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Heavy quark production in very high energy hadron collisions

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Abstract

The hadroproduction of heavy quarks is studied in the kinematic regime in which $\sqrt{s}$ is very much greater than the mass of the produced heavy quark. We introduce a modification of the normal Lipatov equation which allows a factorization between the short distance and long distance physics. Numerical results are provided using the cross section for b quark production as an example.

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1 Introduction

As the energy of hadron-hadron and lepton-hadron colliders increases, an important new regime of QCD physics opens up: that of "semihard" processes[1]. By definition, these contain a hard scattering whose scale, Q, is much less than the total center-of-mass energy $\sqrt{s}$. To say that there is a hard scattering means that Q is large compared with $\Lambda_{\overline{\rm MS}}$. At the Tevatron and at HERA, semihard processes include the production of charm and bottom quarks and the production of jets of E_T of a few GeV. At the SSC ($\sqrt{s} = 40\,\text{TeV}$) even the production of W and Z bosons is a semihard process.

One reason that semihard processes are important is that their cross sections are large compared with the intrinsic cross section (of order Q^{-2}) of the hard scattering, because of the large number of gluons that can participate. As is well-known, minijet cross sections are a significant fraction of the total cross section at center-of-mass energies around a TeV.

An important challenge to QCD is to predict the properties and cross sections for semihard processes. Difficulties with conventional calculations arise because of large higher order corrections, both in the short-distance cross sections (coefficient functions) and in the kernel of the Altarelli-Parisi evolution equation[2]. For example, calculations of bottom quark production at hadron colliders have next-to-leading order corrections that are larger than the lowest order cross section, and the problem gets worse as s/m^2 gets larger [3,4]. (Here m is the mass of the heavy quark.) In this paper we make some first calculations of heavy quark production in hadron-hadron collisions using a formalism[5] we have developed to resum the large order corrections.

Our formalism is set up so that standard methods for analytic calculations of inclusive cross sections can be used. Higher order corrections can be incorporated systematically. Our procedure is therefore *potentially* more powerful than leading logarithm Monte-Carlo calculations.

The central element is a modification[5] of the Lipatov equation[6]. The original version of this equation was applied to complete cross sections, and thus entwined together infrared and ultraviolet behavior. Our modification of the equation applies directly to the short distance coefficients and to the evolution kernel of the Altarelli-Parisi equation. Thus only contributions on a specified scale are important in our equation. The solution of our modified Lipatov equation can then be made with a fixed coupling instead of a running coupling, and a substantial reduction in labor results. Moreover, the infrared-dependent part of the calculation resides solely in the initial conditions for the Altarelli-Parisi equation (with a resummed kernel). The evolution of parton distributions at small x has been considered by Kwiecinski[7].

Catani *et al.*[8] have developed a different, but undoubtedly equivalent formalism. When this is converted to a Monte-Carlo algorithm[9,10] problems occur because of sensitivity to the infrared cutoff. Preliminary results on the hadroproduction of heavy quarks have also been presented by Levin, Ryskin, Shabelski and Shuvaev[11].

To simplify our first attempt at numerical calculations, we work to leading order throughout. We also neglect quarks, since the large s/m^2 behavior is dominated by the effects of gluons on the evolution. The three-gluon vertex is of course the dominant physical object here. Our approximation should be sufficient to indicate the expected size of the corrections to the corresponding calculation using Born graphs.

The outline of our paper is as follows. In Sec. 2, we summarize the equations in our formalism, starting with the conventional factorization theorem. Then, in Sec. 3, we explain our ladder equation, and show how to solve it analytically. In Sec. 4, we explain the definition of the impact factor, and, in Sec. 5, we present our algebraic calculations of the lowest order impact factor for heavy quark production in hadron-hadron collisions. In Sec. 6, we show our results for the modification of the kernel of the Altarelli-Parisi equation. In Sec. 7, we present numerical calculations for the evolution of the gluon density and for the cross section for heavy quark production in hadron-hadron collisions, and compare the results with those from conventional calculations. Finally, in Sec. 8, we state our conclusions.

2 Basic formalism

The standard factorization formula for the production of a heavy quark of mass m in a collision of hadrons H_1 and H_2 with momenta p_1 and p_2 is

$$\sigma(s,4m^2) = \int_0^1 dx_1 \int_0^1 dx_2\; f_{i/H_1}(x_1,\mu)\; f_{j/H_2}(x_2,\mu)\; \hat{\sigma}_{ij}(\hat{s},m,\mu,\alpha_S(\mu)), \qquad (2.1)$$

where $\hat{s} = x_1 x_2 s$, and μ is an arbitrary renormalization scale that should be chosen to be of order the mass of the heavy quark. The hard cross section $\hat{\sigma}$ will contain the kinematic constraint $\hat{s} \equiv x_1 x_2 s \geq 4m^2$, so that the integrals over x_1 and x_2 effectively have nonzero lower limits. For the purposes of this paper, we will consider only the integrated cross section for heavy quark production. We could also consider a differential single-heavy-quark-inclusive cross section.

The parton densities $f_{i/H}(x,\mu)$ satisfy the Altarelli-Parisi equation

$$\frac{d}{d\ln\mu^2} f_{i/H}(x,\mu) = \sum_j \int_x^1 \frac{d\xi}{\xi}\, \gamma_{ij}\left(\frac{x}{\xi}, \alpha_S(\mu)\right)\, f_{j/H}(\xi,\mu). \tag{2.2}$$

For the remainder of this paper, we will consider the only light partons to be gluons, so that the indices in Eqs. (2.1,2.2) will only have the values $i = j = g$.

When $s \gg 4m^2$, there are important contributions in Eq. (2.1) from regions where $x_1 \ll 1$, $x_2 \ll 1$, or $\hat{s} \gg 4m^2$. Consequently, we need to know $\hat{\sigma}(\hat{s}, m)$ for $\hat{s} \gg 4m^2$, and to know the Altarelli-Parisi kernel $\gamma(x/\xi, \alpha_S)$ for $x \ll \xi$. In both cases, higher order terms in the expansion in powers of α_S have logarithms of the large ratios. Thus fixed low order perturbation theory becomes a poor approximation at high enough energy.

Our result with these large corrections resummed, can be expressed in the form

$$\sigma(s) = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty d\mathbf{k}_1^2 \int_0^\infty d\mathbf{k}_2^2\, \mathcal{F}(x_1, \mathbf{k}_1, \mu)\mathcal{F}(x_2, \mathbf{k}_2, \mu) I_2(x_1 x_2 s, \mathbf{k}_1, \mathbf{k}_2), \tag{2.3}$$

where $\mathbf{k}_1$ and $\mathbf{k}_2$ are transverse momentum variables. In the following, vectors in the transverse plane will always be denoted by boldface letters. Since we choose to work with unpolarized hadrons and with an azimuthally averaged cross section, we have performed the integrals over the azimuths of the transverse momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ before writing Eq. (2.3). I_2 is called the impact factor. It is, in effect, an off-shell, but gauge-invariant, cross section, and we will define it and calculate the lowest order contribution in Secs. 4 and 5. When $\hat{s} \to \infty$, the impact factor falls off like a power of $\hat{s}$, even when higher order corrections are added, unlike the complete hard scattering cross section. The ranges of integration in Eq. (2.3) are restricted by kinematic constraints which are implicit in the impact factor. The Lipatov resummation of the large $\hat{s}$ region for the hard scattering coefficient generates gluons of transverse momenta comparable to m. This is accounted for by $\mathcal{F}(x, \mathbf{k}, \mu)$, which is to be considered as the number density for a parton with longitudinal momentum fraction

x, transverse momentum $\mathbf{k}$ and renormalization scale μ.

The function $\mathcal{F}$ is the convolution of the ordinary parton density and the solution of our ladder equation, and it satisfies

$$f(x,\mu) = \int_0^{\mu^2} d\mathbf{k}^2\ \mathcal{F}(x,\mathbf{k},\mu). \tag{2.4}$$

The precise form of the ladder equation and its solution are given in Sec. 3.

Since the integrals extend down to small $\mathbf{k}$, it is inappropriate to use fixed coupling methods in Eq. (2.3) as it stands. Moreover the integrals become almost singular at $\mathbf{k}=0$, so that numerical calculations are not well behaved. It is therefore convenient to rewrite Eq. (2.3) by adding and subtracting the $\mathbf{k}\to 0$ pieces:

$$\begin{aligned}\sigma(s) \ =\ & \int_0^1 dx_1 \int_0^1 dx_2 \Bigg\{ \int_0^\infty d\mathbf{k}_1^2 \int_0^\infty d\mathbf{k}_2^2\ \mathcal{F}(x_1,\mathbf{k}_1,\mu)\mathcal{F}(x_2,\mathbf{k}_2,\mu) I_2^S(x_1x_2s,\mathbf{k}_1,\mathbf{k}_2) \\ & + \int_0^\infty d\mathbf{k}_2^2\ f(x_1,\mu)\mathcal{F}(x_2,\mathbf{k}_2,\mu)\Big[I_1(x_1x_2s,\mathbf{k}_2) - \theta(\mu^2-\mathbf{k}_2^2)I_0(x_1x_2s)\Big] \\ & + \int_0^\infty d\mathbf{k}_1^2\ \mathcal{F}(x_1,\mathbf{k}_1,\mu)f(x_2,\mu)\Big[I_1(x_1x_2s,\mathbf{k}_1) - \theta(\mu^2-\mathbf{k}_1^2)I_0(x_1x_2s)\Big] \\ & + f(x_1,\mu)f(x_2,\mu)I_0(x_1x_2s)\Bigg\},\end{aligned} \tag{2.5}$$

where

$$\begin{aligned} I_0(x_1x_2s) &= I_2(x_1x_2s,0,0), \\ I_1(x_1x_2s,\mathbf{k}) &= I_2(x_1x_2s,\mathbf{k},0), \\ I_2^S(x_1x_2s,\mathbf{k}_1,\mathbf{k}_2) &= I_2(x_1x_2s,\mathbf{k}_1,\mathbf{k}_2) + \theta(\mu^2-\mathbf{k}_1^2)\theta(\mu^2-\mathbf{k}_2^2)I_0(x_1x_2s) \\ & \quad -\theta(\mu^2-\mathbf{k}_1^2)I_1(x_1x_2s,\mathbf{k}_2) - \theta(\mu^2-\mathbf{k}_2^2)I_1(x_1x_2s,\mathbf{k}_1). \end{aligned} \tag{2.6}$$

We regard Eq. (2.5) as the most fundamental equation in our formalism. It is the one for which we expect to be able to systematically calculate higher order corrections.

