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Color Decomposition of One-Loop Amplitudes In Gauge Theories

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Abstract

We present a decomposition of one-loop amplitudes in pure $SU(N)$ gauge theories into a sum of products of traces of charge matrices and color-independent kinematic factors. The decomposition is the one-loop extension of the cyclic color decomposition that has come into widespread use in tree-level amplitudes.

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1. Introduction

The computation of perturbative amplitudes in quantum chromodynamics is important to our understanding both of experimental results on jet production at hadron colliders, and of possible backgrounds to new physics. In recent years, a great deal of progress has been made in computing tree-level amplitudes. Three main ingredients contributed to these advances: use of a spinor helicity basis, such as that of Xu, Zhang, and Chang [1], for gluon polarization vectors; the Berends-Giele recurrence relations for amplitudes [2, 3]; and the color decomposition of the amplitudes [4,5,6,7]. In this paper, we shall derive a color decomposition for one-loop amplitudes in a pure $SU(N)$ gauge theory.

Open string theories provide an easy way to derive the tree-level decomposition and associated properties of the kinematical factors, as we shall review in the next section. They can also be used to obtain a heuristic understanding of the decomposition of one-loop amplitudes, as we shall discuss in section 3. The formal derivation of the representation for loop amplitudes, presented in sections 4 and 5, will rely on appropriately constructed heterotic string theories. In section 6, we discuss the decoupling equations, which yield constraints on the kinematic factors that are the one-loop analogs of the tree-level 'twist' [8,9] 'dual Ward' [5], or 'cyclic' [2] identity, and display the form of solutions to the decoupling equations for the four-, five-, and six-gluon amplitudes. We present the general form of the one-loop decoupling equations in section 7; these decoupling equations are an important tool for checking practical calculations. In section 8, we discuss the expansion of the tree-level cross section in inverse powers of the number of colors, and in section 9, we give the corresponding form for the next-to-leading ($\mathcal{O}(\alpha^{n+1})$) correction to the squared matrix element. In section 10, we summarize the decomposition and the decoupling equations. In the appendices, we discuss a number of string-theoretic and group-theoretic issues, and outline the derivation of the color decomposition from a Feynman diagram approach.

Some aspects of the color decomposition of gauge-theory loop amplitudes have previously been discussed by Mangano [7]. Color decompositions along different lines were discussed by Cvitanovic, Lauwers, and Scharbach [10] and by Zeppenfeld [11].

2. Tree Amplitudes

In an open string theory, the full on-shell amplitude for the scattering of n massless vector mesons can be written as the sum over non-cyclic permutations of the external legs of Chan-Paton factors [12] times Koba-Nielsen partial amplitudes [13],

$$\mathcal{A}_n^{\rm string}(\{k_i,\varepsilon_i,a_i\}) = g^{n-2} \sum_{\sigma\in S_n/Z_n} {\rm Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}\cdots T^{a_{\sigma(n)}}) A_n^{\rm KN}(k_{\sigma(1)},\varepsilon_{\sigma(1)};\ldots;k_{\sigma(n)},\varepsilon_{\sigma(n)}) \tag{2.1}$$

where k_i, ε_i, and a_i are respectively the momentum, polarization vector, and color index of the i-th external gluon. The T^a are the set of hermitian traceless $N\times N$ matrices (normalized so that ${\rm Tr}(T^aT^b)=\delta^{ab}$), and S_n/Z_n is the set of non-cyclic permutations of $\{1,\ldots,n\}$.

In the infinite-tension limit, a $U(N)$ string theory reduces to a $U(N)$ gauge theory, where the matter content depends on the type of the original string theory. For tree amplitudes where all the external legs are gluons, however, the matter content is irrelevant, since the matter fields cannot appear as internal lines. Thus one can use the open bosonic string, the simplest of all string constructions, at tree level. The decomposition of the string amplitude leads immediately to a decomposition of on-shell n-gluon amplitudes,

$$\mathcal{A}_n(\{k_i,\varepsilon_i,a_i\}) = g^{n-2} \sum_{\sigma\in S_n/Z_n} {\rm Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}) A_n(k_{\sigma(1)},\varepsilon_{\sigma(1)};\ldots;k_{\sigma(n)},\varepsilon_{\sigma(n)})\,. \tag{2.2}$$

The *partial amplitudes* A_j possess a number of nice properties that follow immediately from the properties of the Koba-Nielsen amplitudes. Each is gauge invariant on shell, that is invariant under the substitution $\varepsilon_i\to\varepsilon_i+\lambda k_i$ for each leg independently. It is also invariant under cyclic permutation of its arguments, and satisfies a reflection identity,

$$A(n,\ldots,1) = (-1)^n A_n(1,\ldots,n)\,, \tag{2.3}$$

where we use the standard notation

$$A_n(1,\ldots,n) = A_n(k_1,\varepsilon_1;\ldots;k_n,\varepsilon_n)\,. \tag{2.4}$$

The $U(1)$ gauge boson is an integral part of the string theory (its presence is necessary for unitarity), but in the infinite-tension limit, it must decouple from $SU(N)$ gauge boson amplitudes. We can use this observation,

$$\mathcal{A}_n(\{k_i,\varepsilon_i,a_i\}_{i=1}^{n-1};k_n,\varepsilon_n,a_{U(1)}) = 0 \tag{2.5}$$

to derive a decoupling identity, simply by extracting the coefficient of ${\rm Tr}(T^{a_1}\cdots T^{a_{n-1}})$, which is

$$\sum_{\sigma\in Z_{n-1}} A_n(\sigma(1),\ldots,\sigma(n-1),n) = 0. \tag{2.6}$$

(This identity can also be derived starting with the twist operator in the open string theory [8]. Mangano, Parke, and Xu [5] term the identity a dual Ward identity.) Substituting additional photons for gluons leads to equations which are linearly dependent on equation (2.6).

