{n;1}$ ), this set contains only one element, the ordering  $\{1, 2, \dots, n\}$ . As another example, the reduced diagrams associated with the mergings  $M(\{1, 2, 3\}; \{4, 5\})$  are depicted in fig. 6.

For each ordered reduced diagram, one must now examine the  $F$  parameter functions in a term to determine the contribution. Terms with no  $F$ -type parameter function must be multiplied by  $2(1 - \delta_R \epsilon/2)$ , where  $\delta_R = 1$  for either the 't Hooft-Veltman or the conventional dimensional regularization schemes, while  $\delta_R = 0$  for the four-dimensional helicity scheme described in section 7. For the remaining terms, the products of the  $F$ -type parameter functions will again reduce to either 2, 1, or 0, depending on their indices. It turns out that one can never have a lone  $G_F$  in a given term; there will always be two or more, and furthermore, each external label on one  $G_F$  will appear on exactly one other  $G_F$ . Pick only one copy of each label appearing on some  $G_F$ , and call this set of indices  $I_F$ . Order the indices according to the ordering of external labels (the element of  $M_{n_i;j_i}$  we are considering). Then

$$I_F = \{i_1, i_2, \dots, i_{f-1}, i_f\} \quad (9.10)$$

where  $f$  is the number of  $F$ -type parameter functions in the term under consideration.

The complete product of  $G_F$ 's can have the form

$$\left(G_F^{a,b}\right)^2, \quad (9.11)$$

(for any  $a$  and  $b$  in  $I_F$ ) in which case the product is replaced by ‘2’; one of the forms

$$G_F^{i_1, i_2} G_F^{i_2, i_3} \dots G_F^{i_{\ell-1}, i_\ell} G_F^{i_1, i_\ell} \quad (9.12a)$$

$$G_F^{i_\ell-1, i_\ell-2} G_F^{i_\ell-2, i_\ell-3} \dots G_F^{i_1, i_\ell} G_F^{i_\ell-1, i_\ell} \quad (9.12b)$$

(for the order of elements in  $I_F$  as given in equation (9.10)) in which case the product is replaced by ‘1’; or some other form, in which case the product vanishes, and the term should be thrown away.

Once the  $F$ -type parameter functions have been replaced according to the above rules, one should replace each  $B$ -type parameter function by

$$\dot{G}_B^{a,b} \rightarrow -\frac{1}{2} + \sum_{m=\text{Ind}(a)+1}^{\text{Ind}(b)} a_m \quad (9.13)$$

where  $\text{Ind}$  denotes the index of the external label in the given ordering (if it appears in position  $m$  in the given ordering, its index is  $m$ ). The  $a_m$  appearing in equation (9.13) are conventional Feynman parameters.

The Feynman denominator for the given ordering of the given diagram is identical to what one would write down in a  $\phi^3$  field theory; in the notation used above, it is

$$\mathcal{D}(\sigma \in M_{n_\ell, j_\ell}) = \left[ -\sum_{j < l}^{n_\ell} K_{L(\sigma_j)} \cdot K_{L(\sigma_l)} \left( \sum_{m=j+1}^l a_m \right) \left( \sum_{m=1}^j a_m + \sum_{m=l+1}^{n_\ell} a_m \right) \right]^{n_\ell-2+\epsilon/2} \quad (9.14)$$

where  $L(a)$  is the sequence of leaves of the leg with that label, and  $K_{L(a)}$  is the total momentum flowing through the leg labelled  $a$  that is attached directly to the loop.

There is an over-all normalization constant,

$$\mathcal{N}_n = i \frac{(4\pi)^{\epsilon/2}}{16\pi^2} (-\sqrt{2}g)^n \mu^{n\epsilon/2} \quad (9.15)$$

as well as additional constant factors  $C_{n_\ell, j_\ell}$ , which are unity unless  $n_\ell = 4$  and  $j_\ell = 3$ :

$$C_{n', j'} = \begin{cases} 2, & n' = 4 \text{ and } j' = 3 \\ 1, & \text{otherwise} \end{cases} \quad (9.16)$$

and a sign depending on the number of legs attached to the loop in the reduced diagram,

$$(-1)^{j_\ell-1} \quad (9.17)$$

With these pieces, the partial amplitude is given by the sum over diagrams, and the sum over orderings for each diagram of a conventional Feynman-parameter integral,

$$A_{n; j} = \mathcal{N}_n \sum_{\text{diagrams}} \sum_{\sigma \in M_{n_\ell, j_\ell}} \int_0^1 \left( \prod_{m=1}^{n_\ell} da_m \right) (-1)^{j_\ell-1} C_{n_\ell, j_\ell} \frac{\delta(1 - \sum_m a_m) \mathcal{K}|_{\text{reduced}}(a_m, \sigma)}{\mathcal{D}(\sigma)} \quad (9.18)$$

where the subscript ‘reduced’ on  $\mathcal{K}$  indicates that it has been transformed according to the rules described above. It includes those factors arising from trees attached to the loop. (In those partial amplitudes possessing an ultraviolet pole, that pole is still present in this expression, and must be removed in accordance with whatever renormalization prescription is to be used.)

## 10. The Finite Helicity Amplitudes

In this section, we present the results of direct calculation for the partial amplitudes  $A_{4;1}(++++)$ ,  $A_{4;3}(++++)$ ,  $A_{4;1}(-+++)$ , and  $A_{4;3}(-+++)$ . There are several simplifying aspects of this calculation: these amplitudes are infrared- and ultraviolet- finite, as noted earlier; and they do not receive contributions from regions with two pinches or from regions with three points pinched together.

For the  $(-+++)$  partial amplitudes, we choose the following reference momenta: for the first gluon,  $k_4$ ; for the remainder,  $k_1$ . The first partial amplitude,  $A_{4;1}(-+++)$ , receives contributions both from the no-pinch region and from the  $\nu_1 \rightarrow \nu_2$  pinch region; the third partial amplitude,  $A_{4;3}(-+++)$ , receives contributions only from the no-pinch region (from all orderings). Adding the various contributions together, we find

$$\begin{aligned} A_{4;1}(1^-, 2^+, 3^+, 4^+) &= -\frac{i}{48\pi^2} \frac{[24]^2 \langle 24 \rangle}{[12] \langle 23 \rangle \langle 34 \rangle [41]} = \frac{i}{48\pi^2} \frac{[24]^2 (t+s)}{[12] \langle 23 \rangle \langle 34 \rangle [41]} \\ A_{4;3}(1^-, 2^+, 3^+, 4^+) &= \frac{i}{8\pi^2} \frac{\langle 12 \rangle [24] \langle 41 \rangle}{\langle 23 \rangle \langle 34 \rangle \langle 24 \rangle} \end{aligned} \quad (10.1)$$

(In these formulæ, and elsewhere,  $s$  and  $t$  are used as abbreviations for  $\langle 12 \rangle$  and  $\langle 23 \rangle$  respectively; one must bear this in mind when evaluating the partial amplitudes for other orderings of the arguments.) We will discuss the step-by-step procedure of evaluating these partial amplitudes in separate publications [46,49].

These satisfy the required decoupling equation,

$$A_{4;3}(1, 2, 3, 4) = \sum_{\sigma \in S_4/Z_4} A_{4;1}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)). \quad (10.2)$$

since

$$\begin{aligned} \sum_{\sigma \in S_4/Z_4} A_{4;1}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) &= \sum_{\sigma \in S_3\{2,3,4\}} A_{4;1}(1^-, \sigma(2)^+, \sigma(3)^+, \sigma(4)^+) \\ &= \frac{i}{48\pi^2} \times 2 \times \left( -\frac{[24]^2 \langle 24 \rangle}{[12] \langle 23 \rangle \langle 34 \rangle [41]} + \frac{[34]^2 \langle 34 \rangle}{[13] \langle 23 \rangle \langle 24 \rangle [41]} + \frac{[23]^2 \langle 23 \rangle}{[12] \langle 24 \rangle \langle 34 \rangle [31]} \right) \\ &= \frac{i}{8\pi^2} \frac{\langle 12 \rangle [24] \langle 41 \rangle}{\langle 23 \rangle \langle 34 \rangle \langle 24 \rangle} \\ &= A_{4;3}(1^-, 2^+, 3^+, 4^+) \end{aligned} \quad (10.3)$$

Those symmetries leaving the pattern of helicities unchanged may also be used as a check on the particular calculation. (The symmetries *not* leaving the pattern of helicities unchanged are used to deduce the values of the partial amplitudes for the new patterns, for example, the value of  $A_{4;1}(1^+, 2^-, 3^+, 4^+)$ .) For  $A_{4;1}$ , we may consider the reflection, followed by an appropriate cyclic permutation; we should find

$$A_{4;1}(1^-, 2^+, 3^+, 4^+) = A_{4;1}(1^-, 4^+, 3^+, 2^+) \quad (10.4)$$

and the reader may verify that this indeed the case. For  $A_{4;3}$  we may consider the element of  $S_{4;3}$  that exchanges the last two arguments; we should find that

$$A_{4;3}(1^-, 2^+, 3^+, 4^+) = A_{4;3}(1^-, 2^+, 4^+, 3^+); \quad (10.5)$$

i.e., that

$$\frac{\langle 12 \rangle [24] \langle 41 \rangle}{\langle 23 \rangle \langle 34 \rangle \langle 24 \rangle} - \frac{\langle 12 \rangle [23] \langle 31 \rangle}{\langle 23 \rangle \langle 43 \rangle \langle 24 \rangle} = 0. \quad (10.6)$$

Using the anticommutativity of the spinor product and momentum conservation, one may see that the required identity is indeed satisfied.

The third partial amplitude can be extracted using the decoupling equation,

$$A_{4;2}(1, 2, 3, 4) = - \sum_{\sigma \in \mathcal{Z}_3\{2,3,4\}} A_{4;1}(1, \sigma(2), \sigma(3), \sigma(4)) = -\frac{1}{2} A_{4;3}(1, 2, 3, 4) \quad (10.7)$$

which gives

$$A_{4;2}(1^-, 2^+, 3^+, 4^+) = -\frac{i}{16\pi^2} \frac{\langle 12 \rangle [24] \langle 41 \rangle}{\langle 23 \rangle \langle 34 \rangle \langle 24 \rangle} \quad (10.8)$$

In the  $(++++)$  case, we choose the same reference momenta as in the  $(-+++)$ ; here, the separate contributions from different pinch regions are infrared divergent, but when added together the infrared divergences cancel as expected, yielding

$$\begin{aligned} A_{4;1}(1^+, 2^+, 3^+, 4^+) &= \frac{i}{48\pi^2} \frac{st}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \\ A_{4;2}(1^+, 2^+, 3^+, 4^+) &= -\frac{i}{16\pi^2} \frac{st}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \\ A_{4;3}(1^+, 2^+, 3^+, 4^+) &= \frac{i}{8\pi^2} \frac{st}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \end{aligned} \quad (10.9)$$

The last partial amplitude is expected to be completely symmetric under interchange of any two arguments; this is indeed true, since

$$\frac{st}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} = \frac{su}{\langle 12 \rangle \langle 24 \rangle \langle 43 \rangle \langle 31 \rangle} = \frac{tu}{\langle 13 \rangle \langle 32 \rangle \langle 24 \rangle \langle 41 \rangle}. \quad (10.10)$$

## 11. The Remaining Helicity Amplitude

For the purposes of computing the next-to-leading correction to the differential cross-section, the interesting partial amplitudes are  $A_{4;1}(- - + +)$  and  $A_{4;1}(- + - +)$ . In FDH or the 't Hooft-Veltman scheme, these are in fact the only partial amplitudes required. (In the conventional dimensional regularization scheme, additional helicity amplitudes, to be discussed in section 13, are required.)

As discussed earlier, the contribution from the three-points pinch is identically zero, because of the cancellation of infrared and ultraviolet divergences. In addition, for appropriate choices of reference momenta, every pinch of two variables vanishes (at the level of the string integrand). As a result, the only contributions to these partial amplitudes come from the no-pinch region and the ultraviolet subtraction term. In any of the dimensional regulators, we choose the  $\overline{\text{MS}}$  subtraction prescription. For the  $n$ -point amplitude at one loop, the appropriate quantity to subtract is thus

$$(n-2)\beta_0 g^2 \frac{1}{\epsilon} \frac{\Gamma^2(1-\epsilon/2)\Gamma(1+\epsilon/2)}{\Gamma(1-\epsilon)} (4\pi)^{\epsilon/2} (\mu^2)^{\epsilon/2} \mathcal{A}_n^{\text{tree}}(1, \dots, n) \quad (11.1)$$

where  $\beta_0 = -11N_c/(3 \cdot 16\pi^2)$  is the leading coefficient in the gauge theory  $\beta$ -function. Thus only  $A_{n;1}$  is affected by the ultraviolet subtraction,

$$A_{n;1}(1, \dots, n) \rightarrow A_{n;1}(1, \dots, n) - (n-2)\hat{\beta}_0 \frac{1}{\epsilon} \frac{\Gamma^2(1-\epsilon/2)\Gamma(1+\epsilon/2)}{\Gamma(1-\epsilon)} (4\pi)^{\epsilon/2} \mathcal{A}_n^{\text{tree}}(1, \dots, n) \quad (11.2)$$

where  $\hat{\beta}_0 = -11/(48\pi^2)$ .

The normalizations and phase conventions are defined by the expansion (2.2) and the decompositions (4.1) and (4.2)

$$\begin{aligned} \mathcal{A}_4(1, 2, 3, 4) &= \mathcal{A}_4^{\text{tree}}(1, 2, 3, 4) + \mathcal{A}_4^{1\text{-loop}}(1, 2, 3, 4) + \dots \\ \mathcal{A}_4^{\text{tree}}(1, 2, 3, 4) &= g^2 \sum_{\sigma \in S_4/Z_4} \text{Tr}(T^{\alpha_{\sigma(1)}} T^{\alpha_{\sigma(2)}} T^{\alpha_{\sigma(3)}} T^{\alpha_{\sigma(4)}}) A_4(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) \\ \mathcal{A}_4^{1\text{-loop}}(1, 2, 3, 4) &= g^4 (\mu^2) \left( \sum_{\sigma \in S_4/Z_4} N_c \text{Tr}(T^{\alpha_{\sigma(1)}} T^{\alpha_{\sigma(2)}} T^{\alpha_{\sigma(3)}} T^{\alpha_{\sigma(4)}}) A_{4;1}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) \right. \\ &\quad + \sum_{\sigma \in S_4/S_{4;2}} \text{Tr}(T^{\alpha_{\sigma(1)}}) \text{Tr}(T^{\alpha_{\sigma(2)}} T^{\alpha_{\sigma(3)}} T^{\alpha_{\sigma(4)}}) A_{4;2}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) \\ &\quad \left. + \sum_{\sigma \in S_4/S_{4;3}} \text{Tr}(T^{\alpha_{\sigma(1)}} T^{\alpha_{\sigma(2)}}) \text{Tr}(T^{\alpha_{\sigma(3)}} T^{\alpha_{\sigma(4)}}) A_{4;3}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) \right) \end{aligned} \quad (11.3)$$

where the matrices  $T^a$  are normalized so that  $\text{Tr}(T^a T^b) = \delta^{ab}$ . All amplitudes are given in  $D = 4 - \epsilon$  dimensions.

The tree-level ordinary helicity partial amplitudes are

$$\begin{aligned}
A_4(1^+, 2^+, 3^+, 4^+) &= A_4(1^-, 2^+, 3^+, 4^+) = 0 \\
A_4(1^-, 2^-, 3^+, 4^+) &= i \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \\
A_4(1^-, 2^+, 3^-, 4^+) &= i \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}
\end{aligned} \tag{11.4}$$

Other helicities are related either by complex conjugation, or by cyclic permutation of the arguments.

The next-to-leading differential cross-section is given by [11]

$$2g^6(\mu^2) (\mu^2)^\epsilon N_c^3(N_c^2 - 1) \sum_{\sigma \in S_4/Z_4} A_4^{\text{tree}*}(\sigma) \text{Disp } A_{4;1}(\sigma) \tag{11.5}$$

for each helicity independently. The absorptive parts are irrelevant because  $A_{4;1}$  is proportional to  $A_4^{\text{tree}}$ .