3 The ladder equation

In Sec. 2 we presented our fundamental equation, Eq. (2.5), that is to be used for calculating semihard processes, and we asserted that the transverse momentum distribution $\mathcal{F}(x, \mathbf{k}, \mu)$ is the convolution of the ordinary parton density $f(x, \mu)$ with the solution of a ladder equation. In this section we write the ladder equation, and show how to solve it and combine it with the parton density to obtain $\mathcal{F}$. The details of the derivation of the ladder equation will not be presented in this paper.

3.1 Equation

Our starting point is an arbitrary off-shell hard scattering coefficient $\hat{X}(\hat{s}, \mathbf{k}; 4m^2)$. This is defined gauge invariantly in an exactly analogous fashion to the impact factor (to be defined in Sec. 4), except that $\hat{X}$ contains no subtractions to remove the large $\hat{s}$ behavior. We use the ladder equation to resum the large logarithms of $\hat{s}/4m^2$ that occur in the perturbation expansion of $\hat{X}$.

The quantity $\hat{X}$ is given by the impact factor coupled to a sum over ladder graphs for each incoming off-shell gluon. To simplify the formulae we explicitly treat the case of a single ladder, as in Eq. (3.2) below, instead of two ladders, as in Eq. (3.3). The ladders are all of the same form.

Because we are calculating a hard scattering coefficient, there will be no mass singularities when $\mathbf{k} \to 0$. According to the rules for computing hard scattering coefficients, subtractions are applied to higher order graphs, and these cancel the mass singularities. When we set $\mathbf{k} = 0$ in $\hat{X}(\hat{s}, \mathbf{k}; 4m^2)$, we obtain the usual hard scattering cross section $\hat{\sigma}$, as in Eq. (2.1).

We define the moments of $\hat{X}$ by

$$\tilde{X}(j, \mathbf{k}; \alpha_S(\mu)) = \int_0^\infty d\rho\, \rho^{j-2} \hat{X}(\hat{s}, \mathbf{k}; 4m^2; \alpha_S(\mu)), \tag{3.1}$$

where $\rho = 4m^2/s$. Functions with a tilde superscript are all defined in the space of moments. Diagrammatically, $\hat{X}$ is a convolution of an impact factor and a sum over ladder-like graphs, so that it can be represented in the form

$$\tilde{X}(j, \mathbf{k}) = \int d\mathbf{l}^2 \left[\tilde{I}(j, \mathbf{l}) - \theta(\mu^2 - \mathbf{l}^2)\tilde{I}(j, 0)\right] \tilde{L}(j, \mathbf{l}, \mathbf{k}), \tag{3.2}$$

where $\tilde{L}$ is the sum over the ladder graphs, and $\tilde{I}$ is the moment of the appropriate impact factor. Since we are working with a short distance coefficient function, we have displayed the subtraction that cancels the collinear mass singularity. When

$\mathbf{k}=0$, graphs for $\hat{X}$ have singularities when some loop transverse momenta go to zero. Within the ladder $\tilde{L}$ and the impact factor $\tilde{I}$, there are subtractions that cancel all the singularities, except for the one from the region where both $\mathbf{l}$ and all the transverse momenta in $\tilde{L}$ go to zero together. The explicit subtraction in Eq. (3.2) cancels this remaining singularity.

In the particular case of heavy quark production, $\tilde{I}$ could be equal to $\tilde{I}_1$, as used in Eq. (2.5). We would also have a similar equation with two ladders attached to $\tilde{I}_2$:

$$\tilde{X}_2(j,\mathbf{k}_1,\mathbf{k}_2)=\int d\mathbf{l}_1^2\int d\mathbf{l}_2^2\;\tilde{L}(j,\mathbf{l}_1,\mathbf{k}_1)\;\tilde{I}_2^S(j,\mathbf{l}_1,\mathbf{l}_2)\;\tilde{L}(j,\mathbf{l}_2,\mathbf{k}_2), \tag{3.3}$$

where the impact factor is the subtracted one defined in Eq. (2.6). (In the general case of a cross section differential in a heavy quark momentum, we would need two Mellin transform variables j_1 and j_2 instead of the single variable j in Eq. (3.3).)

In moment space $\tilde{X}$ satisfies a modified Lipatov equation ($\overline{\alpha}_S\equiv N_c\alpha_S(\mu)/\pi$)

$$\begin{aligned}\tilde{X}(j,\mathbf{k}) \;=\;& \tilde{I}(j,\mathbf{k})+\frac{\overline{\alpha}_S}{(j-1)}\int_0^\infty d\mathbf{l}^2\left\{\frac{\tilde{X}(j,\mathbf{l})-\tilde{X}(j,\mathbf{k})}{|\mathbf{l}^2-\mathbf{k}^2|}+\frac{\tilde{X}(j,\mathbf{k})}{\sqrt{\mathbf{l}^4+4\mathbf{k}^4}}\right.\\ &\left.-\tilde{X}(j,0)\left[\frac{\theta(\mu^2-\mathbf{l}^2)-\theta(\mu^2-\mathbf{k}^2)}{|\mathbf{l}^2-\mathbf{k}^2|}+\frac{\theta(\mu^2-\mathbf{k}^2)}{\sqrt{\mathbf{l}^4+4\mathbf{k}^4}}\right]\right\}\end{aligned} \tag{3.4}$$

By definition, the function $\tilde{I}(j,\mathbf{k})$ is free of poles at $j=1$.

Compared with the usual Lipatov equation, Eq. (3.4) has a subtraction that prevents regions with strong ordering in transverse momenta from contributing, as is appropriate for a hard scattering coefficient. This has the following consequence: Given that $\tilde{I}(j,\mathbf{k})$ is nonsingular at $\mathbf{k}=0$, the solution $\tilde{X}(j,\mathbf{k})$ of the ladder equation is also nonsingular at $\mathbf{k}=0$. Note also that subtraction in the transverse momentum at scale μ corresponds in lowest order with the normal $\overline{\text{MS}}$ scheme.

3.2 Solution

The equation can be solved by making a further Mellin transform of $\tilde{X}$ and $\tilde{I}$ with respect to $\mathbf{k}^2$:

$$\overline{X}(j,\gamma)\;\equiv\;\gamma\int_0^\infty\frac{d\mathbf{k}^2}{\mathbf{k}^2}\left(\frac{\mathbf{k}^2}{\mu^2}\right)^\gamma\tilde{X}(j,\mathbf{k}), \tag{3.5}$$

$$\overline{I}(j,\gamma) \equiv \gamma \int_0^\infty \frac{d\mathbf{k}^2}{\mathbf{k}^2} \left(\frac{\mathbf{k}^2}{\mu^2}\right)^\gamma \tilde{I}(j,\mathbf{k}). \tag{3.6}$$

These integrals are convergent for $0 < \gamma < \gamma_s$, where γ_s is determined by the large $\mathbf{k}$ behavior of $\tilde{X}$ and $\tilde{I}$. Since $\tilde{X}$ and $\tilde{I}$ are finite at $\mathbf{k} = 0$, we have

$$\lim_{\gamma \to 0} \overline{X}(j,\gamma) = \tilde{X}(j,\mathbf{k}=0), \tag{3.7}$$

and similarly for $\tilde{I}$.

After this transformation we notice that functions which are a power of $\mathbf{k}$ are eigenfunctions of the unmodified Lipatov equation. Then Eq. (3.4) becomes

$$\overline{X}(j,\gamma) = \overline{I}(j,\gamma) + \frac{\overline{\alpha}_S \chi(\gamma)}{j-1}\left[\overline{X}(j,\gamma) - \overline{X}(j,0)\right], \tag{3.8}$$

where the function χ is defined as[6]

$$\chi(\gamma) = \int_0^\infty dz \left[\frac{z^{-\gamma}-1}{|z-1|} + \frac{1}{\sqrt{4+z^2}}\right] = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma). \tag{3.9}$$

Here ψ is the digamma function, for which $\psi(1) = -\gamma_E$. The function χ has the following limiting behaviors:

$$\chi(\gamma) \rightarrow 4\log(2) + 14\zeta(3)\left(\gamma - \frac{1}{2}\right)^2 + O\left(\left(\gamma - \frac{1}{2}\right)^4\right), \text{ when } \gamma \rightarrow \frac{1}{2}; \tag{3.10}$$

$$\chi(\gamma) \rightarrow \gamma^{-1} + 2\zeta(3)\gamma^2 + O(\gamma^4), \text{ when } \gamma \rightarrow 0, \tag{3.11}$$

where $\zeta(3) = 1.20206$.

We now introduce the anomalous dimension function $\gamma_c(j,\overline{\alpha}_S)$ which is defined implicitly by the equation

$$j - 1 - \overline{\alpha}_S \chi(\gamma_c(j,\overline{\alpha}_S)) = 0. \tag{3.12}$$

The resummed anomalous dimension γ_c has a branch point singularity at $j = j_l \equiv 1 + 4\overline{\alpha}_S \ln 2$. For j close to the branch point, the behavior is

$$\gamma_c(j,\overline{\alpha}_S) \rightarrow \frac{1}{2} - \sqrt{\frac{j-j_l}{14\overline{\alpha}_S\zeta(3)}}. \tag{3.13}$$

The behavior of γ_c for small $\overline{\alpha}_S$ is given by

$$\gamma_c(j,\overline{\alpha}_S) = \frac{\overline{\alpha}_S}{(j-1)} + 2\zeta(3)\frac{\overline{\alpha}_S^4}{(j-1)^4} + O(\overline{\alpha}_S^6). \tag{3.14}$$

Eq. (3.8) can be rewritten as

$$\overline{X}(j,\gamma) = \frac{(j-1)\overline{I}(j,\gamma) - \overline{\alpha}_S\chi(\gamma)\overline{X}(j,0)}{j-1-\overline{\alpha}_S\chi(\gamma)}. \tag{3.15}$$

One solution is

$$\overline{X}(j,\gamma) = \frac{(j-1)\overline{I}(j,\gamma) - \overline{\alpha}_S\chi(\gamma)\overline{I}(j,\gamma_c(j))}{j-1-\overline{\alpha}_S\chi(\gamma)}, \tag{3.16}$$

where we have used the property of χ given in Eq. (3.11) to compute $\overline{X}(j,0)$. The most general solution of the equation is obtained by adding to Eq. (3.16)

$$\frac{F(j)}{j-1-\overline{\alpha}_S\chi(\gamma)}, \tag{3.17}$$

where $F(j)$ is an arbitrary function of j. Our first solution has no pole at $j = 1+\overline{\alpha}_S\chi(\gamma)$, but the added term does have such a singularity. This pole can be moved to arbitrarily large j by making γ small enough. Such a property appears to be unphysical. Moreover, if one expands the ladders in powers of α_S, then low order terms in Eq. (3.16) reproduce terms in the series that is obtained by direct evaluation of the ladders. Thus one must set $F(j)$ to zero.