The decoupling identity (2.6) is a special case of the more general Berends-Giele identities [14,15,16], which may be derived by considering $U(N)$ currents (rather than amplitudes) of gluons taken from two or more commuting subalgebras of $U(N)$. These vanish identically because any internal line must fall into one of the different subalgebras, and therefore cannot connect the different kinds of gluons. These considerations lead to the equation

$$\sum_{\sigma\in M(1,\ldots,m_1;m_1+1,\ldots,m_2;m_2+1,\ldots;\ldots,n-1)} A(\sigma(1),\ldots,\sigma(n-1),n)=0 \qquad (2.7)$$

where $M(\{a_i\};\{b_j\};\{c_k\};\ldots)$ is the set of all mergings of the sets $\{a_i\}$, $\{b_j\}$, $\{c_k\},\ldots$, that is, all permutations of the set $\{a_i,b_j,c_k,\ldots\}$ that preserve the ordering within each of the subsets $\{a_i\}$, $\{b_j\}$, $\{c_k\},\ldots$ separately.

At loop level, however, these amplitudes will not vanish, because internal lines can carry charges coupling the two sorts of gluons to each other; the general Berends-Giele identities are therefore not expected to have a one-loop counterpart. We should expect only analogs of the photon decoupling equation (2.6).

The partial amplitudes also satisfy tree-level unitarity, which is to say they factorize on poles of a consecutive set of their arguments.

3. Open Strings at Loop Level

Oriented open strings provide a simple intuitive understanding of the color structure of $SU(N)$ gauge theory loop amplitudes. In all open string models, the associated gauge group arises from explicit Chan-Paton factors [12], in contrast to closed strings where the gauge group arises implicitly [17,18]. The Chan-Paton factors can be understood in terms of 'color charges' sitting at the ends of the string. As the string propagates, it sweeps out a world-sheet, the analog to a particle's world-line, and the ends of the string move along the boundaries of the string world-sheet. Interacting strings give rise to a world-sheet with additional boundaries corresponding to the external states. These boundaries also carry color charges appropriate to the given external states. The color structure of the amplitude is thus associated with the boundary of the string world-sheet describing the scattering process.

The rules of open string constructions [19, 20] tell us that each external gluon has an associated Gell-Mann charge matrix which is its Chan-Paton factor. Each partial amplitude corresponds to a given ordering of the gluons along each of the boundaries of open string diagram. The coefficient of the partial amplitude is simply the trace of the product of the gluon charges, taken along each of the boundaries of the string world-sheet. A boundary containing no external gluons gives rise to a trace over the identity matrix, that is a factor of the number of colors, N_c.

For example, the Chan-Paton factor associated with the five-point partial amplitude represented by the planar diagram in fig. 1 is $N_c \operatorname{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5})$ while for the non-planar diagram in fig. 2 the factor is $\operatorname{Tr}(T^{a_1}T^{a_2}T^{a_3})\operatorname{Tr}(T^{a_4}T^{a_5})$. The full amplitude is the sum of the partial string amplitudes, with associated color trace factors, over all permutations of the external legs which do not leave the Chan-Paton traces invariant.

At one loop, the schematic form of the n-point amplitude is

$$\begin{aligned}
&\mathcal{A}_n^{\rm string}(\{a_i, k_i, \epsilon_i\}) = \\
&\qquad \sum_{\sigma\in S_n/Z_n} N_c \operatorname{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}) A_1^{\rm string}(k_{\sigma(1)}, \varepsilon_{\sigma(1)};\ldots;k_{\sigma(n)},\varepsilon_{\sigma(n)}) \\
&\qquad + \sum_m \sum_{\sigma\in S_n/Z_m\times Z_{n-m}} \operatorname{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(m)}}) \operatorname{Tr}(T^{a_{\sigma(m+1)}}\cdots T^{a_{\sigma(n)}}) A_2^{\rm string}(k_{\sigma(1)}, \varepsilon_{\sigma(1)};\ldots;k_{\sigma(n)},\varepsilon_{\sigma(n)}) \\
&\qquad + \mathcal{O}(\alpha')
\end{aligned} \tag{3.1}$$

where the first term appears when all gluons are attached to a single boundary, while the second term appears when gluons are attached to both boundaries. The higher order corrections in the inverse string tension α' (which do contain terms with three or more non-trivial traces) arise from graviton exchange, corresponding to pulling some of the holes of higher loop diagrams into long tubes. Such contributions disappear in the gauge theory (or infinite-tension) limit where the coupling to gravitons and other colorless states vanishes.

