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THE THEORY OF ANYONIC SUPERCONDUCTIVITY: A REVIEW

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1. INTRODUCTION

Particles in two spatial dimensions with fractional statistics known, generically, as anyons, have been of interest to particle physicists for nearly ten years. A major change in the direction of research occurred when it was discovered that anyons could play a role as quasiparticles in condensed matter systems. This was originally discovered to be the case in systems exhibiting the Fractional Quantum Hall Effect.[1] The application of anyons to condensed matter systems received yet another boost when it was discovered by Laughlin[2-6] that even an ideal gas of anyons was a superfluid and, as a result, a gas of charged anyons would be a superconductor. This led immediately to attempts[2-21] to explain the superconductivity of high T_c materials which are layered ceramics in terms of anyons. The main challenge was to find a reasonable model for these materials which had quasiparticles obeying anyonic statistics. The goal of this article is to review the theory of anyonic superconductivity and its possible relation to high T_c materials. The emphasis in this review is on field theoretical methods.

We begin in chapter 2 by explaining what an anyon is and how it can be modeled mathematically. In chapter 3 we discuss the possible relationship between anyons and high T_c materials. We review several of the attempts to obtain anyonic quasiparticles from the Hubbard model which is commonly used to describe these materials. This discussion is rather brief since a comprehensive review by Balachandran et al[22] is already available.

Chapter 4 describes the mathematical modeling of anyons in terms of their interaction with an abelian gauge field with a Chern–Simons term. This description of anyons is used extensively in this article. In chapter 5 we begin considering an ideal gas of anyons. We discuss how to determine whether this system is a superfluid (or, if charged, a superconductor). In other words we discuss the possible criteria for superconductivity in anyonic systems with particular emphasis on criteria which would be useful in the Chern–Simons description. In chapter 6 we demonstrate that a neutral gas of anyons has a massless "Goldstone" mode. A detailed discussion of a field theoretic method[23] is presented as well as a brief description of the RPA[16] and "inhomogeneous expansion"[24] derivations. Chapter 7 contains a discussion of a physical picture of anyonic superconductivity. Spontaneous breaking of an algebra[15,25-26] is also addressed.

In chapter 8 we present the proof of a nonrenormalization theorem which shows that the results of chapter 6 are valid to all orders in a perturbative expansion about

mean field theory. An attempt to address the non-perturbative renormalization[27] is briefly described. The finite temperature analysis is discussed in chapter 9.

In chapter 10 we present the details of the low energy Landau–Ginsburg effective action for anyonic superconductors. This effective action is directly relevant to experimental tests of the model. The experimental tests are also briefly discussed. We conclude with a summary in chapter 11.

2. ANYONS

Anyons is the name given by Wilczek[28] to denote particles exhibiting fractional statistics in quantum mechanical systems with 2+1 spacetime dimensions. The first general description of anyons was given by Leinaas and Myrheim,[29] Goldin et al,[30] and Wilczek.[28] We will not attempt here to review all of the subsequent literature which has grown up concerning fractional statistics; instead, this section describes the basic concepts essential to an understanding of anyon superconductivity. Readers desiring a more detailed description of anyon fundamentals should consult the recent book by Wilczek,[31] the review by Stone,[32] or the original literature.

The possibility of fractional statistics occurs in a quantum mechanical system when the space of periodic trajectories is not simply or doubly connected. This occurs when the configuration space has non–contractable loops. A very simple example is a system of two particles in $d+1$ dimensions with hard–core repulsion. The relative configuration space is R_d with the point $x=x_1-x_2=0$ removed. This configuration space has non–contractable loops for $d=2$. For a system of two *identical* hard–cores in $d+1$ dimensions, the configuration space is $(R_d-\{0\})/S_2$, where S_2 is the permutation group of 2 objects. This configuration space (which is equivalent to a circular cone[29]) has non–contractable loops for any d. Specifically, a trajectory representing the physical interchange of the two particles cannot be continuously deformed into a trajectory where nothing happens. For $d>2$, that is the end of story: there are only two equivalence classes of trajectories, so the space is doubly–connected. However for $d=2$, trajectories representing n interchanges are topologically inequivalent to trajectories with m interchanges, for all $n\neq m$. Thus, for $d=2$, the space of periodic histories is infinitely–connected.

To quantize such systems one needs to specify a prescription for combining the disconnected parts. General arguments[33-36] indicate that prescriptions with consistent composition rules are representations of the fundamental group $\Pi_1(M)$ of the configuration space M. In the example we are discussing, $\Pi_1(M)=S_2$ for

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$d>2$. From the two one–dimensional representations of S_2, representing relative phases $+1$ or -1 between the two equivalence classes, we obtain ordinary bose or fermi statistics. For $d>2$ fractional statistics can only be obtained by choosing higher–dimensional representations of the fundamental group, which implies additional "internal" degrees of freedom. This route to fractional statistics, called parastatistics, has nothing to do with anyons. Anyons arise in the case $d=2$, for which $\Pi_1(M)=B_2$, the *braid group* of two objects. B_2 has an infinite number of distinct one–dimensional representations, which can be parametrized by a continuous "statistics" parameter γ, defined modulo 2π. For $\gamma=0$ and π, we again obtain a system of (hard–core) bosons or fermions, but for other values we have a quantum system of particles with fractional statistics, i.e. anyons.

Anyons obey a generalization of the usual spin–statistics connection[26,37,38]: a statistical phase $\exp(i\gamma)$ implies spin $\gamma/2\pi$. In 3+1 dimensions, of course, the unitary representations of the rotation group restrict angular momentum to integer or half–integer values. For the 2+1 dimensional case, however, the rotation group $SO(2)$ admits unitary representations with continuous values of angular momentum. Thus there is no inconsistency in anyons with fractional spin.

Anyons have a precise analog, called *cyons*, which exist in 3+1 dimensions and make the properties of anyons much more intuitive.[26,39] A cyon is a charged, infinitely thin, infinitely long solenoid. When one cyon winds by an angle φ around another, an Aharonov-Bohm phase is induced in the wavefunction:

$$e^{2iqe\int A\cdot d\ell} = \exp\left(\frac{iqe\Phi\varphi}{\pi}\right) \tag{2.1}$$

where q is the charge (measured in units of e) and Φ is the flux (measured in units of $\hbar c/e$) carried by each cyon. The extra factor of two is present since the phase occurs in the combined wave function of the two cyons and since we have two charges moving relative to two flux tubes. Thus (provided the flux is not quantized precisely in units of $1/q$) these Aharonov-Bohm phases can simulate fractional statistics. In addition, the field contributions to the total angular momentum can simulate fractional spin.[40] Indeed if we replace (by hand) the Coloumb interaction between cyons by hard-core repulsion, and confine our observations to a planar slice, then cyons *are* anyons. There is clearly much to be gained by thinking about the strange properties of anyons in the familiar language of abelian gauge interactions.

One should keep in mind that cyons have nothing to do with charges and solenoids in the real world. For any real solenoid there is a return flux, whose effects precisely cancel the apparent fractional angular momentum and phases. Put another way, fractional statistics requires non-contractable loops, which arise because the idealized flux tubes of cyons are line singularities in coordinate space. These can not be obtained as a limit of finite solenoids.

It is convenient to distinguish the charge and flux carried by cyons from real electromagnetic charges and fluxes. We will denote these as "statistics" or "fictitious" charge and flux, to emphasize that they are artificial constructs introduced simply to mock up the fractional statistics of anyons. One can consider cyon/anyons which carry both a statistics charge and a real electromagnetic charge. Such charged anyons are responsible, in fact, for anyon superconductivity. To avoid confusion in this case we will use a_μ to denote the statistics vector potential, and A_μ to denote the electromagnetic potential. It is convenient, however, to measure both statistics and electromagnetic charge in the same units e.

With a symmetric gauge choice, the fictitious gauge potential associated with an anyon with unit statistical charge and with statistical flux γ located at the origin can be written[41]:

$$\mathbf{a}(\mathbf{r}) = \frac{\gamma}{2\pi} \frac{\hbar c}{e} \hat{z} \times \frac{\mathbf{r}}{r^2} = \frac{\gamma}{2\pi} \frac{\hbar c}{e} \frac{1}{r} \hat{\theta} \qquad (2.2)$$

where θ is the azimuthal angle. (In this review we shall often use units in which $\hbar = c = 1$.) Note that $\mathbf{a}(\mathbf{r})$ is the gauge potential due to a magnetic flux of strength γ/e located precisely at the origin. Thus to each anyon, we can assign a magnetic flux of strength γ/e. The many–anyon generalization of this is:

$$\mathbf{a}(\mathbf{r}_i) = \frac{\gamma}{2\pi} \frac{\hbar c}{e} {\sum_j}' \nabla_i \theta_{ij} \qquad (2.3)$$

where $\theta_{ij} = tan^{-1}((y_j - y_i)/(x_j - x_i))$ is the relative angle between i and j, and the primed sum indicates that $j = i$ is excluded. From this one can write down a many-body hamiltonian for a gas of free anyons with statistics parameter γ and mass m:

$$H(\gamma) = \sum_i \frac{1}{2m} \left(\mathbf{p}_i - \frac{e}{c} \mathbf{a}(\mathbf{r}_i) \right)^2 \qquad (2.4)$$

Although this hamiltonian represents free anyons, there is an effective long–range (nonlocal) interaction due to fractional statistics. The free anyon gas is,

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effectively, a highly nontrivial dynamical system. Of course, we can eliminate the azimuthal statistics vector potential by a (singular) gauge transformation.[24] However in this gauge, called *anyon gauge*, the wavefunctions will not be single-valued.

An important feature of anyon systems is that they generically violate parity and time–reversal invariance. This is because the effect of either a P or T transformation is to flip the sign of the relative angle between two anyon trajectories. Thus a system of anyons with statistics parameter γ is transformed into a system with statistics $-\gamma$, which is inequivalent except in the cases $\gamma=0$ or π. This has the immediate and important phenomenological implication that anyon superconductivity, if it exists, is linked to spontaneous P and T breaking.[7] It should be noted, however, that it is possible to avoid this link in a system with "mirror" anyons.[42]

As we have seen, anyons can exist as fundamental particles only in 2+1 space-time dimensions. In addition, though, there is nothing to prevent anyons from existing as quasiparticles in quasi–planar condensed matter systems. In fact, there is considerable experimental evidence for the existence of anyonic quasiparticles in Fractional Quantum Hall heterojunctions.[43-46] It is the existence of anyon quasiparticles in real systems that makes anyonic superconductivity an exciting field, and not merely a dry exercise in mathematical physics. To some extent anyon superconductivity is best studied in a particular condensed matter context, such as variants of the Hubbard model, in which it is hypothesized to occur. Certainly many details of anyon behavior, and perhaps even important qualitative features, can only be uncovered in this way. As we will demonstrate in this review, however, there is much that can be learned by studying anyon systems in a 2+1–dimensional point particle idealization. In particular, the Chern–Simons formalism, described in this section, gives a powerful and intuitive description how anyon superconductivity occurs, independent of the of the underlying physics that produces anyons.

3. ANYONS AND HIGH Tc SUPERCONDUCTIVITY

In his pioneering work on high T_c superconductivity, Anderson[46] pointed out the two-dimensional nature of the copper oxide materials, the importance of the strong magnetic ordering, and the possibility of exotic excitations. Immediately afterwards Kivelson, Rokhsar and Sethna[47] showed that Anderson's "resonating valence bond" picture implies the existence of "spinons", neutral spin 1/2 fermions, and "holons", charged spinless bosons. Condensation of the holons was proposed as a possible mechanism for high T_c superconductivity. Dzyaloshinskii, Polyakov and Wiegmann[48] proposed another description involving bosonic composites consisting of charge carriers and solitons of a σ model with a Hopf term. A major development in the field occurred with the works of Laughlin and his collaborators,[2-8] who proposed that high T_c superconductivity could result from condensation of bosonic composites of *anyonic* quasiparticles. Several models were subsequently proposed which contained such anyonic quasiparticles. The most common such model was based on the so called flux phase of the Hubbard Model first studied by Affleck and Marston.[49] In this chapter, following the latter reference, we describe some of these suggestions as to how anyons may arise in high T_c materials.

The suggestion that high T_c materials should be described by a tight binding model such as the Hubbard model was first suggested by Anderson.[46] In the Hubbard model we imagine electrons with their two possible spin orientations tightly bound to atomic sites. The hamiltonian includes two terms. One term allows hopping of the electrons between sites and the other discourages electrons (even those of opposite spin) of occupying the same site. Let $c_{i\sigma}^{\dagger}$ and $c_{i\sigma}$ be the creation and annihilation operators for an electron of spin σ at site i. The Hubbard hamiltonian is then given by:

$$H = t \sum_{<i,j>} (c_{i\sigma}^{\dagger} c_j^{\sigma} + h.c.) + (\frac{U}{2}) \sum_i (n_i - 1)^2 \qquad (3.1)$$

where $n_i = c_{i\sigma}^{\dagger} c_i^{\sigma}$ is the number operator for electrons at site i. Although the model as written clearly prefers precisely one electron per site (neither less nor more) we could easily alter this situation by adding a chemical potential to the hamiltonian proportional to $\sum_i n_i$ which would shift the preferable value of n_i to any other value. Physically the cases of interest are close to the case of half filling, $n_i = 1$. High T_c materials at half filling are not superconductors. Instead these materials are generically antiferromagnets. In order to make a superconductor one

must dope these materials. The net effect of this doping is to remove some free electrons from the system. Thus in order to discuss superconductivity we should consider a Hubbard model somewhat away from half filling.

It is well known [50] that in the limit of large U (i.e. $U >> t$) the Hubbard model, at half filling, reduces to the Heisenberg antiferromagnet with an antiferromagnetic coupling $J = 4t^2/U$. It is often useful to discuss a hybrid of the Heisenberg model and the Hubbard model with a hamiltonian of the form:

$$H = t \sum_{<i,j>} (c_{i\sigma}^{\dagger} c_j^{\sigma} + h.c.) + (\frac{U}{2}) \sum_i (n_i - 1)^2 + J \sum_{<i,j>} S_i \cdot S_j \qquad (3.2)$$

where $S_i = \frac{1}{2} c_i^{\dagger} \sigma c_i$ are the electron spin operators at site i and σ are the Pauli spin matrices. This hybrid model is often called the t-J model.

The Hubbard model and its extensions such as the t-J model are quite difficult to solve in 2+1 dimensions. Even the large U limit of the half filled Hubbard model which, as we have said, is equivalent to a Heisenberg antiferromagnetic model, cannot be solved analytically. It is relatively recently that numerical studies have confirmed that the ground state of the quantum Heisenberg antiferromagnet in 2+1 dimensions is, in fact, antiferromagnetic i.e. that the ground state is the Néel state with a mean, nonzero, staggered magnetization. This state violates both parity P and time reversal invariance T but it is not a candidate for an anyonic model since parity, combined with translation by one site, remains a symmetry of the model. In particular in the continuum limit of the theory both P and T will be good symmetries and no anyonic statistics will be present. In order to get anyons we need a model which violates P and T at large distances as well.

Since the Hubbard model is so difficult to analyze it has been useful to extend it and study the possible phases of the model as a function of the strengths of several other possible interactions. The extension to the t-J model is one such possibility. Other extensions which have proven very useful are the inclusion of a biquadratic spin-spin interaction proportional to $\sum (S_i \cdot S_j)^2$ and the inclusion of a next to nearest neighbor antiferromagnetic interaction which tends to frustrate the antiferromagnetic order. The biquadratic interaction has been used by Affleck and Marston [49] to stabilize the P, T invariant flux phase of the large N Hubbard model (i.e. the Hubbard model generalized to a 'spin' group SU(N) for large N). The next to nearest neighbor interaction leads to the best known way of generating the P, T noninvariant flux phase discussed by Wen, Wilczek and Zee.[9]

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To understand the possible origin of a P,T noninvariant flux phase we shall first describe briefly the mean field theory which leads to the P,T *invariant* flux phase of Affleck and Marston.[49] The first step is to rewrite the t-J hamiltonian, using Pauli matrix identities, in the following form:

$$H = t \sum_{<i,j>} (c_{i\sigma}^{\dagger} c_j^{\sigma} + h.c.) + (\frac{U}{2}) \sum_i (c_{i\sigma}^{\dagger} c_i^{\sigma} - 1)^2 - (J/2) \sum_{<i,j>} (c_{i\sigma}^{\dagger} c_j^{\sigma}) \cdot (c_{j\beta}^{\dagger} c_i^{\beta}) \quad (3.3)$$

where an additional "chemical potential" term proportional to $\sum n_i$ has been suppressed. We next write the partition function in the path integral form as follows:

$$\mathcal{Z} = \int \mathcal{D}c \mathcal{D}c^{\dagger} \exp\left(- \int_0^{\beta} d\tau \left[H + \sum_i c_{i\sigma}^{\dagger} \frac{dc_i^{\sigma}}{d\tau} \right] \right) \quad (3.4)$$

The next step is to replace the (quartic) four-fermion terms in the action with (cubic) Yukawa interactions by introducing two new scalar fields: a real scalar field ϕ_i at each site and a complex scalar field χ_{ij} on each link. The path integral (3.4) can now be replaced with an integral over the fields c, $c^{\dagger}$, ϕ and χ with the lagrangian

$$L = \sum_j c_{j\sigma}^{\dagger} \frac{dc_j^{\sigma}}{d\tau} + t \sum_{<j,l>} (c_{j\sigma}^{\dagger} c_l^{\sigma} + h.c.) + \sum_j (\frac{1}{2U} \phi_j^2 + i\phi_j (c_{j\sigma}^{\dagger} c_j^{\sigma} - 1))$$

$$+ \sum_{<j,m>} (\frac{2}{J} |\chi_{jm}|^2 + \{\chi_{mj} c_{j\sigma}^{\dagger} c_m^{\sigma} + h.c.\}) \quad (3.5)$$

The action (3.4) can be obtained from this action by performing the Gaussian integration over ϕ and χ. This transformation is known as the Hubbard-Stratonovich transformation.[51] Note that ϕ_i is related to n_i and that χ_{ij} is related to $c_i^{\dagger} c_j$. An additional chemical potential term can be added either as a term proportional to n_i or as a term proportional to ϕ_i. This lagrangian (3.5) forms the basis for the mean field analyses of the Hubbard model. (Of course we may still want to add to it additional terms such as biquadratic spin interactions and next-to-nearest neighbor antiferromagnetic interactions on which variants of the Hubbard-Stratonovich transformation can be performed.)

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It is useful to note that the lagrangian (3.5) has a U(1) gauge symmetry in the Heisenberg limit $t \to 0$, $U \to \infty$ with J fixed. In fact ϕ_i acts like a temporal gauge field and χ_{ij} acts like a spatial link operator. The gauge symmetry is simply $c_j \to \exp(i\theta_j)c_j$ with $\phi_j \to \phi_j - d\theta_j/d\tau$ and $\chi_{jm} \to \chi_{jm}\exp(i(\theta_j - \theta_m))$.

The lagrangian (3.5) is analyzed in the mean field approximation by first formally integrating out the electron fields c and $c^{\dagger}$. The result is an effective action which is a function of the scalar fields ϕ and χ. The next step is to do a saddle point approximation to the integral over ϕ and χ by attempting to minimize the effective action with respect to ϕ and χ. This approximation can be justified in a large N limit (see ref. [49]). The simplest way to find such extrema is to make an ansatz for ϕ and χ assuming certain symmetry properties for them and then to see whether the resulting fields are in fact extrema and whether these extrema are stable. The ansatz which is relevant in discussions concerning anyons results in the so called flux phase. We let the magnitude of χ be constant throughout the lattice and we adjust the phases of χ so that there is a flux π through each plaquette. More precisely the product Π of χ around any plaquette is equal to $e^{\pm i\pi} = -1$. In the Heisenberg limit we cannot allow χ itself to have a nonzero expectation value since it is not gauge invariant but the plaquette operator Π is gauge invariant. It is shown in ref. [49] that for some small nonzero doping, and when a nonzero biquadratic interaction is included, the above ansatz leads to a stable extremum of the action.