3.3 Definition of $\mathcal{F}$

We use $\hat{X}$ in a factorization equation similar to Eq. (2.1). For the case of a process with a single parton external leg the cross section is

$$\sigma(s,4m^2) = \int_0^1 dx f(x,\mu)\hat{X}(xs,0,4m^2). \tag{3.18}$$

After taking the Mellin transform of the cross section the convolution integral becomes a simple product of moments

$$\tilde{f}(j,\mu)\tilde{X}(j,0). \tag{3.19}$$

Here the definition of the moments of the parton distributions f is

$$\tilde{f}(j,\mu) = \int_0^1 dx\, x^{j-1} f(x,\mu) \tag{3.20}$$

and $\tilde{X}$ is given by Eq. (3.1). From Eq. (3.19) we therefore need the limit of $\tilde{X}$ as $\mathbf{k}^2 \to 0$. After Mellin transformation on $\mathbf{k}$, this is equal to the $\gamma \to 0$ limit, *cf.* Eq. (3.7). Using the solution Eq. (3.16), we find that

$$\tilde{X}(j,0) = \overline{X}(j,0) = \overline{I}(j,\gamma_c(j)). \tag{3.21}$$

Since the ladder part of our solution for $\tilde{X}$, Eq. (3.2), is universal, it is convenient to combine it with the parton density $f(x)$ by defining

$$\begin{aligned} \tilde{\mathcal{F}}(j,\mathbf{k},\mu) &\equiv \tilde{L}(j,\mathbf{k},0)\tilde{f}(j,\mu) \\ &= \gamma_c(j,\alpha_S)\frac{1}{\mathbf{k}^2}\left(\frac{\mathbf{k}^2}{\mu^2}\right)^{\gamma_c(j,\alpha_S)} \tilde{f}(j,\mu). \end{aligned} \tag{3.22}$$

Then

$$\tilde{f}(j)\tilde{X}(j,0) = \int d\mathbf{k}^2\, \tilde{\mathcal{F}}(j,\mathbf{k},\mu)\, [\tilde{I}(1) - \theta(\mu - \mathbf{k})\tilde{I}(0)]. \tag{3.23}$$

Notice that the definition Eq. (3.22) satisfies

$$\int_0^{\mu^2} d\mathbf{k}^2\, \tilde{\mathcal{F}}(j,\mathbf{k},\mu) = \tilde{f}(j,\mu). \tag{3.24}$$

Expanding in $\overline{\alpha}_S$, we may define the lowest order perturbative result for the parton distribution with off-shellness

$$\tilde{\mathcal{P}}(j,\mathbf{k},\mu) = \frac{\overline{\alpha}_S}{(j-1)\mathbf{k}^2}\tilde{f}(j,\mu). \tag{3.25}$$

This is the explicit form used in the perturbative calculations of ref. [12].

4 Theory of Impact Factor

We now explain how we define the impact factor for hard scattering. Although all our considerations are rather general, we will treat the case of production of heavy quarks in a hadron-hadron collision for the sake of definiteness.

In a general Feynman graph for heavy quark production there will be logarithms from integrals over transverse momenta and over rapidity. The ordinary factorization theorem, Eq. (2.1), plus the Altarelli-Parisi equation, Eq. (2.2), solve the problem of disentangling the transverse momentum logarithms, as is well known. Given this decomposition, our modified Lipatov equation separates out the logarithms from the rapidity integrals. So far, we have shown how this works for the hard scattering, in Sec. 3. In Sec. 6, we will perform the corresponding task for the Altarelli-Parisi kernel.

After this is all done, it remains to calculate the impact factor that appears in Eqs. (2.3) and (2.5). Fundamentally, the impact factor is an off-shell cross section in which subtractions have been made to cancel the logarithms from the rapidity integrals and the transverse momentum integrals. It is necessary to ensure that its definition is gauge invariant, since off-shell quantities in a gauge theory are not, in general, gauge invariant. Note that in lowest order for heavy quark production, there are no subtractions to be made, but that gauge invariance is, a priori, a problem.

We will parameterize a general momentum k^μ by a Sudakov decomposition:

$$k^\mu = \alpha p_1^\mu + \beta p_2^\mu + \mathbf{k}^\mu, \tag{4.1}$$

where $\mathbf{k}^\mu$ is a vector transverse to both of the momenta of the incoming hadrons, p_1 and p_2. It will be convenient to make p_1 and p_2 exactly light-like: this will simplify the technical details of our discussion. Then the actual momenta of the incoming hadrons will differ slightly from p_1 and p_2.

4.1 Definition of impact factor

The impact factor in Eq. (2.3) contains the core of the hard scattering, and it is attached to each of the parton densities $\mathcal{F}$ by a pair of gluons, in lowest order. The definition of the impact factor requires an approximation to the coupling of these gluons that is valid when the gluons connect lines in the impact factor to lines of very different rapidity. It should remain a good approximation when the gluon has a virtuality or a transverse momentum comparable with the kinematic scale of the hard scattering, viz., m. The approximation should make enough simplification in the structure of the loop integrals that we can derive the ladder equation that we use

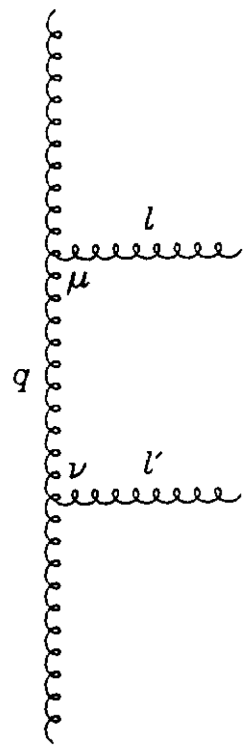


Figure 1: Gluon connecting lines of very different rapidity.

to resum the rapidity logarithms.

It does not matter whether the approximation is good outside the region with a large rapidity gap. There is no large logarithm in such a region. We compensate for the error in the approximation when we calculate the subtraction term for the next order graphs. Thus the error is genuinely higher order in α_S. We will set up the approximation so that it is exact in the limit that the external gluons have zero transverse momentum. This will ensure that we do not need to add correction terms to make our fancy factorization theorem, Eq. (2.5), agree with the standard one, Eq. (2.1), when we are not in the semihard regime.

In Fig. 1, we show one particular gluon that connects a line in the impact factor to one of the lines in the attached ladders. We need to investigate the situation when the lines l and l' have very different rapidity. First, we decompose the momenta as follows:

$$\begin{aligned} q^\mu &= \alpha_q p_1^\mu + \beta_q p_2^\mu + \mathbf{q}^\mu, \\ l^\mu &= \alpha_l p_1^\mu + \beta_l p_2^\mu + \mathbf{l}^\mu, \\ l'^\mu &= \alpha_{l'} p_1^\mu + \beta_{l'} p_2^\mu + \mathbf{l}'^\mu. \end{aligned} \tag{4.2}$$

We choose l' to be the momentum of a line in the impact factor, and l to be the momentum of a line in the ladder attached to hadron H_1. It is sufficient to assume that all three transverse momenta are of order the scale of the hard scattering: $Q = O(m)$. Then, because l and l' are final state momenta, both $\alpha_l \beta_l$ and $\alpha_{l'} \beta_{l'}$ are of order Q^2/s. Moreover α_q is of order $\alpha_{l'}$, while β_q is of order β_l. The strong ordering in rapidity means that $\alpha_l \gg \alpha_{l'}$ and $\beta_l \ll \beta_{l'}$, so that $q^2 = -\mathbf{q}^2$ up to power law corrections.

From Fig. 1 we may write the contribution of one rung as

$$V^\mu(l,q)\, N_{\mu\nu}(q) \frac{i}{q^2}\, V'^\nu(q,l'), \tag{4.3}$$

where $N^{\mu\nu}(q)$ is the numerator factor of the gluon propagator and V^μ and V'^ν represent the parts of the graph above and below the exchanged gluon. We now define momenta q_1 and q_2. These are obtained from q by setting one of the longitudinal components to zero:

$$q_1^\mu \equiv \alpha p_1^\mu + \mathbf{q}^\mu, \tag{4.4}$$

$$q_2^\mu \equiv \beta p_2^\mu + \mathbf{q}^\mu. \tag{4.5}$$

Then to a good approximation, in the region of strongly ordered rapidity, Eq. (4.3) is

$$V^\mu(l,q_2) N_{\mu\mu'}(q_2) p_2^{\mu'}\ \frac{i}{\mathbf{q}^2\, p_1 \cdot p_2}\ p_1^{\nu'} N_{\nu'\nu}(q_1) V^\nu(q_1,l'). \tag{4.6}$$

Our ladder equation, Eq. (3.4), is exactly true for ladders in which the above approximation is made for all gluons that connect lines of widely different rapidities, if the ladder rung is taken to lowest order in α_S.