This discussion is only a heuristic one, however, for in contrast to the case of tree level amplitudes, at loop level all string states which couple to the gauge bosons can circulate inside the loops; thus detailed control of the massless spectrum is required. Although it is possible to build some consistent open superstring models in four dimensions [21], the technology is not as advanced as for heterotic strings, and thus we cannot construct completely consistent open string models which contain a pure $SU(N)$ gauge theory in the low-energy limit. It is possible to construct a range of models, so the conclusions drawn from the open-string representation are not necessarily specific to any particular model; but we cannot use open strings to *prove* the $SU(N)$ color decomposition. We give a proof of the decomposition at one-loop order based on a heterotic string construction in the following two sections (the reader who does not wish to indulge in the technicalities of heterotic strings may skip the details of the next two sections), and return to applications of the decomposition in section 6. It is also possible to give a derivation based on Feynman diagrams, which we outline in appendix VI. Such a derivation (and discussion elsewhere [9]) makes clear that each partial amplitude receives contributions from many Feynman diagrams, and thus does not necessarily provide an efficient method of calculating the partial amplitudes. In contrast, the heterotic string approach has the advantage of providing a concrete and direct formula for the partial amplitudes. This allows, for example, the full utilization of the power of the spinor-helicity basis and in other ways provides an efficient means of evaluating the partial amplitudes. The details of one such calculation will be presented elsewhere [22].

Beyond one loop, the open string picture (fig. 3) suggests that in the gauge theory limit of an L-loop amplitude exactly $L+1$ traces appear in the coefficient of any given partial amplitude, including traces over the identity for boundaries with no attached gluons, and that the gauge theory amplitude is simply the sum over all independent partial amplitudes with exactly $L+1$ accompanying traces. While the broad outline of this decomposition — the appearance of $L+1$ or fewer *non-trivial* traces — is in agreement with a diagrammatic derivation along the lines of appendix VI, the latter derivation allows that coefficients of the terms with fewer than $L+1$ traces might carry fewer powers of N_c than suggested by the string picture. (For example, at two loops, the diagrammatic derivation does not rule out the appearance of terms with a single trace and no powers of N_c. Such terms might conspire to cancel, and do for the two-point function, but it is not clear that will happen in general.) If a more precise statement than the one emerging from the diagrammatic derivation can be made, it must await the multi-loop analogs of the formalism presented in the following sections.

4. Heterotic String Loop Amplitudes

If we wish to write an $SU(N)$ gauge theory loop amplitude as the infinite-tension limit of a string amplitude, we have to control the massless matter content of the string theory, because colored massless matter particles (if any) can run around the loops. It is possible to build heterotic string theories whose infinite-tension limit is a non-Abelian gauge theory where one of the factors is an $SU(N)$ with no matter fields. The technology needed for such a construction is precisely the one used to construct four-dimensional string models. We have discussed the construction of such models, for which we use the fermionic formulation of Kawai, Lewellen, and Tye (KLT), in previous work [23]; a sample model is summarized in appendix I. (Bosonic strings always contain unwanted massless scalars and tachyons, while four-dimensional type II [19,24] and type I [21] superstrings do not have a rich enough variety of models for our purposes.)

The three basic types of four-dimensional heterotic superstring constructions are the bosonic formulations [25], fermionic formulations [26,27], and direct superconformal field theory constructions [28]. These constructions have generated a huge class of consistent four-dimensional superstring models; the freedom in constructing models is so large that strings with virtually any low energy gauge group with rank 22 or less can be built. In this paper, we again use the fermionic formulation in the notation of Kawai, Lewellen and Tye [26]. We have found the fermionic formalism to be particularly straightforward to use for constructing models as well as scattering amplitudes although one could use the other formalisms as well.

These constructions of the four-dimensional string models are based on analyses of the one-loop partition string function. In their construction, Kawai et al. require world-sheet reparameterization invariance, world-sheet supersymmetry, freedom from the conformal anomaly, one-loop modular invariance invariance [29], and a physically sensible projection on the spectrum. (The last condition is actually equivalent to two- and higher-loop modular invariance [27,30].)

In the fermionic formulation all internal degrees of freedom (which carry the non-abelian gauge charges) are taken to be world-sheet fermions [17]. In general each of the world-sheet fermions can have independent boundary conditions on the one-loop world sheet torus,

$$\psi_l(t+2\pi n, \sigma+2\pi m) = e^{2\pi i(\alpha_l n+\beta_l m)}\psi_l(t,\sigma)\,. \tag{4.1}$$

Corresponding to each set of boundary conditions on the n_f fermions is a one-loop partition function $Z^{\vec{\alpha}}_{\vec{\beta}} = \prod_{l=1}^{n_f} Z^{\alpha_l}_{\beta_l}$. (The explicit value of these partition functions in terms of ϑ-functions is given in Appendix III.) The world-sheet fermionic contribution to the full string partition function is a linear combination

$$Z_{\text{fermion}} = \sum_{\alpha,\beta} C^{\vec{\alpha}}_{\vec{\beta}} Z^{\vec{\alpha}}_{\vec{\beta}} \tag{4.2}$$

where the KLT coefficients $C^{\vec{\alpha}}_{\vec{\beta}}$ must be chosen so that the full fermionic partition function is modular invariant and represents a physically sensible projection on the space of states. Rules for choosing such coefficients have been discussed in detail in refs. [26,27].