As mentioned above, the only contribution to this helicity amplitude comes from the no-pinch diagram, fig. 7; following the rules in section 9 and performing a bit of algebra to simplify the polynomial in the Feynman parameters, one obtains

$$\begin{aligned}
A_{4;1}(1^-, 2^-, 3^+, 4^+) &= \frac{i}{8\pi^2} (4\pi\mu^2)^{\epsilon/2} \frac{st}{8} \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \int_0^1 \prod_{i=1}^4 da_i \delta\left(1 - \sum_{i=1}^4 a_i\right) \\
&\quad \times a_3 \frac{8a_1a_3 + 12a_2a_3a_4 + 8a_2a_3^2 + 8a_3^2a_4 + 4a_3^3 - 4a_2a_4 - a_3 - 5a_1 - 3}{(-sa_1a_3 - ta_2a_4)^{2+\epsilon/2}}
\end{aligned} \tag{11.6}$$

One can integrate this expression using the integral table in appendix V. Integrating equation (11.6) and its counterpart for  $A_{4;1}(1^-, 2^+, 3^-, 4^+)$ , one obtains the dispersive parts of the one-loop partial amplitudes needed for the next-to-leading corrections (dropping all terms of  $\mathcal{O}(\epsilon)$ ),

$$\begin{aligned}
\text{Disp } A_{4;1}(1^-, 2^-, 3^+, 4^+) &= i \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{\Gamma^2(1 - \epsilon/2)\Gamma(1 + \epsilon/2)}{8\pi^2\Gamma(1 - \epsilon)} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon/2} \\
&\quad \times \left(-\frac{8}{\epsilon^2} - \frac{22}{3\epsilon} + \frac{11}{6}l_Q(\mu^2) + \frac{2}{\epsilon}(l_Q(s) + l_Q(t)) - l_Q(s)l_Q(t) + \frac{11}{6}l_Q(t) + \frac{\pi^2}{2} - \frac{32}{9} - \frac{\delta_R}{6}\right) \\
\text{Disp } A_{4;1}(1^-, 2^+, 3^-, 4^+) &= i \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{\Gamma^2(1 - \epsilon/2)\Gamma(1 + \epsilon/2)}{8\pi^2\Gamma(1 - \epsilon)} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon/2} \\
&\quad \times \left(-\frac{8}{\epsilon^2} - \frac{22}{3\epsilon} + \frac{11}{6}l_Q(\mu^2) + \frac{2}{\epsilon}(l_Q(s) + l_Q(t)) - \frac{(u^2 - st)^2}{2u^4}(l_Q(s) + l_Q(t))^2\right. \\
&\quad \left.+ \frac{s^4 + t^4}{2u^4}(l_Q^2(s) + l_Q^2(t)) - \frac{11u^2 - 3st}{6u^3}(tl_Q(s) + sl_Q(t))\right. \\
&\quad \left.- \frac{st}{2u^3}(sl_Q(s) + tl_Q(t)) - \frac{st}{2u^2} + \frac{\pi^2}{2} - \frac{\pi^2}{2}\widehat{\Theta}(u)\frac{st(2u^2 - st)}{u^4} - \frac{32}{9} - \frac{\delta_R}{6}\right)
\end{aligned} \tag{11.7}$$

where  $\mu^2$  is the renormalization scale,  $Q^2$  is a completely arbitrary scale introduced in order to simplify the comparison to the result of Ellis and Sexton [14],  $l_Q(x) = \ln|x/Q^2|$ ,  $\widehat{\Theta}(x > 0) = 1$ ,  $\widehat{\Theta}(x < 0) = 0$ , and

$$\delta_R = \begin{cases} 0, & \text{FDH scheme,} \\ 1, & \text{HV scheme,} \end{cases} \quad (11.8)$$

and where evaluation in the physical region is assumed (that is, only one of (12), (13), and (14) may be positive).

Using the following functions,

$$\begin{aligned} \overline{\Theta}(x) &= \frac{1}{i\pi} \log \arg(x) \\ \Theta(x) &= \frac{1}{i\pi} \log \arg(-x) \end{aligned} \quad (11.9)$$

which are discussed more thoroughly in appendix V, along with the shorthand notation

$$\begin{aligned} l_\mu(jl) &= \ln \left| \frac{(jl)}{\mu^2} \right| \\ \Theta(jl) &= \Theta[(jl)] \end{aligned} \quad (11.10)$$



we can write the whole of the partial amplitudes in the form

$$\begin{aligned}
A_{4;1}(1^-, 2^-, 3^+, 4^+) &= i \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{(4\pi)^{\epsilon/2} \Gamma^2(1 - \epsilon/2) \Gamma(1 + \epsilon/2)}{8\pi^2 \Gamma(1 - \epsilon)} \\
&\times \left( -\frac{8}{\epsilon^2} - \frac{22}{3\epsilon} + \frac{2}{\epsilon} (l_\mu(12) + l_\mu(23)) - l_\mu(12) l_\mu(23) + \frac{11}{6} l_\mu(23) \right. \\
&\quad + \frac{\pi^2}{2} (1 + 2\Theta(12) \Theta(23)) - \frac{32}{9} - \frac{\delta_R}{6} \\
&\quad \left. + i\pi \left[ \frac{2}{\epsilon} [\Theta(12) + \Theta(23)] - \Theta(12) l_\mu(23) - \Theta(23) l_\mu(12) + \frac{11}{6} \Theta(23) \right] \right) \\
A_{4;1}(1^-, 2^+, 3^-, 4^+) &= i \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{(4\pi)^{\epsilon/2} \Gamma^2(1 - \epsilon/2) \Gamma(1 + \epsilon/2)}{8\pi^2 \Gamma(1 - \epsilon)} \\
&\times \left( -\frac{8}{\epsilon^2} - \frac{22}{3\epsilon} + \frac{2}{\epsilon} [l_\mu(12) + l_\mu(23)] - \frac{[(13)^2 - (12)(23)]^2}{2(13)^4} [l_\mu(12) + l_\mu(23)]^2 \right. \\
&\quad + \frac{(12)^4 + (23)^4}{2(13)^4} [l_\mu^2(12) + l_\mu^2(23)] - \frac{11(13)^2 - 3(12)(23)}{6(13)^3} [(23)l_\mu(12) + (12)l_\mu(23)] \\
&\quad - \frac{(12)(23)}{2(13)^3} [(12)l_\mu(12) + (23)l_\mu(23)] \\
&\quad + \frac{\pi^2}{2} \left( 1 - \frac{(12)(23) [2(13)^2 - (12)(23)]}{(13)^4} (1 - \Theta^2(12) - \Theta^2(23)) \right. \\
&\quad \left. + \frac{2 [(13)^2 - (12)(23)]^2}{(13)^4} \Theta(12) \Theta(23) \right) \\
&\quad - \frac{(12)(23)}{2(13)^2} - \frac{32}{9} - \frac{\delta_R}{6} \\
&\quad + i\pi \left[ \frac{2}{\epsilon} [\Theta(12) + \Theta(23)] - \frac{[(13)^2 - (12)(23)]^2}{(13)^4} [\Theta(12) l_\mu(23) + \Theta(23) l_\mu(12)] \right. \\
&\quad - \frac{(12)(23) [2(13)^2 - (12)(23)]}{(13)^4} [\Theta(12) l_\mu(12) + \Theta(23) l_\mu(23)] \\
&\quad - \frac{[\Theta(12)(23) + \Theta(23)(12)] [11(13)^2 - 3(12)(23)]}{6(13)^3} \\
&\quad \left. \left. - \frac{(12)(23) [\Theta(12)(12) + \Theta(23)(23)]}{2(13)^3} \right] \right)
\end{aligned} \tag{11.11}$$

(Note that the absorptive parts could have been extracted by replacing the logarithm of each

momentum invariant  $l_\mu(s_{a,b}) \rightarrow l_\mu(s_{a,b}) + i\pi\Theta(s_{a,b})$ ; with the standard field-theory conventions,  $\Theta(s) = 1$ , and  $\Theta(t) = \Theta(u) = 0$  in the physical region.)

The absorptive parts are irrelevant in the four-point cross-section (because  $A_{4;1}$  is proportional to  $A_4$ ), but may be extracted if desired from the dispersive parts by using the appropriate  $i\epsilon$  prescription on the momentum invariants. The absorptive parts are, of course, important in checks of the optical theorem as we shall discuss in section 15.

With the aid of the decoupling equations (4.9), we can write down  $A_{4;3}$  for these helicities,

$$\begin{aligned}
A_{4;3}(1^-, 2^-, 3^+, 4^+) &= i \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{(4\pi)^{\epsilon/2} \Gamma^2(1 - \epsilon/2) \Gamma(1 + \epsilon/2)}{8\pi^2 \Gamma(1 - \epsilon)} \\
&\times \left( -\frac{4}{\epsilon} \frac{1}{\langle 13 \rangle} ((12) l_\mu(12) + (13) l_\mu(13) + (23) l_\mu(23)) \right. \\
&\quad - \frac{(23) [2(12)^2 - (13)(23)]}{(12)^3} (l_\mu^2(23) + l_\mu^2(13)) - 2l_\mu(12) l_\mu(23) - 2 \frac{(23)}{\langle 13 \rangle} l_\mu(12) l_\mu(13) \\
&\quad - 2 \frac{[(12)^2 - (13)(23)]^2}{(12)^3 \langle 13 \rangle} l_\mu(23) l_\mu(13) - \frac{(23) [(12) + 2(23)]}{(12)^2} (l_\mu(23) - l_\mu(13)) \\
&\quad - \frac{(23)}{\langle 12 \rangle} + \pi^2 \left[ -\frac{(23) [2(12)^2 - (23)(13)]}{(12)^3} (1 - \Theta^2(23) - \Theta^2(13)) + 2\Theta(12) \Theta(23) \right. \\
&\quad \left. + 2 \frac{\Theta(12) \Theta(13) (23)}{\langle 13 \rangle} + 2 \frac{[(12)^2 - (13)(23)]^2}{(12)^3 \langle 13 \rangle} \Theta(23) \Theta(13) \right] \\
&\quad + i\pi \left[ -\frac{4}{\epsilon} \frac{1}{\langle 13 \rangle} (\Theta(12) (12) + \Theta(13) (13) + \Theta(23) (23)) \right. \\
&\quad - 2 \frac{(23)}{\langle 13 \rangle} (\Theta(13) l_\mu(12) + \Theta(12) l_\mu(13)) - 2 (\Theta(12) l_\mu(23) + \Theta(23) l_\mu(12)) \\
&\quad - 2 \frac{(23) [2(12)^2 - (23)(13)]}{(12)^3} (\Theta(23) l_\mu(23) + \Theta(13) l_\mu(13)) \\
&\quad - 2 \frac{[(12)^2 - (13)(23)]^2}{(12)^3 \langle 13 \rangle} (\Theta(13) l_\mu(23) + \Theta(23) l_\mu(13)) \\
&\quad \left. - \frac{(23) [(12) + 2(23)]}{\langle 13 \rangle} [\Theta(23) - \Theta(13)] \right] \Bigg) \\
A_{4;3}(1^-, 2^+, 3^-, 4^+) &= A_{4;3}(1^-, 3^-, 2^+, 4^+) \\
A_{4;2}(1^-, 2^-, 3^+, 4^+) &= -\frac{1}{2} A_{4;3}(1^-, 2^-, 3^+, 4^+) \\
A_{4;2}(1^-, 2^+, 3^-, 4^+) &= A_{4;2}(1^-, 3^-, 2^+, 4^+) - \frac{1}{2} A_{4;3}(1^-, 2^+, 3^-, 4^+).
\end{aligned} \tag{11.12}$$

## 12. Infrared Structure

The answers obtained for the  $(--++)$  and  $(-+-+)$  partial amplitudes in the previous section are quite similar for both the FDH and the 't Hooft-Veltman dimensional regularization

(HV) schemes; indeed the pole pieces are identical, and the expressions differ only in the finite pieces. It is possible to understand this similarity by examining the corresponding tree amplitudes with one unobserved gluon, and that is our purpose here.

The infrared singularities appearing in the one-loop amplitude must cancel when eventually forming a physical quantity, for example, a next-to-leading differential cross-section for *observed* partons. Such a quantity has three different contributions, each of which is separately infrared divergent: the one-loop corrected differential cross-section for the  $2 \rightarrow 2$  process; the differential cross-section for the  $2 \rightarrow 3$  process, with one final-state gluon taken to be soft, collinear with another final-state gluon, collinear with an initial-state gluon, or collinear and soft; and the one-loop corrected parton distribution function. When combined, the infrared divergences cancel (and one is left with explicit logarithmic dependence on the minimum energy and minimum angle ‘experimental’ cuts). It should be noted that in QCD, unlike QED, in general the final-state infrared singularities do not separate cleanly from initial-state radiation singularities.

The one-loop corrected parton distribution function has only a single pole in  $\epsilon$  (proportional to the Altarelli-Parisi splitting function), and so the pole structure is of course the same for both the FDH and HV regularizations. Now, the leading  $(1/\epsilon^2)$  singularity is universal, and thus scheme-independent; but why is the subleading pole the same?

The identity may be understood by examining the helicity amplitudes for the  $2 \rightarrow 3$  process. The incoming gluons, and two of the three final-state gluons, are observed and are thus kept in four dimensions in either scheme. The lone unobserved gluon must be treated in  $4 - \epsilon$  dimensions; for the FDH scheme, there are still only two helicities to worry about, but for the HV scheme, there is an additional ‘ $[\epsilon]$ ’ helicity [21] to take into account. However, because there is only *one* unobserved gluon, the  $\epsilon$  helicity amplitude vanishes in the spinor helicity basis, and the differential  $2 \rightarrow 3$  cross-section is in fact the same for both schemes. Thus all pole contributions other than the one-loop correction are the same for both schemes; in order that the infrared divergences cancel in both schemes, the one-loop corrections must have identical pole structure — and they do.

The above arguments are in fact not specific to the  $2 \rightarrow 2$  process; one expects the pole structure for the FDH and HV schemes to be identical at next-to-leading order for any  $2 \rightarrow n$  gluon process.

In the CDR regularization scheme used by Ellis and Sexton, the answers for these helicities are of course the same as in the HV scheme. The two schemes differ in their treatment of the external or *observed* partons, and this difference shows up in the appearance of additional  $\epsilon$  helicity amplitudes, which we discuss in the next section.

### 13. Comparison to Previous Results

It is clear that in the conventional dimensional regularization scheme, one must compute additional helicity amplitudes beyond those needed for the 't Hooft-Veltman or FDH dimensional regulators, and thus additional work is involved. What is less obvious is that these additional helicity amplitudes are typically much more time-consuming to compute, because they necessarily involve more of the  $\epsilon \cdot \epsilon$  terms, which tend to be more complicated than other terms (even though one only needs these terms through  $\mathcal{O}(\epsilon)$  rather than  $\mathcal{O}(\epsilon^0)$ ).

The use of either the HV or FDH schemes is thus clearly preferable for practical calculations. Indeed, in the particular case of the four-point function, the extra work needed for the  $\epsilon$  helicities is even more pointless than the above discussion might indicate, because after helicity-averaging, the answers obtained for the CDR and HV schemes are in fact identical thru  $\mathcal{O}(\epsilon^0)$ .

The tree-level four-point functions computed in the two schemes provide a foretaste of this phenomenon. Using, for example, the helicity amplitudes given in section 11 (or from elsewhere in the literature [20]), and the tree-level  $[\epsilon]$ -helicity partial amplitudes,

$$\begin{aligned}
 A_4(1^{[\epsilon]}, 2^{[\epsilon]}, 3^+, 4^+) &= A_4(1^{[\epsilon]}, 2^+, 3^{[\epsilon]}, 4^+) = 0 \\
 A_4(\text{odd number of } [\epsilon]) &= 0 \\
 A_4(1^{[\epsilon]}, 2^{[\epsilon]}, 3^-, 4^+) &= -i\delta_{(-\epsilon)}^{12} \frac{\langle 13 \rangle^2 \langle 23 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \\
 A_4(1^{[\epsilon]}, 2^-, 3^{[\epsilon]}, 4^+) &= -i\delta_{(-\epsilon)}^{13} \frac{\langle 12 \rangle^2 \langle 32 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \\
 A_4(1^{[\epsilon]}, 2^{[\epsilon]}, 3^{[\epsilon]}, 4^{[\epsilon]}) &= i\delta_{(-\epsilon)}^{12} \delta_{(-\epsilon)}^{34} \frac{\langle 13 \rangle \langle 24 \rangle \langle 14 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} - i\delta_{(-\epsilon)}^{13} \delta_{(-\epsilon)}^{24} \frac{\langle 12 \rangle \langle 34 \rangle \langle 14 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \\
 &\quad + i\delta_{(-\epsilon)}^{14} \delta_{(-\epsilon)}^{23} \frac{\langle 12 \rangle \langle 34 \rangle \langle 13 \rangle \langle 24 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle},
 \end{aligned} \tag{13.1}$$

one finds that

$$\begin{aligned}
 \sum_{\substack{\text{colors} \\ \text{helicities}}} |\mathcal{A}_4^{\text{tree}}|_{\text{HV}}^2 &= 4N_c^2(N_c^2 - 1)g^4 \frac{(s^4 + t^4 + u^4)(s^2 + t^2 + u^2)}{s^2 t^2 u^2} \\
 \sum_{\substack{\text{colors} \\ \text{helicities}}} |\mathcal{A}_4^{\text{tree}}|_{\text{CDR}}^2 &= 4N_c^2(N_c^2 - 1)g^4 (1 - \epsilon/2)^2 \frac{(s^4 + t^4 + u^4)(s^2 + t^2 + u^2)}{s^2 t^2 u^2}
 \end{aligned} \tag{13.2}$$

so that if we average over incoming helicities — 2 for each incoming gluon in the HV scheme,  $2(1 - \epsilon/2)$  in the CDR scheme, we find the same expression.