The operation of making this replacement we call the Lipatov approximation. It involves making an approximation to the attachment to the upper part of the graph

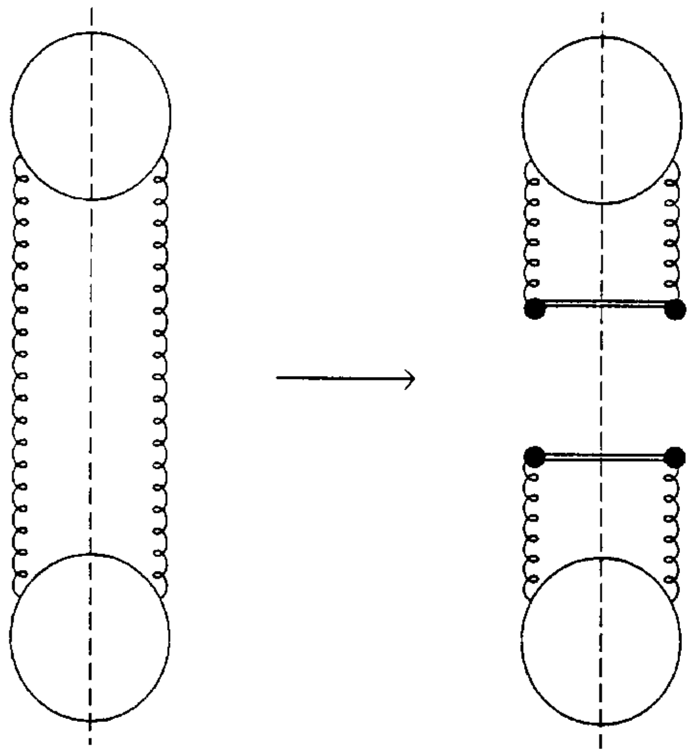


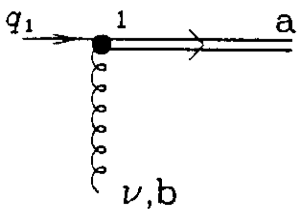
Figure 2: Action of Lipatov approximation.

and a conjugate approximation at the lower end. The result is very conveniently summarized in an operator language, which is pictured in Fig. 2. There, we consider a contribution to the cross section, so we have an amplitude such as we have just considered times a hermitian conjugated amplitude.

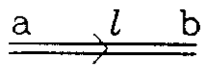
Consider the gluon q as coming from the bottom line and attaching to the top line. Instead, let it be absorbed by the following gluon operator, which carries the burden of projecting out the correct polarization and momentum components:

$$O_1^a(q_1) \equiv -i \int d^4z \; e^{-iq_1\cdot z} \left\{ \mathrm{Pexp}\left[-ig \int_C dz'^{\kappa} A_\kappa^b(z') T^b \right] \right\}_{aa'} \frac{p_2^\mu p_1^\nu}{p_2 \cdot p_1} \frac{\partial}{\partial z^\mu} A_\nu^{a'}(z). \quad (4.7)$$

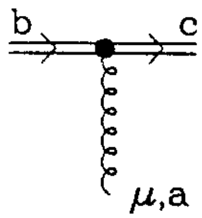
Here T^b is a generator of the SU(3) group in the adjoint representation, and C, the path of integration in the path-ordered exponential, is a straight line that starts at



$$= \frac{q_1 \cdot p_2}{p_1 \cdot p_2}\, p_1{}^{\nu}\, \delta^{ab}$$



$$= \frac{i}{l \cdot p_1 + i\epsilon}\, \delta^{ab}$$



$$= -ig\ (T^a)_{cb}\ p_1{}^{\mu}$$

Figure 3: Feynman rules and typical diagrams for the operator $O_1^a(q_1)$. The label '1' on the top vertex is to distinguish the vertex for the operator O_1 from the vertex for the operator O_2.

the space-time point z^μ and goes out to infinity in a direction parallel to the vector p_1. Because $q_1 \cdot p_1 = 0$ and because we have projected the gluon field with p_1, this operator is, in fact, gauge invariant, except for a surface term at infinity. The surface term will cancel against the corresponding term in the hermitian conjugate operator that is on the opposite side of the final-state cut.

Operators like this, but with a slightly different momentum, appear in the gauge-invariant definition of the parton densities given in Ref. [15]. From the Feynman rules given in that paper, we can extract the Feynman rules for our operator O_1^a. Note, however, that our convention for the coupling g is reversed compared with Ref. [15].

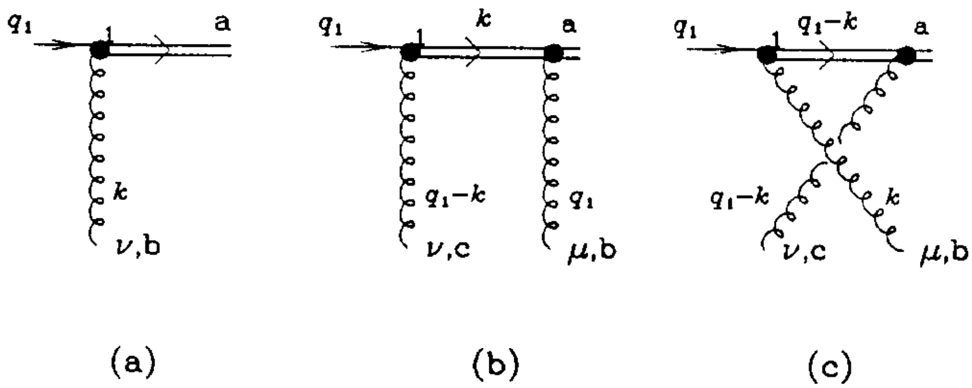


Figure 4: Examples of application of Feynman rules for $O_1^a(q_1)$.

The Feynman rules are as given in [16,17]. The Feynman rules for O_1 are shown in Fig. 3. These contain a vertex for the $A_\nu p_1^\nu$ factor, into which the momentum q_1 flows. To this vertex is attached a double line that represents the path-ordered exponential. There are any number of $p_1 \cdot A$ vertices on the double line; they are joined by eikonal propagators. The lowest order vertex reproduces the right-hand factor in Eq. (4.6).

We show some examples of the application of these Feynman rules in Fig. 4. The values of the graphs in this figure are:

$$\begin{aligned} \text{Graph (a)} &= \frac{q_1 \cdot p_2}{p_1 \cdot p_2} p_1^\nu \delta_{ab} \\ &= x_1 p_1^\nu \delta_{ab}, \end{aligned} \tag{4.8}$$

$$\begin{aligned}
\text{Graph (b)} &= -ig(T^b)_{ac} p_1^\mu \frac{i}{k\cdot p_1 + i\epsilon} p_1^\nu \frac{(q_1-k)\cdot p_2}{p_1\cdot p_2} \\
&= g f_{abc} p_1^\mu p_1^\nu \frac{i}{k\cdot p_1 + i\epsilon} \frac{(q_1-k)\cdot p_2}{p_1\cdot p_2}, \qquad (4.9)\\
\text{Graph (c)} &= -ig(T^c)_{ab} p_1^\nu \frac{i}{(q_1-k)\cdot p_1 + i\epsilon} p_1^\mu \frac{k\cdot p_2}{p_1\cdot p_2}, \\
&= g f_{abc} p_1^\mu p_1^\nu \frac{i}{k\cdot p_1 - i\epsilon} \frac{k\cdot p_2}{p_1\cdot p_2}. \qquad (4.10)
\end{aligned}$$

In these examples, we have chosen $q_1 = x_1 p_1^\mu + \mathbf{q}_1$, so that $p_1 \cdot q_1 = 0$.

A corresponding operator is used at the other end of the gluon line q:

$$O_2^a(q_2) \equiv -i \int d^4z\, e^{-iq_2\cdot z} \left\{ \mathrm{P}\exp\left[-ig \int_C dz'^\kappa A_\kappa^b(z') T^b \right] \right\}_{aa'} \frac{p_1^\mu p_2^\nu}{p_1\cdot p_2} \frac{\partial}{\partial z^\mu} A_\nu^{a'}(z), \tag{4.11}$$

where now the path C starts at z^μ and goes out to infinity in a direction parallel to the vector p_2.

We can now make a gauge-invariant definition of the impact factor for gluon-gluon fusion to heavy quarks, by attaching the gluons to these path-ordered exponentials. To lowest order, the impact factor is the square of an amplitude given by the sum of the graphs in Fig. 5. Schematically, we can represent this as the lowest order approximation to the following Green's function:

$$\mathcal{N} \sum_X \langle 0|\overline{T}\, O_1^{a\dagger}(k_1) O_2^{b\dagger}(k_2)|X\rangle\langle X|T O_2^b(k_2) O_1^a(k_1)|0\rangle / [(2\pi)^4 \delta^{(4)}(0)], \tag{4.12}$$

where the sum over final states is over those that contain the heavy quark that defines the cross section we are calculating, and $\mathcal{N}$ is a normalization factor that we will define in a moment. In lowest order, we mean to define the impact factor by exactly the above equation. But in higher order we must impose subtractions that cancel all the transverse momentum and rapidity logarithms.

In accordance with the prescription for defining the operators O_1 and O_2, the external momenta of the impact factor have the forms

$$k_1 = x_1 p_1 + \mathbf{k}_1, \tag{4.13}$$

$$k_2 = x_2 p_2 + \mathbf{k}_2, \tag{4.14}$$

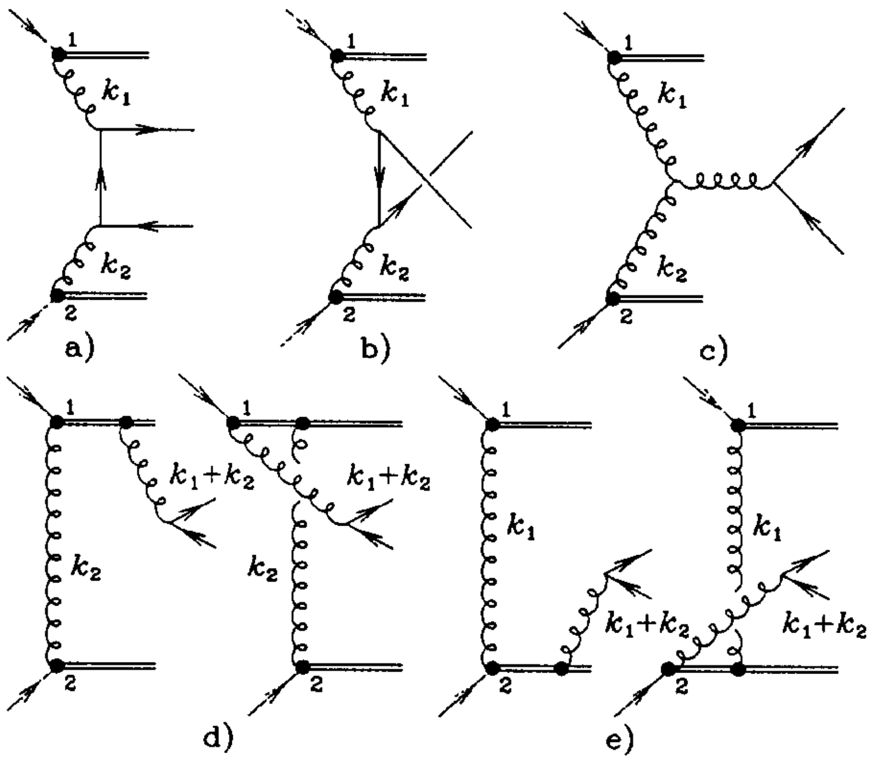


Figure 5: Lowest order graphs for amplitude for impact factor for production of heavy quarks, in gauge-invariant formalism.

so that $p_1 \cdot k_1 = 0 = p_2 \cdot k_2$.