The full one-loop partition function of the string is the product of the above fermionic partition function with the bosonic partition function Z_B given in Appendix III.

Once the partition function of a particular model has been determined the procedure for computing a scattering amplitude is straightforward using the vertex operators [31,20] of the theory. The amplitude is given by the expectation value of the vertex operators using the world-sheet action for free fermions and bosons,

$$\mathcal{A}_n^{\text{string}}\left(\{a_i,k_i,\varepsilon_i\}\right) \sim \int [DX][D\psi]\exp\left[-S\right]\,V^{a_1}(k_1,\varepsilon_1)\cdots V^{a_n}(k_n,\varepsilon_n)\,. \tag{4.3}$$

For a bosonic string, the world-sheet action $S = (4\pi\alpha')^{-1}\int d^2\xi\;\eta_{\mu\nu}\partial_\beta X^\mu\partial^\beta X^\nu$; it is a bit more complicated for a four-dimensional heterotic string (see, for example, ref. [26]). Using Wick's theorem and expressions for the Green functions, these the expectation values can be computed explicitly. (The details of performing such computations in the operator formalism may be found in ref. [20].)

We are interested in an amplitude with n external gluons. The one-loop string amplitude can be written as follows*,

$$
\begin{aligned}
\mathcal{A}_n = {} & \frac{1}{2(16\pi^2)}\lambda^{n/2-2}(\sqrt{2}g)^n\, T^{a_1}{}_{m_1}{}^{\widehat{m}_1}\cdots T^{a_n}{}_{m_n}{}^{\widehat{m}_n} \\
& \int\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\int\left(\prod_{i=1}^{n} d\theta_{i1}\, d\theta_{i2}\, d\theta_{i3}\, d\theta_{i4}\right)\int\left(\prod_{i=1}^{n} d^2\nu_i\right)\sum_{\vec\alpha,\vec\beta} C^{\vec\alpha}_{\vec\beta}\mathcal{Z}^{\vec\alpha}_{\vec\beta}(\tau) \\
& \prod_{i=1}^{n}\exp\left[-\theta_{i1}\theta_{i2}\delta^{m_i}{}_{\widehat{m}_i} S_F\begin{bmatrix}\alpha_{m_i}\\ \beta_{\widehat{m}_j}\end{bmatrix}\right] \\
& \prod_{i<j}\exp\Bigg[\lambda k_i\cdot k_j\, G_B(\nu_i-\nu_j) \\
& \qquad - \theta_{i1}\theta_{j2}\delta^{m_i}{}_{\widehat{m}_j} G_F\begin{bmatrix}\alpha_{m_i}\\ \beta_{\widehat{m}_j}\end{bmatrix}(\nu_i-\nu_j) - \theta_{i2}\theta_{j1}\delta^{m_j}{}_{\widehat{m}_i}\hat{G}_F\begin{bmatrix}\alpha_{m_j}\\ \beta_{\widehat{m}_i}\end{bmatrix}(\nu_i-\nu_j) \\
& \qquad - \theta_{i3}\theta_{j3}\lambda k_i\cdot k_j\, G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i-\overline{\nu}_j) \\
& \qquad + i\sqrt{\lambda}(\theta_{i3}\theta_{j4}k_i\cdot\varepsilon_j + \theta_{i4}\theta_{j3}k_j\cdot\varepsilon_i)\, G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i-\overline{\nu}_j) \\
& \qquad - i\sqrt{\lambda}(\theta_{i3}\theta_{i4}k_j\cdot\varepsilon_i - \theta_{j3}\theta_{j4}k_i\cdot\varepsilon_j)\, \dot{G}_B(\overline{\nu}_i-\overline{\nu}_j) \\
& \qquad + \theta_{i4}\theta_{j4}\varepsilon_i\cdot\varepsilon_j\, G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i-\overline{\nu}_j) \\
& \qquad + \theta_{i3}\theta_{i4}\theta_{j3}\theta_{j4}\varepsilon_i\cdot\varepsilon_j\, \ddot{G}_B(\overline{\nu}_i-\overline{\nu}_j)\Bigg]
\end{aligned}
\tag{4.4}
$$

where $\lambda = \pi\alpha'$ is the inverse string tension; the $\theta_{i,j}$ are Grassman integration variables, and the ν_i are integrated over the torus specified by the modular parameter τ; $\vec\alpha$ and $\vec\beta$ are the various boundary conditions for the world-sheet fermions, over which one must sum into order to obtain a modular-invariant answer; $\mathcal{Z}^{\vec\alpha}_{\vec\beta}$ is the partition function for a given set of boundary conditions; $G_F\left[{\alpha_m \atop \beta_{\widehat{m}}}\right]$ are the left-mover fermionic Green functions, with α_m and $\beta_{\widehat{m}}$ the boundary conditions on the torus of world-sheet fermions associated with the gauge group of interest; $G_F\left[{\alpha_\uparrow \atop \beta_\uparrow}\right]$ are the right-mover fermionic Green functions, with $\alpha_\uparrow$ and $\beta_\uparrow$ the boundary conditions of the world-sheet fermions carrying the space-time index (which occupies the first right-mover postion in the world-sheet boundary condition vector); G_B are the bosonic Green functions (dots indicate derivatives

* This form is valid in all string sectors which do not contain world-sheet zero-modes, such as the sectors of interest to us — those containing the gluons.

with respect to ν); and $S_F\left[{\alpha_m \atop \beta_{\widehat{m}}}\right]$ are the self-contractions of the left-mover fermions. Our convention is that the left-movers corrspond to the bosonic string while the right-movers correspond to the superstring. Detailed expressions for all of these quantities are given in the appendices. (The form of the amplitude given here differs slightly from the standard form, in that we have chosen to integrate over all νs, and have compensated by dividing by the volume of the torus, rather than fixing ν_n.)