The flux phase described above does not lead to anyonic statistics in the continuum limit since it violates neither P nor T. In ref. [9] a modification of the flux phase of ref. [49] is proposed which does violate P and T. Imagine a plaquette on a square lattice with points labeled 1 through 4 in a clockwise direction beginning with the lower left hand corner. The plaquette operator Π described above is given by the product $\chi_{12}\chi_{23}\chi_{34}\chi_{41}$ with each χ_{ij} being related, via the Gaussian integral (3.5), to $c_{i\sigma}^{\dagger}c_j^{\sigma}$. We could introduce a diagonal operator χ_{31} equal to $c_{3\sigma}^{\dagger}c_1^{\sigma}$. We can then consider the product of the χ's (i.e. the flux through) the triangle [1,2,3,1]. This is given by $\chi = \chi_{12}\chi_{23}\chi_{31}$. One now considers a modified flux phase in which the expectation value of χ is nonzero and equals $\exp(i\pi/2)$. There is now a flux $\pi/2$ through each of these triangles. Such a ground state breaks parity and time reversal invariance even in the continuum limit and is thus a candidate for an anyonic model. It is shown in ref. [9] that the introduction of a sufficiently strong next to nearest neighbor antiferromagnetic interaction can stabilize this phase. In such a case one can analyze the electronic spectrum about the extremum for χ. One finds that the electrons acquire a parity noninvariant mass. The resulting low energy

effective theory is that of two massive electrons (actually quasiparticles with the electron's quantum numbers) coupled to a fictitious gauge field. Since the theory is not parity invariant, a Chern–Simons term will be generated in one loop leading to anyonic statistics for the quasiparticles. It is expected that these quasiparticles will have a statistics parameter $\frac{1}{2}$ and they are thus called semions.

It should be clear from the above discussion that although anyons are a possible outcome of modified Hubbard models, there is no compelling reason to suppose that these models do in fact violate P and T and thus lead to anyons. It is merely the possibility that an anyonic system may occur in such cases combined with the fact that anyonic systems are superconductors which makes the subject of anyonic superconductivity so fascinating.

4. CHERN–SIMONS DESCRIPTION OF ANYONS

In chapter 2 we saw that an anyon with statistics parameter γ could be described by introducing a fictitious "statistical" gauge field a_μ and by assigning to each anyon a "statistical" magnetic flux of strength γ/e. We wrote down the hamiltonian (2.4) for a system of (otherwise) noninteracting anyons in terms of the gauge potentials generated by these magnetic fluxes. The statistical gauge field has no dynamics except for the dynamics implied by its being attached to a dynamical anyon.

The above considerations allows for a very compact way of implementing anyonic statistics in quantum field theory. We begin by describing anyons in the language of relativistic quantum field theory. For non-interacting anyons, the non-relativistic limit of such a theory will precisely describe a nonrelativistic system of anyons. The idea is to begin with a single massive two–component fermion field ψ and with a statistical gauge field a_μ. The lagrangian will have a kinetic term for the fermion field but none for the gauge field since it is not dynamical. Anyonic statistics are implemented by introducing a Chern–Simons term into the action. The lagrangian is given by[52,53]

$$\mathcal{L} = i\overline{\psi}(\slashed{D} - m)\psi - \frac{\theta}{2}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda. \tag{4.1}$$

where $D_\mu = \partial_\mu - iea_\mu$ and θ is called the Chern–Simons coefficient. Note that the Maxwell term for a_μ is absent. The equation of motion which is derived by varying

a_0 is simply

$$b = \frac{e}{\theta}\rho \tag{4.2}$$

where $b = \nabla \times a$ is the magnetic field associated with a and $\rho = \psi^\dagger \psi$ is the fermion number density operator. Thus to every fermion (with $\rho V = 1$) there is attached a "magnetic" flux of magnitude e/θ. It follows that each fermion is actually an anyon. When two such fermions are interchanged one picks up a phase $\exp(-i\pi)$ since the particles are fermions but one also picks up the extra phase due to the magnetic flux of the fermions. This extra phase is only half the usual Aharonov–Bohm phase, due to the presence of the Chern–Simons term.[37] Thus the statistics parameter becomes

$$\gamma = -\pi + \frac{e^2}{2\theta} = -\pi(1 - \frac{1}{N}) \tag{4.3}$$

where the parameter $N = 2\pi\theta/e^2$ has been introduced for convenience. Note that when $N=1$ we have bosons. When $N=2$ the particles are called semions and when $N \to \infty$ we recover fermions. In principle there is no reason for N to be an integer but we shall see that superconductivity occurs only for integer N. The case $N=2$ when the anyons are semions is the case of most interest in applications to real materials.

The argument presented above for the relationship between anyons and Chern Simons theory is quite heuristic. In fact the Chern–Simons theory above describes anyons with an additional interaction due to the "other" equation of motion $\vec{\mathcal{E}} \propto \vec{J}$ which is obtained by varying a_i. There is, however, a wealth of literature on this subject.[54–60] Most authors agree that the Chern–Simons lagrangian above describes a system of anyons although there is still some remaining controversy on this subject. In this paper we shall take the Chern–Simons theory as our definition of an anyonic system and we shall be concerned exclusively with studying this model.

The next important step in studying an anyonic system is to introduce a finite density of anyons. The simplest way to do this in the context of the field theory above is to introduce a chemical potential term into the hamiltonian replacing the hamiltonian H with $H - \mu\psi^\dagger\psi$. This analysis is most conveniently implemented in the Euclidean formulation of the quantum field theory in which we write the zero temperature partition function, Z, evaluated by the path integral expression

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}a_\mu \, exp(-S_E) \tag{4.4}$$

where S_E is the Euclidean action given by

$$S_E = \int d^3x \left(\overline{\psi}(\not{D} - m)\psi + i\frac{\theta}{2}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda - \mu\psi^\dagger\psi \right) \qquad (4.5)$$

Note that the chemical potential term has been introduced into the action to implement a finite density of anyons. This particular description of anyons will be used extensively in this review, although other descriptions will also be considered.

5. THE CRITERION FOR ANYONIC SUPERCONDUCTIVITY

Now that the notion of an anyon is established and the field theoretic description presented, the next question to address is whether an anyonic gas is a superfluid, or a superconductor if it is electromagnetically charged. This is the central topic of this review. Before delving into detailed analyses of this question we should first clarify precisely what we mean by superconductivity of anyonic systems. A related question is whether we can find a simple criterion which can be used to determine whether a gas of anyons is a superfluid (or a superconductor). The ultimate test of whether a material is a superconductor is whether it admits dissipationless current flow and whether it has a Meissner effect. A superfluid should, of course, have dissipationless flow as well as other features of superfluids such as the fountain effect and vortex formation. In conventional (such as BCS) superconductors and in conventional superfluids (such as ^{4}He) the generic signal of superconductivity and superfluidity is the spontaneous breaking of a $U(1)$ symmetry. In BCS theory this results in the breaking of the electromagnetic $U(1)$ symmetry which leads to a mass for the photon and a Meissner effect. In liquid Helium the symmetry breaking leads to a Goldstone boson which is the sound wave in the superfluid. In this case, a generic feature of the theory is the presence of a pole in the current-current correlator due to the Goldstone mode. In the case of the superconductor, this pole is present only in the one–particle–irreducible current-current correlator. The full correlator has a massive pole due to the acquired photon mass.

As a result of the generic presence of symmetry breaking in superfluids and superconductors, it is tempting to use this as a definition of these phenomena. This is inadequate since in 1+1 dimensional systems as well as in 2+1 dimensional systems at nonzero temperature, symmetry breaking cannot occur due to the Coleman-Mermin-Wagner theorem.[61] But despite this theorem the Goldstone mode is still

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present. It simply destroys the long range order. Thus the presence of a Goldstone mode seems like a better criterion for superfluidity than simply symmetry breaking. Furthermore we shall see during our detailed analysis of anyonic systems that there is no obvious symmetry which is broken. Yet the system exhibits a Meissner effect and supercurrent flow. The massless mode in the current-current correlator is, nonetheless, present. Of course the presence of a massless mode (even one with a dispersion relation $\omega \propto k$ for small k) is clearly not sufficient for superfluidity. We must, of course, demand that the system be a liquid or a gas and not a solid (i.e. that translation symmetry be unbroken). Furthermore, if the system contains fermions, there must also (generically) be a gap in the fermion spectrum. (We shall not concern ourselves with gapless superconductors which are an exception to this rule.) This gap serves two purposes. First of all it stabilizes the Goldstone mode against decay into fermion–hole pairs. Secondly it ensures that the production of fermion–hole pairs does not dissipate the superflow as would occur in Fermi-liquid theory.

In summary, a reasonable criterion for superfluidity is the presence of a massless mode in the current-current correlation function combined with a gap in the fermion spectrum. In the case of charged anyons this will naturally lead to a Meissner effect but, of course, we should show this as well.

We shall see in chapter 6 that it is quite straightforward to show that there is a gap in the fermion spectrum when $N = 2\pi\theta/e^2$ is an integer. In the remainder of this chapter we would like to establish a simple field theoretical criterion first proposed by Banks and Lykken[14] which ensures the presence of a massless pole in the current–current correlator.

5.1. The renormalized Chern–Simons term and superfluidity

A massless pole in the current-current correlator will occur if there exists a mode (which couples to the current) whose dispersion relation $\omega(k)$ is such that $\omega \to 0$ as $k \to 0$ or, more precisely, $\omega \propto k$ for small k. We are thus interested in studying the small ω and k regime of the theory. Our technique for finding a suitable criterion for superfluidity (or superconductivity) is to consider the "bare" action (4.1) or, its Euclidean version, (4.5), and to evaluate the low energy effective action for this theory. There are many ways to do this but, for definiteness, we imagine integrating over all the fermionic degrees of freedom and then evaluating the effective action for the gauge fields. This effective action can be evaluated as a series in powers of the gauge fields, and, if desired in a derivative (or low

momentum) expansion. In perturbation theory, the coefficient of the term in the effective action which is of order j in powers of a_μ is equal to the sum of all (amputated) one–particle–irreducible graphs with j gauge field legs (plus, of course, any such term which is present in the action to begin with). In particular the one–particle–irreducible current-current correlation function is the coefficient of the $a_\mu a_\nu$ (i.e. 2 "photon") term in the effective action. Thus by studying the low momentum effective action for this theory we can determine the one–particle–irreducible current-current correlation from which one can deduce the possible existence of a massless mode in a straightforward manner.

It turns out that we can say quite a lot about the effective action for the theory given by the action (4.5), by symmetry considerations alone. The lowest order renormalization of the effective action is of the form

$$a^\mu(k)\Pi_{\mu\nu}(k)a^\nu(-k) \tag{5.1}$$

where $\Pi_{\mu\nu}(k)$ is the vacuum polarization (one–particle–irreducible current-current correlation function). $\Pi_{\mu\nu}$ can be split into an even and an odd part:

$$\Pi_{\mu\nu}(k) = \Pi^e_{\mu\nu}(k) + \epsilon_{\mu\nu\lambda}k^\lambda \Pi_{odd}(k) \tag{5.2}$$

The even part corresponds to an induced term which is analogous to the Maxwell term in electrodynamics (but including a general momentum dependence) whereas the odd part corresponds to a renormalization of the Chern–Simons term in the effective action. At zero temperature and density the system is Lorentz invariant (but, of course, not parity invariant). This constrains $\Pi^e_{\mu\nu}$ to be of the form:

$$\Pi^e_{\mu\nu}(k) = \Pi_e(k^2)(g_{\mu\nu}k^2 - k_\mu k_\nu) \tag{5.3}$$

At nonzero temperature and density the system is no longer Lorentz invariant but it remains invariant under spatial rotations. The odd part of Π in equation (5.2) is unchanged in this case. This is related to the fact that there is no difference between the form of the Chern–Simons term in the nonrelativistic and in the relativistic $(T=\mu=0)$ case. The reason for this is that there is no gauge invariant way to split the Chern–Simons term. This, in turn, is an outcome of the fact that the Chern–Simons term is a topological (metric independent) term. The even part of Π is, of

course, modified. The most general form of Π which is rotationally invariant and depends only on the momentum k is:

$$\Pi_{\mu\nu}^e(k) = \Pi_e^a\left(\omega, \overrightarrow{k}^2\right)(g_{\mu\nu}k^2 - k_\mu k_\nu) + \Pi_e^b\left(\omega, \overrightarrow{k}^2\right)(\delta^{ij}\overrightarrow{k}^2 - k^i k^j)\delta_\mu^i \delta_\nu^j$$
$$+ \Pi_e^N\left(\omega, \overrightarrow{k}^2\right)\Theta_{\mu\nu} \tag{5.4}$$

where $\omega = k^0$ and where $\Theta_{\mu\nu}$ is an additional gauge and rotation invariant tensor which can be constructed in 2+1 dimensions and which depends on $^*k_i = \epsilon_{ij}k_j$ the dual vector of k_i. $\Theta\mu\nu$ is given by:

$$\Theta_{\mu\nu} = \begin{pmatrix} 0 & -^*k_i\overrightarrow{k}^2 \\ -^*k_j\overrightarrow{k}^2 & \omega(^*k_ik_j + k_i{}^*k_j) \end{pmatrix} \tag{5.5}$$

We are now ready to look at the contribution of Π to the effective action. Since we are interested in the low momentum behavior of the theory we now consider the leading contributions of each of the above terms at small momentum. Let us assume that each of Π_e^a, Π_e^b and Π_e^N have a nonsingular limit as $k \to 0$. This will certainly be the case when $\mu=T=0$ since the fermions have a mass. We shall see in chapter 6 that it is also true when $\mu \neq 0$ provided the Chern–Simons coefficient N is an integer so that there is a gap in the fermion spectrum. This assumption may, however, fail at nonzero temperature which somewhat complicates that analysis. The leading low momentum behavior of each of the terms in equation (5.4) is then given by replacing the various Π's by their values at $k=0$. Let us define:

$$\Pi_a = \Pi_e^a(k = 0); \quad \Pi_b = \Pi_e^b(k = 0); \quad \Pi_N = \Pi_e^N(k = 0); \Pi_o = \Pi_{odd}(k = 0) : \tag{5.6}$$

Let us also define the renormalized Chern–Simons coefficient $\theta_R = \theta - \Pi_o$ where θ is the bare Chern–Simons coefficient. Using equations (5.1), (5.4) and (5.5) and including the bare Chern–Simons term from the action (4.5) the leading low momentum terms in the effective lagrangian are given by:

$$\mathcal{L}_{eff} = \frac{1}{4}\Pi_a f^{\mu\nu}f_{\mu\nu} + \frac{1}{4}\Pi_b f^{ij}f_{ij} + \frac{i}{2}\theta_R\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda - \frac{i}{2}\Pi_N\partial_i f_{0i}\epsilon_{jk}f_{jk} \tag{5.7}$$

Note that a Maxwell term has been induced by quantum and statistical corrections and that the value of the Chern–Simons coefficient is renormalized. The last term

in equation (5.7) is higher order in derivatives and we include it here both for completeness and because it turns out to lead to important low energy consequences which are discussed in chapter 10.

The central result of this section is that *a necessary and sufficient condition for the existence of a massless pole in the current-current correlator is that θ_R vanish*; assuming, of course, that there are no singularities in the vacuum polarization at zero momentum which could introduce nonlocal terms into the effective action. There are several ways to see this result. The simplest is to consider the (Euclidean) equation of motion for the statistical gauge field in the presence of a current density J_μ in the long wavelength and linear approximation. From equation (5.7) it is given by:

$$\Pi_a \partial_\mu f^{\mu\nu} + \Pi_b \delta_j^\nu \partial_i f^{ij} + i\theta_R \epsilon^{\nu\alpha\beta} \partial_\alpha a_\beta = J^\nu \tag{5.8}$$

When $\theta_R = 0$ this is the equation of motion of a massless (gauge) particle with a dispersion relation (in Minkowski space)

$$\omega = vk \quad \text{with} \quad v^2 = \left(1 + \frac{\Pi_b}{\Pi_a}\right) \tag{5.9}$$

Now according to equation (5.8) this massless particle couples to the statistical current J^μ. It thus follows that there if θ_R vanishes there will be a massless pole in the current-current correlation. Note, however that, as discussed above, the regularity of the vacuum polarization is essential. For example a term in Π_e^a $\propto 1/k^2$ as $k \to 0$ would lead to a term $\propto a^\mu$ in equation (5.8) which would lead to a Higgs–like mass for this mode even when $\theta_R = 0$. If, on the other hand, $\theta_R \neq 0$ then equation (5.8) is precisely the nonrelativistic generalization of the equation of motion for a topologically massive gauge field in 2+1 dimensions.[52,53] Both charges and currents are screened. The mass of the mode is θ_R/Π_a and its spin is $\theta_R/|\theta_R|$. It follows that the current–current correlator cannot have a massless pole since, if it does, the statistical photon Green's function would have a massless pole which lead to long range forces which are not present in equation (5.8).

There are several other ways of demonstrating this relationship between the vanishing of θ_R and the occurrence of a massless excitation coupled to the current. In particular it is possible to demonstrate this perturbatively. By summing the geometric series which relates the one–particle–irreducible vacuum polarization to the full current–current correlator it can be shown that a pole exists in the full correlator if and only if the renormalized Chern–Simons term vanishes (assuming the now familiar regularity conditions). This is discussed in detail in ref. [23].

5.2. THE RENORMALIZED CHERN–SIMONS TERM AND SUPERCONDUCTIVITY

Having established a reasonably simple criterion for anyonic superfluidity we can now show that this condition, namely the vanishing of θ_R, is also a necessary and sufficient condition for superconductivity if the anyons are electrically charged and thus coupled to the photon field. Electromagnetism is introduced into the system by introducing the electromagnetic field A_μ and its field strength $F_{\mu\nu}$ and by replacing the action (4.5) with the modified Euclidean action

$$S_E = \int d^3x \left(\bar{\psi}(\slashed{D} - m)\psi + i\frac{\theta}{2}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \mu\psi^\dagger\psi \right) \quad (5.10)$$

with $D_\mu = \partial_\mu - iea_\mu - igeA_\mu$ where g is the electromagnetic charge of the anyon measured in units of e. We can now proceed, as before, by integrating out the fermion fields and considering the low energy effective action for both the electromagnetic and the statistical gauge fields. As discussed above we shall concentrate on those terms which are quadratic in the gauge fields. Since both a_μ and A_μ couple to the current in an identical fashion except with a different coupling strength, it is the same vacuum polarization tensor $\Pi_{\mu\nu}$ which couples a to a, A to A and a to A. $\Pi_{\mu\nu}$ is, of course, just the one–particle–irreducible fermion current-current correlator. We again assume that Π has the decomposition (5.4) and that the various coefficients are nonsingular at zero momentum. Using the definitions (5.6) we can write the low momentum Euclidean effective lagrangian as:

$$\mathcal{L}_{eff} = \frac{1}{4}(1 + g^2\Pi_a)F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}g^2\Pi_b F_{ij}F^{ij} + \frac{1}{4}\Pi_a f^{\mu\nu}f_{\mu\nu} + \frac{1}{4}\Pi_b f^{ij}f_{ij}$$

$$+\frac{i}{2}\theta_R\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda - \frac{i}{2}g^2\Pi_o\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda - \frac{i}{2}g\Pi_o\epsilon^{\mu\nu\lambda}(A_\mu\partial_\nu a_\lambda + a_\mu\partial_\nu A_\lambda)$$

$$+\frac{1}{2}g\Pi_a F_{\mu\nu}f^{\mu\nu} + \frac{1}{2}g\Pi_b F_{ij}f^{ij} - \frac{i}{2}\Pi_N\partial_i(f_{0i} + gF_{0i})\epsilon_{jk}(f_{jk} + gF_{jk}) \quad (5.11)$$

Note that, as expected, the parity violation in the anyonic sector now appears also in the electromagnetic sector.