We define the normalization factor to be

$$\mathcal{N} = \frac{\mathbf{k}_1^2 \mathbf{k}_2^2}{4x_1 x_2 p_1 \cdot p_2 V^2}. \tag{4.15}$$

Here $V \equiv N_c^2 - 1 = 8$, so that the factor $1/V^2$ represents an average over the colors of the incoming gluons. $\mathcal{N}$ is chosen so that when we take the limit that $\mathbf{k}_1 = \mathbf{k}_2 = 0$, we obtain a properly normalized cross section, which is exactly the usual gluon-gluon fusion cross section, when we work in lowest order. The argument is a little subtle, because the external gluons have polarizations proportional to p_1 or p_2, and are therefore longitudinal when $\mathbf{k}_1 = \mathbf{k}_2 = 0$.

The first three graphs in Fig. 5 are the conventional ones for gluon-gluon fusion, with some appropriate polarization vectors, and with off-shell gluons. The last two graphs are needed to preserve gauge invariance, and contain couplings to the path-ordered exponentials in O_1 and O_2.

We can eliminate graphs (d) and (e) and put the others in a form that more closely resembles the calculation of a cross section with transverse polarizations, by applying Ward identities, as follows. First, consider graphs (a) to (d) where only one gluon attaches to the lower eikonal. These graphs have the form

$$x_2 p_2^\mu G_\mu(k_2) \text{ where } G^\mu = G^\mu_{(a)} + G^\mu_{(b)} + G^\mu_{(c)} + G^\mu_{(d)} \tag{4.16}$$

G is the sum of the upper parts of the four graphs. Since G is gauge invariant, at this order, there is a Ward identity

$$k_2^\mu G_\mu(k_2) = 0. \tag{4.17}$$

Since k_2 has no component proportional to p_1, this implies that the contribution of the sum of the first four graphs may be written as

$$x_2 p_2^\mu G_\mu(k_2) = -\mathbf{k}_2^\mu G_\mu(k_2). \tag{4.18}$$

However since $G^\mu_{(d)} \propto x_1 p_1^\mu$ in Feynman gauge, graph (d) makes no contribution to the right hand side of the above equation. We apply the same Ward identity argument to the other external gluon line in the sum of graphs (a), (b) and (c). In this fashion we express the result as transverse projections of graphs (a), (b) and (c) plus a single term left over in the Ward identity that exactly cancels graph (e), in Feynman gauge.

The final result, at lowest order, is that the impact factor is given by the off-shell graphs for gluon-gluon fusion , that is graphs (a) to (c). They are to be computed in Feynman gauge with amputated external gluon propagators. The incoming gluon momenta satisfy $p_2 \cdot k_2 = p_1 \cdot k_1 = 0$. The gluon polarization vectors are $\mathbf{k}_1/|\mathbf{k}_1|$ for k_1 and $\mathbf{k}_2/|\mathbf{k}_2|$ for k_2, there is a color average for the incoming gluons, and there is an overall flux factor of $1/(2x_1x_2s)$. This implies that the impact factor is normalized appropriately for a cross section, so that when $\mathbf{k}_1 = \mathbf{k}_2 = 0$ we recover the usual gluon-gluon fusion result for on-shell gluons.

There is somewhat of a convention in our normalization. It is defined so that the flux factor for the off-shell cross section is $1/(2x_1x_2s)$ rather than a more complicated expression such as $1/[2(k_1+k_2)^2]$. This makes our algebra a little simpler.

When we put the appropriate compensating factors into the ladder graphs, the remaining factors in the modified factorization can be interpreted as parton number densities.

5 Calculation of Impact factor

We now summarize the lowest order calculation of the cross section for two off-shell gluons to make a pair of heavy quarks (the impact factor). First, we make a Sudakov decomposition for the incoming gluon momenta k_1, k_2 and the momenta of the produced heavy quarks, p_3, p_4, (which have mass m):

$$\begin{aligned} k_1 &= x_1 p_1 + \mathbf{k}_1, \\ k_2 &= x_2 p_2 + \mathbf{k}_2, \\ p_3 &= a_3 x_1 p_1 + b_3 x_2 p_2 + \mathbf{k}_3, \\ p_4 &= a_4 x_1 p_1 + b_4 x_2 p_2 + \mathbf{k}_4. \end{aligned} \tag{5.1}$$

As explained in Sec. 4, the longitudinal component of k_1, (k_2) is only in the direction of the light-like vector p_1, (p_2). This together with the choice of gluon polarizations to be in the transverse plane ensures that the impact factor is gauge invariant. Momentum conservation gives $a_3 + a_4 = 1, b_3 + b_4 = 1$ and $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$.

We define the longitudinal contribution to the square of the total incoming four momentum to be

$$s_L = 2x_1x_2\, p_1 \cdot p_2 = x_1 x_2 s. \tag{5.2}$$

It is convenient to introduce a notation for the rescaled propagators of the graphs, using s_L to set the scale,

$$\sigma = \frac{(k_1+k_2)^2}{s_L}, \quad \tau_{ij} = \frac{m^2 - (k_i - p_j)^2}{s_L}, \tag{5.3}$$

so that by momentum conservation $\tau_{13} = \tau_{24}$ and $\tau_{23} = \tau_{14}$. We further define

$$r = \frac{4m^2}{s_L}, \quad \kappa_1 = \frac{\mathbf{k}_1^2}{s_L}, \quad \kappa_2 = \frac{\mathbf{k}_2^2}{s_L}. \tag{5.4}$$

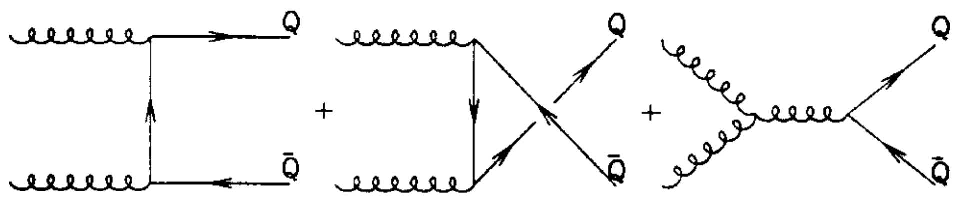


Figure 6: Graphs for impact factor for gluon–gluon fusion to heavy quarks.

As described in Sec. 4, we are interested in a definite polarization projection of the invariant matrix element squared. The matrix element M is calculated using the diagrams of Fig. 6. The result can be expressed in terms of the function A_2,

$$A_2 = \frac{1}{g^4 \mathbf{k}_1^2 \mathbf{k}_2^2} \overline{\sum} |M_{\mu\nu}(k_1, k_2, p_3, p_4)\ \mathbf{k}_1^\mu\ \mathbf{k}_2^\nu|^2 \,. \tag{5.5}$$

Here, $\overline{\sum}$ indicates an average over the $V \equiv N_c^2 - 1 = 8$ colors of both incoming gluons, with $N_c = 3$ being the number of quark colors. The result for A_2 is

$$A_2 = \frac{1}{4\kappa_1\kappa_2 N_c V} \left[N_c^2(Y_1 + Y_2) - 2Y_2 \right], \tag{5.6}$$

where Y_1 is

$$Y_1 = \left[\kappa_1\kappa_2\left(\frac{4}{\sigma} - \frac{1}{\tau_{13}\tau_{23}} + \frac{(a_3 - a_4 - b_3 + b_4)}{\sigma}\left(\frac{1}{\tau_{13}} - \frac{1}{\tau_{23}}\right)\right) - \Delta^2\right], \tag{5.7}$$

with

$$\Delta = \left[\frac{b_3 a_4}{\tau_{13}} - \frac{b_4 a_3}{\tau_{23}} + \frac{1}{\sigma}\Big(\tau_{13} - \tau_{23} + (b_4 - b_3)(1-\kappa_1) + (a_3 - a_4)(1-\kappa_2)\Big)\right], \tag{5.8}$$

and Y_2 is

$$Y_2 = \left[\frac{\kappa_1\kappa_2}{\tau_{13}\tau_{23}} - \left(1 - \frac{b_3 a_4}{\tau_{13}} - \frac{b_4 a_3}{\tau_{23}}\right)^2\right]. \tag{5.9}$$

If we take the limit of zero $\mathbf{k}_1$, and if we average over the transverse directions of $\mathbf{k}_1$, we obtain the invariant matrix element with one gluon off-shell:

$$\left\langle\frac{\mathbf{k}_1^\mu \mathbf{k}_1^\nu}{\mathbf{k}_1^2}\right\rangle = \frac{1}{2}\left(-g^{\mu\nu} + \frac{p_1^\mu p_2^\nu + p_1^\nu p_2^\mu}{p_1\cdot p_2}\right), \tag{5.10}$$

$$A_1 = \frac{1}{g^4\mathbf{k}_2^2}\overline{\sum}|M_{\mu\nu}(p_1,k_1,p_3,p_4)\,\epsilon^\mu(p_1)\mathbf{k}_2^\nu|^2\,. \tag{5.11}$$

$\overline{\sum}$ now indicates an average over spins and colors of line one and an average over colors of line two. The result for A_1 is

$$\begin{aligned} A_1 \;=\; & \left(\frac{1}{2b_3b_4N_c} - \frac{N_c}{V}\right)\left[(a_3^2 + a_4^2) + \frac{r}{8b_3b_4\kappa_2}(a_3 + b_3 - a_4 - b_4)^2\right] \\ & + \frac{N_c}{2Vb_3b_4}\frac{\kappa_2}{(1-\kappa_2)}\left[(a_3^2 + a_4^2 + r)(a_3b_4 + a_4b_3) + r(2b_3b_4 - 1) + \frac{b_3b_4r}{(1-\kappa_2)}\right] \\ & - \frac{N_c}{4Vb_3b_4}(b_3 - b_4)(a_3 + b_3 - a_4 - b_4)(a_3^2 + a_4^2 + r). \end{aligned} \tag{5.12}$$