What about the next-to-leading corrections to the four-point matrix element? The difference in the pole terms, if any, can be understood by comparing the singular regions of the  $2 \rightarrow 3$  matrix

element. The two schemes lead to the following results for the five-point matrix element (the CDR result is from ref. [14]),

$$\sum_{\substack{\text{colors} \\ \text{helicities}}} |A_5^{\text{tree}}|^2 = N_c^3(N_c^2 - 1)g^6 \sum_{\sigma \in S_5/Z_5} \sum_{\text{helicities}} |A_5^{\text{tree}}(\sigma)|^2 \quad (13.3)$$

where

$$\begin{aligned} \sum_{\text{helicities}} |A_5^{\text{tree}}|_{\text{HV}}^2 &= \frac{2}{(12)(23)(34)(45)(51)} \sum_{\substack{1 \leq j_1 \leq 5 \\ j_1 < j_2 \leq 5}} (j_1 j_2)^4 + \dots \\ \sum_{\text{helicities}} |A_5^{\text{tree}}|_{\text{CDR}}^2 &= \frac{2}{(12)(23)(34)(45)(51)} \\ &\times \left[ (1 - \epsilon/2)^2 \sum_{\substack{1 \leq j_1 \leq 5 \\ j_1 < j_2 \leq 5}} (j_1 j_2)^4 + 3\epsilon(3 + \epsilon/2) \sum_{z \in Z_5} (f_1(z) + f_2(z) + f_3(z)) \right] \end{aligned} \quad (13.4)$$

in which

$$\begin{aligned} f_1(1, 2, 3, 4, 5) &= -(12)^2(23)^2 \\ f_2(1, 2, 3, 4, 5) &= 2(12)(23)^2(34) \\ f_3(1, 2, 3, 4, 5) &= -2(12)(23)(34)(45), \end{aligned} \quad (13.5)$$

and where the dots represent terms that make only a finite contribution to cross sections integrated over unobserved particles. Thus after helicity averaging, the difference between the two schemes is given by

$$\begin{aligned} &\left[ \frac{1}{(2 - \epsilon)^2} \sum_{\text{helicities}} |A_5^{\text{tree}}|^2 \right]_{\text{CDR}} - \left[ \frac{1}{4} \sum_{\text{helicities}} |A_5^{\text{tree}}|^2 \right]_{\text{HV}} = \\ &\frac{3\epsilon(6 + \epsilon)}{(2 - \epsilon)^2 (12)(23)(34)(45)(51)} \left( \sum_{z \in Z_5} f_1(z) + f_2(z) + f_3(z) \right) + \text{finite} \end{aligned} \quad (13.6)$$

Because of the explicit power of  $\epsilon$  in front, the only differences in pole contributions must come from those regions of phase space which produce a double pole in  $\epsilon$ , that is in which one of the final-state momenta must be soft (and in addition collinear to another gluon). Consider, for example, the case when particle 5 becomes soft. The phase space measure behaves like

$$E_5^{1-\epsilon} dE_5 \quad (13.7)$$

while the coefficient in front of the parentheses in equation (13.6) behaves like  $1/E_5^2$ , so that the only singular part of the integral comes from setting  $E_5$  (and therefore  $k_5$ ) to zero inside the parentheses.

This yields

$$\frac{3\epsilon(6 + \epsilon)}{(2 - \epsilon)^2 (12)(23)(34)(45)(51)} \left( -(12)^2(23)^2 - (23)^2(34)^2 + 2(12)(23)^2(34) \right) = 0 \quad (13.8)$$

so that there are no differences between the pole contributions of the integrations over unobserved real particles in the  $2 \rightarrow 3$  matrix elements in the two schemes, and there should thus be *no* difference in the pole pieces of the four-point matrix element in the two schemes.

This is indeed what one finds. Explicit computation gives (we have dropped the  $\mathcal{O}(\epsilon^0)$  terms since they are not needed)

$$\begin{aligned}
\text{Disp } A_{4;1}(1^{[\epsilon]}, 2^{[\epsilon]}, 3^-, 4^+) &= -i\delta_{(-\epsilon)}^{12} \frac{\langle 13 \rangle^2 \langle 23 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{\Gamma^2(1 - \epsilon/2)\Gamma(1 + \epsilon/2)}{8\pi^2\Gamma(1 - \epsilon)} \left( \frac{4\pi\mu^2}{Q^2} \right)^{\epsilon/2} \\
&\times \left( -\frac{8}{\epsilon^2} - \frac{22}{3\epsilon} + \frac{2}{\epsilon} (l_Q(s) + l_Q(t)) \right) \\
\text{Disp } A_{4;1}(1^{[\epsilon]}, 2^-, 3^{[\epsilon]}, 4^+) &= -i\delta_{(-\epsilon)}^{13} \frac{\langle 12 \rangle^2 \langle 32 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{\Gamma^2(1 - \epsilon/2)\Gamma(1 + \epsilon/2)}{8\pi^2\Gamma(1 - \epsilon)} \left( \frac{4\pi\mu^2}{Q^2} \right)^{\epsilon/2} \\
&\times \left( -\frac{8}{\epsilon^2} - \frac{22}{3\epsilon} + \frac{2}{\epsilon} (l_Q(s) + l_Q(t)) \right) \\
\text{Disp } A_{4;1}(1^{[\epsilon]}, 2^{[\epsilon]}, 3^{[\epsilon]}, 4^{[\epsilon]}) &= \\
&\left( i\delta_{(-\epsilon)}^{12}\delta_{(-\epsilon)}^{34} \frac{\langle 13 \rangle \langle 24 \rangle \langle 14 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} - i\delta_{(-\epsilon)}^{13}\delta_{(-\epsilon)}^{24} \frac{\langle 12 \rangle \langle 34 \rangle \langle 14 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} + i\delta_{(-\epsilon)}^{14}\delta_{(-\epsilon)}^{23} \frac{\langle 12 \rangle \langle 34 \rangle \langle 13 \rangle \langle 24 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \right) \\
&\times \frac{\Gamma^2(1 - \epsilon/2)\Gamma(1 + \epsilon/2)}{8\pi^2\Gamma(1 - \epsilon)} \left( \frac{4\pi\mu^2}{Q^2} \right)^{\epsilon/2} \left( -\frac{8}{\epsilon^2} - \frac{22}{3\epsilon} + \frac{2}{\epsilon} (l_Q(s) + l_Q(t)) \right)
\end{aligned} \tag{13.9}$$

for the  $[\epsilon]$ -helicity partial amplitudes, so that for all helicity configurations of the  $A_{4;1}$ , one can write for the pole pieces,

$$A_{4;1}^{\text{singular}}(1, 2, 3, 4) = \frac{\Gamma^2(1 - \epsilon/2)\Gamma(1 + \epsilon/2)}{8\pi^2\Gamma(1 - \epsilon)} \left( \frac{4\pi\mu^2}{Q^2} \right)^{\epsilon/2} \left( -\frac{8}{\epsilon^2} - \frac{22}{3\epsilon} + \frac{2}{\epsilon} (l_Q(s) + l_Q(t)) \right) \times A_4^{\text{tree}}(1, 2, 3, 4) \tag{13.10}$$

After helicity averaging, the pole pieces are indeed identical for the HV and CDR schemes. Moreover, the finite pieces are identical up to order  $\mathcal{O}(\epsilon)$ , since that is the only difference which can be introduced by the helicity averaging.

If we now compute the next-to-leading correction to the differential cross-section in the CDR scheme,

$$\begin{aligned}
\sum_{\lambda_i=(+,-,[\epsilon])} \sum_{\text{colors}} [A_4^* \mathcal{A}_4]_{\text{NLO}} &= \\
2g^6 N_c^3 (N_c^2 - 1) \text{Re} \sum_{\sigma \in S_4/Z_4} A_4^{\text{tree}*}(\sigma(1)^{\lambda_{\sigma(1)}}, \sigma(2)^{\lambda_{\sigma(2)}}, \sigma(3)^{\lambda_{\sigma(3)}}, \sigma(4)^{\lambda_{\sigma(4)}}) &\tag{13.11} \\
\times A_{4;1}(\sigma(1)^{\lambda_{\sigma(1)}}, \sigma(2)^{\lambda_{\sigma(2)}}, \sigma(3)^{\lambda_{\sigma(3)}}, \sigma(4)^{\lambda_{\sigma(4)}}), &
\end{aligned}$$

and add in the leading-order contributions, we find complete agreement with the result  $d(s, t, u)$  of Ellis and Sexton, equations (2.25–2.26) of ref. [14] (note that those authors compute the answer in  $4 - 2\epsilon$  rather than  $4 - \epsilon$  dimensions).

## 14. Consistency Checks

There are a variety of checks that can be applied to the calculation pursued in this paper. One of the more important is the check on the gauge invariance of the partial amplitudes. There are two ways to perform such a check. One may choose a different set of reference momenta for a partial amplitude with given physical helicities, and verify that one obtains the same answer as with the original set of reference momenta (we have performed checks of this sort). One may also calculate a longitudinal amplitude, replacing one of the polarization vectors in equation (5.7) with the corresponding momentum, then picking physical helicities and reference momenta for the remaining legs, and performing the calculation in the standard way. In the case of the four-point amplitude, there are three independent longitudinal forms one may calculate:  $A_{4;1}(1^{[L]}, 2^+, 3^+, 4^+)$ ,  $A_{4;1}(1^{[L]}, 2^-, 3^+, 4^+)$ , and  $A_{4;1}(1^{[L]}, 2^+, 3^-, 4^+)$ , where ‘[L]’ denotes a longitudinal external leg. (The other configurations of helicities are related by the various symmetries of the partial amplitude, and the longitudinal values of the other partial amplitudes are related by the decoupling equations.) It turns out that the first and last of this trio are sensitive to the prescription  $(p^2)^\epsilon|_{p^2=0} \rightarrow 0$  for the case where the loop is isolated on an external leg, and thus can be used to verify it. (The same prescription is used in field theory calculations; to our knowledge, no-one has shown explicitly that it is consistent and gauge invariant.) We shall discuss this point in greater detail elsewhere [17]. With an appropriate choice of reference momenta —  $k_3$  for leg 2, and  $k_2$  for legs 3 and 4 — the remaining longitudinal partial amplitude is independent of this prescription, and thus serves as an independent check on gauge invariance.

If one employs the spinor-helicity basis and performs the Grassmann integrations in the right-mover contributions (5.10) to the longitudinal partial amplitude  $A_{4;1}(1^{[L]}, 2^-, 3^+, 4^+)$ , one find an expression proportional to

$$\dot{G}_B(\bar{\nu}_{14}) \left( G_F \begin{bmatrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{bmatrix} (\bar{\nu}_{34})^2 - \dot{G}_B^2(\bar{\nu}_{34}) \right) \left( s\dot{G}_B(\bar{\nu}_{12}) + u\dot{G}_B(\bar{\nu}_{13}) + t\dot{G}_B(\bar{\nu}_{14}) \right) \quad (14.1)$$

which is a total derivative with respect to  $\bar{\nu}_1$  after multiplication by the common factors (5.11), and thus vanishes identically in the string theory. Taking the infinite-tension limit of this object, using the sorts of expansions derived in section 8 (along with a prescription [36,18] for the contributions with the loop isolated on an external leg), one should still obtain zero, of course; this is effectively a check on the algebra yielding the final answer from the kinematic tensor expression (9.1). In this particular case, the contribution from the three-points pinch is in fact identically zero, independent of the prescription on  $(p^2)^\epsilon|_{p^2=0}$ ; indeed, the only non-vanishing contributions



come from the diagram with no pinches, fig. 8a,

$$\begin{aligned} \frac{C}{st} & \left( \frac{3}{\epsilon^2} (t-s) + \frac{1}{6\epsilon} (9s \ln |t/\mu^2| - 9t \ln |s/\mu^2| + 10t - 12s + 3\delta_R(t-s)) \right. \\ & \left. + \frac{1}{72} (-s \ln |t/\mu^2| (27 \ln |t/\mu^2| - 72 - 18\delta_R) + t \ln |s/\mu^2| (27 \ln |s/\mu^2| - 72 - 18\delta_R) \right. \\ & \left. - 27\pi^2 (\Theta^2(s)t - \Theta^2(t)s) - 96s + 40t + \delta_R(84t - 72s)) \right) \end{aligned} \quad (14.2)$$

the  $1 \leftrightarrow 2$  pinch, diagram 8b,

$$\begin{aligned} \frac{C}{st} & \left( -\frac{3}{\epsilon^2} t + \frac{1}{6\epsilon} (9t \ln |s/\mu^2| - 10t - 3\delta_R t) \right. \\ & \left. + \frac{1}{72} (-27t \ln |s/\mu^2| (\ln |s/\mu^2| - 60 - 18\delta_R) + 27\pi^2 \Theta^2(s)t - 40t - 84\delta_R t) \right) \end{aligned} \quad (14.3)$$

and the  $1 \leftrightarrow 4$  pinch, diagram 8c,

$$\begin{aligned} \frac{C}{st} & \left( \frac{3}{\epsilon^2} s + \frac{1}{6\epsilon} (-9s \ln |t/\mu^2| + 12s + 3\delta_R s) \right. \\ & \left. + \frac{1}{72} (27s \ln |t/\mu^2| (\ln |t/\mu^2| - 72 - 18\delta_R) - 27\pi^2 \Theta^2(t)s + 96s + 72\delta_R s) \right) \end{aligned} \quad (14.4)$$

where  $C$  is a normalization factor (which includes spinor products), and where we have retained only the dispersive parts (the vanishing of the absorptive parts would follow trivially from the vanishing of the dispersive parts). The reader may verify that the sum of the contributions in equations (14.2–14.4) vanishes, as it should.

In addition to such checks on gauge invariance, one may also check various symmetry properties of the partial amplitudes and the decoupling equations they must satisfy; we have given examples of these checks in previous sections.

Another check is on the cancellation of infrared and collinear divergences with those arising from singular regions of phase space of an appropriate  $2 \rightarrow 3$  process. We will not discuss this cancellation in detail, but it can be done most conveniently using the universal soft and collinear functions of Giele and Glover [16], and the poles do indeed cancel as expected\*.

A final consistency check on the answers comes from unitarity, that is, the optical theorem, and it is to this that we turn next.

## 15. Optical Theorem

In its most commonly-used form, the optical theorem relates the imaginary part of a forward amplitude to the integral over all phase space of the amplitude squared. To leading order, this

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\* The appropriate  $2 \rightarrow 3$  process for strict cancellation is in fact an unphysical, crossed process, with all legs treated as final-state particles. What should happen in a physical process, such as that arising in the scattering of gluonic partons within two incoming hadrons, is the cancellation of soft and final-state collinear divergences, and the factorization of remaining collinear divergences into (universal) gluon distribution functions. J. Guillet has informed us that he has verified this factorization for our four-point helicity amplitudes in the CDR scheme.

amounts to relating the imaginary part of a loop amplitude to the integral of a tree amplitude squared. We will in fact check a more general form of the optical theorem, which relates the imaginary part of an amplitude to the interference of amplitudes with different initial states (but common final state).

In the calculation at hand, there are several subtleties. The simplest is the fact that the spinor helicity basis introduces additional phases (associated with the external legs) into the amplitude; as a result, we must replace ‘real’ and ‘imaginary’ with ‘dispersive’ and ‘absorptive’, respectively, as distinguished by their origin in cuts of logarithms.

A more serious subtlety has to do with infrared regularization. In perturbation theory, a massless gauge theory leads to infinite-range forces, and thus to a divergent total cross section. Both sides of the optical theorem relation must therefore be calculated using an infrared regulator, for which we shall again pick dimensional regularization.

The optical theorem tells us that

$$2 \text{Absp} \frac{1}{i} \mathcal{A}_4(1, 2, \hat{1}, \hat{2}) = \frac{1}{2} \sum_{\substack{\text{colors}(3,4) \\ \text{helicities}(3,4)}} \int d^{4-\epsilon} \text{LIPS}(3, 4) \mathcal{A}_4^*(\hat{1}, \hat{2}, 3, 4) \mathcal{A}_4(1, 2, 3, 4). \quad (15.1)$$

where  $d^{4-\epsilon} \text{LIPS}(3, 4)$  denotes the  $(4 - \epsilon)$ -dimensional Lorentz-invariant phase space of particles 3 and 4, excluding the symmetry factor for identical particles which we have specified explicitly. (The factor of  $1/i$  on the left-hand side divides out the additional relative phase between an amplitude and the usual convention employed in Feynman diagrams.)

We shall now proceed to show that the results for the partial amplitude  $A_{4;1}$  satisfies this constraint.

Extracting the  $\mathcal{O}(g^4)$  coefficient of  $\text{Gr}_{4;1}(1, 2, \hat{1}, \hat{2}) = N_c \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4})$  from the right-hand side of equation (15.1), and dividing by  $g^4 (\mu^2)^{\epsilon/2}$ , we find that it is

$$\begin{aligned} I_{4;1}(1^{\lambda_1}, 2^{\lambda_2}, \hat{1}^{\lambda_1}, \hat{2}^{\lambda_2}) = \\ \frac{1}{2} (\mu^2)^{\epsilon/2} \sum_{\text{helicities}(3,4)} \int d^{4-\epsilon} \text{LIPS}(3, 4) \left[ A_4^{\text{tree}*}(\hat{2}^{-\lambda_2}, \hat{1}^{-\lambda_1}, 3, 4) A_4^{\text{tree}}(1^{\lambda_1}, 2^{\lambda_2}, 3, 4) \right. \\ \left. + A_4^{\text{tree}*}(\hat{1}^{-\lambda_1}, \hat{2}^{-\lambda_2}, 3, 4) A_4^{\text{tree}}(2^{\lambda_2}, 1^{\lambda_1}, 3, 4) \right] \end{aligned} \quad (15.2)$$

where  $A_4^{\text{tree}*}(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4})$  means  $[A_4^{\text{tree}}(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4})]^*$ . Using the symmetry properties of  $A_4^{\text{tree}}$  and relabelling integration variables, we can rewrite

$$\begin{aligned} I_{4;1}(1^{\lambda_1}, 2^{\lambda_2}, \hat{1}^{\lambda_1}, \hat{2}^{\lambda_2}) = \\ (\mu^2)^{\epsilon/2} \sum_{\lambda_3, \lambda_4=(+,-)} \int d^{4-\epsilon} \text{LIPS}(3, 4) A_4^{\text{tree}*}(\hat{2}^{\lambda_2}, \hat{1}^{\lambda_1}, 3^{\lambda_3}, 4^{\lambda_4}) A_4^{\text{tree}}(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}) \end{aligned} \quad (15.3)$$

$$\begin{aligned}
I_{4;1}(1^+, 2^+, \hat{1}^+, \hat{2}^+) &= (\mu^2)^{\epsilon/2} \int d^{4-\epsilon} \text{LIPS}(3, 4) \left[ A_4^{\text{tree}*}(\hat{2}^-, \hat{1}^-, 3^+, 4^+) A_4^{\text{tree}}(1^+, 2^+, 3^+, 4^+) \right. \\
&\quad + A_4^{\text{tree}*}(\hat{2}^-, \hat{1}^-, 3^-, 4^+) A_4^{\text{tree}}(1^+, 2^+, 3^-, 4^+) \\
&\quad + A_4^{\text{tree}*}(\hat{2}^-, \hat{1}^-, 3^+, 4^-) A_4^{\text{tree}}(1^+, 2^+, 3^+, 4^-) \\
&\quad \left. + A_4^{\text{tree}*}(\hat{2}^-, \hat{1}^-, 3^-, 4^-) A_4^{\text{tree}}(1^+, 2^+, 3^-, 4^-) \right] \\
&= 0
\end{aligned} \tag{15.4}$$

Similarly, we find  $I_{4;1}(1^-, 2^+, 3^+, 4^+) = 0$ . The results in section 10 show that the absorptive parts of the corresponding partial amplitudes also vanish, satisfying the optical theorem (15.1) to the required order in perturbation theory.