Just as in the case of neutral anyons there are several ways of analyzing this effective action. Since the result which we need, namely that for $\theta_R = 0$ the photon acquires a mass, is a central argument in the discussion of anyonic superconductivity, we shall discuss several of these approaches to analyzing this problem. The

most straightforward way is, just as before, to study the equations of motion for the two gauge fields. If we are interested only in the dynamics of the electromagnetic field (for example if we want to study the possibility of a Meissner effect) it is simplest to solve the field equations for a_μ as a functional of A_ν and then to consider the effective action for the electromagnetic field alone. (This is equivalent to having integrated both the fermion fields and the statistical gauge fields out at the beginning and considering an effective action for the electromagnetic field alone.) If we were to follow the analysis for neutral anyons faithfully we would now ignore the terms in equation (5.11) which are proportional to Π_N since they are cubic in derivatives. It turns out, however, that this is not such a good idea since these terms contribute to the effective action for A in the same order as some of the lower order terms. The derivation of the effective action for A for general θ_R is carried out in detail in ref. [42]. In this chapter, however, we shall restrict our attention to the case when $\theta_R = 0$ and we shall provide several arguments to show that, in this case, the photon acquires a mass term which leads to the Meissner effect and to superconductivity. The converse of this is demonstrated in ref. [42][†]

The procedure just outlined is straightforward, though quite tedious. Fortunately there is a very useful trick for solving for a in terms of A. The idea is to change the dependent variables in the problem from $a_\mu(x)$ to $f_{\mu\nu}$. This can be done[14,26] by introducing a lagrange multiplier field $\tilde{\varphi}$ which imposes the Bianchi identity constraint $\partial_\alpha \, ^*f^\alpha = 0$ (where $^*f^\alpha = \frac{1}{2} \epsilon^{\mu\nu\alpha} f_{\mu\nu}$ is the dual of f) so that $\mathcal{L}_{eff} \to \mathcal{L}_{eff} + \tilde{\varphi}\partial_\alpha \, ^*f^\alpha$.[††] The trick now is to solve only the equation of motion for the statistical gauge field, which now becomes $\delta S_{eff}/\delta f_{\mu\nu} = 0$, in terms of $\tilde{\varphi}$ and A_μ to obtain a new effective action which depends on both A and on $\tilde{\varphi}$. To simplify the discussion as much as possible at this stage we shall neglect the terms proportional to Π_N in equation (5.11) and simply mention what effect they have. More details will be presented in chapter 10. Thus, setting $\theta_R = 0$ in equation (5.11), neglecting terms $\propto \Pi_N$, solving for $f_{\mu\nu}$ and plugging the solution back into

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[†] The method of solving the equations of motion is very closely related to an alternate approach which treats the low energy effective action as a quantum action. One then performs a saddle point (or, if possible a Gaussian) integral over the gauge fields to obtain a similar result. The method which we have been discussing is more appropriate for the effective action defined as a Legendre transform of the energy.

[††] In ref. [14] a slightly different method was used to derive $\mathcal{L}_{eff}(A_\mu, \tilde{\varphi})$. There, using a path integral approach, an insertion of $1 = \int DZ_{\mu\nu}\delta(Z_{\mu\nu} - f_{\mu\nu})$ into the path integral for the effective action, leads to a similar final answer.

equation (5.11) results in the following modified effective lagrangian:

$$L_{eff}(A_\mu, \varphi) = \frac{1}{4}(1 + g^2\Pi_a)F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}g^2\Pi_b F_{ij}F^{ij} - \frac{i}{2}g^2\theta\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda$$

$$+ \frac{1}{2}\left(\partial_0\varphi + CA_0\right)^2 + \frac{1}{2}v^2(\partial_i\varphi + CA_i)^2 \tag{5.12}$$

$$+ a(\partial_0\varphi + CA_0)^*F_0 + b(\partial_i\varphi + CA_i)^*F_i$$

where we have defined $\varphi^2 = \bar{\varphi}^2/(\Pi_b + \Pi_a)$ and where v is the velocity of the Goldstone mode of equation (5.9): $v^2 = (1 + \Pi_b/\Pi_a)$; $C = \theta g/\sqrt{\Pi_a + \Pi_b}$ and $a=b = \left(g\sqrt{\Pi_a + \Pi_b}\right)$. $^*F$ is, of course, just the dual of F, and we have used the fact that $\theta_R = 0$ several times to replace Π_o with θ. Note that neither gauge invariance nor rotational invariance requires the coefficients a and b to be equal. In fact had we kept the terms $\propto \Pi_N$ in equation (5.11) we would have found that a and b are not equal. We shall discuss this further in chapter 10.

The effective lagrangian (5.12) is invariant under the gauge transformation $\varphi \rightarrow \varphi - C\Lambda$ and $A_\mu \rightarrow A_\mu + \partial_\mu\Lambda$. In fact the terms $\frac{1}{2}\left(\partial_0\varphi + CA_0\right)^2 + \frac{1}{2}v^2(\partial_i\varphi + CA_i)^2$ together with the Maxwell terms are precisely the Higgs effective action in a form known as the Stuckelberg form. In fact in the conventional Higgs mechanism with a scalar field $\chi = \rho e^{i\phi}$, the phase ϕ of the field plays precisely the role which φ plays in the above effective action. Note also that explicit photon mass terms of the form A_0A^0 and A_iA^i are present in the above effective action. Furthermore if we wish, we can now solve the equation of motion for φ to get an effective action for A alone which would also exhibit a Higgs mass for the photon.

We thus see that a massless statistical photon (i.e. one with $\theta_R = 0$) leads to a *mass* for the real photon. It also leads to P and T violating terms in the effective action.

Equation (5.12) has one additional feature which requires attention. Besides the "Higgs" mass for the photon, the photon has also acquired a "topological" mass due to the induced Chern–Simons term $\frac{1}{2}g^2\Pi_o\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda$. This mass term is different from the Higgs mass term in many respects. The most striking difference is that it violates parity and time reversal invariance. Furthermore it's presence neither results in nor implies a massless pole in the one–photon–irreducible vacuum polarization. It is simply due to an additional tensor structure which appears due to parity violation. Thus even if the *Higgs* mass is related to the breaking

of some unknown symmetry and/or Bose condensation the Chern–Simons mass is unrelated to these effects and simply reflects the parity violation in this system. A consequence of this is that the Chern–Simons mass term is present even if $\theta_R \neq 0$ (provided that Π_o does not vanish).

Why then do we not accept the presence of a Chern–Simons mass for the photon as a criterion for superconductivity? There are many reasons. The simplest explanation is that the Higgs terms in the effective action lead to the conventional phenomenology of the Landau–Ginsburg theory of superconductivity. If only a Chern–Simons term were present, the phenomena would differ significantly. For example in the absence of a Higgs field such as φ there is no natural reason for vortices to occur. Vortices are, of course, a generic feature of superconductors. Note that although there is a Meissner effect even if only a topological mass is present in the sense that a constant magnetic field is *not* a solution to the field equations its origin is quite different from the conventional effect. Furthermore in the presence of a topological mass term alone one can compute the response of the system to a constant electric field by computing the current–current correlator and one finds that there is no divergence in the current. Thus even in the absence of impurities the system does not have a supercurrent. A related problem is that if θ_R does not vanish (so that there is no Higgs mass) then a neutral system of anyons is *not* a superfluid. Thus if the anyons are charged, we do not expect superconductivity even if a topological mass is present. An additional problem with a criterion based solely on the presence of a topological mass is that such a mass is present in 2+1 dimensions only. It is difficult to see how such a mass term could be modified so that it would lead to a Meissner effect in a realistic 3+1 dimensional system[14]. This is not a problem with the Higgs form of the Meissner effect.[†] We shall thus call a system an anyonic superconductor only if a Higgs mass is present which, as we have seen, is related to the vanishing of θ_R.

This relationship between a massless statistical photon in the absence of electromagnetic coupling and a massive electromagnetic photon can also be seen perturbatively (i.e. diagrammatically). To see this, let $K_{\mu\nu}$ be the full current-current correlator in the absence of electromagnetic coupling. The presense of a Goldstone mode implies that K has a massless pole of the form $(g_{\mu\nu}q^2 - q_\mu q_\nu)/q^2$. When

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† Note that both forms of the mass term can expel magnetic fields in a direction perpendicular to the 2 dimensional plane only since this is the only magnetic field present in the 2+1 dimensional model. The Meissner effect parallel to the plane will likely result from superconductivity between the 2 dimensional planes which may be caused by some kind of a tunneling phenomenon.

electromagnetic coupling is present, K can be treated as the leading order (in the electromagnetic coupling g) one–photon–irreducible vacuum polarization for the photon. The photon propagator is estimated by summing the geometric series shown in Figure 1. In Feynman gauge the photon propagator has a term proportional to $g_{\mu\nu}$ with a coefficient $1/(q^2 - \tilde{K})$ where $\tilde{K}$ is the coefficient of $g_{\mu\nu}$ in K. Clearly if K has a massless pole then $\tilde{K}$ approaches a nonzero constant as $q \to 0$. The photon propagator will then have no massless pole and will in fact have a pole at a nonzero value of q^2. This is, of course, the Meissner effect.

An alternate but closely related way of seeing this "duality" between the fictitious and the electromagnetic gauge fields was presented by Wen and Zee.[62] Using $J^\mu = \theta \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda$ we can relate the current-current correlator to the propagator of the fictitious gauge field $C_{\mu\nu}$ as follows:

$$K_{\mu\nu} \equiv\; <J_\mu(k)J_\nu(-k)> \;= \epsilon_{\mu\alpha\beta}\epsilon_{\nu\gamma\delta}k^\alpha k^\gamma C^{\beta\delta}(k) \tag{5.13}$$

Now for $\theta_R = 0$ we showed that there is a massless pole in the current-current correlator and that the statistical photon's propagator is that of a massless photon (for small momentum). Thus in the long wavelength approximation

$$C_{\mu\nu} \sim \frac{(g_{\mu\nu} - k_\mu k_\nu / k^2)}{k^2}; \qquad K_{\mu\nu} \sim \frac{(g_{\mu\nu}k^2 - k_\mu k_\nu)}{k^2} \tag{5.14}$$

When coupling J_μ to the electromagnetic field A_μ we get in momentum space the effective action

$$\int d^3k\, K^{\mu\nu} A_\mu(k) A_\nu(-k) \sim \int d^3k\, g_{\mu\nu} A^\mu(k) A^\nu(-k) \tag{5.15}$$

We thus get a mass term for the photon. For $\theta_R \neq 0$, on the other hand, we find

$$C_{\mu\nu} \sim \frac{(g_{\mu\nu} - k_\mu k_\nu / k^2)}{k^2 + m^2}; \qquad K_{\mu\nu} \sim \frac{(g_{\mu\nu}k^2 - k_\mu k_\nu)}{m^2} \tag{5.16}$$

where m is given by θ_R/Π_e. The photons will now acquire an effective Maxwell term for without a mass term.

$$\int d^3k\, K^{\mu\nu} A_\mu(k) A_\nu(-k) \sim \int d^3k \frac{(g_{\mu\nu}k^2 - k^\mu k^\nu) A^\mu(k) A^\nu(-k)}{m^2}$$
$$\sim \int d^3k \frac{F_{\mu\nu}F^{\mu\nu}}{m^2} \tag{5.17}$$

This phenomenon of the "duality" between the two gauge fields has been studied extensively.[62–64]

Various other derivations of the effective action have been suggested.[15,24,55−64]
A brief summary of some of these methods will also be included in chapter 6.

To summarize, it was shown in this section that the vanishing of the renormalized Chern–Simons term is a necessary and sufficient condition for having a massless pole in the current current correlator. This result holds for both the relativistic and non-relativistic domains i.e. both at zero and at finite density. The only possible loophole occurs at finite temperature where there is the possibility of singularities in the even part of the vacuum polarization. We have also shown that the existence of this massless mode leads to a mass term for the photons and to a superconductivity in the low energy effective action for the theory which is, in fact, the Landau–Ginsburg effective action.

6. THE MASSLESS MODE

Our goal in this chapter is to show the existence of a massless pole in the anyonic current-current correlator. In the previous chapter we saw that in the context of a field theoretic formulation of anyons (i.e. the Chern–Simons theory) a condition for the existence of such a massless pole is that the *renormalized* Chern–Simons term for this theory vanish. In other words if the quantum and statistical corrections to the Chern–Simons coefficient precisely cancel the coefficient in the lagrangian, we are assured that a pole is present in the current-current correlation function.

We begin this chapter with the calculation of the one loop renormalization of the Chern–Simons term at nonzero density. This is presented in detail in section 6.1. We show that the renormalized Chern–Simons term at nonzero density vanishes when $2\pi\theta/e^2$ is an integer. This proof is extended to all orders in perturbation theory in chapter 8. In section (6.2) the renormalization of the CS term is rederived using Schwinger's proper time integral method. The remaining sections contain a brief discussion of two other distinct, though not unrelated, methods of deducing the presence of the massless mode. In section 6.3 we discuss the random phase approximation (RPA), which was originally carried out in ref. [5] and was later presented in ref. [15]. The RPA is closely related to the method of section 6.1. We then discuss briefly the derivation based on an "inhomogeneous expansion".[24] It should be noted that Fradkin[65] has obtained results using lattice methods which are in complete agreement with the field theoretic results reviewed below.

6.3. Renormalization of the Chern–Simons term at one loop

In this section we consider a system of anyons described by a single two-component massive fermion field coupled to a CS fictitious gauge field a_μ. We introduce some finite, nonzero density of anyons by adding a chemical potential term to the action. The Euclidean path integral expression at zero temperature ($T{=}0$) for the partition function of the Chern–Simons theory at finite chemical potential μ is given by:

$$Z = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}a_\mu \, exp(-S_E) \qquad (6.1)$$

with

$$S_E = \int d^3x \left(\overline{\psi}(\slashed{D} - m)\psi + i\frac{\theta}{2}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda - \mu\psi^\dagger\psi \right) \qquad (6.2)$$

where $D_\mu = \partial_\mu - iea_\mu$. We shall work throughout with a nonnegative chemical potential μ. Our ultimate goal is to show that the renormalized value of θ vanishes at nonzero density whenever $2\pi\theta/e^2$ is an integer

Feynman Rules

Our first step is to derive Feynman rules for this system for a perturbation expansion in e^2/θ. We choose to work in Coulomb gauge ($\partial_i a^i{=}0$). We proceed by integrating out the a_0 field, which simply gives the Gauss law constraint $\delta(\mathcal{B} - \frac{e}{\theta}\psi^\dagger\psi)$. This delta function now allows us to do the integral over a_1 and a_2 by setting

$$a_i = -\frac{e}{\theta}\epsilon_{ij}\frac{\partial^j}{\nabla^2}\psi^\dagger\psi \qquad (6.3)$$

This leads to the following effective 4-fermi theory:

$$\int \mathcal{D}\psi \mathcal{D}\overline{\psi} \, exp[- \int d^3x \left(\overline{\psi}(\slashed{\partial} - m - \mu\gamma^0)\psi + \frac{ie^2}{\theta}(\overline{\psi}\gamma^i\psi)\frac{\epsilon_{ij}\partial^j}{\nabla^2}(\overline{\psi}\gamma^0\psi) \right)] \quad (6.4)$$

We use the gamma matrices $\gamma_1{=}\sigma_1$, $\gamma_2{=}\sigma_2$, and $\gamma_0{=}\sigma_3$ where σ_i are the Pauli spin matrices. Notice that the effect of the chemical potential is simply to replace ∂_0 by $\partial_0{-}\mu$. We thus define $\tilde{\partial}_\nu$ to be equal to ∂_ν unless $\nu{=}0$ in which case $\tilde{\partial}_0 = \partial_0{-}\mu$.

We begin by studying the fermion propagator $S(x, x')$ for this theory. The bare fermion propagator $S_0(x, x')$ is simply $1/(\bar{\partial} - m)$. Perturbative corrections to this propagator will be computed from the Feynman rules which arise from the path integral expression above. The vertices for this theory are 4-point fermion vertices which connect a current J_i to a density J_0. Each such vertex carries a factor $-i\frac{e^2}{\theta}(\epsilon_{ij}\partial^j/\nabla^2)$. This vertex is shown in Figure 2a. It is often convenient to represent this vertex via the diagram of Figure 2b in which the standard QED vertex with a value $+ie\gamma_\mu$ is used and in which a Chern–Simons propagator is explicitly shown. This propagator (in Coulomb gauge) is nonzero only when connecting a γ_i vertex with a γ_0 vertex and then has a value $\frac{i}{\theta}(\epsilon_{ij}\partial^j/\nabla^2)$.

Tadpole Corrected Perturbation Theory

Our first observation when evaluating perturbative corrections to the fermion propagator is that, in the presence of a nonzero chemical potential $\mu \neq 0$, the tadpole graphs such as those of Figure 3 do not vanish. These tadpoles are nonvanishing since $< J_0 > = \rho_0$ is nonzero when μ is nonzero. Note that the amputated tadpole is precisely equal to the mean density ρ_0. We can thus compute the entire contribution of a single tadpole to the propagator as a function of the mean density ρ_0. We shall see that the net effect is to modify the fermion propagator to the propagator of a fermion (at finite μ) in a constant magnetic field proportional to the density.

Using the Feynman rules described above each tadpole contributes an amount

$$-i\frac{e^2}{\theta}\gamma^i\epsilon_{ij}\frac{\partial_j}{\nabla^2}\rho_0 \tag{6.5}$$

to the fermion propagator. This single tadpole contribution can be written in the suggestive form $ie\gamma^i\mathcal{A}_i$ where

$$\mathcal{A}_i = -\frac{e}{\theta}\epsilon_{ij}\frac{\partial_j}{\nabla^2}\rho_0 \tag{6.6}$$

Notice that $\mathcal{A}_i$ is precisely the gauge potential one would obtain from a constant fictitious magnetic field $\mathcal{B} = \frac{e}{\theta}\rho_0$.

The contribution of all the tadpoles to the fermion propagator is now computed by summing the geometric series of Figure 3. We call the resulting object the

tadpole-corrected propagator and we denote it by S_T. The result of the calculation is

$$S_T = (S_0^{-1} - ie\gamma^i \mathcal{A}_i)^{-1} = [\gamma^\mu (\partial_\mu - ie\mathcal{A}_\mu) - m - \mu\gamma^0]^{-1} \qquad (6.7)$$

where we have made the definition $\mathcal{A}_0 = 0$. This leads us to the important conclusion that the tadpole-corrected propagator S_T is precisely the Green's function for a free fermion in a constant magnetic field $\mathcal{B} = \frac{e}{g}\rho_0$ and with chemical potential μ. We shall thus reorganize our perturbation expansion as follows. All propagators will be fully tadpole–corrected propagators. The vertices will be the same vertices as those for the basic theory, and, of course, no additional propagators are included. We shall call this "tadpole–corrected perturbation theory". We shall be able to use this reorganized expansion to prove some very powerful results about the Chern–Simons theory. In fact for many quantities of interest only one–loop effects will contribute.

The Fermion Propagator in a Constant Magnetic Field

The fermion Green's function in a constant magnetic field $\mathcal{B}$ is evaluated by choosing from among the many possible gauge potentials which are consistent with coulomb gauge. We discuss the propagator in an asymmetric gauge in which $\mathcal{A}_y = \mathcal{B}x$, $\mathcal{A}_z = 0$, $\mathcal{A}_0 = 0$. The fermion propagator S_T is simply the inverse of the operator $\tilde{D}\!\!\!/ - m$ where $\tilde{D}_\mu = \tilde{\partial}_\mu - ie\mathcal{A}_\mu$. This operator is inverted by using the relation:

$$\begin{aligned}
S_T &= [\tilde{D}\!\!\!/ - m]^{-1} = (\tilde{D}\!\!\!/ + m)[(\tilde{D}\!\!\!/ - m)(\tilde{D}\!\!\!/ + m)]^{-1} \\
&= (\tilde{D}\!\!\!/ + m)[\tilde{D}^2 - m^2 + e\sigma_3 \mathcal{B}]^{-1}
\end{aligned} \qquad (6.8)$$

The propagator is thus found in two steps. First the operator $Q = [\tilde{D}^2 - m^2 + e\sigma_3 \mathcal{B}]$ is inverted by finding its eigenvalues and eigenfunctions. The operator $\tilde{D}\!\!\!/ + m$ is then applied to the resulting expression to obtain the propagator.

The eigenfunctions Φ for the operator Q, which are 2-component spinors, are found by fourier transforming in y and t:

$$\Phi(x, y, t) = e^{-i\omega t} e^{-ip_y y} \tilde{\Phi}(x, p_y, \omega) \qquad (6.9)$$

The operator Q when acting on this eigenfunction gives

$$-[(\omega - i\mu)^2 - \partial_x^2 + e^2 \mathcal{B}^2 (x + \frac{p_y}{e\mathcal{B}})^2 + m^2 - e\mathcal{B}\sigma_3] \qquad (6.10)$$

The functions $\tilde{\Phi}$ are thus eigenfunctions of a harmonic oscillator with unit mass and frequency $e\mathcal{B}$. More precisely for each normalized eigenfunction Ψ_n of the

harmonic oscillator there are two eigenfunctions of Q given by

$$e^{-i\omega t}e^{-ip_y y}\begin{pmatrix}\Psi_n(x + \frac{p_y}{eB})\\ 0\end{pmatrix} \quad and \quad e^{-i\omega t}e^{-ip_y y}\begin{pmatrix}0\\ \Psi_n(x + \frac{p_y}{eB})\end{pmatrix} \qquad (6.11)$$

with eigenvalues

$$-[(\omega - i\mu)^2 + (2n + 1)eB \mp eB] \qquad (6.12)$$

respectively, where n is a nonnegative integer. If we set

$$d_n = [(\omega - i\mu)^2 + 2neB + m^2] \qquad (6.13)$$

Then these eigenvalues are simply equal to $-d_n$ and $-d_{n+1}$ respectively.