A rather striking feature of the amplitude is that it is valid for arbitrary numbers of gluon legs. In contrast, the usual Feynman rules do not yield a comparable concrete formula in any straightforward manner.

It will be helpful to distinguish three pieces of the integrand: the left-mover contributions (which are a function of the ν_i alone),

$$\begin{aligned} L(\{\nu_i\}) &= \int \left(\prod_{i=1}^{n} d\theta_{i1}\ d\theta_{i2}\right) \\ &\prod_{i=1}^{n} \exp\left[-\theta_{i1}\theta_{i2}\delta^{m_i}{}_{\widehat{m}_i} S_F\begin{bmatrix}\alpha_{m_i}\\ \beta_{\widehat{m}_i}\end{bmatrix}\right] \\ &\prod_{i\neq j} \exp\left[-\frac{1}{2}\theta_{i1}\theta_{j2}\delta^{m_i}{}_{\widehat{m}_j} G_F\begin{bmatrix}\alpha_{m_i}\\ \beta_{\widehat{m}_j}\end{bmatrix}(\nu_i-\nu_j) - \frac{1}{2}\theta_{i2}\theta_{j1}\delta^{m_j}{}_{\widehat{m}_i} \hat{G}_F\begin{bmatrix}\alpha_{m_j}\\ \beta_{\widehat{m}_i}\end{bmatrix}(\nu_i-\nu_j)\right] \end{aligned} \tag{4.5}$$

the right-mover contributions (which are a function of the $\overline{\nu}_i$ alone),

$$\begin{aligned} R(\{\overline{\nu}_i, k_i, \varepsilon_i\}) = \int &\left(\prod_{i=1}^{n} d\theta_{i3}\ d\theta_{i4}\right) \\ \prod_{i<j} \exp \Bigg[&-\theta_{i3}\theta_{j3}\lambda k_i \cdot k_j\, G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i-\overline{\nu}_j) \\ &+ i\sqrt{\lambda}(\theta_{i3}\theta_{j4}k_i\cdot\varepsilon_j + \theta_{i4}\theta_{j3}k_j\cdot\varepsilon_i)\, G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i-\overline{\nu}_j) \\ &- i\sqrt{\lambda}(\theta_{i3}\theta_{i4}k_j\cdot\varepsilon_i - \theta_{j3}\theta_{j4}k_i\cdot\varepsilon_j)\, \dot{G}_B(\overline{\nu}_i-\overline{\nu}_j) \\ &+ \theta_{i4}\theta_{j4}\varepsilon_i\cdot\varepsilon_j\, G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i-\overline{\nu}_j) \\ &+ \theta_{i3}\theta_{i4}\theta_{j3}\theta_{j4}\varepsilon_i\cdot\varepsilon_j\, \ddot{G}_B(\overline{\nu}_i-\overline{\nu}_j)\Bigg] \end{aligned} \tag{4.6}$$

and the partition function and parts common to both movers,

$$E(\{\nu_i, \overline{\nu}_i, k_i, \varepsilon_i\}) = \sum_{\vec{\alpha},\vec{\beta}} C^{\vec{\alpha}}_{\vec{\beta}} Z^{\vec{\alpha}}_{\vec{\beta}}(\tau) \prod_{i<j}^{n} \exp\left[\lambda k_i\cdot k_j G_B(\nu_i-\nu_j)\right] . \tag{4.7}$$

The overall normalization will be denoted by $\mathcal{N}$.

If we perform the θ_{i1} and θ_{i2} integrals, the $\delta^m{}_n$ will contract the indices of the charge matrices, and we will obtain an expression for the string amplitude as a sum of kinematic coefficients,

$$\begin{aligned} \mathcal{A}_n^{\rm string} = \sum_i \mathrm{Tr}(T^{a_1}\cdots T^{a_j})\cdots \mathrm{Tr}(T^{a_l}\cdots T^{a_m}) \\ \times \mathcal{N}\int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\int d^2\nu_i\, L(\{\nu\})E(\{\nu,k\})R(\{\overline{\nu},k,\varepsilon\}) \end{aligned} \tag{4.8}$$

where the sum runs over all trace structures with up to n traces, and all inequivalent ways of ordering the charge matrices in any given trace structure. (For $SU(N)$ amplitudes, only amplitudes with at least two charge matrices per trace, and thus up to $\lfloor n/2 \rfloor$ traces, will survive; but amplitudes containing a $U(1)$ gauge boson will not vanish in the full string theory, and will be important in the consideration of the decoupling equations.) In the four-point string amplitude, for example, the different types of trace structures are