For the remaining partial amplitudes, the relation is non-trivial. Let us begin with the partial amplitudes in the FDH scheme. We have

$$\begin{aligned}
I_{4;1}(1^-, 2^-, \hat{1}^+, \hat{2}^+) &= (\mu^2)^{\epsilon/2} \int d^{4-\epsilon} \text{LIPS}(3, 4) A_4^{\text{tree}*}(\hat{2}^-, \hat{1}^-, 3^+, 4^+) A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+) \\
&= (\mu^2)^{\epsilon/2} \int d^{4-\epsilon} \text{LIPS}(3, 4) \frac{\langle 12 \rangle^4 [\hat{2}\hat{1}]^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle [\hat{2}4] [43] [3\hat{1}] [\hat{1}\hat{2}]} \\
I_{4;1}(1^-, 2^+, \hat{1}^-, \hat{2}^+) &= (\mu^2)^{\epsilon/2} \int d^{4-\epsilon} \text{LIPS}(3, 4) \left[ A_4^{\text{tree}*}(\hat{2}^-, \hat{1}^+, 3^+, 4^-) A_4^{\text{tree}}(1^-, 2^+, 3^+, 4^-) \right. \\
&\quad \left. + A_4^{\text{tree}*}(\hat{2}^-, \hat{1}^+, 3^-, 4^+) A_4^{\text{tree}}(1^-, 2^+, 3^-, 4^+) \right] \\
&= (\mu^2)^{\epsilon/2} \int d^{4-\epsilon} \text{LIPS}(3, 4) \frac{\langle 14 \rangle^4 [\hat{2}4]^4 + \langle 13 \rangle^4 [\hat{2}3]^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle [\hat{2}4] [43] [3\hat{1}] [\hat{1}\hat{2}]}
\end{aligned} \tag{15.5}$$

The integrals are evaluated in appendix VI; with those results,

$$\begin{aligned}
I_{4;1}(1^-, 2^-, \hat{1}^+, \hat{2}^+) &= \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 2\hat{1} \rangle \langle \hat{1}\hat{2} \rangle \langle \hat{2}1 \rangle} \frac{\Gamma^2(1 - \epsilon/2) \Gamma(1 + \epsilon/2)}{4\pi \Gamma(1 - \epsilon)} \left( \frac{4\pi\mu^2}{(12)} \right)^{\epsilon/2} \left( \frac{2}{\epsilon} - \ln \left[ -\frac{(\hat{1}2)}{(12)} \right] \right) + \mathcal{O}(\epsilon) \\
I_{4;1}(1^-, 2^+, \hat{1}^-, \hat{2}^+) &= \frac{\langle 1\hat{1} \rangle^4}{\langle 12 \rangle \langle 2\hat{1} \rangle \langle \hat{1}\hat{2} \rangle \langle \hat{2}1 \rangle} \frac{\Gamma^2(1 - \epsilon/2) \Gamma(1 + \epsilon/2)}{4\pi \Gamma(1 - \epsilon)} \left( \frac{4\pi\mu^2}{(12)} \right)^{\epsilon/2} \\
&\quad \times \left( -\frac{2}{\epsilon} + \frac{[(1\hat{1})^2 - (12)(2\hat{1})]^2}{(1\hat{1})^4} \ln \left[ -\frac{(\hat{1}2)}{(12)} \right] \right. \\
&\quad \left. - \frac{(2\hat{1}) [11(1\hat{1})^2 - 3(12)(2\hat{1}) + 3(12)^2]^2}{3(1\hat{1})^3} \right) + \mathcal{O}(\epsilon)
\end{aligned} \tag{15.6}$$

From equation (11.11), we find the following expressions for the  $s$ -channel absorptive parts of the one-loop partial amplitudes (that is, the coefficient of  $\Theta(12)$  in the absorptive part),

$$\begin{aligned}
\text{Absp}_s \frac{1}{2} A_{4;1}(1^-, 2^-, 3^+, 4^+) &= \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{\Gamma^2(1 - \epsilon/2) \Gamma(1 + \epsilon/2)}{8\pi\Gamma(1 - \epsilon)} \left( \frac{4\pi\mu^2}{(12)} \right)^{\epsilon/2} \\
&\quad \times \left( \frac{2}{\epsilon} - \ln \left| \frac{\langle 23 \rangle}{\langle 12 \rangle} \right| \right) \\
\text{Absp}_s \frac{1}{2} A_{4;1}(1^-, 2^+, 3^-, 4^+) &= \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \frac{\Gamma^2(1 - \epsilon/2) \Gamma(1 + \epsilon/2)}{8\pi\Gamma(1 - \epsilon)} \left( \frac{4\pi\mu^2}{(12)} \right)^{\epsilon/2} \\
&\quad \times \left( \frac{2}{\epsilon} - \frac{[(13)^2 - (12)(23)]^2}{(13)^4} \ln \left| \frac{\langle 23 \rangle}{\langle 12 \rangle} \right| \right. \\
&\quad \left. + \frac{(23) [11(13)^2 - 3(12)(23) + 3(12)^2]}{6(13)^3} \right)
\end{aligned} \tag{15.7}$$

The reader may substitute these various expressions into equation (15.1), and thereby verify that it is indeed satisfied.

This completes the check of the optical theorem for the FDH regularization scheme. The reader will have noted that the absorptive parts are identical in both the FDH and HV schemes; this can be understood as follows. In the HV scheme, we must also sum over internal ‘ $[\epsilon]$ ’ helicities, so there are additional contributions to  $I_{4;1}$ ,

$$\begin{aligned}
\delta I_{4;1}^{\text{HV}}(1^{\lambda_1}, 2^{\lambda_2}, \hat{1}^{\lambda_1}, \hat{2}^{\lambda_2}) &= \\
(\mu^2)^{\epsilon/2} \sum_{(\lambda_3, \lambda_4) = \left\{ \begin{array}{l} ((\epsilon), +), ((\epsilon), -) \\ (+, [\epsilon]), (-, [\epsilon]) \\ ((\epsilon), [\epsilon]) \end{array} \right\}} \int d^{4-\epsilon} \text{LIPS}(3, 4) A_4^{\text{tree}*}(\hat{2}^{\lambda_2}, \hat{1}^{\lambda_1}, 3^{\lambda_3}, 4^{\lambda_4}) A_4^{\text{tree}}(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}) \\
&= (\mu^2)^{\epsilon/2} \int d^{4-\epsilon} \text{LIPS}(3, 4) A_4^{\text{tree}*}(\hat{2}^{\lambda_2}, \hat{1}^{\lambda_1}, 3^{[\epsilon]}, 4^{[\epsilon]}) A_4^{\text{tree}}(1^{\lambda_1}, 2^{\lambda_2}, 3^{[\epsilon]}, 4^{[\epsilon]})
\end{aligned} \tag{15.8}$$

where the other terms drop out because partial amplitudes with only one ‘ $[\epsilon]$ ’ helicity vanish. But

$$\begin{aligned}
\delta I_{4;1}^{\text{HV}}(1^-, 2^-, \hat{1}^+, \hat{2}^+) &= (\mu^2)^{\epsilon/2} \int d^{4-\epsilon} \text{LIPS}(3, 4) A_4^{\text{tree}*}(\hat{2}^+, \hat{1}^+, 3^{[\epsilon]}, 4^{[\epsilon]}) A_4^{\text{tree}}(1^-, 2^-, 3^{[\epsilon]}, 4^{[\epsilon]}) \\
&= 0
\end{aligned} \tag{15.9}$$

because  $A_4^{\text{tree}}(1^-, 2^-, 3^{[\epsilon]}, 4^{[\epsilon]})$  vanishes. Furthermore,

$$\begin{aligned}
\delta I_{4;1}^{\text{HV}}(1^-, 2^+, \hat{1}^-, \hat{2}^+) &= (\mu^2)^{\epsilon/2} \int d^{4-\epsilon} \text{LIPS}(3, 4) A_4^{\text{tree}*}(\hat{2}^+, \hat{1}^-, 3^{[\epsilon]}, 4^{[\epsilon]}) A_4^{\text{tree}}(1^-, 2^+, 3^{[\epsilon]}, 4^{[\epsilon]}) \\
&= -\epsilon (\mu^2)^{\epsilon/2} \int d^{4-\epsilon} \text{LIPS}(3, 4) \frac{\langle 31 \rangle^2 \langle 41 \rangle^2 [3\hat{1}]^2 [4\hat{1}]^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle [\hat{2}4] [43] [3\hat{1}] [\hat{1}\hat{2}]}
\end{aligned} \tag{15.10}$$

As shown in appendix VI, the integral is finite, so that the whole contribution is  $\mathcal{O}(\epsilon)$ , and as a result, both sides of the optical theorem are identical in the HV and FDH schemes.

## 16. Summary

A major difficulty of the conventional Feynman diagram approach to perturbative QCD calculations is the proliferation of terms as the number of loops or external legs increases. For example, at the starting point of a conventional Feynman diagram computation of the one-loop four-point amplitude, one would be faced with over ten thousand terms.

In this paper we have presented a new approach for computing the pure gauge contributions to the one-loop helicity amplitude; these contributions are by far the most difficult to compute by conventional techniques because of the complexity of the non-abelian vertices. Our approach makes use of the reorganization of the amplitude inherent in string theory. This string-based approach bypasses much of the algebra associated with Feynman diagram computations. Although string theory lies behind the derivations of the new approach, a practical set of rules, such as that presented in section 9, makes no reference to string theory.

The string-based method meshes naturally with the use of the spinor helicity formalism. The loop momentum has already been integrated out at the starting point of the string-based computation, so that all invariants are already expressed solely in terms of the external momenta and polarizations. The spinor helicity basis makes possible vast simplifications at this very first point of the computation. Another important ingredient is the color decomposition of the amplitude into smaller gauge-invariant partial amplitudes, eliminating many of the large cancellations inherent in Feynman diagram computations. The string also provides a systematic and compact expression for the  $n$ -point amplitude, arising from the fact that at each order in perturbation theory there is only a single string diagram.

We have constructed string versions of ordinary field-theory dimensional regulator schemes to handle the usual infrared divergences that appear in gluon amplitudes.

We also presented the first computation of the one-loop four-point helicity amplitudes as well as a variety of checks verifying the correctness of this calculation in the string-based method.

The extension of this method to include virtual fermions is trivial; the extension to include external massless fermions is straightforward. We believe that it should be possible to extend this new computational technique to the multi-loop case, and to the case of massive fermions as well.

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## Appendix I. String Model

In practical computations for QCD it is not important to have a fully consistent model, although it is important to have an explicit example of a consistent model with the required properties in order to ensure that no extraneous difficulties enter. In this appendix we provide an explicit example of a heterotic string model containing an  $SU(9)$  pure gauge theory in the infinite string tension limit. There is no particular significance to nine colors; it just happens to be an easy model to construct and analyze. It is not difficult to construct such models with other gauge groups, including  $SU(3)$ . The model we present is *not* space-time supersymmetric since it contains a pure nonabelian gauge theory in the field theory limit. Once the field theory limit of amplitude has been computed in one particular model the extension to all  $SU(N_c)$  amplitudes is straightforward since the amplitude satisfies a systematic color decomposition. To construct an appropriate four-dimensional string model, we follow the fermionic formulation of Kawai, Lewellen, and Tye (KLT) [29].

The four-dimensional model at hand is specified by the five “basis” vectors,

$$\begin{aligned}
W_0 &= \left( \frac{1}{2}{}^{22} \left| \frac{1}{2}{}^{10} \right. \right) \\
W_1 &= \left( \frac{1}{3}{}^{18} \quad 0^4 \left| 0^{10} \right. \right) \\
W_2 &= \left( 0^9 \frac{1}{2}{}^9 \frac{1}{2}{}^{000} \left| \frac{1}{2} \left( \frac{1}{2} \frac{1}{2} \frac{1}{2} \right) \left( \frac{1}{2}{}^{00} \right) \left( \frac{1}{2}{}^{00} \right) \right. \right) \\
W_3 &= \left( 0^9 \frac{1}{2}{}^9 \quad 0 \frac{1}{2}{}^{00} \left| \frac{1}{2} \left( 0 \frac{1}{2} 0 \right) \left( \frac{1}{2} \frac{1}{2} \frac{1}{2} \right) \left( 0 \frac{1}{2} 0 \right) \right. \right) \\
W_4 &= \left( 0^9 \frac{1}{2}{}^9 \quad 00 \frac{1}{2} 0 \left| \frac{1}{2} \left( 00 \frac{1}{2} \right) \left( 00 \frac{1}{2} \right) \left( \frac{1}{2} \frac{1}{2} \frac{1}{2} \right) \right. \right)
\end{aligned} \tag{I.1}$$

where  $l^n$  signifies  $n$  contiguous components with value  $l$ . The triplet grouping for internal right-movers arises from requiring world-sheet supersymmetry [29]. Each component of a refers to the world-sheet boundary conditions on the torus of a particular complex fermion. For example, the

1/2's refer to antiperiodic (Neveu-Schwarz) boundary conditions while the 0's refer to periodic (Ramond) boundary conditions. The gauge group  $G$  that we will be interested in corresponds to the first nine left-mover oscillators, while the space-time index for vectors is carried by the first right-mover complex fermion. It is straightforward to show that this model satisfies the KLT constraint equations presented in ref [29] and hence is modular invariant.

We do not present a complete analysis here; it is straightforward though tedious to verify the properties of the model. (A more detailed discussion was presented in earlier work [47].) Here we are interested in the spectrum of massless particles, since only these will survive in the infinite-tension limit. Sectors containing massless particles must have both left and right vacuum energies that are zero or negative. There are seventeen such sectors; sixteen of these are easily eliminated, as exciting a gauge oscillator in those sectors would necessarily yield a massive state. These sectors also contain no tachyons. The remaining sector is the Neveu-Schwarz ( $W_0$ ) sector, which is the one containing the graviton and the gauge bosons. The KLT coefficients of various terms in the generalized GSO [43] projection in the Neveu-Schwarz sector are given by  $C_\beta^{1/2} = -\cos 2\pi\beta_1$  where  $\beta_1$  is the boundary condition of the first right-mover. By summing over the terms in the GSO projection states are either removed or kept. In particular, the generalized GSO projection condition on the states in the Neveu-Schwarz ( $W_0$ ) sector is

$$W_i \cdot N_{W_0} = s_i \tag{I.2}$$

where  $N_{W_0}$  is the number of excited oscillators and the statistical factor  $s_i = W_{iR}^1$ , the first right-mover component; any state which does not satisfy the projection condition (I.2) is removed from the spectrum. Since the projection conditions depend on the choice of boundary condition basis vectors we can control the spectrum of the model by choosing an appropriate and consistent set of vectors. In particular, the generalized GSO projections in this model imply that the only massless particles which carry gauge charge of the  $SU(9) \times SU(9)$  subgroup are the gauge bosons themselves. The gauge group of interest  $G$  is specified by the first nine positions in the basis vectors, while the secondary gauge group  $G'$  is specified by the second nine positions. (The above properties hold independent of the choice of consistent KLT 'structure constants'  $k_{ij}$ .)

Thus, this string model yields a tachyon-free pure gauge theory in the infinite-tension limit.

## Appendix II. Decoupling of Secondary Gauge Group

For the sample model of appendix I, the the Neveu-Schwarz ( $W_0$ ) sector of the string model (which is where the gauge bosons of interest live) consists of a direct product gauge group,  $SU(N_c) \times SU(N'_c) \times U(1)^m$  (where  $N_c = N'_c = 9$  for the particular model). The  $\mathcal{O}(\hat{q}^{-1/2})$  left-mover

contributions to the string partition function can therefore be grouped into three types: those associated with the  $SU(N_c)$  gauge group  $G$  of interest those associated with the secondary  $SU(N'_c)$  gauge group  $G'$  and those associated with any remaining string gauge groups.

In this appendix we demonstrate that the secondary gauge group decouples in the field theory limit as claimed. The decoupling of the secondary gauge group is a bit trickier because its decoupling depends on summing over boundary conditions generated by the world-sheet boundary conditions generated by  $W_2, W_3$  and  $W_4$ , which also control the number of space-time states in the string model. Ultimately, these string intricacies are irrelevant in practical QCD computations, since the only relevant fact is the the unwanted secondary gauge group decouples as expected; in a computation the contributions from the unwanted gauge groups are simply dropped because they anyway will not contribute.

Labeling the the world-sheet boundary conditions associated with gauge group of interest  $\beta_G$  and those associated with the secondary gauge group as  $\beta_{G'}$  the  $\mathcal{O}(\hat{q}^{-1/2})$  terms in the partition function (8.8) can be rewritten as

$$-2\hat{q}^{-1/2} e^{\pi i \text{Re } \tau} \sum_{i=1}^{22} \cos 2\pi \beta_{L_i} = -2\hat{q}^{-1/2} e^{\pi i \text{Re } \tau} \left( N_c \cos(2\pi \beta_G) + N'_c \cos(2\pi \beta_{G'}) + \sum_{i=N_c+N'_c+1}^{22} \cos 2\pi \beta_{L_i} \right) \quad (\text{II.1})$$

where we have explicitly separated out the contributions from both the gauge group of interest  $G$  and secondary gauge group  $G'$ .