Having found the eigenvalues the inverse of Q can now be written as

$$
\begin{aligned}
\frac{1}{Q} &= \frac{1}{[\bar{D}^2 - m^2 + e\sigma_3 B]}\\
&= -\sum_{n=0}^{\infty}\int\frac{d\omega}{2\pi}\int\frac{dp_y}{2\pi}\{[\tfrac{1}{2}\left(\frac{1}{d_{n+1}} + \frac{1}{d_n}\right) - \tfrac{1}{2}\left(\frac{1}{d_{n+1}} - \frac{1}{d_n}\right)\sigma_3]
\end{aligned}
\qquad (6.14)
$$

$$e^{-i\omega(t-t')}e^{-ip_y(y-y')}\Psi_n(x + \frac{p_y}{eB})\Psi_n{}^*(x' + \frac{p_y}{eB})\}$$

The tadpole-corrected propagator S_T is then given by

$$S_T = [\bar{\not p} - m]^{-1} = (\bar{\not p} + m)Q^{-1} \qquad (6.15)$$

Recall the *definition* of B as $B = \frac{e}{\theta}\rho_0$. The propagator is thus a function of both ρ_0 and μ. but the density ρ_0 depends itself on μ. The next step is then to find the relationship between ρ_0 and μ. This is done perturbatively using the tadpole-corrected propagator S_T. The lowest order contribution to the mean density ρ_0 is the one loop diagram of Figure 4. higher order corrections to the fermion propagator will be taken into account separately and will be higher order in our expansion. We shall see in chapter 8 that due to a nonrenormalization theorem the result of this lowest order calculation of the density is, in fact, an exact result. The strategy is to first compute ρ_0 for fixed μ and B. Having done this we can use the fact that B itself depends on ρ_0 to find ρ_0 as a function of μ.

Calculation of the Density

The calculation of ρ_0 proceeds as follows:

$$\rho_0 = <\psi^\dagger(r)\psi(r)> = -Tr[\gamma_0 S_T(r,r)]$$

$$= \sum_{n=0}^{\infty} \int \frac{d\omega}{2\pi} \int \frac{dp_y}{2\pi} |\psi_n(x + \frac{p_y}{eB})|^2 \left[-i(\omega - i\mu)(\frac{1}{d_{n+1}} + \frac{1}{d_n}) - m(\frac{1}{d_{n+1}} - \frac{1}{d_n})\right]$$

(6.16)

The integral over p_y can now be done since the functions Ψ_n are normalized eigenfunctions for the harmonic oscillator. This integral simply yields a factor eB. Thus

$$\rho_0 = \frac{-ieB}{2\pi} \sum_{n=0}^{\infty} \int \frac{d\omega}{2\pi} \left[(\omega - i\mu)(\frac{1}{d_{n+1}} + \frac{1}{d_n}) - im(\frac{1}{d_{n+1}} - \frac{1}{d_n})\right]$$

(6.17)

The integrals over ω are now done using contour integral techniques. The integral $\int d\omega \{\frac{1}{d_n}\}$ is standard, and is performed by closing the contour along a semicircle of very large radius in the complex plane either above or below the real axis. The integral $\int d\omega \{\frac{(\omega - i\mu)}{d_n}\}$ can be done using a cutoff regulator. It is convergent despite initial appearances. It is evaluated by shifting the contour from the real axis to the line $-\infty + i\mu < \omega < \infty + i\mu$. The vertical parts of the contour at infinity vanish, and the resulting integral over the new contour vanishes by antisymmetry. All that remains are possible poles in the region $0 < Im(\omega) < \mu$. The results are

$$\int_{-\infty}^{\infty} \frac{d\omega}{(\omega - i\mu)^2 + 2neB + m^2} = \frac{\pi}{\sqrt{2neB + m^2}}\Theta(\sqrt{2neB + m^2} - |\mu|)$$

$$\int_{-\infty}^{\infty} d\omega \frac{\omega - i\mu}{(\omega - i\mu)^2 + 2neB + m^2} = \frac{\mu}{|\mu|}\pi i \Theta(|\mu| - \sqrt{2neB + m^2})$$

(6.18)

Using these results we obtain the final result for the density as a function of μ at fixed B:

$$\rho_0 = \frac{eB}{4\pi}\frac{\mu}{|\mu|} \sum_{n=0}^{\infty} \left[\Theta(|\mu| - \sqrt{2neB + m^2}) + \Theta(|\mu| - \sqrt{2(n+1)eB + m^2}\right]$$

$$+ \frac{eB}{4\pi}\frac{m}{|m|}\Theta(|m| - |\mu|)$$

(6.19)

$$= \frac{eB}{2\pi}\frac{\mu}{|\mu|}\left[Int\left(\frac{\mu^2 - m^2}{2eB}\right) + \frac{1}{2}\right]\Theta(|\mu| - |m|) + \frac{eB}{4\pi}\frac{m}{|m|}\Theta(|m| - |\mu|)$$

where Int stands for the integer part of its argument.

Note that ρ_0 does not vanish when $\mu=0$. To understand this point first note that this is *not* the density for the anyon gas as a function of μ since we have not yet applied the constraint $B = \frac{e}{\theta}\rho_0$. It is, instead the density for the fermion gas in a constant magnetic field. Since the presence of the magnetic field breaks the charge conjugation invariance of the system there is no guarantee that the mean density will vanish. In fact, for any given μ the value of the mean density i.e. the expectation value of $\psi^\dagger(r)\psi(r)$ is regularization dependent. For $\mu = 0$ it is precisely equal to the spectral asymmetry of the Dirac operator. This spectral asymmetry does not vanish for a single massive fermion in $2+1$ dimensions when an ultraviolet cutoff is used as a regulator. This nonvanishing of ρ_0 at $\mu = 0$ is closely related to a similar ambiguity in the $\mu = 0$ renormalization of the Chern–Simons term for this theory.[52] In fact if Pauli–Villars regularization were used the density would vanish and the Chern–Simons term would not renormalize at $\mu = 0$. Even though ρ_0 is nonzero at $\mu = 0$ it is clearly this ρ_0 which is related to B via the relation $B = \frac{e}{\theta}\rho_0$. However, in order to get a physical picture of what is going on, it is useful to consider the "physical" density $\rho_{ph} = \rho_0(\mu)-\rho_0(\mu=0)$. ρ_{ph} is the expectation value of the properly renormalized density operator.

In Figure 5 ρ_{ph} is plotted versus μ for fixed (positive) B when the mass $m>0$. It must be emphasized that this is a plot of ρ as a function of μ *for fixed B*. The self consistency condition which is required by the definition of B in terms of ρ_0 has not yet been imposed. When the density is *not* an integer multiple of $eB/2\pi$ the density can increase with no cost in chemical potential. Thus a new particle can be added to the system at no cost in energy. This corresponds to the filling of a Landau level. When the density reaches an integer multiple of $eB/2\pi$, the level is filled and a discrete jump in chemical potential is required before the next level can be filled. The asymmetry between positive and negative μ reflects the spectral asymmetry of the theory. Evidently the two signs of μ correspond to filling the Landau levels with particles and with antiparticles which have opposite spin. We see from Figure 5 that for $\mu>0$ one requires a chemical potential $\mu^2=m^2+2eB$ to begin filling the first Landau level. For $\mu<0$ the spins point in the opposite direction. The interaction energy of the spin with the magnetic field B precisely cancels its orbital energy in the first Landau level leading to a zero energy mode. Thus one begins to fill the first Landau level at $\mu = -m$. Figure 5 thus corresponds precisely to the usual filling of Landau levels which occurs for fermions in a magnetic field.

The definition of $B = \frac{e}{\theta}\rho_0$ can now be used to find ρ_0 as a function of μ for any given value of θ (which is assumed positive). If, as we shall assume, e, B and θ are positive, then ρ_0 will also be positive. Let us consider the case $m>0$ in which case

$\rho_0 = \rho_{ph} + eB/4\pi$. Suppose ρ_0 is fixed at some physical value (say by fixing ρ_{ph}). We see from Figure 5 that it is useful to write ρ_0 as $eB/4\pi$ plus an integer number of steps of magnitude $eB/2\pi$ plus some remainder.

$$\rho_0 = \frac{eB}{2\pi}[N + \tfrac{1}{2} + \gamma]; \quad 0 \leq \gamma < 1 \tag{6.20}$$

where N is an integer. Now ρ_0 is also equal to $\frac{\theta}{e}B$. Thus

$$\frac{2\pi\theta}{e^2} = N + \tfrac{1}{2} + \gamma \tag{6.21}$$

Note that γ is determined entirely by θ/e^2. Thus fixing the Chern–Simons coefficient simply tells us how many Landau levels are filled and what fraction of the first unfilled level is occupied. The values of N and γ are determined entirely from the values of e and θ.

The resulting value of μ can now be determined from Figure 5. When $\gamma \neq 0$ (i.e. when there is an unfilled Landau level) the value of μ is uniquely determined to be

$$\mu^2 = m^2 + 2(N+1)\frac{2\pi\rho_0}{N + \tfrac{1}{2} + \gamma} = m^2 + 2(N+1)\frac{2\pi\rho_{ph}}{N+\gamma} \tag{6.22}$$

or

$$\rho_{ph} = \frac{\mu^2 - m^2}{4\pi}\frac{N + \gamma}{N+1} \tag{6.23}$$

when $\gamma \neq 0$[†]. On the other hand, when $\gamma = 0$ and a Landau level is filled the value of μ is ambiguous with

$$m^2 + 2N\frac{2\pi\rho_{ph}}{N} < \mu^2 < m^2 + 2(N+1)\frac{2\pi\rho_{ph}}{N} \tag{6.24}$$

We would expect that all physical quantities will turn out to be independent of which value of μ is chosen in this range.

---

† Note that this differs only slightly from the result for fermions for which $\rho = (\mu^2 - m^2)/4\pi$.

First note that the desired result $\Pi_{odd}(q=0)=\theta$ i.e. the vanishing of the renormalized Chern–Simons term does *not* occur at zero density.[14]It may seem strange that θ_R vanishes for *any* nonzero density but not in the zero density limit. This is because the zero density limit of this theory is extremely singular since for any finite density there are a fixed number of occupied Landau levels. This number depends only on θ but not on the density. At precisely zero density there are of course no levels occupied.

The strategy for showing that $\Pi_o=\theta$ is as follows: One first calculates Π_o in the one–loop tadpole–corrected perturbation theory keeping both μ and $\mathcal{B} = (e/\theta)\rho_0$ as variables. One then inserts the appropriate value for μ which was derived above eqn. (6.24). to obtain our desired result. Finally one invokes a non-renormalization theorem which is discussed in chapter 8 to show that the result holds to all orders in tadpole-corrected perturbation theory.

The detailed calculation of Π_o is given in ref. [23]. The final expression for Π_{odd} was found to be

$$
\begin{aligned}
\Pi_{odd} = &-\frac{e^2}{2\pi} \sum_{n=0}^{\infty} \int \frac{d\omega}{2\pi} \{ [m(\frac{1}{d_{n+1}} - \frac{1}{d_n}) + i(\omega - i\mu)(\frac{1}{d_n} + \frac{1}{d_{n+1}})] \\
&- i4neB \frac{(\omega - i\mu)}{[(\omega - i\mu)^2 + M(n)^2]^2} \}
\end{aligned}
\tag{6.26}
$$

where $M(n)^2 = 2neB+m^2$. Fortunately there is an interesting theorem which related Π_o to ρ which greatly simplifies the proof of the result. To any order in tadpole-corrected perturbation theory there is a general relation between the diagrams which contribute to $\Pi_{odd}(q=0)$ and those which contribute to $\rho_0(\mu,\mathcal{B})$. We begin by considering an arbitrary diagram which contributes to $\rho_0(\mu,\mathcal{B})$. We now take $\delta/\delta\mathcal{B}$ at fixed μ of any such diagram. This has the effect of removing precisely one tadpole insertion and replacing it by $ie\gamma^i\epsilon_{ij}\partial_j/\nabla^2$ which is applied to the graph and the result is evaluated at $q=0$. This is shown pictorially in Figure 6. The resulting graph is one–particle–irreducible in terms of tadpole-corrected lines since clearly all diagrams for ρ_0 are one–particle–irreducible. It is thus related to the vacuum polarization Π_{0j}. In fact

$$
\frac{\delta e \rho_0}{\delta \mathcal{B}}|_\mu = i\epsilon_{ij} \frac{\partial_j}{\nabla^2} \Pi_{i0}|_{q=0}
\tag{6.27}
$$

Now the odd part of Π goes to zero linearly with q as q goes to zero whereas the

Notice that the condition of having precisely N filled Landau levels occurs when $2\pi\rho_{ph}/e\mathcal{B} = N$. This then implies that $(2\pi\theta/e^2)-\frac{1}{2} = N$. Naively we might have expected the '$\frac{1}{2}$' to be missing. The reason for the presence of this term is that there is a renormalization of the Chern–Simons coefficient in this theory *at zero density* which occurs in one–loop and which is not renormalized by higher loops The renormalized Chern–Simons term at zero density has been calculated in the literature.[62,69,70] It is given by $\theta_R = \theta-(m/|m|)(e^2/4\pi)$ when cutoff regularization is used. Although this relationship is regularization dependent, it is only the renormalized CS term which is physical. It is in fact this *renormalized* Chern–Simons term at zero density which determines the statistics of the anyons. Our condition for having N filled levels now becomes (for $m>0$)

$$\frac{2\pi\theta_R}{e^2} = N \tag{6.25}$$

which corresponds to a statistics parameter $\pi(1-1/N)$ where N is an integer. Such a statistics parameter corresponds to the horizontal parts of Figure 5 whereas a non–integer N corresponds to its vertical parts. It must be emphasized that the θ_R in equation (6.25) is the renormalized value of θ at **zero density**. We shall often simply call in θ to avoid confusion with the finite density renormalized θ which we have not yet computed.

The Renormalized Chern–Simons Coefficient

We are now ready to discuss the calculation of the renormalized Chern–Simons coefficient. The first step is to consider the (one–particle–irreducible) vacuum polarization $\Pi_{\mu\nu}$. As discussed previously, in this parity non–invariant theory in three dimensions, $\Pi_{\mu\nu}$ can be split up into an even and an odd part:

$$\Pi_{\mu\nu}(q) = \Pi^e_{\mu\nu}(q) + \epsilon_{\mu\nu\lambda}q^\lambda\Pi_{odd}(q)$$

where Π^e is symmetric under interchange of μ and ν. Recall that gauge invariance requires the odd part of Π to have the above covariant form even at finite density in which case Lorentz invariance is lost. Our goal is to show that the renormalized Chern–Simons coefficient θ_R at nonzero density vanishes. This turns out *not* to be true for arbitrary values of $\theta_R(\mu{=}0)$. We shall see that θ_R at finite density vanishes if and only if $N = 2\pi\theta_R(\rho{=}0)/e^2$ is an integer. As discussed in chapter 5 (see equation (5.6)) θ_R is defined as $\theta_R = \theta - \Pi_o$ with $\Pi_o = \Pi_{odd}(q{=}0)$. Thus the condition for the vanishing of θ_R is that $\Pi_o = \theta$. We now proceed to discuss this result.

even part of Π vanishes quadratically[†]. Thus

$$\frac{\delta e \rho_0}{\delta B}\Big|_\mu = i\epsilon_{ij}\frac{\partial_j}{\nabla^2}\epsilon_{ik}(-i)\frac{\partial_k}{\nabla^2}\Pi_{odd}\Big|_{q=0} = \Pi_{odd}(q=0) \tag{6.28}$$

Thus Π_{odd} at $q=0$ is simply evaluated by differentiating ρ_0 with respect to B at fixed μ. We can do this either via eqn. (6.19) or more simply from Figure 5. Note that on the vertical sections (i.e. when $(\mu^2-m^2)/2eB$ is an integer) the above derivative is divergent. It is only convergent on the horizontal sections which correspond to case of completely filled Landau levels. In this case ρ_0/B is independent of B . Thus $\delta\rho_0/\delta B = \rho_0/B$. But ρ_0 is related to B via the tadpole relation $\rho_0 = \frac{\theta}{e}B$. Thus

$$\Pi_{odd}(q=0) = \frac{\delta e \rho_0}{\delta B} = \theta \tag{6.29}$$

This is the result which was claimed above. It implies that the one loop (tadpole–improved) renormalized Chern–Simons term vanishes, and thus there is pole in the current–current correlation for this theory. It occurs if and only if the *zero density* renormalized CS coefficient θ_R is such that some integer number N of Landau levels are precisely filled. As discussed above eqn. (6.25) this occurs (for $m>0$) when $2\pi\theta_R/e^2 = N$.

In chapter 8 we show that these results hold to all orders in perturbation theory. This is done by extending previous nonrenormalization theorems at zero density to the case of finite density. It can be seen from the above discussion that proving a nonrenormalization theorem for ρ_0 is sufficient since the corresponding result for Π_{odd} can be derived as a its consequence.

Summarizing this section, we have shown that in the one loop approximation, when $2\pi\theta/e^2$ is an integer N so that N Landau levels are filled, the renormalized Chern–Simons term at nonzero density vanishes. This then implies that there is a gapless excitation coupled to the current. This, combined with the presence of a gap in the fermion spectrum due to the filled Landau levels results in superfluidity of the anyonic system. In chapter 8 this result is shown to hold to any finite order in perturbation theory.

[†] This, of course assumes that there are no singularities in the even part of the vacuum polarization as the external momentum tends to zero. This will be valid for the case of filled Landau levels since the spectrum will have a gap.

6.4. SCHWINGER'S PROPER TIME INTEGRAL METHOD

Another field theoretical method to derive the renormalization of the CS term uses Schwinger's proper time integral.[23] The reason we present this computation here is two-fold: as a confirmation of the Feynman diagrams results of the last section, and as a possible framework to handle more complicated models where, e.g., $\partial_i \mathcal{B} \neq 0$.

The tadpole-corrected fermion propagator was shown in the previous section to be equivalent to that of a fermion in a constant fictitious magnetic field, which is defined via the density: $\mathcal{B} = e\rho/\theta$. We now calculate this propagator using Schwinger's proper time integral:

$$S_T(x, x') = <x| \int\limits_0^\infty ds\, e^{-Hs} (\not{\mathcal{D}} + m)|x'>$$ (6.30)

where $H = -(\not{\mathcal{D}} \not{\mathcal{D}}) = D^2 + e\mathcal{B}\sigma_3$. We differ from the original calculation[71] in using a Euclidean metric and working in three dimensions. The matrix element of the operator $U(s) = e^{-Hs}$ is thus given[71] by

$$
\begin{aligned}
<x|U(s)|x'> = \frac{C(x, x')}{s^{3/2}} e^{-L(-is)} e^{-\frac{e s \mathcal{B}\sigma_3}{2}} \\
\times e^{-\frac{1}{4}(x-x')_\mu [eF\cot(eFs)]^{\mu\nu}(x-x')_\nu}
\end{aligned}
$$ (6.31)

where $C(x - x') = \frac{1}{8\pi^{3/2}} e^{i \int_{x'}^x d\xi^\mu A_\mu(\xi)}$ and $e^{-L(-is)} = \frac{e\mathcal{B}s}{\sinh(e\mathcal{B}s)}$. Expanding the various factors in (6.31) we get the following expression for the tadpole-corrected fermion propagator:

$$
\begin{aligned}
S_T(x, x') = \frac{e\mathcal{B}}{8\pi^{3/2}} e^{i \int_{x'}^x d\xi^\mu A_\mu(\xi)} \int\limits_0^\infty ds \frac{e^{-m^2 s}}{s^{1/2}\sinh(e\mathcal{B}s)} e^{-\frac{(t-t')^2}{4s}} \\
\times e^{-\frac{1}{4}e\mathcal{B}\cot(e\mathcal{B}s)(x-x')_i (x-x')^i} \times [G_1 + G_0\sigma_3 + G^i\sigma_i]
\end{aligned}
$$ (6.32)

where G_1, G_0 and G^i are given by

$$
\begin{aligned}
G_1 &= m\cosh(e\mathcal{B}s) + \frac{1}{2s}(t - t')\sinh(e\mathcal{B}s) \\
G_0 &= -m\sinh(e\mathcal{B}s) - \frac{1}{2s}(t - t')\cosh(e\mathcal{B}s) \\
G^i &= -\frac{(x^i - x'^i)}{2\sinh(e\mathcal{B}s)}
\end{aligned}
$$ (6.33)

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Note that here we supressed the chemical potential. In the following computations before integrating over the frequency we first analytically continue $\omega \to \omega - i\mu$.