$$\begin{aligned} &\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}), \quad \mathrm{Tr}(T^{a_1})\,\mathrm{Tr}(T^{a_2}T^{a_3}T^{a_4}), \quad \mathrm{Tr}(T^{a_1}T^{a_2})\,\mathrm{Tr}(T^{a_3}T^{a_4}), \\ &\mathrm{Tr}(T^{a_1})\,\mathrm{Tr}(T^{a_2})\,\mathrm{Tr}(T^{a_3}T^{a_4}), \quad \mathrm{Tr}(T^{a_1})\,\mathrm{Tr}(T^{a_2})\,\mathrm{Tr}(T^{a_3})\,\mathrm{Tr}(T^{a_4}). \end{aligned} \tag{4.9}$$

The on-shell amplitude of equation (4.4) is manifestly gauge-invariant, and as a result the coefficient of each trace term — the one-loop partial amplitudes — are also invariant under on-shell gauge transformations [20]. (We shall return to this point at the end of the section.) The amplitude as a whole is of course also Bose symmetric, that is, invariant under the simultaneous interchange of $\{a_i,k_i,\varepsilon_i\} \leftrightarrow \{a_j,k_j,\varepsilon_j\}$. This can be seen most easily in the integral by observing that the integrand is invariant under the simultaneous interchange of $\{\nu_i,\overline{\nu}_i,a_i,k_i,\varepsilon_i\} \leftrightarrow \{\nu_j,\overline{\nu}_j,a_j,k_j,\varepsilon_j\}$; and of course any permutation of the ν_i and $\overline{\nu}_i$ does not affect the integral, since it is merely a relabeling of integration variables. What are the symmetries of the various pieces? Since the $\delta^m{}_{\widehat{m}}$ are attached to left-mover Green functions with the same external indices, a simultaneous permutation of external color indices a and of integration variables ν simply carries left-mover contributions along with the corresponding trace structure. Consider, for example, the exchange $\{a_x,\nu_x\} \leftrightarrow \{a_y,\nu_y\}$; the part of the exponential of Green functions involving ν_x and ν_y transforms as

$$\begin{aligned} &\exp\left[-\theta_{x1}\theta_{y2}\delta^{m_x}{}_{\widehat{m}_y} G_F\begin{bmatrix}\alpha_{m_x}\\ \beta_{\widehat{m}_y}\end{bmatrix}(\nu_x-\nu_y) - \theta_{x2}\theta_{y1}\delta^{m_y}{}_{\widehat{m}_x}\hat{G}_F\begin{bmatrix}\alpha_{m_y}\\ \beta_{\widehat{m}_x}\end{bmatrix}(\nu_x-\nu_y)\right] \\ \longrightarrow &\exp\left[-\theta_{x1}\theta_{y2}\delta^{m_y}{}_{\widehat{m}_x} G_F\begin{bmatrix}\alpha_{m_y}\\ \beta_{\widehat{m}_x}\end{bmatrix}(\nu_x-\nu_y) - \theta_{x2}\theta_{y1}\delta^{m_x}{}_{\widehat{m}_y}\hat{G}_F\begin{bmatrix}\alpha_{m_x}\\ \beta_{\widehat{m}_y}\end{bmatrix}(\nu_x-\nu_y)\right] \end{aligned} \tag{4.10}$$

since $\hat{G}_F[{\alpha \atop \beta}](-\nu) = -G_F[{\alpha \atop \beta}](\nu)$, with similar behavior (exchange of indices $\{m_x,n_x\} \leftrightarrow \{m_y,n_y\}$) for the parts involving only one of ν_x and ν_y in a given Green function. The rest of L is invariant, and on the charge matrices the transformation amounts simply to exchanging the indices of T^{a_μ} with those of T^{a_ν}.

In particular, if the permutation of the color indices a leaves the trace structure invariant, the corresponding permutation on the ν will leave the left-mover contribution invariant as well. The right-mover pieces R are invariant under a simultaneous permutation of the momenta k_i, the polarization vectors ε_i, and the integration variables $\bar{\nu}_i$; while the common parts E are invariant under simultaneous permutations of the momenta k_i and all integration variables, ν_i and $\bar{\nu}_i$. Thus if we consider the coefficient of a given trace term in the sum of equation (4.8), we find that it is invariant under those permutations of the external momenta and polarization vectors that leave the trace structure invariant when applied to the color indices.

To express this structure a bit more formally, we must introduce a bit of notation. Denote by $\mathrm{Gr}_{n;m_1,m_2,\ldots,m_x}(i_1,i_2,\ldots,i_N)$ a term with $x+1$ traces, starting at the m_1-th element of the i_k, m_2-th element, and so on:

$$\mathrm{Gr}_{n;m_1,m_2,\ldots,m_x}(i_1,i_2,\ldots) = \mathrm{Tr}(T^{a_{i_1}}\cdots T^{a_{i_{m_1-1}}})\,\mathrm{Tr}(T^{a_{i_{m_1}}}\cdots T^{a_{i_{m_2-1}}})\cdots\mathrm{Tr}(T^{a_{i_{m_x}}}\cdots T^{a_{i_n}})\,. \tag{4.11}$$

A trivial trace (when $m_l = m_{l+1}$ or $m_1 = 1$) is replaced by the number of colors N_c. (The reason for the explicit appearance of the number of colors will become more obvious when we consider the infinite-tension limit; for the moment, the reader may treat it as an arbitrary convention.) In the four-point case, we have in the string amplitude the trace structures $\mathrm{Gr}_{4;1}$; $\mathrm{Gr}_{4;2}$; $\mathrm{Gr}_{4;3}$; $\mathrm{Gr}_{4;2,3}$; and $\mathrm{Gr}_{4;2,3,4}$.