As discussed in section 8, for the partial amplitudes associated with a single color trace, the non-vanishing contributions in the field theory limit arise from only interferences between the left-mover fermionic Green functions and the left-mover partition function. (The double trace partial amplitudes which receive contributions from only an interference between two 'cycles' (each of which contribute a single power of  $\hat{q}^{1/2}$ ) of left-mover Green functions are independent of the secondary gauge group and therefore are trivially decoupled from the secondary gauge group.)

From equation (8.13) or (8.16) we find that for the single trace partial amplitudes the color factors are of the form

$$\sum C_{\vec{\beta}}^{\vec{\alpha}_0} \left[ N_c (e^{2\pi i \beta_G} + e^{-2\pi i \beta_G}) + N'_c (e^{2\pi i \beta_{G'}} + e^{-2\pi i \beta_{G'}}) + \sum_{i=N_c+N'_c+1}^{22} (e^{2\pi i \beta_{L_i}} + e^{-2\pi i \beta_{L_i}}) \right] \left[ -e^{-2\pi i \beta_G} \right] \quad (\text{II.2})$$

where the terms in the first bracket are from the string partition function and the terms in the second bracket are from the left-mover Green functions.

The sum over the boundary conditions induced by world sheet boundary condition basis vector  $W_1$  removes contributions to the field theory limit from all gauge bosons except from the  $SU(N_c) \times SU(N'_c)$  subgroup, as discussed in section 8; contributions where the phases obtained from the  $W_1$



boundary condition vector do not cancel will vanish. That is, terms proportional to  $e^{\pm 2\pi i \beta_G}$  lead to a vanishing result while terms proportional to unity or  $e^{\pm 2\pi i (\beta_G - \beta_{G'})}$  will not vanish when summed over these boundary conditions. This then leaves

$$\sum_{\vec{\beta}} C_{\vec{\beta}}^{\vec{\alpha}_0} \left( N_c + N'_c e^{2\pi i (\beta_{G'} - \beta_G)} \right) \quad (\text{II.3})$$

as potential contributions left-mover contributions.

We now show that the interference factors  $e^{\pm 2\pi i (\beta_G - \beta_{G'})}$  between the  $G$  and  $G'$  gauge groups will vanish from the sum over the  $W_2, W_3$  and  $W_4$  boundary conditions. To see how this works we must take into account the right-mover phase contributions in (8.27) and multiply them by the factor of  $e^{\pm 2\pi i (\beta_G - \beta_{G'})}$  found in equation (II.3). This means that the sum over boundary conditions will be of the form

$$\sum_{\vec{\beta}} \cos(2\pi \beta_{\tau}) \cos(2\pi \beta_{R_i}) e^{\pm 2\pi i (\beta_G - \beta_{G'})}, \quad i = 1, 2, \dots, 10 \quad (\text{II.4})$$

for the sample model of appendix I. For  $i = 1$  this becomes

$$\frac{1}{48} \sum_{n_i} (-1)^{n_0+n_2+n_3+n_4} (-1)^{n_0+n_2+n_3+n_4} (-1)^{n_2+n_3+n_4} = 0 \quad (\text{II.5})$$

while for, e.g.  $i = 2$  we get

$$\frac{1}{48} \sum_{n_i} (-1)^{n_0+n_2+n_3+n_4} (-1)^{n_0+n_2} (-1)^{n_2+n_3+n_4} = 0 \quad (\text{II.6})$$

It is not difficult to see that all other potential terms vanish since it is not possible to cancel all factors of  $(-1)^{n_i}$ . Thus, we see that the gauge group  $G'$  decouples as expected.

### Appendix III. Notation and Normalizations

We define theta functions for general twisted boundary conditions by

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\nu | \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i (n + \alpha - 1/2)^2 \tau} e^{2\pi i (n + \alpha - 1/2)(\nu - \beta - 1/2)} \quad (\text{III.1})$$

Then

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\nu | \tau) = e^{\pi i \alpha^2 \tau} e^{2\pi i \alpha (\nu - \beta - 1/2)} \vartheta_1(\nu + \alpha \tau - \beta | \tau) \quad (\text{III.2})$$

where  $\vartheta_1 = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  is the conventional first Jacobi theta function.

We remind the reader of the definition of the Dedekind  $\eta$  function,

$$\begin{aligned}\eta(\tau) &= e^{\pi i \tau / 12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) \\ &= \sqrt[3]{\vartheta' \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0 | \tau)} / 2\pi\end{aligned}\tag{III.3}$$

where the prime indicates differentiation with respect to the first argument.

The bosonic partition function is

$$\mathcal{Z}_B(\tau) = (\eta(\tau)\bar{\eta}(\tau))^{-2} (\text{Im } \tau)^{\epsilon/2-1} [F_c(\tau)]^\epsilon\tag{III.4}$$

where  $D = 4 - \epsilon$ ; as discussed in the text, to maintain modular invariance the torroidal compactification function

$$F_c(\tau) = \sum_{n,m \in \mathbb{Z}} e^{-2\pi i n m \text{Re } \tau} e^{-\pi(n^2 + m^2) \text{Im } \tau}$$

should be retained, although in the gauge theory limit of the string it reduces to a factor of unity. [For simplicity we have set the compactification radius to  $\sqrt{\alpha'} = \sqrt{\lambda/\pi}$ ].

We define  $\mathcal{Z}_1 \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\tau)$  to be the partition function for a single left-moving complex fermion with  $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$  boundary conditions,

$$\mathcal{Z}_1 \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\tau) = \text{Tr} \left[ e^{2\pi i \hat{H}_\alpha \tau} e^{2\pi i (\frac{1}{2} - \beta) \hat{N}_\alpha} \right] = \frac{e^{-2\pi i (1/2 - \alpha)(1/2 + \beta)} \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (0 | \tau)}{\eta(\tau)}\tag{III.5}$$

where the phase is present in order to be consistent with the KLT definition [29]. It is really irrelevant, and could be absorbed into the definitions of the summation coefficients  $C_{\bar{\beta}}^{\bar{\alpha}}$ .

Putting the pieces together, the complete partition function for the set of fermions with  $\begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \end{bmatrix}$  boundary conditions is

$$\begin{aligned}\mathcal{Z}_{\bar{\alpha}}^{\bar{\beta}}(\tau) &= \mathcal{Z} \begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \end{bmatrix}(\tau) = \mathcal{Z}_B(\tau) \mathcal{Z}_F \begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \end{bmatrix}(\tau) = \mathcal{Z}_B(\tau) \prod_{i=1}^{\text{len } \alpha_L} \mathcal{Z}_1 \begin{bmatrix} \alpha_{Li} \\ \beta_{Li} \end{bmatrix}(\tau) \prod_{i=1}^{\text{len } \alpha_R} \mathcal{Z}_1 \begin{bmatrix} \alpha_{Ri} \\ \beta_{Ri} \end{bmatrix}(\tau) \\ &= (\eta(\tau)\bar{\eta}(\tau))^{-2} (\sqrt{\text{Im } \tau})^{-2+\epsilon} [F_c(\tau)]^\epsilon \\ &\quad \prod_{i=1}^{\text{len } \alpha_L} \frac{e^{-2\pi i (1/2 - \alpha_{Li})(1/2 + \beta_{Li})} \vartheta \begin{bmatrix} \alpha_{Li} \\ \beta_{Li} \end{bmatrix} (0 | \tau)}{\eta(\tau)} \prod_{i=1}^{\text{len } \alpha_R} \frac{e^{2\pi i (1/2 - \alpha_{Ri})(1/2 + \beta_{Ri})} \bar{\vartheta} \begin{bmatrix} \alpha_{Ri} \\ \beta_{Ri} \end{bmatrix} (0 | \tau)}{\bar{\eta}(\tau)}.\end{aligned}\tag{III.6}$$

The bosonic correlation function,  $G_B(\nu)$ , is defined via

$$\langle X^\mu(\nu_1, \bar{\nu}_1) X^\nu(\nu_2, \bar{\nu}_2) \rangle_\tau = \eta^{\mu\nu} G_B(\nu = \nu_1 - \nu_2).\tag{III.7}$$

It can be expressed in terms of theta functions,

$$G_B(\nu) = -\frac{1}{\pi} \ln \left| 2\pi e^{-\pi(\text{Im } \nu)^2 / \text{Im } \tau} \frac{\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\nu | \tau)}{\vartheta' \begin{bmatrix} 0 \\ 0 \end{bmatrix}(0 | \tau)} \right|. \quad (\text{III.8})$$

A dotted variable, for our purposes, will always be taken to signify differentiation with respect to  $\bar{\nu}$ ,

$$\dot{X} = \partial_{\bar{\nu}} X. \quad (\text{III.9})$$

In a slight abuse of notation, we write the correlation function for right-movers as  $\dot{G}_B(\bar{\nu})$  although in fact it is equal to  $\overline{\dot{G}_B(\nu)}$ .

We thus have

$$\dot{G}_B(\bar{\nu}) = \frac{i \text{Im } \nu}{\text{Im } \tau} - \frac{1}{2\pi} \frac{\vartheta' \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\bar{\nu} | -\bar{\tau})}{\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\bar{\nu} | -\bar{\tau})}. \quad (\text{III.10})$$

The fermionic particle correlation function  $G_F \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\nu)$  and anti-particle correlation function  $\hat{G}_F \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\nu)$  are defined as follows (excluding the case  $\alpha = \beta = 0$ ):

$$\begin{aligned} \langle \Psi^{i\dagger}(\nu_1) \Psi^j(\nu_2) \rangle_{\beta; \tau}^{\alpha} &= \delta^{ij} G_F \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\nu = \nu_1 - \nu_2) \\ \langle \Psi^i(\nu_1) \Psi^{j\dagger}(\nu_2) \rangle_{\beta; \tau}^{\alpha} &= \delta^{ij} \hat{G}_F \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\nu = \nu_1 - \nu_2) \end{aligned} \quad (\text{III.11})$$

where here the expectation value is understood to exclude a factor of the partition function.

These correlation functions can also be expressed in terms of theta functions,

$$\begin{aligned} G_F \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\nu) &= \frac{\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\nu | \tau) \vartheta' \begin{bmatrix} 0 \\ 0 \end{bmatrix}(0 | \tau)}{2\pi \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\nu | \tau) \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(0 | \tau)} \\ \hat{G}_F \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\nu) &= G_F \begin{bmatrix} 1 - \alpha \\ 1 - \beta \end{bmatrix}(\nu) = -G_F \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(-\nu) \\ G_F \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\bar{\nu}) &= \frac{\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\bar{\nu} | -\bar{\tau}) \vartheta' \begin{bmatrix} 0 \\ 0 \end{bmatrix}(0 | -\bar{\tau})}{2\pi \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\bar{\nu} | -\bar{\tau}) \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(0 | -\bar{\tau})} \\ \hat{G}_F \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\bar{\nu}) &= G_F \begin{bmatrix} 1 - \alpha \\ 1 - \beta \end{bmatrix}(\bar{\nu}) = -G_F \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(-\bar{\nu}) \end{aligned} \quad (\text{III.12})$$

where the last equality derives from a theta function identity.

The case  $\alpha = \beta = 0$  demands special treatment, because of the presence of the zero mode; however, in this paper since we are only concerned with the pure gluon contributions, which live in the Neveu-Schwarz or  $W_0$  sector of the string, we do not encounter fermionic zero-modes.

The self-contractions are obtained by taking  $\nu \rightarrow 0$  but with the pole piece subtracted [50] so that

$$S_F \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{\vartheta' \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(0)}{2\pi \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}(0)}. \quad (\text{III.13})$$

Although these self-contraction pieces are important for deriving the  $U(1)$  decoupling equations [11], in practical computations with gluons, the self-contractions drop out trivially because they are associated with the trace over a single  $SU(N_c)$  generator,  $\text{Tr}(T^a) = 0$ .

With the modular transformation properties of the theta functions, one can derive the modular transformation properties of the partition function and the various correlation functions. With these properties the modular invariance of the string amplitude (5.7) can be verified for a consistent string model although we shall not do so here.

The vertex operator for emission of a gauge boson, in the  $F_1$  picture for the right-movers, is

$$V^a(\varepsilon, k; \nu, \bar{\nu}) = -\sqrt{2}g\sqrt{\lambda}T^{a,i,j} : \Psi^{i\dagger}(\nu)\Psi_j(\nu) \varepsilon \cdot \left( \partial_{\bar{\nu}}\bar{X}(\bar{\nu}) + i\sqrt{\lambda}\bar{\Psi}(\bar{\nu}) k \cdot \bar{\Psi}(\bar{\nu}) \right) e^{i\sqrt{\lambda}k \cdot (X(\nu, \bar{\nu}))} : \quad (\text{III.14})$$

or, using Grassmann variables to put it into an exponential form,

$$V^a(\varepsilon, k; \nu, \bar{\nu}) = \sqrt{2}g\sqrt{\lambda}T^{a,i,j} : \int d\theta_1 d\theta_2 d\theta_3 d\theta_4 \exp \left( i\sqrt{\lambda}k \cdot X(\nu, \bar{\nu}) + \theta_1 \Psi^{i\dagger}(\nu) + \theta_2 \Psi_j(\nu) + \theta_3 \theta_4 \varepsilon \cdot \dot{\bar{X}}(\bar{\nu}) + i\sqrt{\lambda}\theta_3 k \cdot \bar{\Psi}(\bar{\nu}) + \theta_4 \varepsilon \cdot \bar{\Psi}(\bar{\nu}) \right) : . \quad (\text{III.15})$$

The  $N$ -point string amplitude in dimensional regularization in  $4 - \epsilon$  dimensions is then given by

$$\mathcal{A}_n = i \frac{(4\pi)^{\epsilon/2}}{2(16\pi^2)} \frac{1}{\lambda^2} \int \frac{d^2\tau}{(\text{Im}\tau)^2} (\text{Im}\tau) \int d^2\nu_1 \cdots \int d^2\nu_{n-1} \mathcal{Z}_B(\tau) \sum_{\vec{\alpha}, \vec{\beta}} C_{\vec{\beta}}^{\vec{\alpha}} \mathcal{Z}_F \left[ \begin{matrix} \vec{\alpha} \\ \vec{\beta} \end{matrix} \right] (\tau) \langle V^{a_1}(\nu_1) \cdots V^{a_n}(\nu_n) \rangle_{\vec{\beta}; \tau} . \quad (\text{III.16})$$

where the different versions of dimensional regularization are determined by the prescriptions used

in  $\mathcal{Z}_F$  as discussed in section 7. Evaluating the correlation functions gives (in Minkowski space)

$$\begin{aligned}
cal A_n &= i \frac{(4\pi)^{\epsilon/2}}{2(16\pi^2)} \lambda^{n/2-2} (\sqrt{2g\mu^{\epsilon/2}})^n T^{a_1}_{m_1} \widehat{m}_1 \dots T^{a_n}_{m_n} \widehat{m}_n \\
&\int \frac{d^2\tau}{(\text{Im}\tau)^{2+\epsilon/2}} \int \left( \prod_{i=1}^n d\theta_{i1} d\theta_{i2} d\theta_{i3} d\theta_{i4} \right) \int \left( \prod_{i=1}^n d^2\nu_i \right) \sum_{\vec{\alpha}, \vec{\beta}} C_{\vec{\beta}}^{\vec{\alpha}} \mathcal{Z}_{\vec{\beta}}^{\vec{\alpha}}(\tau) \\
&\prod_{i=1}^n \exp \left[ -\theta_{i1} \theta_{i2} \delta^{m_i}_{\widehat{m}_i} S_F \left[ \begin{matrix} \alpha_{m_i} \\ \beta_{\widehat{m}_i} \end{matrix} \right] \right] \\
&\prod_{i < j} \exp \left[ \lambda k_i \cdot k_j G_B(\nu_i - \nu_j) \right. \\
&\quad - \theta_{i1} \theta_{j2} \delta^{m_i}_{\widehat{m}_j} G_F \left[ \begin{matrix} \alpha_{m_i} \\ \beta_{\widehat{m}_j} \end{matrix} \right] (\nu_i - \nu_j) - \theta_{i2} \theta_{j1} \delta^{m_j}_{\widehat{m}_i} \hat{G}_F \left[ \begin{matrix} \alpha_{m_j} \\ \beta_{\widehat{m}_i} \end{matrix} \right] (\nu_i - \nu_j) \\
&\quad - \theta_{i3} \theta_{j3} \lambda k_i \cdot k_j G_F \left[ \begin{matrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{matrix} \right] (\bar{\nu}_i - \bar{\nu}_j) \\
&\quad + i\sqrt{\lambda} (\theta_{i3} \theta_{j4} k_i \cdot \varepsilon_j + \theta_{i4} \theta_{j3} k_j \cdot \varepsilon_i) G_F \left[ \begin{matrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{matrix} \right] (\bar{\nu}_i - \bar{\nu}_j) \\
&\quad - i\sqrt{\lambda} (\theta_{i3} \theta_{i4} k_j \cdot \varepsilon_i - \theta_{j3} \theta_{j4} k_i \cdot \varepsilon_j) \hat{G}_B(\bar{\nu}_i - \bar{\nu}_j) \\
&\quad + \theta_{i4} \theta_{j4} \varepsilon_i \cdot \varepsilon_j G_F \left[ \begin{matrix} \alpha_{\uparrow} \\ \beta_{\uparrow} \end{matrix} \right] (\bar{\nu}_i - \bar{\nu}_j) \\
&\quad \left. + \theta_{i3} \theta_{i4} \theta_{j3} \theta_{j4} \varepsilon_i \cdot \varepsilon_j \check{G}_B(\bar{\nu}_i - \bar{\nu}_j) \right]. \tag{III.17}
\end{aligned}$$

This formula is valid in all sectors except Ramond–Ramond, where the fermionic zero mode demands special treatment. However, that sector enters only into parity-violating amplitudes, and so is not relevant to any of the calculations in this paper. The normalization of the amplitude has been calculated by Polchinski [51] and Sakai and Tanii [52].