The next task is to calculate the fermion density. This is achieved by substituting $S_T(x-x')$ from (6.33) into

$$\rho_0 = -Tr[\gamma_0 S_T(x,x)] \tag{6.34}$$

We have to integrate over $d^2(x-x')\delta(x_i - x'_i)$ and then over $d(t-t')\delta(t-t')$. The last delta function is written as $\delta(t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(t-t')}$. After the first integration we find

$$
\rho = -\frac{eB}{8\pi^{3/2}} \int d(t-t') \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega(t-t')} \int ds \frac{e^{-ms^2}}{s^{1/2}}
$$

$$
\times e^{\frac{-(t-t')^2}{4s}} [-m - \frac{1}{2s}(t-t')(coth(eBs))] \tag{6.35}
$$

$$
= \frac{2eB}{4\pi^2} \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-s(m^2+\omega^2)} [m + i\omega coth(eBs)]
$$

If we now write

$$
coth(eBs) = \frac{1 + e^{-2eBs}}{1 - e^{-2eBs}} = (1 + e^{-2eBs}) \sum_{n=0}^{\infty} e^{-2neBs} \tag{6.36},
$$

substitute it into (6.35) and replace ω with $\omega - i\mu$ we find exactly the same expression as the one given in eqn. (6.19).

The third step is to calculate the odd part of the vacuum polarization Π_o. We substitute (6.32) into $\Pi_{\mu\nu}(x,y) = e^2 Tr[\gamma_\mu S_T(x,y)\gamma_\nu S_T(y,x)]$ and we extract the term proportional to $\epsilon_{ij}\omega$. The resulting term is:

$$
\Pi_o^{ij}(q) = \frac{2ie^2(eB)^2}{8^3\pi^6} \epsilon_{ij} \int d^3p \int d^3x \int d^3x' e^{ipx+(p-q)x'} \int\limits_0^{\infty} \frac{ds_1}{\sqrt{s_1}} \int\limits_0^{\infty} \frac{ds_1}{\sqrt{s_2}}
$$

$$
\times \frac{e^{m^2(s_1^2+s_2^2)}}{sinh(eBs_1)sinh(eBs_2)} e^{\frac{-t^2}{4s_1}} e^{\frac{-t'^2}{4s_2}} \times e^{[-\frac{1}{4}eB[coth(eBs_1)x^2 + coth(eBs_2)x'^2]]}
$$

$$
[(\frac{tt'}{4s_1s_2} - m^2)sinh[eB(s_1 - s_2)] - m(\frac{t}{2s_1} - \frac{t'}{2s_2})cosh[eB(s_1 - s_2)]].
$$

$$\tag{6.37}$$

Now we do the x and x' integrations, the p integration, substitute $k^ik_i = 0$, and

perform the t and t' integrations. The result is:

$$\Pi_o^{ij}(0, q_0) = \frac{ie^2(eB)}{4\pi^2}\epsilon_{ij} \int d\omega \int_0^\infty \frac{ds_1}{\sqrt{s_1}} \int_0^\infty \frac{ds_1}{\sqrt{s_2}}$$

$$\times \frac{e^{m^2(s_1^2+s_2^2)}}{sinh(eBs_1)sinh(eBs_2)} e^{-\omega^2 s_1} e^{-(\omega-q_0)^2 s_2}$$

$$[[-(\omega(\omega-q_0)-m^2)sinh[eB(s_1-s_2)] - imq_0cosh[eB(s_1-s_2)]] \cdot \quad (6.38)$$

Replacing hyperbolic trignometric functions with sums and integrating over the proper time variables s_1 and s_2 one gets the same result as given in eqn. (6.26).

6.5. THE RANDOM PHASE APPROXIMATION

The massless pole in the current-current correlator was discovered in the original work of Fetter, Hanna and Laughlin.[8] The method used in that work was the random phase approximation (RPA). The calculation was repeated and extended in the work of Chen, Halperin, Wilczek and Witten.[18]Here we summarize this approach following the paper of Chen et al.

A non-relativistic anyonic gas is described by the following second quantized hamiltonian

$$H = \int d^2r \Psi^\dagger \frac{1}{2m}|p + a(r)|^2 \Psi \quad (6.39)$$

where Ψ is a spinless fermionic field. a is the gauge field which is expressed in terms of the density of particles using Gauss's law in the Coulomb gauge (see section (6.1)) as follows:

$$a_i(r) = \bar{a}_i + \frac{1}{N} \int d^2r' \frac{\epsilon_{ij}(r - r')^j}{|r - r'|^2}[\rho(r') - \bar{\rho}] \quad (6.40)$$

with $\rho(r) = \Psi^\dagger \Psi(r)$. (Note that the charge e is absorbed into a in this section.) In the above the gauge field has been decomposed into a background field $\bar{a}_i = (\pi\bar{\rho}/N)\epsilon_{ij}r^j$ and a fluctuating field $a_i - \bar{a}_i$. The system of fermions with density $\bar{\rho}$ in the presence of the field $\bar{a}$ is taken as the unperturbed "reference" system and the interaction hamiltonian is rewritten as

$$H_I = H - H_0 = H_1 + H_2$$

$$H_1 = \frac{1}{N} \int d^2r \int d^2r' j_i(r) \frac{\epsilon_{ij}(r-r')^j}{|r-r'|^2} [(\rho - \bar{\rho})(r')]$$

$$H_2 = \frac{1}{2mN^2} \int d^2r \int d^2r' \int d^2r'' \rho(r) \frac{(r-r')_i(r-r'')^i}{|r-r'|^2|r-r''|^2} [\rho(r') - \bar{\rho}][\rho(r'') - \bar{\rho}] \quad (6.41)$$

where the currents are given by: $j_i(r) = \Psi^\dagger(r)\frac{1}{m}(p_i + \bar{a}_i)\Psi(r)$. It is argued[18] that the Coulomb potential which results from the substitution of $\bar{\rho}$ for $\rho(r)$ in H_2 is important in two aspects: (i) The long range attraction between particles and holes is ultimately responsible for the gapless mode which can be viewed as a zero-mass bound state. (ii) The repulsion between "vortices" which are intimately related to the fermions is responsible for the anyonic superconductivity being type II. Neither of these statements are actually proven in ref. [15].

Neglecting the fluctuating part of $\rho(r)$ in H_2 the interaction hamiltonian yields a two-body interaction term which can be rewritten as:

$$H_I = \frac{1}{2} \int d^2r \int d^2r' j_\mu(r) V_{\mu\nu}(r,r') j_\nu(r'), \quad (6.42)$$

where $j_0 = \rho(r) - \bar{\rho}$ and where the Fourier transform of $V_{\mu\nu}$ is given by:

$$V_{\mu\nu}(q) = \int d^2r' V_{\mu\nu} e^{iq(r-r')} = \frac{2\pi}{Nmq} \begin{pmatrix} \frac{1}{N}\bar{\rho}\frac{2\pi}{q} & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad (6.43)$$

Next, one defines the current-current correlator $\bar{K}_{\mu\nu} = -i < T[j_\mu(1)j_\nu(2)] >$, denoting the unperturbed correlator by $\bar{K}^0_{\mu\nu}$. One finds that

$$\bar{K} = (1 - \bar{K}^0 V)^{-1} \bar{K}^0 \quad (6.44)$$

The explicit computation of $\bar{K}^0_{\mu\nu}$ is presented in the appendix of ref.[15]. We briefly summarize it here. The electron wave function Ψ is expanded in terms of Landau

wave functions eqn. (6.11)

$$\Psi(r,t) = \sum_{n=0}^{\infty} \int \frac{d\omega}{2\pi} \int \frac{dp_y}{2\pi} e^{-i\omega t} e^{-ip_z z} \Psi_n\left(y + \frac{p_x}{eB}\right) \qquad (6.45)$$

following the conventions of ref.[15] where $a_x = -\hbar eB$, $a_y = 0$ rather then those of the previous section. Inserting now the unperturbed currents expressed in terms of these wave functions one gets a complicated expression for $\bar{K}_{00}^0$ [18]. After some straightforward but tedious algebra the following simplified form is derived,

$$\bar{K}_{00}^0 = \frac{m}{2\pi\hbar eB} q^2 \Sigma_0$$

$$\Sigma_0 = \Sigma_{l=0}^{n-1} \Sigma_{m=n}^{\infty} \frac{l!}{m!} \frac{m-l}{(\frac{\omega}{\omega_c})^2 - (m-l)^2} e^{-X} X^{m-l-1} [L_l^{m-1}(X)]^2 \qquad (6.46)$$

where $X = \frac{q^2}{2m eB}$, $\omega_c = \frac{\hbar eB}{m}$ and L_l^{m-l} is are Laguerre polynomials. Using the same method for calculating the other components one gets

$$\bar{K}^0 = \frac{N}{2\pi\omega_c}\begin{pmatrix} q^2\Sigma_0 & q\omega\Sigma_0 & -iq\omega_c\Sigma_1 \\ q\omega\Sigma_0 & (\omega^2\Sigma_0 - \omega_c^2\Sigma_3) & -iq\omega\omega_c\Sigma_1 \\ iq\omega_c\Sigma_1 & i\omega\omega_c\Sigma_1 & \omega_c^2\Sigma_2 \end{pmatrix} \qquad (6.47)$$

Approximate expressions for Σ_i are given below. The e.m. response function $K_{\mu\nu}^*$ is defined by the two point function of the true currents $J_\mu = \Psi^\dagger(p_\mu + a_\mu)\Psi$ and not the one used in eqn. (6.41). $K_{\mu\nu}^*$ can be deduced approximately from $\bar{K}_{\mu\nu}$ as follows: $K_{\mu\nu}^* = (1 + \bar{p}u^\dagger)\bar{K}_{\mu\nu}(1 + \bar{p}u)$, where $-i\frac{Nmq}{2\pi}u$ is a matrix with all zeros apart from 1 in the $(0, y)$ component. For the "physically" relevant response function one has to add also the contribution from the contact term (the a^2 term in the action). Thus, finally the response function reads

$$K = \frac{N}{2\pi det}\begin{pmatrix} \frac{q^2}{\omega_c}\Sigma_0 & \frac{q}{\omega_c}\Sigma_0 & iq\Xi \\ q\frac{\omega}{\omega_c}\Sigma_0 & \frac{\omega^2}{\omega_c}\Sigma_0 & i\omega\Xi \\ -iq\Xi & -i\omega\Xi & \omega_c(\Xi - \Sigma_1 + \Sigma_2 + det) \end{pmatrix} \qquad (6.48)$$

where $\Xi = -\Sigma_1 - \Sigma_1^2 + \Sigma_0\Sigma_2 + \Sigma_0$ and $det = 1 - \Sigma_0 + 2\Sigma_1 + \Sigma_1^2 - \Sigma_0\Sigma_2$.

Three approximation are involved in the derivation of the response function:
(i) in the interaction hamiltonian, (ii) in the inverse propagator $\tilde{K}^{-1}$ and (iii) in the
passage from $\tilde{K}$ to K^*. These approximations are essentially replacing the actual
density with the mean density and thus is justified in the large N limit. For small
q and ω, $\Sigma_i = -1 - (\frac{\omega}{\omega_c})^2 + \eta_i N \frac{\hbar q^2}{m\omega_c}$ with $\eta_i = (\frac{3}{8}, \frac{3}{4}, 1)$ for $i = 0, 1, 2$ respectively.
Inserting these expressions into eqn.(6.48) one finds that $det = -(\frac{\omega}{\omega_c})^2 + N\frac{\hbar q^2}{m\omega_c}$. It
is therefore evident that the current-current correlator has a massless pole, namely,
a pole at $(\frac{\omega}{\omega_c})^2 = N\frac{\hbar q^2}{m\omega_c}$.

6.6. THE "INHOMOGENEOUS EXPANSION TECHNIQUE"

Panigrahi, Ray and Sakita[24] calculate the effective action $\mathcal{L}_{eff}$ from the cur-
rents which are derived using a perturbation method, the "inhomogeneity expan-
sion technique". Starting with the non-relativistic analog of eqns. (4.4)-(4.5) they
write for the vacuum expectation values of the currents

$$< J_\mu(x) > = \frac{\delta \mathcal{L}_{eff}}{\delta a^\mu} = \frac{e}{2m} < x | \tilde{\mathcal{P}}_\mu \frac{1}{\tilde{\mathcal{P}}} + \frac{1}{\tilde{\mathcal{P}}} \tilde{\mathcal{P}}_\mu | x > \qquad (6.49)$$

where $\tilde{\mathcal{P}}_0 = im$, $\tilde{\mathcal{P}}_i = p_i - ea_i(\hat{x})$ and $\tilde{\mathcal{P}} = -p_0 + a_0(\hat{x}) + \frac{\tilde{\mathcal{P}}_i^2}{2m} - \mu$. The currents are reg-
ulated using the Pauli-Villars regularization. The method includes expanding $a_\mu(\hat{x})$
around $a_\mu(x)$, translating x to the origin and invoking unitary transformations to
express the currents in terms of field strengths as far as possible. One then finds
that the denominator can be expanded around the hamiltonian of a two dimen-
sional electron in a constant magnetic field, namely: $\tilde{\mathcal{P}} = -\hat{P}_0 - \hat{\Delta}_0 + \frac{\hat{\mathcal{P}}_i \hat{\Delta}_i + \hat{\Delta}_i \hat{\mathcal{P}}_i + \hat{\Delta}_i^2}{2m}$,
with $\hat{P}_0 = \tilde{\mathcal{P}} + \hat{H}_0 - \mu + a_0$, where $\hat{H}_0 = \frac{[p_x^2 + (p_y - eB\hat{x})^2]}{2m}$, and where $\hat{\Delta}_\mu$ are pertur-
bations of the gauge fields around the constant magnetic field. The expectation
values eqn. (6.49) are shifted to $|x = 0 >$. As was shown already in section 6.1
the basic commutation relation of the Landau problem can be expressed in terms
of creation and annihilation operators of a harmonic oscillator. With $a \equiv \frac{\hat{\mathcal{P}}_x - i\hat{\mathcal{P}}_y}{\sqrt{2}}$,
and center of mass coordinates $\hat{X} = \hat{x} - \frac{\mathcal{P}_y}{eB}$ and $\hat{Y} = \hat{y} + \frac{\mathcal{P}_x}{eB}$, one finds that

$$[\hat{X}, \hat{\mathcal{P}}_i] = [\hat{Y}, \hat{\mathcal{P}}_i] = [\hat{X}, \hat{H}_0] = [\hat{Y}, \hat{H}_0] = 0 \qquad [\hat{X}, \hat{Y}] = \frac{i}{eB} \qquad (6.50)$$

In the basis $|n, X > = |n > |X >$ where $|n >$ is the occupation number and $\hat{x}$ is
diagonal in $|x >$, using some calculational tricks it was found that to lowest order

40

in the perturbation around $\hat{P}_0$ the expectation values of the currents are:

$$< J_i > = \frac{e^2}{2\pi}\tilde{\gamma}[\epsilon_{ij}f^{0j} - \frac{2+\tilde{\gamma}}{2m}\epsilon_{ij}\partial^j \mathcal{B}] \qquad (6.51)$$

and a similar expression for $< J_0 >$. $\tilde{\gamma}(e\mathcal{B},\mu,m)$ is a staircase function given in Fig 4. It is now straightforward to obtain the effective action:

$$\mathcal{L}_{eff}(a) = -\frac{e^2}{2\pi}\tilde{\gamma}a_0\mathcal{B} + \frac{e^2}{8\pi m}\tilde{\gamma}^2\mathcal{B}^2 + +\frac{e^2 m}{4\pi}\frac{\tilde{\gamma}}{e\mathcal{B}}\mathcal{E}^2 \qquad (6.52)$$

for positive $e\mathcal{B}$, and for negative $e\mathcal{B}$ the first and third terms flips their signs. The fermionic integration, thus, produced an effective CS term which adds to the tree level CS term. For electromagnetically interacting anyons, namely fermions coupled to a fictitious CS term and a Maxwell term, an effective action $\mathcal{L}_{eff}(a_\mu, A_\mu)$ is derived following the same lines as above. The resulting expression is that of eqn.(6.52) where $\mathcal{B}$, $\mathcal{E}$ and a_0 are replaced with $\mathcal{B} + B$, $\mathcal{E} + E$ and $a_0 + A_0$ respectively. Leaving only the CS terms (lowest powers of derivatives) one finds that the free energy has a minimum at zero magnetic field $B = 0$, namely, a Meissner effect. The minimum corresponds to a cancellation between the tree level and the induced CS terms. It is thus the same result as the criterion of superconductivity that was presented in chapter 5.

The existence of a massless pole in the current-current correlator is shown[24] using a saddle point approximation. Starting with the partition function $\delta(B) \int DB e^{\tilde{S}_{eff}(B)}$ with

$$\tilde{S}_{eff} = \frac{1}{\pi}\int d^2x[BA_0 - \frac{B^2}{2m} + \frac{m}{2\pi|B|}[E_iE_i - 2\epsilon_{ij}\frac{\partial_0\partial_j}{\nabla^2}BE_i + (\frac{\partial_0\partial_j}{\nabla^2}B)^2] \qquad (6.53)$$

Constant $B = B_0$ extremizes the partition function so that for the fluctuating field $\tilde{B}$ one has

$$Z \sim \delta(B)e^{\tilde{S}_{eff}(B_0)} \int D\tilde{B}e^{\tilde{S}_{eff}(\tilde{B})}$$

$$\tilde{S}_{eff}(\tilde{B}) = -\frac{1}{\pi m}\int d^2x\tilde{B}[1 + \frac{m^2}{|B_0|}\frac{\partial_0^2}{\nabla^2}]\tilde{B} \qquad (6.54)$$

It is thus apparent that the rescaled fluctuation field $\frac{1}{\sqrt{\pi m}}\frac{1}{|q|}\tilde{B}(q\omega)$ has a propagator corresponding to a massless mode $(q^2 + \frac{m^2}{B_0}\omega^2)^{-1}$. Of course this is the same result as that of the RPA calculation.

7. THE PHYSICAL PICTURE OF ANYON SUPERCONDUCTORS

There is a simple physical picture of anyon superfluidity in the fermionic CS description, which makes intuitive all of its essential properties. This picture was described in detail by Fradkin,[65] based upon work of previous authors.[14,15,20,62] The importance of this picture is that it makes clear why the Goldstone collective mode is *essential* for anyon superconductivity, and why a pairing mechanism and a local order parameter are not needed to exhibit superfluid and superconducting behavior.

As we have seen, the anyon gas, condensed to its superfluid phase, can be thought of as a system of planar fermions moving in a uniform fictitious magnetic field $B=e\rho_0/\theta$ perpendicular to the plane. The first N Landau bands are completely filled, and there is an energy gap $\Delta E=eB/m$ (in the nonrelativistic limit) to the first unfilled level. The presence of an energy gap is familiar from BCS superconductors. Indeed it plays a similar role here, that of stabilizing the superfluid state against fluctuations. However, the energy gap, while necessary, is not sufficient to produce a superfluid. This is obvious when one realizes that insulators all have gaps in their electron spectra and they are certainly not superconductors. Furthermore the system just described is identical to the integer quantum Hall (IQH) system,[1] save only that for IQH the magnetic field is a real one and is independent of the fermion density. The IQH system, of course, is not a superfluid. Although it supports dissipationless currents, it does not exhibit a Meissner effect. This is because the fermion fluid in this case is *incompressible*, since the charged fermions resist crossing real magnetic field lines. Compressibility of the fermi fluid, *in addition* to the gap, is an essential ingredient for superfluidity.

Why then, do we have a compressible fermion system in the CS description of anyons, but an incompressible one for the IQH system? The difference arises precisely because, for the anyon system, the fictitious magnetic field is tied to the fermion density by the linear relation $B=e\rho_0/\theta$. If, in some region, we add some fermions to this system, both the density *and* the fictitious field B are increased. The increase of B in turn implies an increase in the Landau band degeneracies, which is just enough to accommodate the added fermions in the (previously full) first N levels provided they are added in multiples of N. This is the origin of a gapless mode, and the compressibility for this fermi system.

Let us examine this argument in more detail.[65] Fermions in Landau levels can be regarded in a classical picture as moving in Landau orbits, which are circles of radii equal to the cyclotron radius $r_c=mv/eB$, which can also be written

$r_c = n/\sqrt{\pi\rho_0}$, where n labels the Landau levels $1,...N$. Imagine then a set of N filled Landau orbits, with radii r_c, and centers equally spaced to fill up the plane. Now imagine a density fluctuation, whereby a fermion is removed from an orbit centered around one point x_0, and placed in a new orbit around a different point x_1. At the point x_0 the density has now *decreased*, B has also *decreased*, and thus the cyclotron radius r_c has *increased*. Conversely, at the point x_1 the density has now *increased*, B has also *increased*, and so the orbit radius is *decreased*. Thus we have a classical picture of a wave in which the fermions move on closed orbits whose radii execute an oscillatory motion as the wave goes by. The motion of the fermions is *transverse* to the direction of propagation of the wave. This wave is nothing but a classical description of the gapless collective mode; in other words, this wave *is* the Goldstone mode.