If we further take the limit $\mathbf{k}_2 \to 0$ in Eq. (5.12) and average over the transverse plane, we recover the normal on-shell matrix element,

$$A_0 = \frac{1}{g^4}\overline{\sum}|M_{\mu\nu}(p_1,p_2,p_3,p_4)\,\epsilon^\mu(p_1)\epsilon^\nu(p_2)|^2\,, \tag{5.13}$$

where $\overline{\sum}$ now indicates an average over colors and spins of both initial gluons. We find the standard result[13,14],

$$A_0 = \left(\frac{1}{2b_3b_4N_c} - \frac{N_c}{V}\right)\left[b_3^2 + b_4^2 + r - \frac{r^2}{4b_3b_4}\right]. \tag{5.14}$$

The impact factors for total heavy quark production are defined in analogy with normal cross sections:

$$
\begin{aligned}
I_0(s_L) &= \frac{1}{2s_L}\frac{\alpha_S^2}{2}\int d\cos\theta\, d\phi\, A_0, \\
I_1(s_L,\mathbf{k}_1) &= \frac{1}{2s_L}\frac{\alpha_S^2}{2}\int d\cos\theta\, d\phi\, A_1, \\
I_2(s_L,\mathbf{k}_1,\mathbf{k}_2) &= \frac{1}{2s_L}\frac{\alpha_S^2}{2}\int d\cos\theta\, d\phi\, \frac{d\phi_{12}}{2\pi}\, A_2.
\end{aligned}
\tag{5.15}
$$

Here, we have chosen the flux factor to be $2s_L$, and θ and ϕ are the scattering angles in the center of mass system of the two incoming gluons. In defining I_2, we have averaged over the angle between the two incoming gluons in the transverse plane, ϕ_{12}. The first moment of I_1, which is relevant in $O(\alpha_S^3)$ perturbation theory, was calculated in ref. [12]. I_0 is the normal total cross section. Impact factors for the differential cross sections may be obtained directly from A_0, A_1 and A_2. In the present paper we concentrate on the integrated cross sections. Using the explicit expressions given in Eqs. (5.6,5.12,5.13), it is straightforward to derive analytic expressions for the impact factors for the total production of heavy quarks. These expressions are too complicated to publish in closed form. Fortran routines for the integrated impact factors I_i are available on request.

6 Evolution Equation

In analogy with the off-shell cross section, we can define a generalized off-shell anomalous dimension $\gamma(x/\xi,\mathbf{k};\alpha_S)$. We will mostly work with its Mellin transform:

$$
\begin{aligned}
\tilde\gamma(j,\mathbf{k};\alpha_S) &= \int_0^1 dx\, x^{(j-1)}\gamma(x,\mathbf{k};\alpha_S), \\
\gamma(x,\mathbf{k};\alpha_S) &= \frac{1}{2\pi i}\int_C dj\, x^{-j}\tilde\gamma(j,\mathbf{k};\alpha_S).
\end{aligned}
\tag{6.1}
$$

At zero $\mathbf{k}$ this is the ordinary anomalous dimension — the Altarelli-Parisi kernel. It enters in the renormalization group equation (Altarelli-Parisi equation) for the parton distribution function:

$$
\left[\frac{\partial}{\partial \ln\mu^2} + \beta(\alpha_S)\frac{d}{d\alpha_S}\right]\tilde f_i(j,\mu^2) = \sum_k \tilde\gamma_{ik}(j,0;\alpha_S)\tilde f_k(j,\mu^2). \tag{6.2}
$$

where $i, k = g$ in the present paper.

In moment space, the off-shell anomalous dimension satisfies a modified Lipatov equation:

$$\begin{aligned}\tilde{\gamma}(j,\mathbf{k}) = \quad & \tilde{\gamma}_I(j,\mathbf{k}) + \frac{\bar{\alpha}_S}{(j-1)} \int_0^\infty d\mathbf{l}^2 \left\{ \frac{\tilde{\gamma}(j,\mathbf{l}) - \tilde{\gamma}(j,\mathbf{k})}{|\mathbf{l}^2 - \mathbf{k}^2|} + \frac{\tilde{\gamma}(j,\mathbf{k})}{\sqrt{\mathbf{l}^4 + 4\mathbf{k}^4}} \right. \\ & \left. -\tilde{\gamma}(j,0) \left[\frac{\theta(\mu^2 - \mathbf{l}^2) - \theta(\mu^2 - \mathbf{k}^2)}{|\mathbf{l}^2 - \mathbf{k}^2|} + \frac{\theta(\mu^2 - \mathbf{k}^2)}{\sqrt{\mathbf{l}^4 + 4\mathbf{k}^4}} \right] \right\}. \end{aligned} \tag{6.3}$$

The function $\tilde{\gamma}_I(j, \mathbf{k})$ is defined in an analogous fashion to the impact factor for the hard scattering cross section. It is free of poles at $j = 1$.

After defining moments of $\tilde{\gamma}$ with respect to $\mathbf{k}^2$,

$$\overline{\gamma}(j,f) = f \int_0^\infty \frac{d\mathbf{k}^2}{\mathbf{k}^2} \left(\frac{\mathbf{k}^2}{\mu^2}\right)^f \tilde{\gamma}(j,\mathbf{k}), \tag{6.4}$$

we obtain from Eq. (6.3) the transformed equation

$$\overline{\gamma}(j,f) = \overline{\gamma}_I(j,f) + \frac{\bar{\alpha}_S \chi(f)}{(j-1)} \left[\overline{\gamma}(j,f) - \overline{\gamma}(j,0) \right]. \tag{6.5}$$

This leads to the following solution, at zero $\mathbf{k}$, *cf.* Eq. (3.21):

$$\tilde{\gamma}(j,0) = \overline{\gamma}(j,0) = \overline{\gamma}_I\left(j, \gamma_c(j, \overline{\alpha}_S)\right). \tag{6.6}$$

In perturbation theory γ_I has the expansion:

$$\tilde{\gamma}_I(j,\mathbf{k}) = \mu^2 \delta(\mathbf{k}^2 - \mu^2) + \sum_{n=1}^{\infty} \overline{\alpha}_S^n \, \tilde{P}_I^{(n)}(j,\mathbf{k}). \tag{6.7}$$

Using Eq. (6.4) this gives in transform space

$$\overline{\gamma}_I(j,f) = f \left[1 + \sum_{n=1}^{\infty} \overline{\alpha}_S^n \int_0^\infty \frac{d\mathbf{k}^2}{\mathbf{k}^2} \left(\frac{\mathbf{k}^2}{\mu^2}\right)^f \tilde{P}_I^{(n)}(j,\mathbf{k}) \right]. \tag{6.8}$$

This solution of the ladder equation for the anomalous dimension may be written as

$$\tilde{\gamma}(j,0) \quad = \quad \gamma_c(j) + \sum_{n=1}^{\infty} \overline{\alpha}_S^n \left\{ \tilde{P}_I^{(n)}(j,0) \right.$$

$$+\gamma_c(j)\int_0^\infty \frac{d\mathbf{k}^2}{\mathbf{k}^2}\left(\frac{\mathbf{k}^2}{\mu^2}\right)^{\gamma_c(j)}\left(\tilde{P}_I^{(n)}(j,\mathbf{k}^2)-\theta(\mu^2-\mathbf{k}^2)\tilde{P}_I^{(n)}(j,0)\right)\Bigg\} \tag{6.9}$$

The third term is nonleading both in $\overline{\alpha}_S$ and in $\overline{\alpha}_S/(j-1)$ and will hence be neglected in the following. For the purposes of this paper we restrict the sum in Eq. (6.9) to the $n=1$ term:

$$\tilde{\gamma}(j,0)\to\gamma_c(j)+\overline{\alpha}_S\tilde{P}_I^{(1)}(j,0). \tag{6.10}$$

Note that if we were to include higher order corrections, we would also need the higher order corrections to our ladder equation, in order to be completely consistent in the accuracy of our perturbative approximations.

By comparing the expansion of Eq. (6.10) with the standard perturbative result for the one loop anomalous dimension[17] we can identify the impact factor $\tilde{P}_I^{(1)}$

$$\tilde{P}_I^{(1)}(j,0)=\left(\frac{11}{12}-\frac{2}{j}+\frac{1}{(j+1)(j+2)}+\psi(1)-\psi(j)-\frac{n_f}{6N_c}\right). \tag{6.11}$$

Here, n_f is the number of active light flavors.

7 Numerical Results

7.1 Modified evolution equation

The numerical work which we report will be mainly illustrative in nature. We use the one loop form of the running coupling,

$$\frac{1}{\overline{\alpha}_S(\mu)}=\bar{b}\ln\left(\frac{\mu^2}{\Lambda^2}\right), \tag{7.1}$$

where

$$\bar{b}=\frac{33-2n_f}{36}. \tag{7.2}$$

We work in the limit of zero light quarks ($n_f=0$) but choose $\Lambda=0.215$ GeV. This gives a coupling which at 5 GeV is equal to the two loop coupling calculated with four flavors using $\Lambda(n_f=4)=0.19$ GeV. It therefore gives a reasonable coupling in the range of interest for bottom quark production. We will choose $\mu=m=5\,\mathrm{GeV}$.

In moment space the solution to the Altarelli-Parisi equation, Eq. (2.2) or (6.2) with the lowest order kernel, is

$$\tilde{f}_g^{\rm LO}(j,\mu) = \tilde{f}_g(j,\mu_0)\left[\ln\left(\frac{\mu}{\Lambda}\right)\Big/\ln\left(\frac{\mu_0}{\Lambda}\right)\right]^{d(j)}. \tag{7.3}$$

If we take $\tilde{\gamma}$ to be the standard perturbative anomalous dimension, d is as given below:

$$\tilde{\gamma}(j) = \sum_{n=1}\overline{\alpha}_S^n \tilde{P}^{(n)}(j),\quad d(j) = \frac{\tilde{P}^{(1)}(j)}{\bar{b}}. \tag{7.4}$$

When we substitute for $\tilde{\gamma}$ the solution to the modified Lipatov equation, Eq. (6.10), we obtain

$$\tilde{f}_g(j,\mu) = \tilde{f}_g(j,\mu_0)\left[\ln\left(\frac{\mu}{\Lambda}\right)/\ln\left(\frac{\mu_0}{\Lambda}\right)\right]^{d_I(j)}\exp\left(\frac{J(j,\mu,\mu_0)}{\bar{b}(j-1)}\right), \tag{7.5}$$

where

$$\bar{d}_I(j) = \frac{\tilde{P}_I^{(1)}(j,0)}{\bar{b}}, \tag{7.6}$$

so that

$$d(j) = d_I(j) + \frac{1}{\bar{b}(j-1)}, \tag{7.7}$$

and J is

$$J(j,\mu,\mu_0) = \int_{\gamma_c(j,\mu_0)}^{\gamma_c(j,\mu)} d\gamma\,\gamma\frac{d}{d\gamma}\chi(\gamma), \tag{7.8}$$

with χ as given in Eq. (3.9).