#### Appendix IV. Field Theory Limit Expansions

We need the expansions of the Green functions and the partition function in two limits:  $\nu \rightarrow 0$  and  $\text{Im}\tau \rightarrow \infty$ .

As  $\nu \rightarrow 0$ ,

$$\begin{aligned}
\exp(G_B(\nu)) &\rightarrow |\nu|^{-1/\pi} \times \text{constant} \\
\dot{G}_B(\bar{\nu}) &\rightarrow -\frac{1}{2\pi\bar{\nu}} \\
\ddot{G}_B(\bar{\nu}) &\rightarrow \frac{1}{2\pi\bar{\nu}^2} \\
G_F\left[\begin{array}{c} \alpha \\ \beta \end{array}\right](\nu) &\rightarrow \frac{1}{2\pi\nu} \\
G_F\left[\begin{array}{c} \alpha \\ \beta \end{array}\right](\bar{\nu}) &\rightarrow \frac{1}{2\pi\bar{\nu}}
\end{aligned} \tag{IV.1}$$

while in the limit  $\text{Im } \tau \rightarrow \infty$ ,

$$\begin{aligned}
\mathcal{Z}_1\left[\begin{array}{c} 0 \\ 0 \end{array}\right] &= 0 \\
\mathcal{Z}_1\left[\begin{array}{c} 0 \\ \beta \neq 0 \end{array}\right] &\rightarrow \hat{q}^{1/12} e^{\pi i \text{Re } \tau / 6} (1 - e^{-2\pi i \beta}) (1 - 2\hat{q} e^{2\pi i \text{Re } \tau} \cos 2\pi \beta) \\
\mathcal{Z}_1\left[\begin{array}{c} 0 < \alpha < 1/2 \\ \beta \end{array}\right] &\rightarrow \hat{q}^{(\alpha^2 - \alpha + 1/6)/2} e^{\pi i \text{Re } \tau (\alpha^2 - \alpha + 1/6)/2} \left(1 - \hat{q}^\alpha e^{2\pi i \alpha \text{Re } \tau} e^{-2\pi i \beta} - \hat{q}^{1-\alpha} e^{2\pi i (1-\alpha) \text{Re } \tau} e^{2\pi i \beta}\right) \\
\mathcal{Z}_1\left[\begin{array}{c} 1/2 \\ \beta \end{array}\right] &\rightarrow \hat{q}^{-1/24} e^{-\pi i \text{Re } \tau / 12} \left(1 - 2\hat{q}^{1/2} e^{\pi i \text{Re } \tau} \cos 2\pi \beta + \hat{q} e^{2\pi i \text{Re } \tau}\right) \\
\exp(G_B(\nu)) &\rightarrow \exp\left(\frac{(\text{Im } \nu)^2}{\text{Im } \tau}\right) |\sin \pi \nu|^{-1/\pi} \times \text{constant} \\
\dot{G}_B(\bar{\nu}) &\rightarrow \frac{i \text{Im } \nu}{\text{Im } \tau} - \frac{\cos \pi \bar{\nu}}{2 \sin \pi \bar{\nu}} \\
\ddot{G}_B(\bar{\nu}) &\rightarrow -\frac{1}{2 \text{Im } \tau} + \frac{\pi}{2 \sin^2 \pi \bar{\nu}} \\
G_F\left[\begin{array}{c} 0 \\ \beta \neq 0 \end{array}\right](\nu) &\rightarrow \frac{\sin \pi(\beta - \nu)}{\sin \pi \beta} \left(\frac{1}{2 \sin \pi \nu} + 2\hat{q} e^{2\pi i \text{Re } \tau} (\sin \pi(\nu - 2\beta) - \sin \pi \nu)\right) \\
G_F\left[\begin{array}{c} 0 < \alpha < 1/2 \\ \beta \end{array}\right](\nu) &\rightarrow \frac{e^{2\pi i \nu(\alpha - 1/2)}}{2 \sin \pi \nu} \left(\frac{1 - \hat{q}^\alpha e^{2\pi i \alpha \text{Re } \tau} e^{2\pi i(\nu - \beta)} - \hat{q}^{1-\alpha} e^{2\pi i(1-\alpha) \text{Re } \tau} e^{-2\pi i(\nu - \beta)}}{1 - \hat{q}^\alpha e^{2\pi i \alpha \text{Re } \tau} e^{-2\pi i \beta} - \hat{q}^{1-\alpha} e^{2\pi i(1-\alpha) \text{Re } \tau} e^{2\pi i \beta}}\right) \\
&\quad - 2\hat{q} e^{2\pi i \text{Re } \tau} e^{2\pi i \nu(\alpha - 1/2)} \sin \pi \nu \\
G_F\left[\begin{array}{c} 1/2 \\ \beta \end{array}\right](\nu) &\rightarrow \frac{1}{2 \sin \pi \nu} + 2\hat{q}^{1/2} e^{\pi i \text{Re } \tau} \sin \pi(\nu - 2\beta) + 2\hat{q} e^{2\pi i \text{Re } \tau} \sin \pi(\nu - 4\beta) \\
G_F\left[\begin{array}{c} 0 \\ \beta \neq 0 \end{array}\right](\bar{\nu}) &\rightarrow \frac{\sin \pi(\beta - \bar{\nu})}{2 \sin \pi \beta \sin \pi \bar{\nu}} \\
G_F\left[\begin{array}{c} 1/2 \\ \beta \end{array}\right](\bar{\nu}) &\rightarrow \frac{1}{2 \sin \pi \bar{\nu}} + 2\hat{q}^{1/2} e^{-\pi i \text{Re } \tau} \sin \pi(\bar{\nu} - 2\beta)
\end{aligned} \tag{IV.2}$$

Retaining only the leading terms in  $e^{\text{Im } \nu}$ , we can simplify the Green's functions further,

( $\text{sim}(x) = \text{sign}(\text{Im } x)$ ),

$$\begin{aligned}
\exp(G_B(\nu)) &\rightarrow \exp\left(\frac{(\text{Im } \nu)^2}{\text{Im } \tau} - |\text{Im } \nu|\right) \times \text{constant} \\
\dot{G}_B(\bar{\nu}) &\rightarrow \frac{i}{2} \left( \text{sim}(\bar{\nu}) - \frac{2 \text{Im } \bar{\nu}}{\text{Im } \tau} \right) \\
\ddot{G}_B(\bar{\nu}) &\rightarrow -\frac{1}{2 \text{Im } \tau} \\
G_F \left[ \begin{array}{c} 0 \\ \beta \neq 0 \end{array} \right] (\nu) &\rightarrow -i \text{sim}(\nu) \left( e^{\pi i \text{sim}(\nu) \text{Re } \nu} e^{-\pi |\text{Im } \nu|} - \hat{q} e^{2\pi i \text{Re } \tau} e^{-\pi i \text{sim}(\nu) \text{Re } \nu} e^{\pi |\text{Im } \nu|} \left( e^{2\pi i \text{sim}(\nu) \beta} - 1 \right) \right) \\
G_F \left[ \begin{array}{c} 1/2 \\ \beta \end{array} \right] (\nu) &\rightarrow -i \text{sim}(\nu) \left( e^{\pi i \text{sim}(\nu) \text{Re } \nu} e^{-\pi |\text{Im } \nu|} - \hat{q}^{1/2} e^{\pi i \text{Re } \tau} e^{-\pi i \text{sim}(\nu) \text{Re } \nu} e^{\pi |\text{Im } \nu|} e^{2\pi i \text{sim}(\nu) \beta} \right. \\
&\quad \left. - \hat{q} e^{2\pi i \text{Re } \tau} e^{-\pi i \text{sim}(\nu) \text{Re } \nu} e^{\pi |\text{Im } \nu|} e^{4\pi i \text{sim}(\nu) \beta} \right) \\
G_F \left[ \begin{array}{c} 0 \\ \beta \neq 0 \end{array} \right] (\bar{\nu}) &\rightarrow -i \text{sim}(\bar{\nu}) \left( e^{\pi i \text{sim}(\bar{\nu}) \text{Re } \bar{\nu}} e^{-\pi |\text{Im } \bar{\nu}|} \right) \\
G_F \left[ \begin{array}{c} 1/2 \\ \beta \end{array} \right] (\bar{\nu}) &\rightarrow -i \text{sim}(\bar{\nu}) \left( e^{\pi i \text{sim}(\bar{\nu}) \text{Re } \bar{\nu}} e^{-\pi |\text{Im } \bar{\nu}|} - \hat{q}^{1/2} e^{-\pi i \text{Re } \tau} e^{-\pi i \text{sim}(\bar{\nu}) \text{Re } \bar{\nu}} e^{\pi |\text{Im } \bar{\nu}|} e^{2\pi i \text{sim}(\bar{\nu}) \beta} \right)
\end{aligned} \tag{IV.3}$$

A special case of particular interest for the partition function is (with  $\vec{\alpha}_0 = (\frac{1}{2}^{22} | \frac{1}{2}^{10})$ )

$$\begin{aligned}
\mathcal{Z} \left[ \begin{array}{c} \vec{\alpha}_0 \\ \vec{\beta} \end{array} \right] (\tau) &= (\text{Im } \tau)^{\epsilon/2-1} \eta(\tau)^{-24} \bar{\eta}(\tau)^{-12} F_c(\tau)^\epsilon \prod_{i=1}^{\text{len } \alpha_{0L}} \vartheta \left[ \begin{array}{c} 1/2 \\ \beta_{Li} \end{array} \right] \prod_{i=1}^{\text{len } \alpha_{0R}} \bar{\vartheta} \left[ \begin{array}{c} 1/2 \\ \beta_{Ri} \end{array} \right] \\
&= (\text{Im } \tau)^{\epsilon/2-1} \hat{q}^{-3/2} e^{-\pi i \text{Re } \tau} (1 + 24\hat{q} e^{2\pi i \text{Re } \tau}) F_c^\epsilon(\tau) \\
&\quad \times \left( 1 - 2\hat{q}^{1/2} e^{\pi i \text{Re } \tau} \sum_{i=1}^{\text{len } \alpha_{0L}} \cos 2\pi \beta_{Li} + 4\hat{q} e^{2\pi i \text{Re } \tau} \sum_{i < j=1}^{\text{len } \alpha_{0L}} \cos 2\pi \beta_{Li} \cos 2\pi \beta_{Lj} \right) \\
&\quad \times \left( 1 - 2\hat{q}^{1/2} e^{-\pi i \text{Re } \tau} \sum_{i=1}^{\text{len } \alpha_{0R}} \cos 2\pi \beta_{Ri} \right)
\end{aligned} \tag{IV.4}$$

## Appendix V. Integrals for the Loop Calculation

The class of integrals we need is

$$(-s)^{-2-\epsilon/2} \int_{[0,1]^4} d^4 y \frac{P(\{y_i\}) \delta(\sum y_i - 1)}{[y_1 y_3 + \chi y_2 y_4]^{2+\epsilon/2}} \tag{V.1}$$

where  $P$  is some polynomial, and where  $\chi = t/s$ .

Through the change of variables

$$\begin{aligned}
y_1 &= (1-y)(1-z) \\
y_2 &= z(1-y) \\
y_3 &= y(1-x)
\end{aligned} \tag{V.2}$$

which also implies that  $y_4 = xy$ , we can transform the original integral into

$$(-s)^{-2-\epsilon/2} \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{P((1-y)(1-z), z(1-y), y(1-x), xy)}{[y(1-y)]^{1+\epsilon/2} [(1-x)(1-z) + \chi xz]^{2+\epsilon/2}}. \quad (\text{V.3})$$

The  $y$  integral is now elementary; we may note that only terms symmetric under  $y \rightarrow 1-y$  will survive, so that we can replace the powers of  $y$  in the numerator with powers of  $P_y = y(1-y)$  as follows,

$$\begin{aligned} 1 &\rightarrow 1 \\ y &\rightarrow \frac{1}{2} \\ y^2 &\rightarrow \frac{1}{2} - P_y \\ y^3 &\rightarrow \frac{1}{2}(1 - P_y) \\ y^4 &\rightarrow P_y^2 - 2P_y + \frac{1}{2} \\ y^5 &\rightarrow \frac{1}{2}(5P_y^2 - 5P_y + 1) \\ y^6 &\rightarrow -P_y^3 + \frac{9}{2}P_y^2 - 3P_y + \frac{1}{2} \end{aligned} \quad (\text{V.4})$$

The  $x \rightarrow z$  symmetry can also be used to modify the polynomial.

We might thus focus on the integrals

$$\int_0^1 \int_0^1 dx dz x^n z^m [(1-x)(1-z) + \chi xz]^{-2-\epsilon/2} \quad (\text{V.5})$$

but it will be convenient to consider a somewhat more general integral,

$$\begin{aligned} \mathcal{J}(n, m, l; \xi; \chi) &= \int_0^1 \int_0^1 dx dz x^n z^{n+m} (1-z)^l [(1-x)(1-z) + \chi xz]^\xi \\ &= \frac{\Gamma(1+\xi)}{\Gamma(n+1+\xi)} \partial_x^n \mathcal{J}(0, m, l; n+\xi; \chi) \\ &= \frac{\Gamma(1+\xi)}{\Gamma(n+2+\xi)} \partial_x^n \left[ B(n+l+2+\xi, m+1) {}_2F_1(m+1, 1; n+m+l+3+\xi; 1+\chi) \right. \\ &\quad \left. - \chi^{n+1+\xi} B(n+m+2+\xi, l+1) {}_2F_1(1, n+m+2+\xi; n+m+l+3+\xi; 1+\chi) \right] \end{aligned} \quad (\text{V.6})$$

(Note that any polynomial in  $z$  can be expressed linearly in terms of the  $P_{z;m} = z^m(1-z)$ ,  $z$ , and 1.)



An inspection of the integrands reveals that we require the following set of integrals,

$$\begin{aligned}
\mathcal{J}(0, 0, 0; -2 - \epsilon/2; \chi) & \quad \text{to } O(\epsilon) \\
\mathcal{J}(0, 1, 0; -2 - \epsilon/2; \chi) & \quad \text{to } O(\epsilon) \\
\mathcal{J}(1, 0, 0; -2 - \epsilon/2; \chi) & \quad \text{to } O(1) \\
\mathcal{J}(1, 1, 0; -2 - \epsilon/2; \chi) & \quad \text{to } O(1) \\
\mathcal{J}(2, 0, 0; -2 - \epsilon/2; \chi) & \quad \text{to } O(1) \\
\mathcal{J}(0, 1, 1; -2 - \epsilon/2; \chi) & \quad \text{to } O(\epsilon) \\
\mathcal{J}(0, 2, 1; -2 - \epsilon/2; \chi) & \quad \text{to } O(\epsilon) \\
\mathcal{J}(0, 3, 1; -2 - \epsilon/2; \chi) & \quad \text{to } O(\epsilon) \\
\mathcal{J}(1, 1, 1; -2 - \epsilon/2; \chi) & \quad \text{to } O(1)
\end{aligned} \tag{V.7}$$

To express these integrals, we should first introduce a bit of notation for expressing imaginary parts. Define  $\Theta(\mathbf{x})$  and  $\bar{\Theta}(\mathbf{x})$  via

$$\begin{aligned}
\bar{\Theta}(\mathbf{x}) &= \frac{1}{i\pi} \log \arg(\mathbf{x}) \\
\Theta(\mathbf{x}) &= \frac{1}{i\pi} \log \arg(-\mathbf{x}) \\
\hat{\Theta}(\mathbf{x}) &= |\Theta(\mathbf{x})| = \Theta^2(\mathbf{x})
\end{aligned} \tag{V.8}$$

where  $\mathbf{x}$  is assumed to have an infinitesimal imaginary part in order to make these objects well-defined. The cut in the argument of the logarithm is taken along the negative real axis. That is, in the usual complex plane,

$$\begin{aligned}
\bar{\Theta}(\mathbf{x}) = 0, \quad \Theta(\mathbf{x}) = -1, \quad \mathbf{x} = |\mathbf{x}| + i0^+ \\
\bar{\Theta}(\mathbf{x}) = 0, \quad \Theta(\mathbf{x}) = 1, \quad \mathbf{x} = |\mathbf{x}| + i0^- \\
\bar{\Theta}(\mathbf{x}) = 1, \quad \Theta(\mathbf{x}) = 0, \quad \mathbf{x} = -|\mathbf{x}| + i0^+ \\
\bar{\Theta}(\mathbf{x}) = -1, \quad \Theta(\mathbf{x}) = 0, \quad \mathbf{x} = -|\mathbf{x}| + i0^-
\end{aligned} \tag{V.9}$$

and

$$\hat{\Theta}(\mathbf{x} > 0) = 1, \quad \hat{\Theta}(\mathbf{x} < 0) = 0. \tag{V.10}$$

Note that in the usual  $s$  plane, the branch cut in any amplitude runs along the positive real axis, and by convention the momentum invariant is just below the branch cut. This corresponds to having the momentum invariant just above the branch cut in the usual complex plane, so that  $\Theta(s) = 1$  in the physical region.