An important feature of the simple physical picture presented above is the absence of anyon pairing in the argument leading to the gapless mode. This is interesting since Laughlin's original arguments[2] for anyon superconductivity involved binding pairs of semion quasiparticles to obtain a bosonic condensate, analogous to the Cooper effect in BCS superconductivity. Several authors[2,17,62] have given arguments that, for $\gamma = \pi/N$ type anyons, it is energetically favorable to 'pair' N-anyon states, and that this results in a bosonic condensate which corresponds to the superconducting phase of the anyon gas. In any case it seems quite likely that anyon pairing *does* occur in this system. Nevertheless, anyon pairing, unlike the gap or the Goldstone mode, is not a necessary feature of anyonic superfluidity or superconductivity. Note that if anyon pairing is present, the size of such anyon 'pairs' is comparable to the interparticle separation.[62] In ordinary BCS theory, we know that a mean field description works well due to the large size of the Cooper pairs. In the case of anyon superconductors we expect mean field theory to work as well but in this case the mean field description is *exact*.[23] Another way of seeing this is to note that pairing in a BCS superconductor is really a statement about pairwise correlations in momentum space, namely, that there is a finite probability for pairs of particles to have zero total momentum.[72] This feature *is* exhibited in the Landau level picture presented above. The Landau system is a *coherent state*, which means that all the fermions rotate in phase at the same cyclotron frequency $\omega_c = eB/m$. This implies that the average distance between *any* pair of fermions is constant. The resulting pairwise correlations are the true anyon analog of explicit bound state pairing in BCS systems. Fradkin points out[68] that this coherent state property is essentially a kinematic effect, and in this regard anyon superconductivity strongly resembles the Schwinger mechanism in 1+1 dimensional QED with

massless fermions.

It is also illuminating to compare and contrast anyon superfluids with *fractional quantum Hall* (FQH) systems. The fractional systems differ from the IQH in that the filling factor of Landau bands is a rational fraction ν rather than an integer N. The low energy excitation spectrum, nonetheless, possesses a finite gap, and the FQH systems support dissipationless current flow. Furthermore, in the FQH system, the density, magnetic field, and filling fraction ν, are observed to obey the same relationship as ρ_0, B, and N in our anyon superfluid! So, are FQH systems superfluids? No, they are not.[1] The FQH fermion fluid is incompressible, and the low energy magnetophonons have nonzero mass, i.e., there is no Goldstone mode. The problem, of course, is that in the FQH system the magnetic field is a real external field, not a fictitious field. Thus, for FQH, B can be regarded as fixed, and it is the filling fraction ν that adjusts with the density. For anyons, N is fixed by the anyon statistics, and it is the fictitious B which adjusts with the density.

The FQH system supports finite energy vortex–like excitations that carry fractional charge. These quasiparticles are, in fact, anyons. Thus one may ask whether a gas of FQH anyonic quasiparticles is a superconductor. The answer is no. Unlike the free anyon gas, anyons in a FQH system are subject to a strong external magnetic field. The result is that this anyon system is incompressible, just like the FQH ground state. The FQH system, in fact, defines an entire hierarchy of incompressible quantum fluids.

Neither the anyon superfluid or the FQH system has conventional spontaneous symmetry–breaking, and neither possesses a local order parameter. These properties are well–known for FQH,[73-75] for which explicit nonlocal order parameters have been constructed. For the anyon superfluid, Chen et al have argued[18]that a similar nonlocal order parameter occurs, and its origin can be traced to the noncommutativity of the translation generators in the effective CS description. This phenomena is discussed in the next section. Halperin et al[20] have proposed nonlocal order parameters for anyon superconductors which strongly resemble those employed in the FQH system.

To summarize, anyon superfluids and superconductors can be regarded as compressible cousins of quantum Hall systems. Fradkin[68] has extended this analysis to show that, when the anyon statistics takes values *other* than the π/N, the free anyon gas *is* a novel kind of incompressible quantum Hall system.

Even this is not the end of the story: as we will see in the next chapter, there is yet another remarkable correspondence, this between the CS nonrenormalization

theorem and the topological quantization of the quantum Hall conductance.

7.7. SPONTANEOUS BREAKING OF AN ALGEBRA

The origin of the massless mode in the current-current correlator and hence the superfluidity of an anyon gas according to ref. [15] is the spontaneous breaking of the commutation relation of the spatial translation generators, or in their terminology "spontaneous fact violation". In the phenomenological description of the system, based on quasiparticles and quasiholes, the momentum operators do not commute. In fact the whole Poincaré algebra is not preserved. These models are constructed from some matter fields coupled to a background gauge fields. It is straightforward to check that in this case classically the energy momentum tensor of the matter is not conserved:

$$\partial^{\mu} T_{\mu\nu} = J^{\mu} F_{\mu\nu}, \tag{7.1}$$

where $F_{\mu\nu}$ is the field strength and J_{μ} is the current coupled to the background $U(1)$ gauge field . Moreover upon canonically quantizing the system one finds[20] the following commutation relations for the normal ordered momentum and the hamiltonian:

$$
\begin{aligned}
[P_i, P_j] &= \int d^2x d^2y [T_{0i}(x), T_{0j}(y)] = i\hbar \int d^2x J_0 F_{ij} \\
[P_i, H] &= \int d^2x d^2y [T_{0i}(x), T_{00}(y)] = i\hbar \int d^2x J_k F_{ki}
\end{aligned}
\tag{7.2}
$$

The rest of the Poincaré algebra is modified in a similar way. The case of a constant magnetic field $\partial_k F_{ij} = 0$ is special. In this case one can[16] use modified momentum operators $\tilde{P}_i$ given by

$$\tilde{P}_i = P_i + \int d^2x J_0 F_{ij} x^j \tag{7.3}$$

in such a way that the momentum commutes with the hamiltonian, and is thus conserved quantum mechanically, but the first equation in (7.2) still holds (with a change of sign): $[\tilde{P}_i, \tilde{P}_j] = -i\hbar Q F_{ij}$. The symmetry of a system of quasiparticles coupled to a constant electromagnetic field, is generated by the same number of generators as in the underlying theory, but with an "anomalous" algebra. For that reason the phenomenon is referred to as the spontaneous breaking of the algebra rather then spontaneous symmetry breaking.

For a dynamical gauge field, instead of a gauge background, the situation is completely different, and as expected the Poincaré algebra is restored. When a Maxwell term is added the energy-momentum tensor $T_{\mu\nu}^{(A)} + T_{\mu\nu}^{(M)}$ is classically conserved. The algebra of $P_i = P_i^{(M)} + P_i^{(A)}$ on physical states is given now by:

$$
\begin{aligned}
[P_i, P_j] &= i\hbar \int d^d x [(J_o + \partial_k E^k) F_{ij}] = 0 \\
[P_i, H] &= i\hbar \int d^d x [(J_0 + \partial_k E^k) E_i] = 0
\end{aligned}
\tag{7.4}
$$

where the Gauss law of the combined system $J_0 + \partial_k E^k = 0$ was used. If instead of a Maxwell term a CS term is invoked or added to the former, the Poincaré algebra is also restored.[26]

The connection between the restoration of the original algebra and the existence of a "Goldstone boson" can now be clarified. In the case with no CS term, we can define a new field which is related to the gauge fields by $F_{\alpha\beta} = \epsilon_{\alpha\beta\gamma}\partial^\gamma\tilde{\varphi}$, and therefore

$$
\mathcal{L}_A = -\frac{1}{4} f^{\mu\nu} f_{\mu\nu} = \tfrac{1}{2} \partial_\mu \tilde{\varphi} \partial^\mu \tilde{\varphi}
\tag{7.5}
$$

The field $\tilde{\varphi}$ is the "Goldstone Boson" discussed in ref. [15]. On the other hand a system which contains both the F^2 and the CS terms, is solved by a canonical free massive field φ which is related to the electromagnetic fields in the following way: $E_i = -\epsilon_{ij}\hat{\partial}^j \partial_0 \varphi - \theta \hat{\partial}_i \varphi$, $B = \sqrt{-\nabla^2}\varphi$ where $\hat{\partial}_i = \frac{\partial_i}{\sqrt{-\nabla^2}}$. The field φ has a spin equal to $\theta/|\theta|$.

Thus the restoration of the Poincaré algebra at zero chemical potential may correspond to an addition of a "Goldstone mode", but this is not the unique possibility. In case that the complete effective action includes both a CS term and an F^2 term , then what is added to the quasiparticle and quasihole modes is a massive[52]degree of freedom which has spin ± 1. Hence the appearance of the massless pole can not be attributed only to the restoration of the Poincaré algebra.

The phenomenon of "spontaneous fact violation" received a new interpretation by Wilczek in ref. [25] as a "spontaneous projectivization of an algebra". The main point in this approach is that the realization of symmetries as unitary transformations in the corresponding Hilbert space are only determined up to a phase. In case that this phase can not be shifted away one has a projective realization of the symmetry algebra. For a continuous symmetry, this leads to a modification

of the algebra. In ref. [25] it is argued that if the ground state is realized by a projective representation, one can repeat the arguments of the ordinary Nambu-Goldstone theorem to show the necessary existence of a massless mode. However this statement was not explicitly proven.

8. The Nonrenormalization Theorem and the Stability of Mean Field Results.

The simple physical picture of the anyon superfluid described in the previous section relied on the mean field approximation where fermions exactly fill N Landau levels. In this section we want to examine the validity of the mean field approximation as a starting point for the computation of anyon dynamics, and examine the stability of the mean field results to higher order corrections. We will see that, not only is the Landau level picture a valid starting point for anyon computations, but in addition the mean field derivation of the Goldstone mode becomes *exact* in the cases where N is an integer. This is due to a nonrenormalization theorem of finite density CS theory[23] which holds (at zero temperature) for any integer N, and to all orders in perturbation theory. At the end of this chapter we briefly mention an attempt to understand nonperturbative effects in CS theory.

Several authors[59,62,67] have argued that the Landau level description of the anyon gas is inconsistent and/or unstable at the semiclassical level. The argument goes as follows. The finite anyon density generates a fictitious B field due to the CS field equation

$$\theta B = e\rho \tag{8.1}$$

Thus a constant density generates a constant B which in turn induces the fermions to occupy Landau orbits. This Landau fermions constitute a current, which generates a fictitious electric field $\mathcal{E}$ according to the *other* CS field equation

$$\theta \mathcal{E}^i = -e\epsilon^{ik} j_k \tag{8.2}$$

resulting in a kind of reverse Hall effect. Furthermore (the argument goes), the Lorentz force from this $\mathcal{E}$ field precisely cancels that from the B field. This is easily seen by noting[59] that the CS term, since it is metric–independent, makes no contribution to the energy–momentum tensor; thus conservation of this tensor requires that the total CS force density vanishes. The argument concludes either that the Landau level description is inconsistent,[59] or that consistency is restored by shifting the CS coupling to zero.[62,67]

The problem with this argument is that it assumes that the Landau level picture derives from the local field equation (8.1) while ignoring the effect of the second field equation (8.2). This is not the case, as we can see by a more careful analysis of the field equations. The basic point is that there is a fundamental dichotomy in CS theory between physics on small or large distance scales relative to the inverse CS mass. A good way to appreciate the importance of this dichotomy is to consider the effect of including a Maxwell term for the fictitious gauge field in addition to its CS term. Such a term is generated in any case at one-loop level, and one can argue that it is more proper to include it from the start in writing an effective action for anyons. The addition of a Maxwell term considerably alters the *local* field equations. The fictitious photon becomes dynamical, and physics on small distance scales looks totally different. However, the $\mathcal{E}$ and $\mathcal{B}$ fields are exponentially dumped by the CS mass and thus are short-ranged. Thus physics on long distance scales is insensitive to the presence or absence of the Maxwell term.

The Landau level picture arises from an analysis of the *full* set of field equations (with or without Maxwell contributions) on *long distance scales.* Specifically, one solves the full set of field equations for the fictitious vector potential a_i and then takes the asymptotic form of the solution. The result is[53]

$$a_i \xrightarrow[r \to \infty]{} -\frac{Q}{2\pi\theta}\nabla_i tan^{-1}\frac{y}{x} \qquad (8.3)$$

where Q is the total charge. The mean field approximation is then the replacement of Q/r^2 by the mean density ρ_0. We then have:

$$a_x = -y\frac{\rho_0}{\theta} \quad ; \quad a_y = x\frac{\rho_0}{\theta} \qquad (8.4)$$

Thus on long distance scales, in the mean field approximation, we obtain Landau levels. This analysis already takes into account both field equations and the vanishing of the CS force density.

On the other hand, there is a possible problem with the Landau level picture, although it has nothing to do with the local field equations. The problem is that the cyclotron radius of the Landau orbits is not large compared to the inverse CS mass scale, except when N is large.[15] Thus one may worry that the Landau level description is a poor approximation to the real physics unless N is large. Since $N=2$ is the case of particular interest, it would be disappointing if corrections to the mean field approximation wash out the Landau level picture for small N.

There are several arguments[15,23,65] that the existence of a Goldstone mode for N integer is immune to the effects of fluctuations about mean field theory. One argument appeals to the relation described above between the anyon superfluid and the integer quantum Hall system. As we have seen, the Goldstone mode arises from the vanishing of the effective CS mass for the fictitious gauge field. If higher order corrections generated a finite (there are no infinite renormalizations in CS theory[62,70]) nonzero value for the finite density renormalized CS coupling, then superfluidity would be lost. However this quantity is precisely analogous to the IQH conductance σ_{xy}.[65,67] The IQH conductance can be expressed as a topological invariant, and thus takes quantized values which are completely insensitive to fluctuations, provided only that the Landau gap is preserved and the many–body ground state is nondegenerate.[76,1] Furthermore this property is not dependent on the particular physical boundary conditions which apply at the boundary of the planar system. All of these statements can be taken over as implying the stability of the mean field anyon superfluid. The beauty of this argument is that applies equally to idealized anyon systems (as considered in this review) as well as realistic anyon systems.

One can also show directly that the vanishing of the finite density renormalized CS coupling is unaffected by higher order radiative corrections in tadpole–corrected perturbation theory. This proof takes the form of a nonrenormalization theorem[23] for finite density CS theory, which applies to ρ and $\Pi_{odd}(0)$. This theorem is an extension of the nonrenormalization theorem of Coleman and Hill[69,70] which applies to $\Pi_{odd}(0)$ in the zero density case, to the case of finite density.

Before attempting to extend the Coleman-Hill theorem, let us briefly review it. Consider the Euclidean n-photon effective vertex, at zero density, given by summing all graphs consisting of a single fermion loop with n external photons attached. We denote this by:

$$\Gamma^{(n)}_{\mu_1\dots\mu_n}(k_1\dots k_n) \tag{8.5}$$

All diagrams in vacuum perturbation theory which contribute to $\Pi_{odd}(0)$ can be constructed from the $\Gamma^{(n)}$'s, by sewing together photon lines (see ref.[69] for details). One set of contributions is obtained by sewing together all but two photon lines of a $\Gamma^{(n)}$, and finding the piece of the resulting two-point function which is linear in the external momentum and antisymmetric in the vector indices.[70] The remaining contributions are obtained by sewing together, in all possible one-photon-irreducible ways, two different $\Gamma^{(n)}$'s, such that one external photon line

remains on each. These two types of contributions have the following form:

$$\lim_{k \to 0} \epsilon_{\mu\nu\lambda} \frac{\partial}{\partial k_\lambda} \int dk_3 \cdots dk_n \Gamma^{(n)}_{\mu\nu\lambda_3\ldots\lambda_n} \left(k; -k; k_3; \ldots; -\sum_3^{n-1} k_\ell \right) \mathcal{K}_{\lambda_3\ldots\lambda_n}(k_3; \ldots; k_n)$$

$$\lim_{k \to 0} \epsilon_{\mu\nu\lambda} \frac{\partial}{\partial k_\lambda} \int dk_2 \cdots dl_2 \cdots \Gamma^{(n)}_{\mu\rho\ldots}(k; k_2; \ldots) \Gamma^{(n)}_{\nu\lambda\ldots}(-k; l_2; \ldots) \mathcal{K}_{\rho\ldots\lambda\ldots}(k; k_2; \ldots)$$

$$(8.6)$$

Now for any $\Gamma^{(n)}$ gauge invariance implies

$$k^\mu \Gamma^{(n)}_{\mu\ldots} = 0 \tag{8.7}$$

Differentiating this expression gives

$$\Gamma^{(n)}_{\nu\ldots} + k^\mu \left(\partial/\partial k^\nu \right) \Gamma^{(n)}_{\mu\ldots} = 0 \tag{8.8}$$

Provided that $\Gamma^{(n)}$ is analytic as $k \to 0$ this implies

$$\Gamma^{(n)}_{\ldots}(0; k_2; k_3; \ldots) = 0 \tag{8.9}$$

Furthermore, if $n > 2$, so that k_1 and k_2 are independent variables, then

$$\Gamma^{(n)}_{\ldots}(k_1; k_2; \ldots) = \mathcal{O}(k_1 k_2) \tag{8.10}$$

as k_1, $k_2 \to 0$. These relations imply that all contributions to $\Pi_{odd}(0)$ of two-loop and higher order vanish. This is the Coleman-Hill nonrenormalization theorem.

We now want to extend these arguments to the case of finite fermion density. We thus define a finite density Euclidean n-photon effective vertex, given by summing all graphs consisting of a single *tadpole-corrected* fermion loop with n external photons attached. Order by order in *tadpole-corrected* perturbation theory, the structure of the graphs which contribute to $\Pi_{odd}(0)$ is identical to the zero-density case. Furthermore, we can apply the same construction to obtain all the graphs contributing to ρ. These are obtained by sewing together all but one

external photon line of a $\Gamma^{(n)}$, and have the form:

$$\int dk_2 \cdots dk_n \Gamma^{(n)}_{\mu\lambda_2\ldots\lambda_n}\left(0;k_2;\ldots;-\sum_2^{n-1} k_\ell\right) \mathcal{K}_{\lambda_2\ldots\lambda_n}(k_2;\ldots;k_n) \qquad (8.11)$$

Thus, to prove the desired nonrenormalization theorem for ρ and $\Pi_{odd}(0)$, it suffices to show that, for $k_1, k_2 \to 0$:

$$\begin{aligned} \Gamma^{(n)}_{\ldots}(k_1\ldots) &= \mathcal{O}(k_1), & n > 1 \\ \Gamma^{(n)}_{\ldots}(k_1, k_2, \ldots) &= \mathcal{O}(k_1 k_2), & n > 2 \end{aligned} \qquad (8.12)$$

By gauge invariance and the argument presented above, these relations are true provided that $k \to 0$ is in the region of analyticity of the $\Gamma^{(n)}$.

We prove the nonrenormalization theorem therefore by demonstrating the analyticity of the $\Gamma^{(n)}$ as $k^2 \to 0$ in the Euclidean region. This is obvious for the zero density system, since the physical (Minkowski) threshold for fermion-antifermion pairs begins at $k^2 = 4m^2$. At finite density, however, one must also worry about the production of fermion-hole pairs. In our case, since the $\Gamma^{(n)}$ are defined in tadpole-corrected perturbation theory, this corresponds to a (Minkowski) photon being absorbed by a fermion in a Landau level, causing a transition to an unoccupied state. The Landau levels allow continuous values of momentum but are discretely spaced in energy (with spacing eB/m in the non-relativistic limit). Therefore, when we have N completely filled Landau levels, physical singularities are absent for (Minkowski) $k_0 < eB/m$. Thus as we approach $k^2 \to 0$ from the Euclidean region the $\Gamma^{(n)}$ are analytic, and the nonrenormalization theorem holds precisely for $\theta = Ne^2/2\pi$.

Note that for other values of θ we obtain no definite conclusions; this is similar to the $m = 0$ case of the zero density system. For self-consistency, we should also note that the Goldstone pole, which is the end result of this analysis, does not appear in the individual 1PI diagrams of the $\Gamma^{(n)}$.

The physical content of the finite density nonrenormalization theorem is that the spatially averaged mean density ρ_0 is insensitive to local perturbations of the background fictitious gauge field. This can also be demonstrated directly by a topological argument,[23] which shows that ρ_0 is dependent only on the asymptotic behavior of the background field. This makes the connection between the nonrenormalization theorem and the analogy to IQH quantization.