Numerical work is done by working in the j-plane, and by performing the Mellin transform numerically. We take the starting distributions at $\mu_0 = 2$ GeV to be of the form

$$f_g(x,\mu_0) = \frac{1}{2}\,\frac{\Gamma(8-a)}{\Gamma(2-a)\Gamma(6)}\,x^{-a}(1-x)^5. \tag{7.9}$$

We will consider the two extreme cases $a = 1$ and $a = 1.5$. In both cases, the gluon density is normalized to carry half the total hadron momentum. In Fig. 7 we show evolution starting from the distribution with $a = 1$. The resummed formula introduces a singularity at $j = j_L$. This explains the substantial difference between the modified and the standard AP evolution shown in Fig. 7. Despite the fact that the singularity at $j = j_L$ is quite weak, it has immediate effect because for $a = 1$ all other singularities are at $j = 1$. In Fig. 8 we show evolution starting from the

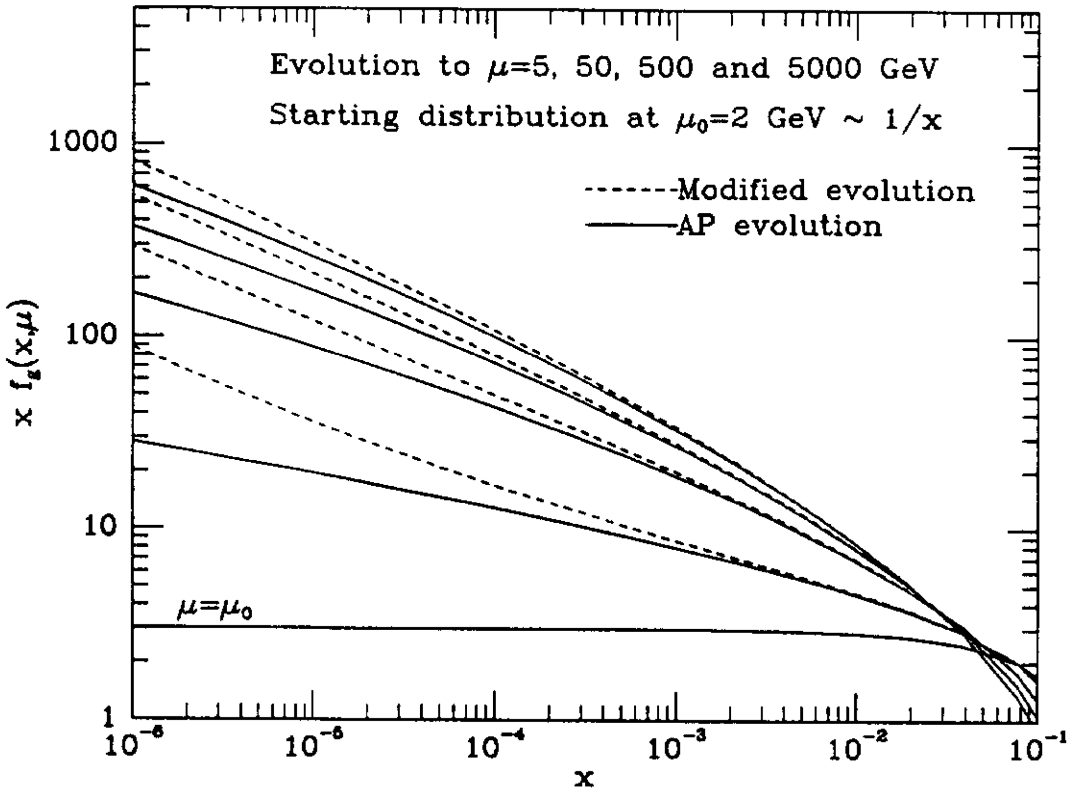


Figure 7: Evolution of gluon from starting distribution with $1/x$ behavior at small x.

distribution with $a = 1.5$. Because the distribution already contains a singularity at $j = 1.5$ the Lipatov singularity is less important.

Note that the rapid growth of the gluon density shown in these figures cannot continue unchecked[1]. When the packing fraction of the gluons within the hadron exceeds unity the partons interact with one another. An estimate of the position of the onset of this saturation is

(No. of partons per unit rapidity) $\times$ (transverse area of parton) $\approx$ (area of hadron)

$$xf_g(x,\mu) \approx \mu^2 R^2, \tag{7.10}$$

where R is the hadron radius. For $\mu = 5\,\text{GeV}$ this limit will occur when $xf_g(x,\mu) \approx 10^3$. We make no attempt to include this saturation effect in our numerical work.

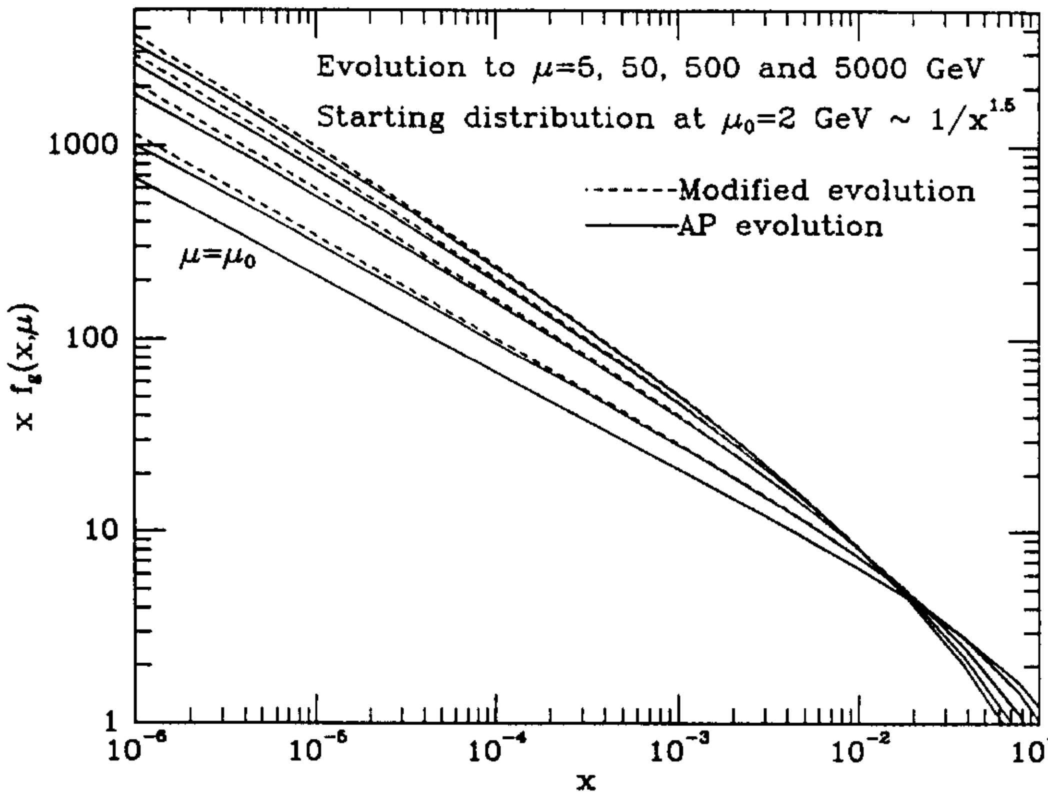


Figure 8: Evolution of gluon from starting distribution with $1/x^{1.5}$ behavior at small x.

The above figures make it clear that the two distributions are very different below $x = 10^{-2}$. This immediately leads to a sizeable uncertainty in the bottom cross-section. Accurate predictions for bottom quark production at the energy of the Tevatron (and above) will require experimental information on the gluon distribution below $x = 10^{-2}$. Alternatively bottom quark production can be used to determine the gluon distribution in this region.

7.2 Cross sections

In Fig. 9 we show the bottom cross section as calculated using the less singular gluon distribution. We show three plots. The highest curve, σ_r is calculated using

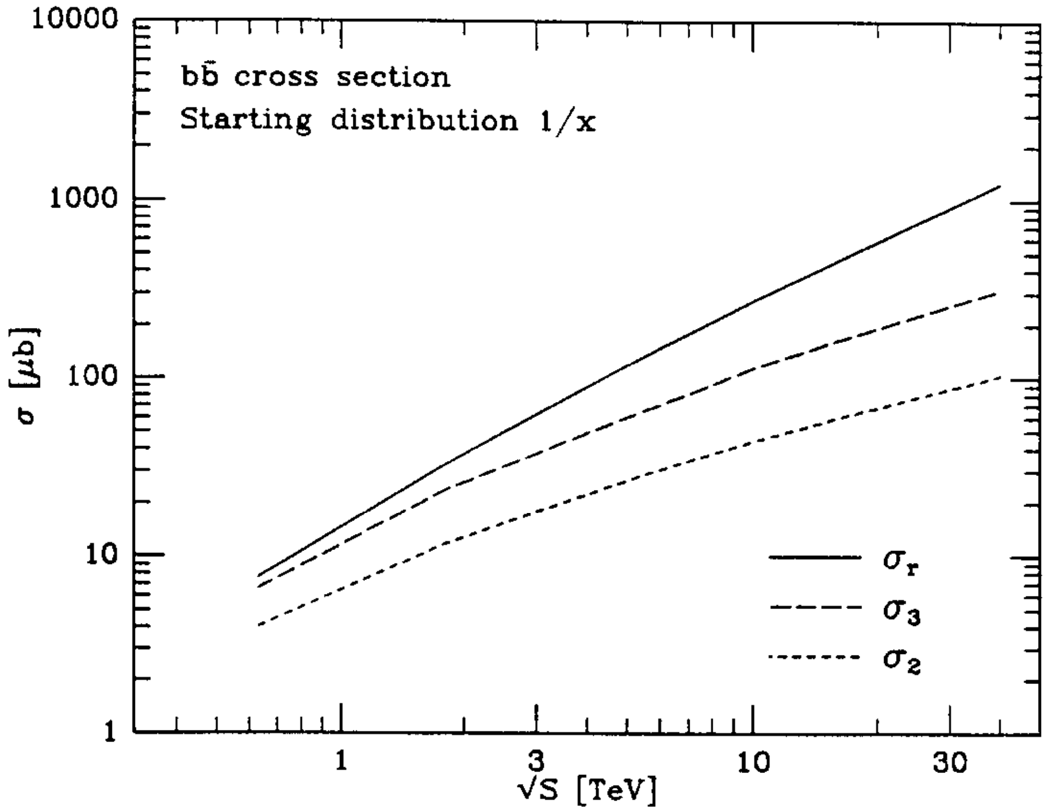


Figure 9: Bottom cross section starting with $1/x$ behavior at small x. σ_2, σ_3, and σ_r are the cross section calculated with respectively the lowest order approximations everywhere, with the order α_S^3 approximation to our resummation, and with the full resummation.