Note that we may write in general

$$\begin{aligned}
\Theta(\mathbf{x})\bar{\Theta}(\mathbf{x}) &= 0 \\
\Theta^3(\mathbf{x}) &= \Theta(\mathbf{x}) \\
\bar{\Theta}^3(\mathbf{x}) &= \bar{\Theta}(\mathbf{x}) \\
\bar{\Theta}(1/\mathbf{x}) &= -\bar{\Theta}(\mathbf{x})
\end{aligned}
\tag{V.11}$$

The correct prescription on its derivative will be

$$\partial_{\mathbf{x}}\Theta(\mathbf{x}) = 0 \tag{V.12}$$

not a  $\delta$  function. We also have the additional properties

$$\begin{aligned}
\bar{\Theta}(-\mathbf{x}) &= \Theta(\mathbf{x}) \\
\Theta(-\mathbf{x}) &= \bar{\Theta}(\mathbf{x}) \\
(\Theta(\mathbf{x}) - \bar{\Theta}(\mathbf{x}))^2 &= 1 = \Theta^2(\mathbf{x}) + \bar{\Theta}^2(\mathbf{x})
\end{aligned}
\tag{V.13}$$

We can relate  $\bar{\Theta}(t/s)$  to  $\Theta(t)$  and  $\Theta(s)$ ,

$$\begin{aligned}
\bar{\Theta}(t/s) &= \bar{\Theta}(t) - \bar{\Theta}(s) \\
&= \Theta(t) - \Theta(s) \\
\Theta^2(t/s) &= 1 - \bar{\Theta}^2(t/s)
\end{aligned}
\tag{V.14}$$

With these definitions, one has

$$\begin{aligned}
\mathcal{J}(0,0,0;-2-\epsilon/2;\chi) &= -\frac{2}{(1+\epsilon/2)\epsilon\chi} \left[ 2 - \frac{\epsilon}{2} \ln|\chi| - \frac{\pi^2}{2} \left(\frac{\epsilon}{2}\right)^2 - i\pi\frac{\epsilon}{2}\bar{\Theta}(\chi) \right] + \mathcal{O}(\epsilon^2) \\
\mathcal{J}(0,1,0;-2-\epsilon/2;\chi) &= -\frac{1}{(1+\epsilon/2)\epsilon\chi} \left[ 2 - \left(\frac{\epsilon}{2}\right)^2 \frac{1}{1+\chi} (\ln^2|\chi| + \pi^2\Theta^2(\chi) + 2i\pi\bar{\Theta}(\chi)\ln|\chi|) \right] + \mathcal{O}(\epsilon^2) \\
\mathcal{J}(1,0,0;-2-\epsilon/2;\chi) &= -\frac{1}{\chi(1+\chi)^2} \left[ -\frac{1}{2}\chi\ln^2|\chi| + (1+\chi)\ln|\chi| - \frac{\pi^2}{2}\chi\Theta^2(\chi) \right. \\
&\quad \left. - i\pi\bar{\Theta}(\chi)(\chi\ln|\chi| - 1 - \chi) \right] + \mathcal{O}(\epsilon) \\
\mathcal{J}(1,1,0;-2-\epsilon/2;\chi) &= \frac{1}{\chi(1+\chi)^3} \left[ (\ln|\chi| + 1 + \chi)(\chi\ln|\chi| - 1 - \chi) + \pi^2\chi\Theta^2(\chi) \right. \\
&\quad \left. + i\pi\bar{\Theta}(\chi)(2\chi\ln|\chi| - 1 + \chi^2) \right] + \mathcal{O}(\epsilon) \\
\mathcal{J}(2,0,0;-2-\epsilon/2;\chi) &= \frac{1}{\chi(1+\chi)^4} \left[ -\chi(\chi-2)(\ln^2|\chi| + \pi^2\Theta^2(\chi)) \right. \\
&\quad \left. + (5\chi^2 + 4\chi - 1)\ln|\chi| + \chi^3 - 3\chi - 2 \right. \\
&\quad \left. - i\pi\bar{\Theta}(\chi)(2\chi^2\ln|\chi| - 4\chi\ln|\chi| - 5\chi^2 - 4\chi + 1) \right] + \mathcal{O}(\epsilon) \\
\mathcal{J}(0,1,1;-2-\epsilon/2;\chi) &= -\frac{1}{(1+\chi)^2} \left[ -\frac{(1+\chi)^2}{\chi} \right. \\
&\quad \left. + \frac{\epsilon}{2} \left( -\frac{1}{2}\ln^2|\chi| + \frac{1+\chi}{\chi}\ln|\chi| - \frac{\pi^2}{2}\Theta^2(\chi) + i\pi\bar{\Theta}(\chi)\left(\frac{1+\chi}{\chi} - \ln|\chi|\right) \right) \right] + \mathcal{O}(\epsilon^2) \\
\mathcal{J}(0,2,1;-2-\epsilon/2;\chi) &= -\frac{\Gamma(1-\epsilon/2)}{(1+\epsilon/2)\Gamma(3-\epsilon/2)} \frac{1}{(1+\chi)^3} \\
&\quad \times \left[ -\frac{(1+\chi)^3}{\chi} + \frac{\epsilon}{2} \left( -\ln^2|\chi| + \frac{1-\chi^2}{\chi}\ln|\chi| + \frac{(1+\chi)^2}{\chi} - \pi^2\Theta^2(\chi) \right. \right. \\
&\quad \left. \left. - i\pi\bar{\Theta}(\chi)(\chi - \chi^{-1} + 2\ln|\chi|) \right) \right] + \mathcal{O}(\epsilon^2) \\
\mathcal{J}(0,3,1;-2-\epsilon/2;\chi) &= \frac{2\Gamma(1-\epsilon/2)}{(1+\epsilon/2)\Gamma(4-\epsilon/2)\chi} \\
&\quad \times \left[ 1 + \frac{\epsilon}{2} \frac{1}{2(1+\chi)^4} \left( -3(1+\chi)^2 + (1+\chi)(\chi^2 + 5\chi - 2)\ln|\chi| + 3\chi\ln^2|\chi| \right. \right. \\
&\quad \left. \left. + 3\pi^2\chi\Theta^2(\chi) + i\pi\bar{\Theta}(\chi)(6\chi\ln|\chi| + (1+\chi)(\chi^2 + 5\chi - 2)) \right) \right] + \mathcal{O}(\epsilon^2) \\
\mathcal{J}(1,1,1;-2-\epsilon/2;\chi) &= \frac{1}{2\chi(1+\chi)^4} \left[ \chi(2\chi-1)(\ln^2|\chi| + \pi^2\Theta^2(\chi)) \right. \\
&\quad \left. + (\chi-5)(1+\chi)\chi\ln|\chi| + 1 - 3\chi^2 - 2\chi^3 \right. \\
&\quad \left. + i\pi\bar{\Theta}(\chi)\chi(2(2\chi-1)\ln|\chi| + (\chi-5)(1+\chi)) \right] + \mathcal{O}(\epsilon)
\end{aligned}$$

(V.15)

## Appendix VI. Integrals for the Optical Theorem

We wish to evaluate the integrals

$$\begin{aligned}
\mathcal{I}_0 &= (\mu^2)^{\epsilon/2} \int d^{4-\epsilon}\text{LIPS}(3,4) \frac{\langle 12 \rangle^4 [\hat{2}\hat{1}]^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle [\hat{2}\hat{4}] [43] [3\hat{1}] [\hat{1}\hat{2}]} \\
\mathcal{I}_{4a} &= (\mu^2)^{\epsilon/2} \int d^{4-\epsilon}\text{LIPS}(3,4) \frac{\langle 13 \rangle^4 [\hat{2}\hat{3}]^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle [\hat{2}\hat{4}] [43] [3\hat{1}] [\hat{1}\hat{2}]} \\
\mathcal{I}_{4b} &= (\mu^2)^{\epsilon/2} \int d^{4-\epsilon}\text{LIPS}(3,4) \frac{\langle 14 \rangle^4 [\hat{2}\hat{4}]^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle [\hat{2}\hat{4}] [43] [3\hat{1}] [\hat{1}\hat{2}]}
\end{aligned} \tag{VI.1}$$

and to show that

$$\mathcal{I}_{4c} = (\mu^2)^{\epsilon/2} \int d^{4-\epsilon}\text{LIPS}(3,4) \frac{\langle 31 \rangle^2 \langle 41 \rangle^2 [3\hat{1}]^2 [4\hat{1}]^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle [\hat{2}\hat{4}] [43] [3\hat{1}] [\hat{1}\hat{2}]} \tag{VI.2}$$

is finite. In the four-point case, we can solve for some of the spinor products, using on-shell conditions and momentum conservation ( $k_1 + k_2 + k_3 + k_4 = 0$ ,  $k_1 + k_2 + k_1 + k_3 = 0$ ):

$$\begin{aligned}
\langle 23 \rangle \langle 41 \rangle &= -\frac{\langle 23 \rangle \langle 12 \rangle \langle 34 \rangle}{\langle 12 \rangle} \\
[3\hat{1}] [\hat{2}\hat{4}] &= \frac{(\hat{1}\hat{2}) (3\hat{1})}{\langle \hat{1}\hat{2} \rangle \langle 34 \rangle}
\end{aligned} \tag{VI.3}$$

so that

$$\begin{aligned}
\mathcal{I}_0 &= (\mu^2)^{\epsilon/2} \langle 12 \rangle^2 [\hat{2}\hat{1}]^2 \int d^{4-\epsilon}\text{LIPS}(3,4) \frac{1}{(32)(3\hat{1})} \\
\mathcal{I}_{4a} &= S_{\mu_1} S_{\mu_2} S_{\mu_3} S_{\mu_4} (\mu^2)^{\epsilon/2} \frac{1}{\langle 12 \rangle^2 [\hat{2}\hat{1}]^2} \int d^{4-\epsilon}\text{LIPS}(3,4) \frac{k_3^{\mu_1} k_3^{\mu_2} k_3^{\mu_3} k_3^{\mu_4}}{(32)(3\hat{1})} \\
\mathcal{I}_{4b} &= S_{\mu_1} S_{\mu_2} S_{\mu_3} S_{\mu_4} (\mu^2)^{\epsilon/2} \frac{1}{\langle 12 \rangle^2 [\hat{2}\hat{1}]^2} \int d^{4-\epsilon}\text{LIPS}(3,4) \frac{k_4^{\mu_1} k_4^{\mu_2} k_4^{\mu_3} k_4^{\mu_4}}{(32)(3\hat{1})} \\
&= S_{\mu_1} S_{\mu_2} S_{\mu_3} S_{\mu_4} (\mu^2)^{\epsilon/2} \frac{1}{\langle 12 \rangle^2 [\hat{2}\hat{1}]^2} \int d^{4-\epsilon}\text{LIPS}(3,4) \frac{k_4^{\mu_1} k_4^{\mu_2} k_4^{\mu_3} k_4^{\mu_4}}{(41)(4\hat{2})} \\
\mathcal{I}_{4c} &= (\mu^2)^{\epsilon/2} \frac{1}{\langle 12 \rangle^2 [\hat{2}\hat{1}]^2} \int d^{4-\epsilon}\text{LIPS}(3,4) \frac{\langle 31 \rangle^2 \langle 41 \rangle^2 [3\hat{1}]^2 [4\hat{1}]^2}{(41)(3\hat{1})}
\end{aligned} \tag{VI.4}$$

where  $S_\mu = \langle 1^- | \gamma_\mu | \hat{2}^- \rangle$ . (Note that we need only the sum of  $\mathcal{I}_{4a}$  and  $\mathcal{I}_{4b}$ .)

In the center-of-mass frame, we can perform the azimuthal integral trivially, so the phase space measure is given by

$$\begin{aligned}
d^{4-\epsilon}\text{LIPS}(3,4) &= \frac{d^{3-\epsilon}\mathbf{k}_3}{(2\pi)^{3-\epsilon} 2E_3} \frac{d^{3-\epsilon}\mathbf{k}_4}{(2\pi)^{3-\epsilon} 2E_4} (2\pi)^{4-\epsilon} \delta^{4-\epsilon}(P - k_3 - k_4) \\
&= \frac{1}{(1-\epsilon)B(1/2, 1-\epsilon/2)} \frac{\Gamma(1-\epsilon/2)}{8\pi \Gamma(1-\epsilon)} \frac{(4\pi)^{\epsilon/2}}{(12)^{\epsilon/2}} \sin^{-\epsilon} \theta d(\cos \theta)
\end{aligned} \tag{VI.5}$$

where  $P = -k_1 - k_2$ , and where  $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$  is the Euler beta function.

The  $\mathcal{I}_0$  integral can be evaluated by standard techniques,

$$\begin{aligned}
\mathcal{I}_0 &= \frac{\Gamma(1 - \epsilon/2)}{4\pi \Gamma(1 - \epsilon)} \left( \frac{4\pi\mu^2}{(12)} \right)^{\epsilon/2} \langle 12 \rangle^2 [\hat{2}\hat{1}]^2 \frac{1}{(12)^2} \\
&\quad \times \left[ \frac{2}{2 + \epsilon} {}_2F_1(1, 1; 2 + \epsilon/2; -(\hat{1}2)/(12)) \right. \\
&\quad \left. + \frac{2}{\epsilon} \left( -\frac{(12)}{(\hat{1}2)} \right)^{1 + \epsilon/2} \Gamma(1 + \epsilon/2)\Gamma(1 - \epsilon/2) {}_1F_0\left(-\epsilon/2; -\frac{(\hat{1}2)}{(12)}\right) \right] \quad (\text{VI.6}) \\
&= -\frac{\Gamma^2(1 - \epsilon/2)\Gamma(1 + \epsilon/2)}{4\pi \Gamma(1 - \epsilon)} \left( \frac{4\pi\mu^2}{(12)} \right)^{\epsilon/2} \frac{\langle 12 \rangle^2 [\hat{2}\hat{1}]^2}{(12)(\hat{1}2)} \left( \frac{2}{\epsilon} - \ln \left[ -\frac{(\hat{1}2)}{(12)} \right] \right) + \mathcal{O}(\epsilon) \\
&= \frac{\langle 12 \rangle^4}{(12)\langle 2\hat{1} \rangle \langle \hat{1}\hat{2} \rangle \langle \hat{2}\hat{1} \rangle} \frac{(4\pi)^{\epsilon/2} \Gamma^2(1 - \epsilon/2)\Gamma(1 + \epsilon/2)}{4\pi \Gamma(1 - \epsilon)} \left( \frac{2}{\epsilon} - l_\mu(\hat{1}2) \right) + \mathcal{O}(\epsilon)
\end{aligned}$$

The basic integral in  $\mathcal{I}_{4a,b}$ ,

$$\mathcal{I}_4^{\mu_1\mu_2\mu_3\mu_4} = (\mu^2)^{\epsilon/2} \int d^4 -\epsilon \text{LIPS}(3, 4) \frac{k_3^{\mu_1} k_3^{\mu_2} k_3^{\mu_3} k_3^{\mu_4}}{(32)(3\hat{1})} \quad (\text{VI.7})$$

is traceless, and thus can be expressed in terms of the traceless tensors  $\hat{T}_{n;m,l}^{\mu_1 \dots \mu_n}$ :

$$\begin{aligned}
T_{n;m,l}^{\mu_1 \dots \mu_n}(2, \hat{1}) &= (-1)^m k_2^{\mu_1} \dots k_2^{\mu_m} k_{\hat{1}}^{\mu_{m+1}} \dots k_{\hat{1}}^{\mu_{m+l}} P^{\mu_{m+l+1}} \dots P^{\mu_n} \\
\hat{T}_{n;m,l}^{\mu_1 \dots \mu_n}(2, \hat{1}) &= T_{n;m,l}^{\mu_1 \dots \mu_n}(2, \hat{1}) - \text{traces} \quad (\text{VI.8})
\end{aligned}$$

To do this, one performs a Passarino-Veltman [19] reduction. One attaches a coefficient to each tensor, and writes

$$\mathcal{I}_4^{\mu_1\mu_2\mu_3\mu_4} = \sum_{m+l \leq 4} C_{4;m,l} \left( -\frac{(\hat{1}2)}{(12)} \right) \hat{T}_{4;m,l}^{\mu_1 \dots \mu_4}(2, \hat{1}) \quad (\text{VI.9})$$

The symmetry of the integrand forces  $C_{4;m,l} = C_{4;l,m}$ , reducing the fifteen possible coefficient functions to nine. One then contracts this equation with the tensors  $T_{4;m,l}$  ( $m \leq l$ ), which yields a set of nine equations for the  $C_{4;m,l}$ . Finally, one solves the equations to obtain the coefficient

so that only  $C_{4;0,0}$  contributes to  $\mathcal{I}_{4b}$ . Thus

$$\begin{aligned}
\mathcal{I}_{4a} + \mathcal{I}_{4b} &= \frac{\langle 12 \rangle^4 [2\hat{2}]^4}{\langle 12 \rangle^2 [\hat{2}\hat{1}]^2 (12)(2\hat{1})} \left( C_{4;0,0} + \sum_{m+l \leq 4} C_{4;m,l} \right) \\
&= \frac{\langle 1\hat{1} \rangle^4}{\langle 12 \rangle \langle 2\hat{1} \rangle \langle \hat{1}\hat{2} \rangle \langle \hat{2}1 \rangle} \frac{\Gamma^2(1-\epsilon/2)\Gamma(1+\epsilon/2)}{4\pi\Gamma(1-\epsilon)} \left( \frac{4\pi\mu^2}{(12)} \right)^{\epsilon/2} \\
&\quad \times \left( -\frac{2}{\epsilon} + \frac{[(1\hat{1})^2 - (12)(2\hat{1})]^2}{(1\hat{1})^4} \ln \left[ -\frac{(\hat{1}2)}{(12)} \right] \right. \\
&\quad \left. - \frac{(2\hat{1}) [11(1\hat{1})^2 - 3(12)(2\hat{1}) + 3(12)^2]^2}{3(1\hat{1})^3} \right) + \mathcal{O}(\epsilon) \\
&= \frac{\langle 1\hat{1} \rangle^4}{\langle 12 \rangle \langle 2\hat{1} \rangle \langle \hat{1}\hat{2} \rangle \langle \hat{2}1 \rangle} \frac{(4\pi)^{\epsilon/2} \Gamma^2(1-\epsilon/2)\Gamma(1+\epsilon/2)}{4\pi\Gamma(1-\epsilon)} \\
&\quad \times \left( -\frac{2}{\epsilon} + \frac{[(1\hat{1})^2 - (12)(2\hat{1})]^2}{(1\hat{1})^4} l_\mu(\hat{1}2) - \frac{(12)(2\hat{1}) [2(1\hat{1})^2 - (12)(2\hat{1})]^2}{(1\hat{1})^4} l_\mu(12) \right. \\
&\quad \left. - \frac{(2\hat{1}) [11(1\hat{1})^2 - 3(12)(2\hat{1}) + 3(12)^2]^2}{3(1\hat{1})^3} \right) + \mathcal{O}(\epsilon)
\end{aligned} \tag{VI.13}$$

As to the remaining integral, we may note that  $\mathcal{I}_{4c}$  is more convergent than the following integral,

$$\mathcal{I}_{4c} = (\mu^2)^{\epsilon/2} \frac{1}{\langle 12 \rangle^2 [\hat{2}\hat{1}]^2} \int d^{4-\epsilon} \text{LIPS}(3,4) \left| \frac{\langle 31 \rangle^2 \langle 41 \rangle^2 [3\hat{1}]^2 [4\hat{1}]^2}{(41)(3\hat{1})} \right| \tag{VI.14}$$

in which we have taken the absolute value of the integrand. But this latter integral is

$$(\mu^2)^{\epsilon/2} \frac{1}{\langle 12 \rangle^2 [\hat{2}\hat{1}]^2} \int d^{4-\epsilon} \text{LIPS}(3,4) |(31)(4\hat{1})| \tag{VI.15}$$

which is manifestly convergent.