8.8. NON-PERTURBATIVE RENORMALIZATION OF THE CS TERM

Morozov and Niemi[27] discussed the effect of instantons on the renormalization of the CS term. The model they considered was a Euclidean three dimensional CP^1 model with a Hopf-invariant term which when expressed in terms of the fictitious gauge fields take the form of a CS term:

$$S(\vec{z}, a_\mu) = \frac{1}{J} \int d^3x |D_\mu \vec{z}|^2 + i\theta CS(a_\mu) \tag{8.13}$$

with $\vec{z}$ a complex two-vector where $|z|^2 = 1$ and $a_\mu = \frac{i}{2}(\bar{z}_i \partial_\mu z_i - \partial_\mu \bar{z}_i z_i)$. It is well known that this model admits soliton configurations since $\pi_2(S^2) = \mathcal{Z}$, but in addition there should be also instantons related to the non-trivial Hopf homotopy $\Pi_3(S^2) = \mathcal{Z}$. The solitons are classified by the winding number, and the instantons, which have not yet been explicitly constructed, by the CS term. In four-dimensional QCD it was shown that instantons renormalize the strong CP θ term as

$$\frac{\partial \theta(\mathcal{M})}{\partial \mathcal{M}} \sim sin\theta \tag{8.14}$$

where $\mathcal{M}$ determines the scale. For decreasing $\mathcal{M}$, $\theta(\mathcal{M})$ tends toward 0 mod 2π and hence it vanishes in the infrared. The authors of ref.[27] argued that since several instanton effects were dimension independent it was reasonable to expect that the same phenomenon existed also for the CS θ. This argument was further supported by the following calculation. They estimated the instanton and anti-instanton contributions to the effective action in the dilute gas approximation:

$$\frac{\delta \mathcal{L}}{\delta \mathcal{M}} \sim \rho_{in}(a)e^{S_{in}} + \rho_{a-in}e^{S_{a-in}} \tag{8.15}$$

with the assumptions that $S_{in} = S_0 + i\alpha\theta$, $\rho_{in}(a) = \bar{\rho}(a) - \zeta CS(a)$, where α and ζ are some parameters and for anti-instantons the sign between the two terms is switched. This yields a beta function of the form

$$\frac{\delta \theta(\mathcal{M})}{\delta \mathcal{M}} \sim -Dsin(\alpha\theta) \tag{8.16}$$

Provided the parameters in the last expression are non-trivial the renormalized CS term vanishes. Using topologically non-trivial configurations which do not solve

the equation of motion, they find that the matrix element $< a_\mu a_\nu | instanton | vac >$ is non-trivial. Hence when inserted back to the variation of the effective action, this implies that the parameters in eqn. (8.16) are indeed non-trivial.

While this discussion of non-perturbative renormalization is clearly speculative, we cannot discount the notion that these or similar effects may be important for a full understanding of anyonic superconductivity.

9. FINITE TEMPERATURE BEHAVIOR

We have seen that a system of anyons at zero temperature is a superfluid if the anyons are neutral and a superconductor if the anyons are electrically charged. In this section we study the finite temperature behavior of these systems. The naive expectation is that the superfluidity and superconductivity of these systems should persist, at least for small temperatures. In fact since at zero temperature there is a massless "Goldstone" mode in the neutral system, at finite temperature we might expect this mode to be thermally occupied, resulting in both a normal and a superfluid component to the system as is usual for superfluids and superconductors. Furthermore, since anyonic systems are 2+1 dimensional, we expect any symmetry breaking which may be present at $T = 0$ to disappear at finite temperature. In fact if such a symmetry exists then even though we expect the massless mode to remain massless at $T \neq 0$ the correlations of any order parameter would vanish as a power of the distance as occurs in two dimensional models such as the x-y model,[77] and as is usually referred to as Kosterliz-Thouless behavior. In analogy with these two dimensional models we might, naively, expect a phase transition at some temperature T_c to a non-superfluid phase which we might expect to have some analogies with the Kosterliz-Thouless transition. We shall see, however, that the finite temperature behavior of anyonic systems is much more complicated.

The most straightforward method of analyzing the finite temperature system is using a finite temperature generalization of the tadpole-improved perturbation analysis which we discussed for the zero temperature case in chapter 6. We shall see using this method that the massless mode does *not* survive to finite temperature and that, in fact, there is an exponentially small renormalized Chern–Simons term and thus a mass for the "Goldstone" mode. This differs entirely from the naive expectations described above. Furthermore, when the anyons are charged, we found, at $T = 0$, that the two dimensional electromagnetic gauge field acquired a Chern–Simons term and a "Higgs" mass term ($A_\mu A^\mu$). The Chern–Simons term survives the finite temperature analysis (though it's coefficient shifts by an

exponentially small amount (at low temperature)) but the "Higgs" mass term is no longer present. This seems to imply that in the purely two dimensional system, there is no superconductivity at nonzero temperature.

Our goal in this section is to first describe this perturbative analysis at nonzero temperature and to establish the results described above. We then discuss the physical interpretation of having a Chern–Simons term but no "Higgs" term in the electromagnetic action especially in light of the fact that the system is, in fact, three dimensional. The consequences of this result for High T_c materials is discussed. We then discuss some nonperturbative approaches to the finite temperature problem.

9.9. RENORMALIZATION OF THE CHERN–SIMONS TERM AT FINITE TEMPERATURE

We begin by discussing the perturbative result. Let us consider neutral anyons. Recall from chapter 5 that the passage to finite temperature does not alter the fact that the condition for the massless mode in the current–current correlator is the vanishing of θ_R unless singularities are present in the even part of the vacuum polarization (Π_e) at low momentum. Since such singularities do generically occur at finite temperature we should see if they could significantly affect the result. Clearly a sufficiently singular Π_e could make the massless mode massive even if θ_R vanishes since, for example, a Higgs mass results simply from a singular vacuum polarization. Note, however, that if $\theta_R \neq 0$ as, we shall see, occurs in our case, there is no way that singular terms in Π_e can make the mode massless.. Thus, in order to check whether the massless mode persists at nonzero temperature we should first evaluate θ_R and determine whether it vanishes. We thus repeat the calculation of ρ and Π_{odd}, which are now also function of the temperature, and check whether we still have $\Pi_{odd} = \frac{e\rho}{B}$ which would imply the vanishing of θ_R. We shall only calculate the one–loop contributions to these quantities (in tadpole–improved perturbation theory). Our proofs of the nonrenormalization theorem do not extend to the finite temperature case and it is nearly certain that the theorem fails to hold. We shall discuss the implications of this limitation to one loop later in this section.

Technically, the standard procedure to pass to the finite temperature calculation in Euclidean space involves the compactification of the (Euclidean) time direction into the range $0 \leq t \leq \beta = \frac{1}{T}$ and the imposition of antiperiodic boundary conditions (in time) for fermions and periodic boundary conditions for bosons. For our calculation this implies replacing the integral over frequencies ω with a

sum over discrete 'Matsubara' frequencies $\omega_l = \frac{2\pi}{\beta}(l + \frac{1}{2})$, l being an integer. In particular we can evaluate the mean density ρ_0 at finite temperature by using equation (6.17) but, instead of integrating over ω, we sum over the discrete frequencies. This results in the expression:

$$\rho_0 = -\frac{ieB}{2\pi\beta}\sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty}[(\omega_l - i\mu)(\frac{1}{d_{n+1}(l)} + \frac{1}{d_n(l)}) - im(\frac{1}{d_{n+1}(l)} - \frac{1}{d_n(l)})] \quad (9.1)$$

where

$$d_n(l) = (\omega_l - i\mu)^2 + M^2(n) = (\omega_l - i\mu)^2 + 2neB + m^2$$

The frequency sums $\sum G(l)$ can be done exactly by evaluating contour integrals of the form

$$\frac{1}{2\pi i}\oint \frac{cos\pi x}{sin\pi x}G(x) \quad (9.2)$$

where the integral is over a contour which surrounds the real axis. When evaluating ρ_0 we get the following answers for the sums:

$$\begin{aligned}\frac{2\pi}{\beta}\sum_l \frac{1}{d_n(l)} &= -\frac{\pi}{2M}\left[tgh[\frac{\beta}{2}(\mu - M)] - tgh[\frac{\beta}{2}(\mu + M)]\right] \\ \frac{2\pi}{\beta}\sum_l \frac{(\omega - i\mu)}{d_n(l)} &= \frac{\pi i}{2}\left[tgh[\frac{\beta}{2}(\mu - M)] + tgh[\frac{\beta}{2}(\mu + M)]\right]\end{aligned} \quad (9.3)$$

Inserting these results into the expression (9.1) for ρ_0 we get the following expression for the density at finite temperature:

$$\rho_o = \frac{eB}{4\pi}\{\sum_{n=0}^{\infty}[tgh[\frac{\beta}{2}(\mu + M(n))] + tgh[\frac{\beta}{2}(\mu - M(n))]] - \frac{m}{|m|}tgh[\frac{\beta}{2}(\mu - |m|)]]\} \quad (9.4)$$

Note that in the limit $\beta \to \infty$ this expression reduces to the zero temperature result given in equation (6.19). At nonzero temperature the density is no longer a step function as the chemical potential is varied. The steps are smoothed out as is shown for specific values of the parameters in Figure 7. This result is, of course well known from the theory of the Quantum Hall Effect.

We now compute the renormalized CS term at finite temperature. The simplest way to do this is to use eqn. (6.29) which is valid at finite temperature. Recall that the renormalized Chern–Simons term is proportional to $d(\rho/B)/d(B)$. In the zero temperature case ρ was (piecewise) proportional to eB and thus the renormalized Chern–Simons term vanished. At finite temperature we see from eqn. (9.4) that this is no longer the case. In fact ρ/eB is a monotonic function of eB. The relation $\Pi_{odd} = e\rho_0/B$ is thus never valid, and the renormalized CS term is nonzero for any finite temperature. In fact by differentiating eqn. (9.4) we find:

$$
\theta_R(\mu, T) = - eBe^2 \frac{d(\rho/eB)}{d(eB)}
$$

$$
= -\frac{\alpha eB}{2}\beta \sum_{n=0}^{\infty} \frac{n}{M(n)} \left[tgh^2[\frac{\beta}{2}(\mu + M(n))] - tgh^2[\frac{\beta}{2}(\mu - M(n))] \right]
$$

$$(9.5)$$

where $\alpha = e^2/4\pi$. Note that the term inside the sum is positive for all values of n as long as μ is nonzero. Thus θ_R is nonzero for any nonzero value of μ. Note that it is not necessary to use eqn. (6.29) to evaluated the renormalized Chern–Simons term. An explicit calculation of Π_{odd} is given in ref. [23].

It is important to emphasize that this result, namely the presence of a nonzero renormalized Chern–Simons term at finite temperature, has only been demonstrated in the one–loop approximation. Although it is difficult to imagine that higher order perturbative effects would force the mass of this mode to vanish, it is, in principle, possible. More reasonably, nonperturbative effects may generate a massless mode in the current–current correlation even at finite temperature. We shall discuss this issue further below.

In order to obtain an estimate of the size of the renormalized Chern–Simons term and of the resulting mass of the "pseudo–Goldstone mode in a realistic system we evaluate θ_R in the low temperature limit. (The precise limit will be described below.) We shall specialize to the case of most interest for which the $T=0$, $\mu=0$ Chern–Simons coefficient is an integer, N, i.e. for which N Landau levels are filled. We assume a density ρ of anyons. Keeping in mind the distinction between ρ_{phys} and ρ_0 which was discussed in section 2, we then require a field $eB=2\pi\rho/N$. If we then assume that the temperature is sufficiently low so that $\beta(M_N - M_{N-1})) >> 1$ (recall that $M_N^2 = 2NeB + m^2$) we can compute the sums in both eqn. (9.4) and eqn. (9.5) since only one term in each sum contributes significantly. We can

express the result in terms of the renormalized value of N, $N_{ren}=2\pi\theta_R/e^2$ as

$$N_{ren} = \frac{2\pi\rho}{N}\beta\left(\frac{N-1}{M_{N-1}} + \frac{N}{M_N}\right)exp\left(-\frac{\beta}{2}(M_N - M_{N-1})\right) \qquad (9.6)$$

In the non-relativistic limit $\rho << m^2$ this becomes

$$N_{ren} = \frac{2\pi\rho}{mN}\beta(2N-1)exp\left(\frac{-\beta\pi\rho}{mN}\right) \qquad (9.7)$$

We see that for integer N and for small temperature, the renormalized Chern–Simons term is exponentially suppressed compared to its unrenormalized value.

The mass of the "pseudo–Goldstone" mode is given by:

$$m_{PG} = \frac{\theta_R}{\Pi_e} = (\frac{2\beta}{N})(\frac{\pi\rho}{mN})^2(2N-1)exp\left(\frac{-\beta\pi\rho}{mN}\right) \qquad (9.8)$$

where for Π_e we have used the estimate derived in ref. [15]. We can get a rough idea of the order of magnitude of this mass by putting in some possible numbers for the mass and density such as may occur in high T_c superconductors. Choosing the density ρ to be $10^{14}cm^{-2}$ and the mass m to be the electron mass and a temperature T of $100^o K$ we find that m_{PG} is approximately 5×10^{-6} ev. This corresponds to a distance scale of roughly 5 cm. This estimate is of course extremely crude since there are large uncertainties in the exponent.

9.10. ANYONIC SUPERFLUIDITY AND SUPERCONDUCTIVITY AT FINITE TEMPERATURE

Since the results of the previous section differ from our expectations and experience about superfluids and superconductors, we should scrutinize these results quite carefully. First note that in conventional superfluids the massless pole in the current-current correlation persists at finite temperature. This happens even in 2+1 dimensions in which case symmetry breaking cannot persist to finite temperature due to the Coleman–Mermin–Wagner theorem[61] since even there the massless mode remains. Thus the results of the previous section seem to imply that superfluidity is lost.

In order to analyze this matter further we first make a comment on the spectrum of the zero temperature theory (of neutral anyons). Viewed perturbatively (via the "tadpole–improved" perturbation theory of chapter 6) we have a massless statistical gauge field a_μ together with a fermion field, in a magnetic field, at finite density, with full Landau levels leading to a gap in its spectrum. This fermion is 'charged' (with a statistical charge) and thus still couples to the statistical photon. The resulting theory has, in the low momentum limit, similarities with Quantum Electrodynamics (QED) is 2+1 dimensions. In particular the self–energy of the fermion is infrared divergent and grows as the logarithm of the size of the system. Two fermions, as well as a fermion–hole pair have a logarithmic interaction at large distances. At exceedingly large distances, the separation energy of a fermion–hole pair is so large that it pays to create another pair across the Landau level gap so as to screen the charges. This screening of the charges is unusual since it occurs without the 'photon' acquiring even an electric mass as can be seen by the fact that two equally charged fermions would *not* be screened. The screening is thus a nonlinear nonperturbative[†] phenomenon. A similar nonlinear phenomenon screens the charge of N anyons (where $N = 2\pi\theta/e^2$ is related to the Chern–Simons coefficient). Even though a priori a collection of N anyons will, perturbatively, have a divergent self–energy, there is a nonlinear mechanism whereby they can redistribute themselves among the Landau levels (since they have now increased the local mean magnetic field) and simply create a density perturbation. (See chapter 7 for more discussion on this point.) Thus these fermions behave very much as do vortices in models such as the x–y model. This adds to the suspicion that the physics here should be similar to that in the x–y model in that the superfluidity should persist to finite temperature and eventually disappear due to a vortex pair unbinding transition. These ideas have been used quite widely in analyzing the finite temperature behavior of anyonic systems.[16,66,64,82] Yet vortices are somehow different. If we consider, for example, a theory of bosons with a hard core repulsion at finite density and low temperature, they do, of course, Bose condense and they do have a massless pole in the current–current correlator which persists at nonzero temperature. The long distance behavior of this theory is, however, very similar to that of an x–y model in that there are vortices with similar interactions. Thus in our theory, the lack of a massless pole in the current–current correlator indicates that this theory differs somewhat from the conventional theories of vortices.

The case of charged anyons has the additional feature of the presence of a

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† By nonperturbative we mean that it cannot be seen in any finite order in perturbation theory

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Chern–Simons term for the electromagnetic photon which is only slightly modified at nonzero temperature. Thus the Meissner effect which results from this term is present at nonzero temperature as well.[21] The conventional Meissner effect due to a Higgs effective lagrangian is not present since the photon is massless (although, at low temperature, this term still causes a large but finite energy penalty for the presence of a constant magnetic fields). There will be no dissipationless current flow but the resistance will be exceedingly small at low temperature.

In summarizing this section we once again point out that since we have no control of nonperturbative effects, one cannot argue convincingly that superfluidity is lost at any finite temperature. What is clear, however, is that one of the main steps in the argument for superfluidity at zero temperature, namely the presence in the RPA approximation of a massless pole in the current–current correlation, is lost at finite temperature.

10. Low Energy Effective Action

The low energy effective action determines the phenomenology of the anyonic superconductors. It is thus very important to determine and analyze this effective action if we are to confront the theory with experimental results. As discussed in the previous chapters, the essential picture that emerges from the various formulations of the model, is that of a gapless density wave coupled to electromagnetism. Hence, it is not surprising that the low energy effective action of the various approaches to the problem are quite similar. Nevertheless, it turns out that there are several differences some of which lead to significant phenomenological differences. In this chapter we shall first present the field theoretical derivation of the low energy effective action and we later compare it to the RPA calculation. We then discuss the implications on several experiments which have been suggested as tests for the hypothesis of anyonic superconductivity.

The low energy effective action of a system of charged anyons was derived in chapter 5. For experimental applications we are, of course, interested in the Minkowski rather than the Euclidean effective action. For the finite density, zero temperature, non-relativistic case we can deduce from equation (5.12) the form of the effective lagrangian in Minkowski space. It is given by:

$$\mathcal{L}_{eff}(A_\mu,\varphi) = -\frac{1}{4}(1 + g^2\Pi_a)F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}g^2\Pi_b F_{ij}F^{ij} - \frac{1}{2}g^2\theta\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda$$

$$+\tfrac{1}{2}\,(\partial_0\varphi + CA_0)^2 - \tfrac{1}{2}\,v^2(\partial_i\varphi + CA_i)^2 \qquad (10.1)$$

$$+a(\partial_0\varphi + CA_0)^*F^0 + b(\partial_i\varphi + CA_i)^*F^i$$

where the parameters v, C, a and b are functions of the magnetic and electric even parts of the vacuum-polarization as follows: v is the velocity of the Goldstone mode of equation (5.9): $v^2 = (1 + \Pi_b/\Pi_a)$; $C = \theta g/\sqrt{\Pi_a + \Pi_b}$, and $a = b = \sqrt{\Pi_a + \Pi_b}$. $^*F$ is the dual of F. As discussed in chapter 5 the "degeneracy" between a and b is in fact lifted by terms proportional to $\tfrac{1}{2}\Pi_N\partial_i(f_{0i} + gF_{0i})\epsilon_{jk}(f_{jk} + gF_{jk})$ in equation (5.11). Recall that Π_N is the zero momentum limit of the coefficient[14] of $\omega(^*k_ik_j + k_i{}^*k_j)$ in the non-relativistic expression for Π_{ij}. In spite of the fact that this gauge and rotationally invariant term is higher in derivatives it conspires with the mixed CS term to give a contribution to the term whose coefficient is a in (10.1). Thus the corrected effective action is identical to equation (10.1) except that a and b are no longer equal. The calculation of a and b in terms of the various Π's is straightforward though tedious. The result is:

$$a = g\sqrt{\Pi^B}[1 + \frac{\theta\Pi_N}{\Pi_E\Pi_B}]; \quad b = g\sqrt{\Pi^B} \qquad (10.2)$$

where $\Pi^E = \Pi_a$ and $\Pi^B = \Pi_a + \Pi_b$.