the full resummation formula, Eq. (2.5), and parton distributions obtained with the modified evolution equation. The lowest curve displays σ_2, which is calculated using lowest order approximations everywhere, without resummation:

$$\sigma_2(s) = \int_0^1 dx_1 \int_0^1 dx_2 f^{\mathrm{LO}}(x_1,\mu) f^{\mathrm{LO}}(x_2,\mu) I_0(x_1 x_2 s). \tag{7.11}$$

Here f^{LO} is the gluon density calculated using the standard Altarelli-Parisi evolution, Eq. (7.3). The Lipatov ladder (with no rungs) first shows up in a fixed order calculation at order α_S^3. It is interesting to try and estimate how much of the effect of the resummation is already included in an order α_S^3 calculation. Therefore the remaining

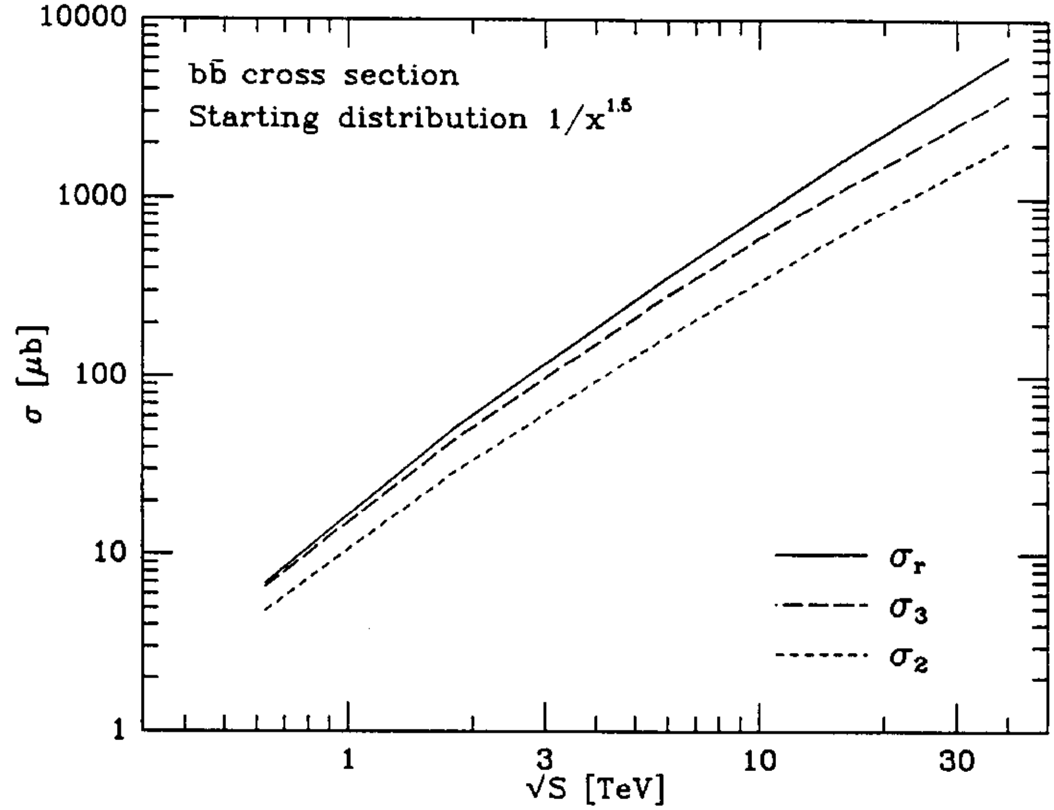


Figure 10: Bottom cross section starting with $1/x^{1.5}$ behavior at small x.

curve shows σ_3

$$\begin{aligned}\sigma_3(s) = & \int_0^1 dx_1 \int_0^1 dx_2 \Big\{ f^{\rm LO}(x_1,\mu) f^{\rm LO}(x_2,\mu) I_0(x_1x_2s) \\ & + \int_0^\infty d\mathbf{k}_2^2\, f^{\rm LO}(x_1,\mu)\mathcal{P}(x_2,\mathbf{k}_2,\mu)\Big[I_1(x_1x_2s,\mathbf{k}_2) - \theta(\mu^2-\mathbf{k}_2^2)I_0(x_1x_2s)\Big] \\ & + \int_0^\infty d\mathbf{k}_1^2\, \mathcal{P}(x_1,\mathbf{k}_1,\mu) f^{\rm LO}(x_2,\mu)\Big[I_1(x_1x_2s,\mathbf{k}_1) - \theta(\mu^2-\mathbf{k}_1^2)I_0(x_1x_2s)\Big]\Big\},\end{aligned} \tag{7.12}$$

with $f^{\rm LO}$ given by Eq. (7.3) and $\mathcal{P}$ given by Eq. (3.25). The results at $\sqrt{s} = 1.8\,\mathrm{TeV}$ are shown in Table 1. We see that σ_3 is smaller than σ_r especially for the less steep starting distribution.

a	$\sigma_2[\mu b]$	$\sigma_3[\mu b]$	$\sigma_r[\mu b]$
1	12	23	33
1.5	29	44	51

Table 1: Cross sections at the Tevatron in various orders in perturbation theory

In Fig. 10 we show the bottom cross section as calculated using the more singular gluon distribution. The cross sections are larger than in the previous case, and the influence of the resummation is smaller. These two figures show that the size of the cross section is dependent on the form of the gluon distribution. Note however that the inclusion of the resummation reduces the sensitivity to the form of the gluon distribution.

8 Conclusions

We have presented an outline of a new method of dealing with small x physics which clearly separates the short and long distance components of the calculation. It represents a generalization of the normal QCD parton model to the case where gluons which have a transverse momentum of the order of the hard interaction scale are important. Our formalism allows the inclusion of higher order effects.

We have presented some illustrative numerical results, using bottom quark production at $\sqrt{s} > 1$ TeV as an example. Our results include neither higher order effects nor light flavors of quarks, both of which are known to be important at the energy of the Tevatron. Furthermore, no attempt has been made to include saturation effects which are important at higher energies. Despite these deficiencies we believe that our numerical results are sufficient to demonstrate two qualitative features about bottom production at the Tevatron. First, resummation effects are already important at the energy of the Tevatron. Secondly, the calculations performed through $O(\alpha_S^3)$ include part but not all of the small-x physics. Bottom quark production at the Tevatron is not adequately described either by fixed order perturbation theory or by the resummed formula without higher order terms.

There are a number of further developments of our methods that must be made.

First, we must extend the formalism to include light quarks as partons. This is an elementary extension of the results presented in this paper, since only gluon exchange generates the rapidity logarithms that we have resummed. Then we must calculate impact factors for other processes of interest, like Drell-Yan and jet production. It would also be useful to do calculations with a more differential heavy quark cross section.

A most important task is to devise convenient methods of calculating higher order corrections in α_S for the impact factors and for the kernels of our ladder equation. We believe that our gauge invariant definition of the impact factors is an important step forward for this.

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References

1) L. V. Gribov, E. M. Levin and M. G. Ryskin, *Phys. Rep.* **100** (1983) 1 .

2) G. Altarelli and G. Parisi, *Nucl. Phys.* **B126** (1977) 298 ;
Yu. L. Dokshitzer *Sov. Phys. JETP* **46** (1977) 641 ;
cf. L. N. Lipatov, *Sov. J. Nucl. Phys.* **20** (1975) 95 ;
V. N. Gribov and L. N. Lipatov, *Sov. J. Nucl. Phys.* **15** (1972) 438 .

3) P. Nason, S. Dawson and R. K. Ellis, *Nucl. Phys.* **B303** (1988) 607 .

4) W. Beenakker *et al.*, *Phys. Rev.* **D40** (1989) 54 .

5) J. C. Collins and R. K. Ellis, Argonne preprint ANL-HEP-CP-90-62, to be published in the proceedings of DESY topical meeting on Small-x behavior of Deep Inelastic Structure functions in QCD, *Nucl. Phys.* **B** Supplements.

6) L. N. Lipatov, in "Perturbative QCD", A. H. Mueller, (editor), (World Scientific, Singapore, 1989);
L. N. Lipatov and G. V. Frolov, *Sov. J. Nucl. Phys.* **13** (1971) 333 ;
E. A. Kuraev, L. N. Lipatov and V. S. Fadin, *Sov. Phys. JETP* **44** (1976) 443 , *ibid.* **45** (1977) 199 ;
Ya. Ya. Balitskii and L. N. Lipatov, *Sov. J. Nucl. Phys.* **28** (1978) 822 .

7) J. Kwiecinski, *Zeit. Phys.* **C29** (1985) 561 .

8) S. Catani, M. Ciafaloni and F. Hautmann, *Phys. Lett.* **B242** (1990) 97 .

9) G. Marchesini and B. R. Webber, *Nucl. Phys.* **B349** (1991) 617 .

10) E. M. Levin *et al.*, Cavendish preprint-HEP-90/9,(1990), *Nucl. Phys.* **B** (in press).

11) E. M. Levin, M. G. Ryskin, Yu. M. Shabelski and A. G. Shuvaev, (private communication).

12) R. K. Ellis and D. A. Ross, *Nucl. Phys.* **B345** (1990) 79 .

13) M. Gluck, J.F. Owens and E. Reya, *Phys. Rev.* **D17** (1978) 2324 ;
B. Combridge, *Nucl. Phys.* **151** (1979) 429 .

14) R. K. Ellis, *Strong Interactions and Gauge Theories,* edited by J. Tran Thanh Van, Editions Frontières, Gif-sur-Yvette, 1986, p. 339.

15) J. C. Collins and D. E. Soper, *Nucl. Phys.* **B194** (1982) 445 .

16) J. C. Collins, *Renormalization*, Cambridge University Press, (1984).

17) G. Altarelli, *Phys. Rep.* **81** (1982) 1 .