We may now write the full string amplitude as

$$\mathcal{A}_n^{\mathrm{string}}(\{a_i,k_i,\varepsilon_i\}) = \sum_{\vec{m}}\sum_{\sigma\in S_n/S_{n;\vec{m}}} \mathrm{Gr}_{n;\vec{m}}(\sigma(1)\cdots\sigma(n))A_{n;\vec{m}}^{\mathrm{string}}(\{k_i,\varepsilon_i\}) \tag{4.12}$$

where $\vec{m}$ ranges over all distinguishable trace structures, for which we use the convention that shorter traces appear first. In this equation, $S_{n;\vec{m}}$ is the symmetry group of the given trace structure, that is those elements of the permutation group S_n which leave the trace structure invariant,

$$S_{n;\vec{m}} = \{\sigma\in S_n \,|\, \mathrm{Gr}_{n;\vec{m}}(\sigma(1)\cdots\sigma(n)) = \mathrm{Gr}_{n;\vec{m}}(1\cdots n)\}\,. \tag{4.13}$$

The sum over σ in equation (4.12) thus instructs us to sum only over inequivalent trace structures. In general, the lengths of the strings of charge matrices inside the traces in equation (4.11) are different, and the symmetry group is simply a product of cyclic transformations on the elements within each trace. In certain cases, however, the lengths of several different traces may be the same, in which case the symmetry group also includes the permutations exchanging the equal-length traces.

From the earlier discussion, we know that the corresponding string partial amplitudes $A^{\rm string}_{n;\vec{m}}$ are invariant under this symmetry group,

$$\forall\sigma \in S_{n;\vec{m}}, \qquad A^{\rm string}_{n;\vec{m}}(\sigma) = A^{\rm string}_{n;\vec{m}}(1\cdots n). \tag{4.14}$$

There is one more symmetry of the $A^{\rm string}_{n;\vec{m}}$ that may be extracted from the integral representation implicit in equation (4.8). We may note that changing the sign of every ν and $\overline{\nu}$ changes the sign of every $G_F\left[{\alpha_\Upsilon \atop \beta_\Upsilon}\right]$ and $\dot{G}_B$, while leaving each $\ddot{G}_B$ and each G_B invariant. Every pair of θs amongst the right movers is thus multiplied by -1, so that the right-mover contribution as a whole is multiplied by $(-1)^n$, but the right-movers and common part are otherwise invariant. The integration measure is unchanged ($(-1)^{2n} = 1$). For the left-movers, this inversion exchanges the $G_F\left[{\alpha_G \atop \beta_G}\right]$ terms in the exponential with the $\hat{G}_F\left[{\alpha_G \atop \beta_G}\right]$ terms, but the interchange of θs absorbs the sign change from the Green function, so for the left-movers, the only effect is on the $\delta^m{}_n$: each $\delta^m{}_n$ tensor has its upper and lower indices interchanged. This has the effect of transposing each charge matrix in the trace structure. Putting these transformations together, we find that

$$\text{Coeff}\left[\text{Tr}(T^{a_1}\cdots T^{a_n})\right] = (-1)^n\text{Coeff}\left[\text{Tr}(T^{a_1T}\cdots T^{a_nT})\right] = (-1)^n\text{Coeff}\left[\text{Tr}(T^{a_n}\cdots T^{a_1})\right] \tag{4.15}$$

and similarly for other trace structures. Introducing a reflection operator,

$$R_{n;m_1,m_2,\ldots,m_s}(i_1,\ldots,i_n) = (i_{m_1-1},\ldots,i_1,i_{m_2-1},\ldots,i_{m_1},\ldots,i_n,\ldots,i_{m_s}) \tag{4.16}$$

we may write the reflection identity as

$$A^{\rm string}_{n;\vec{m}}(R_{n;\vec{m}}(1,\ldots,n)) = (-1)^n A^{\rm string}_{n;\vec{m}}(1,\ldots,n)\,. \tag{4.17}$$

Each of these partial amplitudes is also gauge invariant on shell; replacing a polarization vector with its corresponding momentum gives a vanishing result. Within the string context this is not difficult to prove; starting from the gluon vertex operator (III.15) and for example setting $\varepsilon_1 = k_1$ for the first leg the corresponding vertex operator becomes

$$V(\varepsilon_1 = k_1) = -\sqrt{2}gT^{aj}_i : \Psi^{i\dagger}(\nu_1)\Psi_j(\nu_1)\partial_{\overline{\nu}_1}e^{ik_1\cdot X(\nu_1,\overline{\nu}_1)} : \,. \tag{4.18}$$