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### Figure Captions

Figure 1: The punctured torus represent a string one-loop amplitude.

Figure 2: In the field theory limit the torus reduces to a loop.

Figure 3: The  $\phi^3$ -like diagrams for the one-loop four-point gluon amplitude.

Figure 4: Labeled and ordered loops.

Figure 5: The reduced diagram with trees amputated.

Figure 6: The labeled reduced diagrams for the indices  $\{1, 2, 3\}$  and  $\{4, 5\}$  respectively associated with two distinct color traces.

Figure 7: The sole non-vanishing contribution to  $A_{4;1}(- - + +)$ .

Figure 8: Diagrams which contribute to  $A_{4;1}(1^{[L]}, 2^+, 3^-, 4^+)$

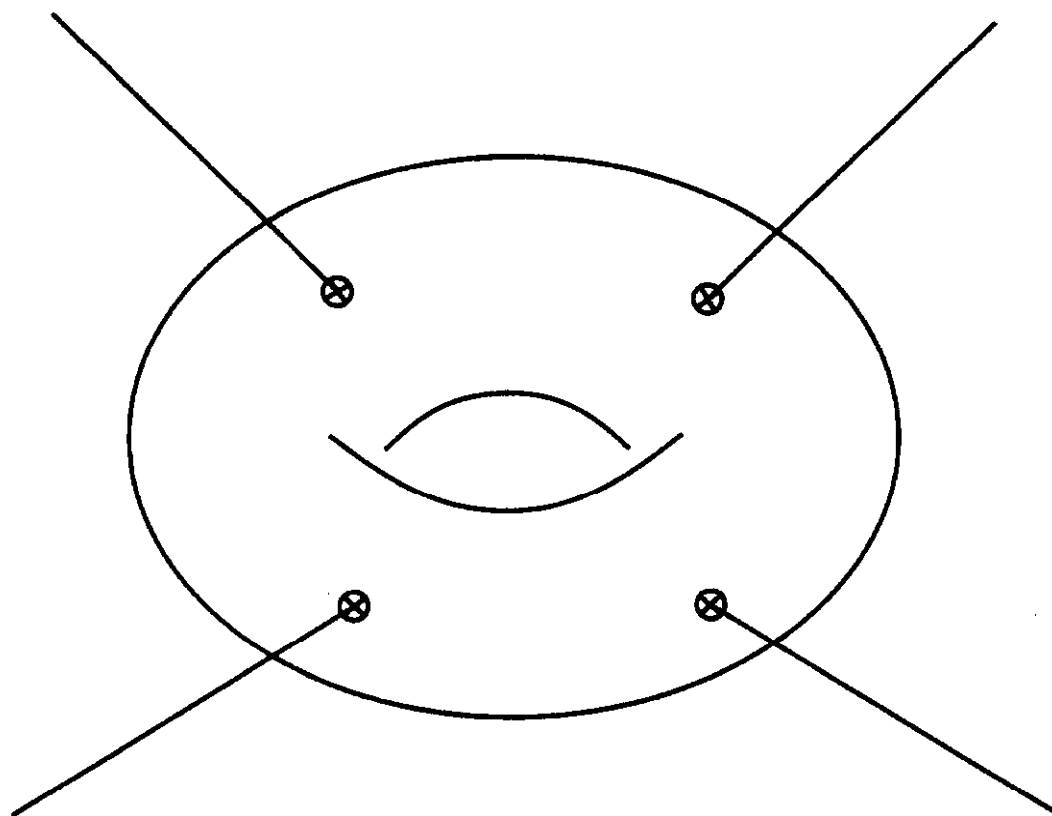
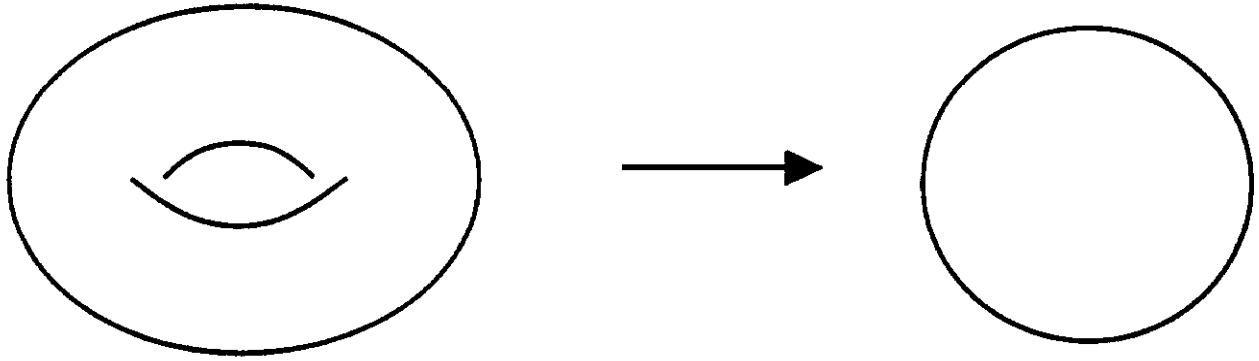
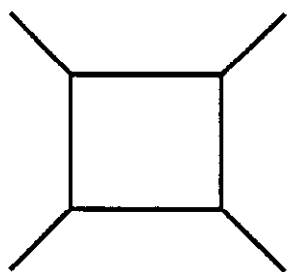


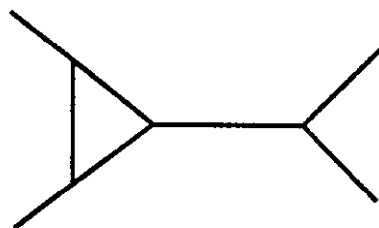
Fig. 1



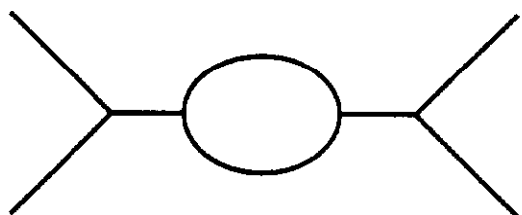
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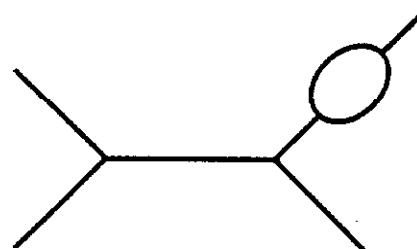
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(b)

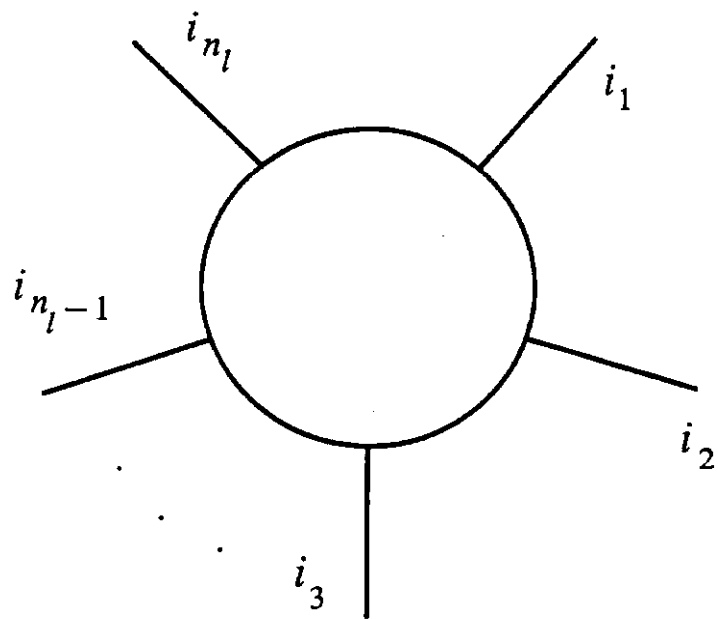


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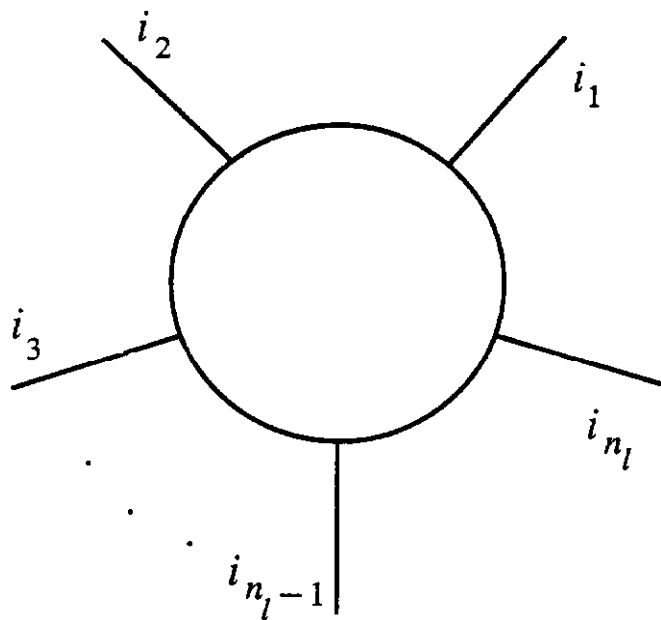


(d)

**Fig. 3**

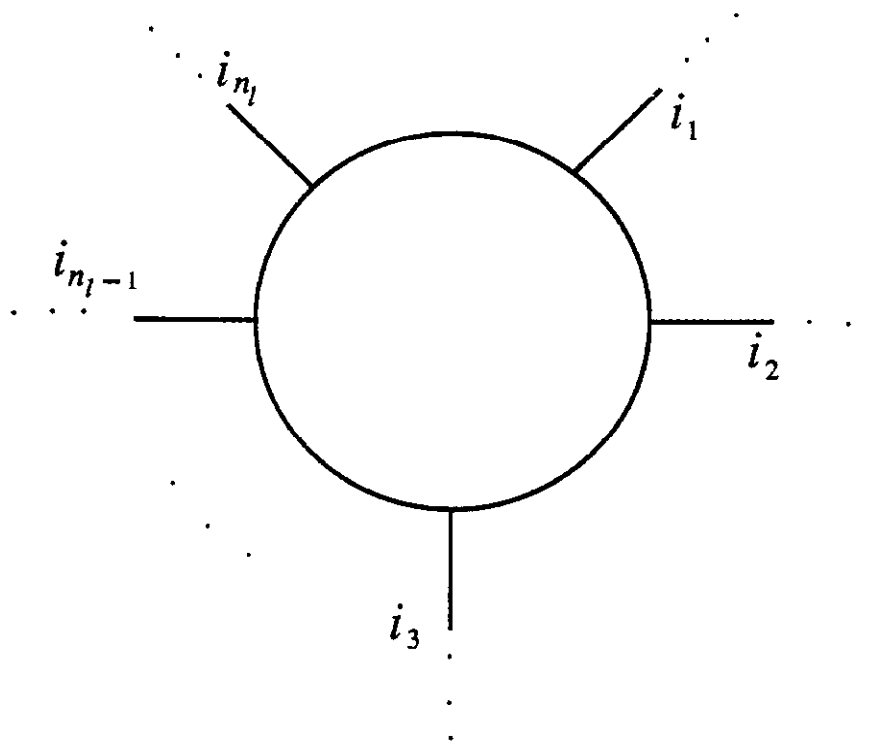


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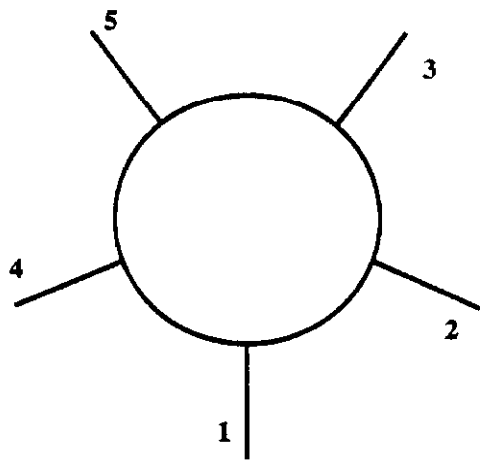


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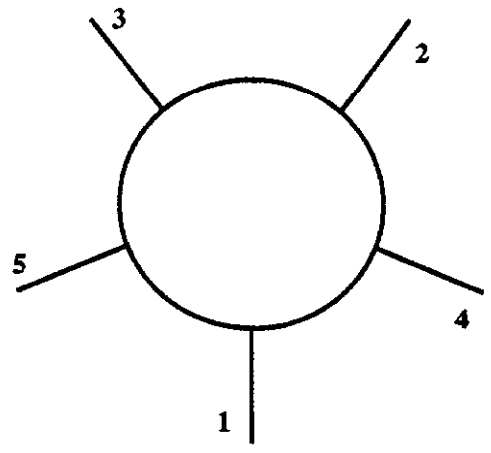
Fig. 4



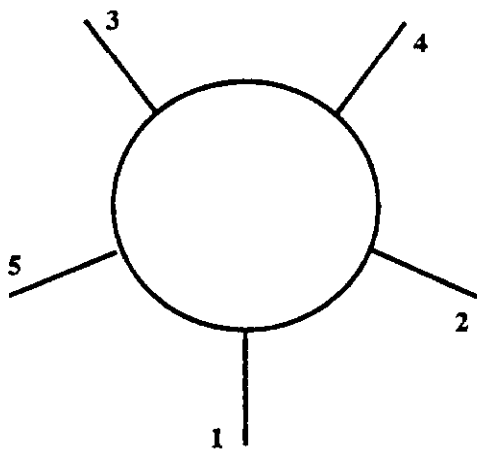
**Fig. 5**



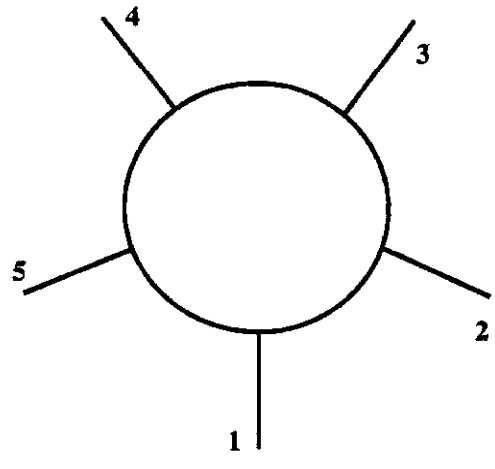
(a)



(b)



(c)



(d)

Fig. 6



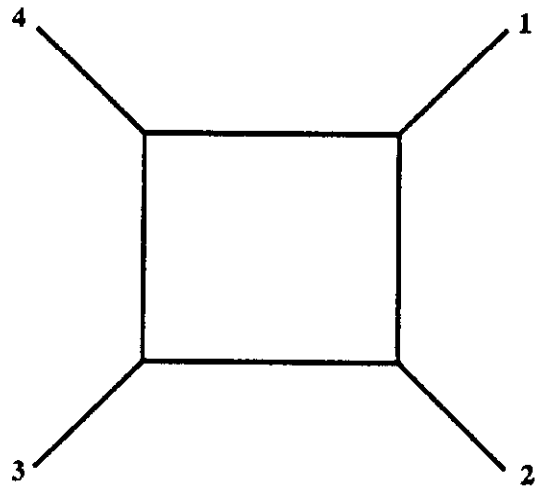
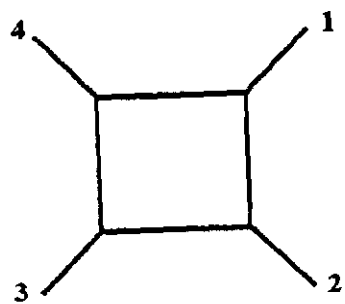
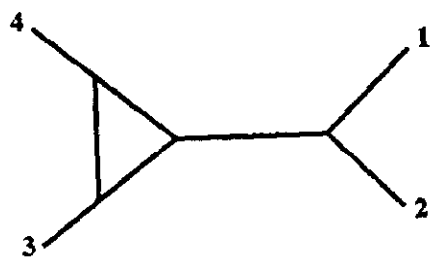


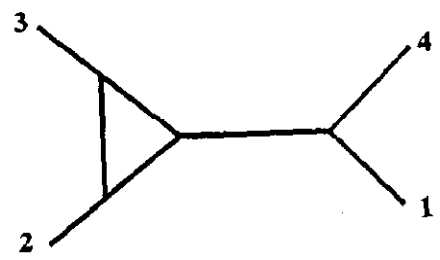
Fig. 7



(a)



(b)



(c)

Fig. 8