Equation (10.1) is a description of the low lying collective excitations of the anyonic system and it exhibits the Higgs mechanism in the Stuckelberg form. In the absence of an electromagnetic field it describes a "sound" wave. This sound wave turns into the longitudinal component of the massive photon when electromagnetism is included. The "anyonic" origin of this action appears via the P and T violating terms in the effective action. There are three such terms. The first is the induced Chern–Simons term for the photon, and the others are the terms in equation (10.1) which are proportional to a and b. If we set aside the F^2 and the Chern–Simons terms then the current and the density *which are generated by the Higgs terms* (both the P, T invariant and non-invariant terms) are given by:

$$J_i = \frac{-\delta\mathcal{L}_{eff}}{\delta A_i} = -v^2C(\partial_i\varphi + CA_i) + a\epsilon_{ij}\partial^j(\partial_0\varphi + CA_0) - b\epsilon_{ij}\partial_0(\partial^j\varphi + CA^j)$$

$$J_0 = \frac{\delta\mathcal{L}_{eff}}{\delta A_0} = C(\partial_0\varphi + CA_0) + \tfrac{1}{2}\,aC\epsilon^{ij}F_{ij}$$

The first term of the current is the standard London supercurrent and the addi-[13][4] tional terms emerge from the P and T violating terms of the action.[20]

The parameters of the Landau-Ginsburg effective action (10.1) can be determined by evaluating Π^E, Π^B and Π_N. These can be calculated, in principle, to any desired order in perturbation theory. Note, however that the nonrenormalization theorem does not apply to the even part of the vacuum polarization and thus the one loop calculation on which we shall base our discussion is only an approximation. The values of these various Π's are evaluated in ref. [15]. Inserting their results into the expressions which appear after equation (10.1) gives

$$
\begin{aligned}
v^2 &= \frac{\Pi^B}{\Pi^E} \sim \frac{\rho}{m^2} & C &= \frac{\theta}{\sqrt{\Pi^B}} \sim e\sqrt{m} \\
b &= \sqrt{\Pi^B} \sim \frac{eN}{\sqrt{m}} & a &= \sqrt{\Pi^B} + \frac{\theta \Pi_N}{\sqrt{\Pi^B \Pi^E}} \sim \frac{eN}{\sqrt{m}}
\end{aligned}
\tag{10.4}
$$

The effective action which is derived from the RPA analyses of Laughlin[8] and Chen et al[16] has the same form as (10.1). However the coefficients deduced by Chen et al and by Halperin et al[20] differ somewhat from the ones which are derived from the field theoretic considerations above. We begin by discussing their results. The parameters of the Landau Ginzburg action as computed by these authors, to lowest order in $\frac{1}{N}$, are

$$
\begin{aligned}
v^2 &= \frac{2\pi\rho\hbar^2}{m^2} & C &= e\sqrt{\frac{m}{2\pi\hbar}} \\
b &= 0 & a &= eN\sqrt{\frac{\hbar}{32\pi m}}
\end{aligned}
\tag{10.5}
$$

Halperin et al[20] suggest that these parameters can be determined from physical arguments which are outlined below.

(i) The velocity of sound is determined by the compressibility, namely, $v^2 = \rho\frac{\partial^2(E/\rho)}{\partial \rho^2}$. Now the energy of an electron gas filling exactly N Landau levels is the same as that of gas of electrons without any magnetic field. The expression for the velocity in equation (10.5) is precisely the value for fermions with no magnetic field. This expression is only expected to be valid for large N. For small N the RPA is no longer a good approximation and we expect to find corrections to this expression. In particular when $N \to 0$ which is the limit of free bosons we expect to get $v^2 = 0$.

(ii) From the expression of the supercurrent (10.3) it is easy to see that the product C^2v^2 is directly related to the penetration depth. Since the anyon gas exhibits perfect diamagnetism it is expected to obey the classic London formula for the response of the current to a slowly varying quasi-static gauge field:

$$J_i = -\frac{e^2\rho}{m^*}A_i \qquad (10.6)$$

Comparing this to the expression for C^2v^2 in equation (10.5) we see that the results for the London current match those from our effective action with the coefficients given by (10.5).

(iii) The 'a' term describes a P and T non-invariant interaction of the form $\rho \times B$. It is thus related to the intrinsic magnetic moment of the anyon superfluid. In the approximation of a constant fictitious magnetic field the fermions are moving in Landau orbits. Since they have an electric charge, this motion produces a real magnetic field. Hence, when the spin of the fermions is ignored, the entire magnetic moment is due to their orbital motion. We thus expect that this magnetic moment should be proportional to the angular momentum per particle. In ref. [20] this intrinsic magnetic moment is calculated in the mean field approximation for n fermions in N exactly filled Landau levels. The result is found to be

$$\frac{n^2}{2N} - \frac{nN}{2} \qquad (10.7)$$

Transforming to the so called "anyon gauge" by multiplying the wave function by the phase factor

$$\prod_{\text{pairs } ij} \frac{(z_i - z_j)^{\frac{1}{N}}}{|z_i - z_j|}$$

the magnetic moment per particle is found to be

$$a = e(N - \frac{1}{N})\sqrt{\frac{\hbar}{32\pi m}} \qquad (10.8)$$

This result agrees with the result presented in equation (10.5) in the large N limit. It has the further advantage of being more accurate for smaller N since, for example, it vanishes for a gas of bosons ($N = 1$) where P and T violations are not expected.

(iv) Using physical arguments it is claimed in ref. [15] that the term proportional to b should vanish. This term is a $J \times E$ interaction as can be seen from equation (10.1). Applying an electric field in the plane will cause a current which is also in the plane but which is perpendicular to the direction of the applied electric field. Since all the anyons have the same charge to mass ratio, it means that the electric field couples in fact to the current carried by the center of mass. The motion of the center of mass is independent of inter-particle interaction. Thus, as for free particles, the transverse current should not exist. In ref. [78] this argument is dressed in the form of a general rule for determining effective actions. It is argued that in addition to preserving the symmetries of the underlying microscopic theory, the effective action has to obey the operator relations that exist in the theory. In the underlying Chern–Simons action the momentum density T_{0i} is linearly related to the current. $T_{0i} = mJ_i/e$. Imposing this relation on the effective low energy action dictates the vanishing of the b term.

Let us now compare the parameters of the effective action which follows from the field theoretical approach (equation (10.4) to the one that emerged from the above physical arguments. The most important difference is the fact that they imply that the b term vanishes, whereas in the field theoretical analysis it does not. This has a significant implication for the analysis of experimental tests of anyonic superconductors. We shall now discuss this point briefly.

The main idea of the experimental tests of anyonic superconductivity is to look for P and T violating effects in high T_c superconductors. The first experiment which was suggested as a test of P and T violating effects was muon spin relaxation Polarized positively charged muons are captured in the material and decay into positrons. The direction of the outgoing positron is correlated with the spin of the muon which precesses in response to the internal magnetic field. The results of this experiment set a limit on the strength of the internal magnetic field. They find that the field is smaller than 0.8 gauss. The theoretical prediction[20] was that the field should be approximately 10 gauss. This experiment poses the most serious problem, at present, for a description of high T_c materials in terms of anyons. One possible way out of this experimental result is to use a parity invariant anyonic model which has been proposed in refs. [42] and [82].

Another experiment was suggested by Wen and Zee.[11] It is the measurement of the rotation of the plane of polarization of linearly polarized light normally incident on the superconductor. Such an effect would be produced by the b term. It is here that the difference between the RPA and the field theoretical effective action is

crucial. According to the latter the effect should exist, whereas the former, at least in the form of an ideal anyon gas, excludes this possibility. There are, at this point, conflicting results from various experiments.[80,81]

It has been suggested by Kitazawa[68] that the seeming contradictions between the field theoretic results and the physical arguments given above can be reconciled. This requires a careful treatment of screening effects and separation of physically relevant length scales. He concludes that the theoretical magnetic moment prediction of ref. [20] was incorrect, while the b term generated optical rotation is a real effect. We rather suspect, however, that this is not the last word on anyonic superconductor effective actions!

11. SUMMARY

In this article we have reviewed the basic theory of anyonic superconductivity. The basic idea was to discuss a novel type of superconductivity which exists, a priori, in two spatial dimensions and which may lead to a better understanding of high T_c superconductors which are layered two dimensional materials.

Theoretically it seems quite certain that a noninteracting gas of electrically neutral anyons at zero temperature is a superfluid. In fact such a system has a gap in its single anyon spectrum together with a gapless mode. This gapless mode couples to the anyonic current and is thus similar to a Goldstone mode which occurs due to the symmetry breaking in conventional superfluids. The system is thus expected to exhibit all the conventional features of superfluidity including dissipationless flow in the presence of impurities and the presence of vortices. It seems unlikely that there is a local order parameter which signals the breaking of a symmetry in this system although such a nonlocal order parameter *may* exist. The situation at nonzero temperature is less clear. The perturbatively improved field theoretic mean field analysis predicts that there is no gapless mode at nonzero temperature. This differs from the usual situation with symmetry breaking in which the gapless mode persists at nonzero temperature. It may be that in a pure anyonic system superfluidity is lost at nonzero temperature (though the mass of the "nearly–gapless" mode is exceedingly small). Many authors, however, claim that this is not the case, and that at nonzero temperature the usual scenario (i.e. a two fluid model) occurs.[16,67,82] Yet another finite temperature scenario is provided by Hetrick et al.[83]

A gas of charged anyons is, at zero temperature, a superconductor. It has all the usual features of superconductivity including the Meissner effect, persistent currents and vortices. The Landau-Ginsburg effective action for an anyonic superconductor is quite interesting. It contains all the features of the Landau-Ginsburg effective action for a usual superconductor including a "Higgs" mass term for the photon, but it also contains several extra terms. The most interesting of these is a "topological" (or Chern–Simons) mass for the photon. In fact the main unique feature of an anyonic superconductor is that it's ground state breaks parity and time reversal invariance. Several experiments have looked for this symmetry breaking but there is no conclusive experimental evidence yet.

If anyons are responsible for high T_c superconductivity, it remains an open challenge to figure out why this is so. Although there are several approaches to high T_c materials based on the Hubbard model[22] which can possibly lead to anyonic quasiparticles, there is no compelling reason, nor any reasonably convincing model, which predicts their presence.

Although much has been learned about anyons and anyonic superconductivity in the past few years, there remain many interesting unsolved problems and controversies in the subject. Many of these have been discussed in this review. The question of whether anyons are responsible for high T_c superconductivity can, in the end, only be decided by experiments.

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Figure Captions

Fig 1: Diagramatic expansion for the electromagnetic photon propagator in terms of the current-current correlator K of the pure Chern-Simons theory. Electromagnetic corrections to the vacuum polarization are not shown.

Fig 2: Two representations of the Feynman diagrams for the Chern-Simons theory.

Fig 3: Tadpole contributions to the fermion propagator. S_F represents the full fermion propagator.

Fig 4: One loop diagram for the density ρ_0 in tadpole-improved perturbation theory.

Fig 5: The physical density $\rho_{ph} = \rho(\mu) - \rho(0)$ is plotted versus $\mu^2 \times sign(\mu)$ at fixed B. Here $sign(\mu)$ is the sign of μ.

Fig 6: Diagramatic representation of the result that $\delta e \rho_0 / \delta B |_\mu = \Pi_{odd}(q = 0)$. Only the simplest class of diagrams are shown.

Fig 7: Numerical results for the physical density as a function of μ for various values of the inverse temperature β.
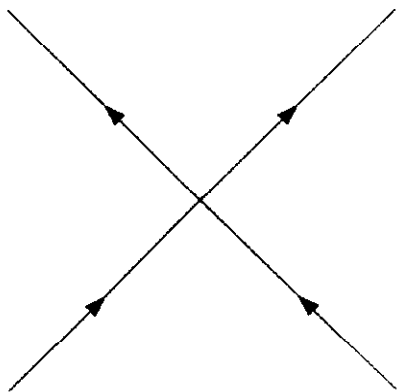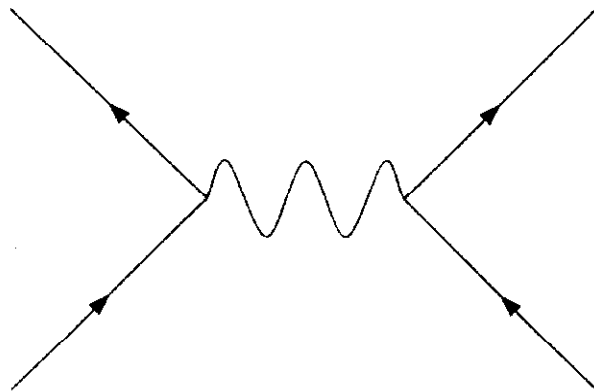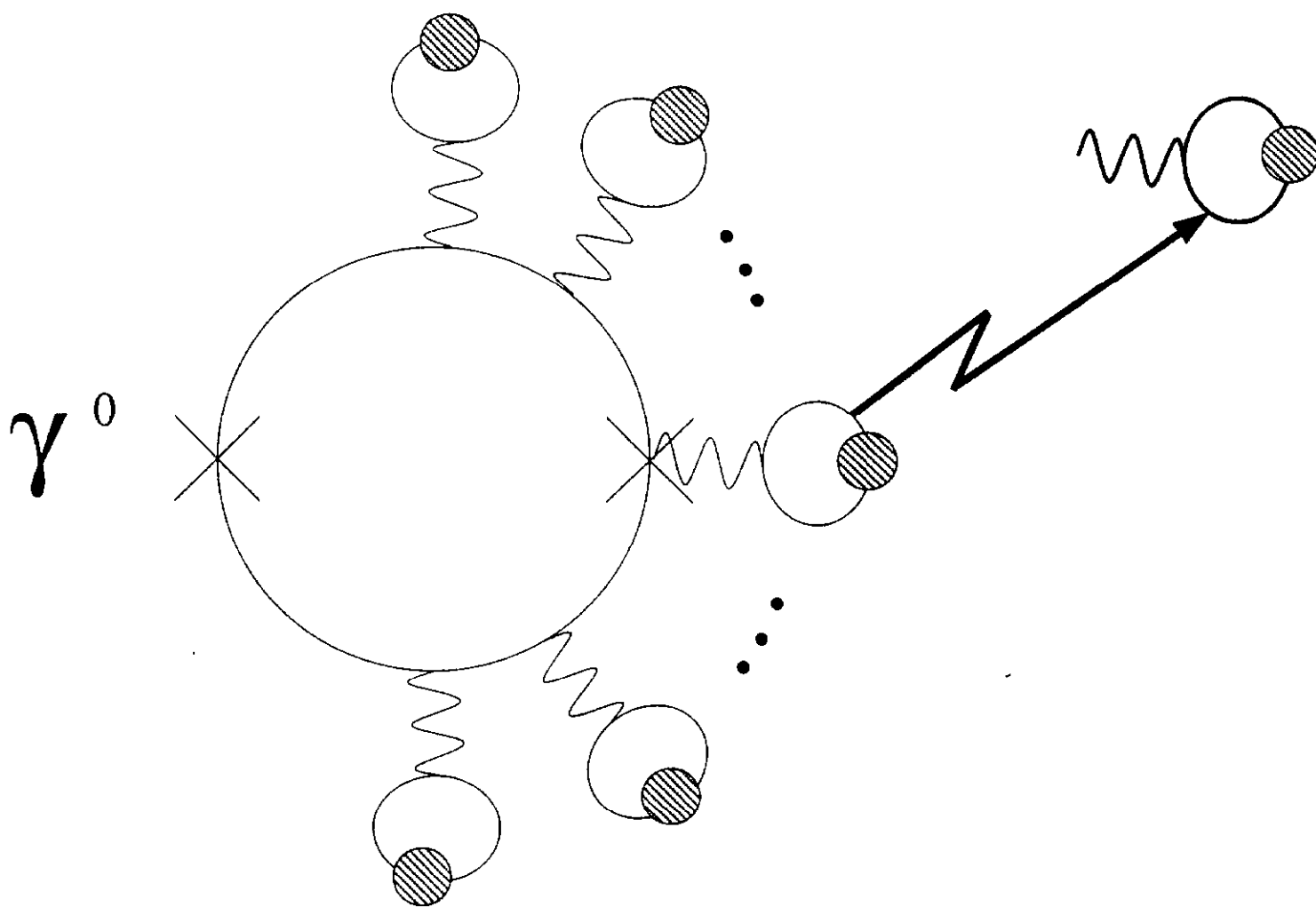
Figure 1

Figure 2 a   *Figure 2 b*
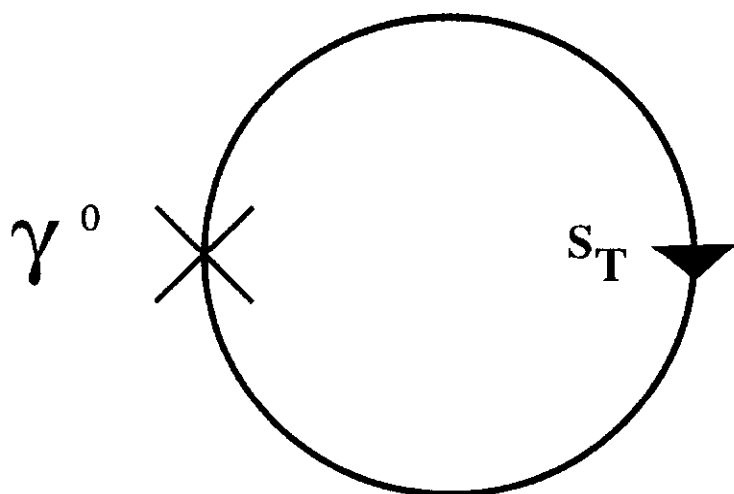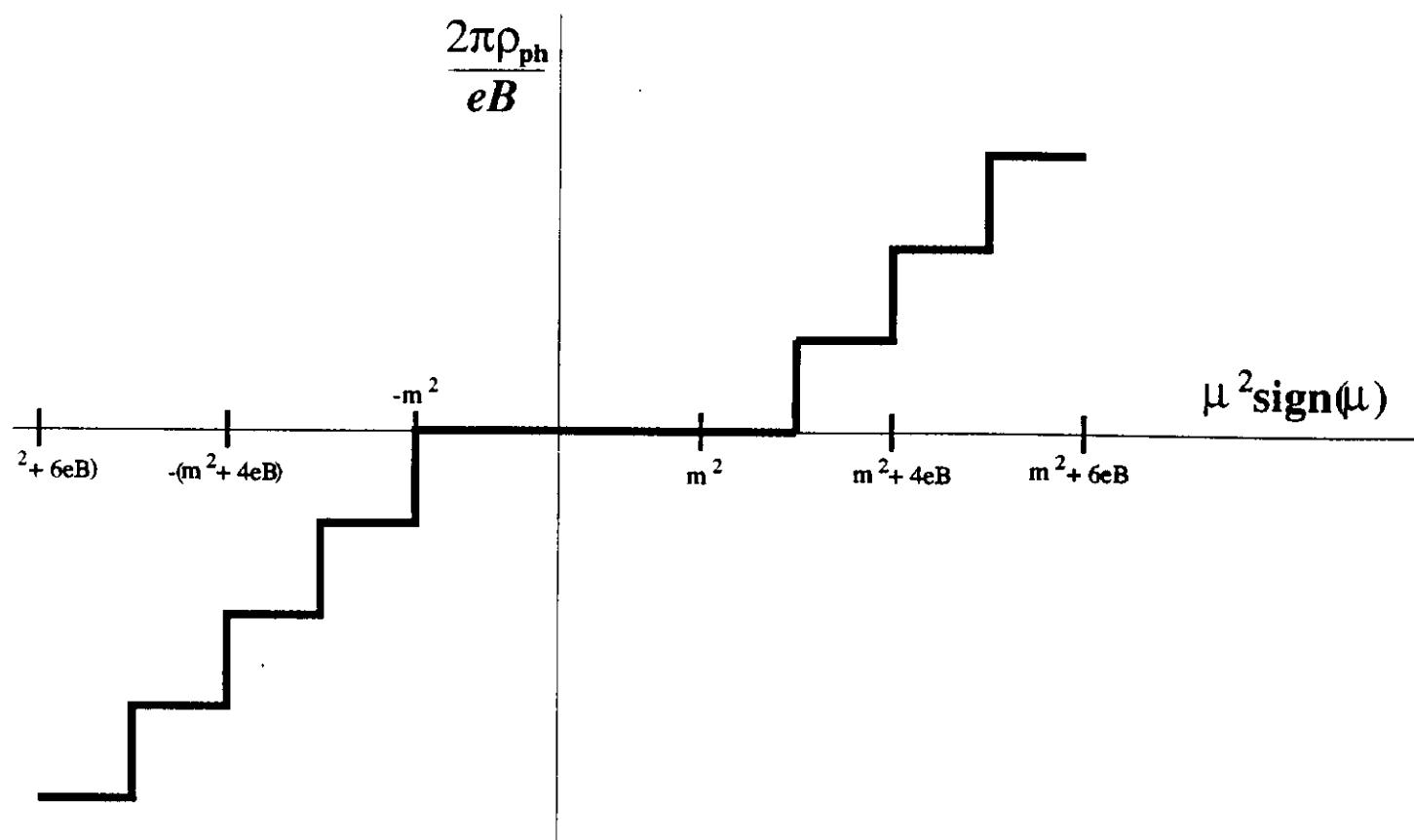
Figure 3

Figure 4

Figure 5

Figure 6

Figure   7