If we compute expectation values using this vertex operator instead of the usual one on the first leg we obtain a result whose right-mover and common exponential part is a total derivative in $\overline{\nu}_1$. Since the torus has no boundaries one might then conclude that the amplitude vanishes. This is almost right, but there are two subtleties to be addressed. The left-mover contains poles, and thus is not necessarily analytic everywhere in ν_1. As a result, when we attempt to integrate by parts in order to prove the vanishing of the resulting longitudinal amplitude, the derivative in $\bar{\nu}_1$ can hit a left-mover Green function, and this in principle gives a non-vanishing contribution. The typical form of the ν_1 dependence in the amplitude near a left-mover pole is

$$(\nu_1 - \nu_j)^{-1+\lambda s_{1j}/4\pi} \tag{4.19}$$

which is not necessarily analytic as $\nu_1 \to \nu_j$ when the momentum invariants are in the physical region. However, an analytic continuation to sufficiently positive values of the s_{ij} renders the expression completely analytic in ν_1 so that the $\bar{\nu}_1$ derivative vanishes. There are further subtleties in the regions where the loop is isolated on the first external leg, and in the region where the loop is isolated at the end of a tadpole. The latter contribution can be eliminated in the string theory using Green-Seiberg [32] type contact terms, and in any event may be shown to drop out [22] in the gauge theory (infinite-tension) limit. In the former case, the momentum invariant in which we want to continue is k_1^2, which vanishes identically on shell. Because of 0/0 ambiguities [33], which we have discussed at length in previous work [34,35], contributions from this region of moduli space must anyway defined by an appropriate 'offsheet' prescription during which the momentum invariant does *not* vanish. (The limit of vanishing momentum invariants would be taken at the end.) This means that the appropriate analytic continuation can then be performed. (It is also amusing to note that within the context of a dimensional regularization scheme, all contributions with a loop isolated on an external leg vanish identically because of a complete cancellation of infrared and ultraviolet contributions [36,22].) The gauge invariance property holds for each partial amplitude since the above argument holds term-by-term for the left-movers. Each color trace structure is associated with a different set of left-mover Green functions, and so the coefficients of the different given color traces, that is the different partial amplitudes, are independently gauge-invariant on shell.

5. The Field Theory Limit

In order to extract gauge-theory amplitudes from the string amplitudes discussed in the previous section, we must take the infinite-tension limit of the string theory, $\lambda \to 0$. If we examine the right-mover terms in equation (4.6), we will see that every factor of a θ_{i3} carries along a factor of $\sqrt{\lambda}$, except for the θ_{i3} multiplying the double derviatives of the bosonic Green function ($\ddot{G}_B$). Thus after performing all the θ_{i3} integrals, each term not containing double derivatives will carry an overall factor of λ^{n-2} (after combining with the factors of λ in the over-all normalization). However, it is possible (as shown in appendix II) to integrate by parts to remove all the double derivatives of the bosonic Green functions. After this is done, the right-mover contribution contains only fermionic Green functions and single derivatives of bosonic Green functions; and all terms in the amplitude have a uniform factor of λ^{n-2} in front. (This form is also preferrable in that it makes the world-sheet supersymmetry of the right-movers manifest, in the sense that disappearance of the poles associated with the fictitious F_1-formalism tachyon is manifest.)

The presence of these explicit powers of λ means that only those regions of the integral which yield an appropriate number of powers of λ^{-1} will survive in the gauge-theory limit of vanishing λ. There are two sources of such powers: the large $\operatorname{Im}\tau$ region of the modular parameter integral, and poles in the differences of the locations of the massless-vector vertex operators, $\nu_{ij} = \nu_i - \nu_j$. Even after extracting as many powers of λ^{-1} as possible from poles in the ν_{ij}, it turns out that surviving contributions come only from the large-$\operatorname{Im}\tau$ region. The ν_{ij} pole contributions yield trees of massless vectors sewn onto a loop; and large $\operatorname{Im}\tau$ means that only the massless particles survive to run around the loop.

In extracting surviving contributions to the amplitude, we must therefore 'pinch' together various sets of νs at a set of locations on the world-sheet torus, and then extract the large-$\operatorname{Im}\tau$ contributions to the modular-parameter integral. In this limit, the theta functions that comprise the various world-sheet Green functions have simple expansions in terms of ordinary transcendental functions, which makes it straightforward to compute the integrals explicitly.

In the pinch limit each single pole in both a ν_{ij} and a $\overline{\nu}_{ij}$ will remove one power of λ from the over-all coefficient since we would get an integral of the form

$$\int d^2\nu_i \frac{|\nu_{ij}|^{\lambda k_i \cdot k_j/\pi}}{|\nu_{ij}|^2} = -\frac{2\pi^2}{\lambda k_i \cdot k_j}\,. \tag{5.1}$$

(Higher integer powers of $|\nu_{ij}|$ in the expansion about $\nu_{ij} = 0$ correspond to the propagation of a massive string state; these yield expressions of the form $(\lambda k_i \cdot k_j - \text{integer})^{-1}$ and thus are not relevant in the field theory limit.) The imaginary parts of the remaining ν_i parts are then integrated over the size of the torus, a region from 0 to $\operatorname{Im}\tau$. If we rescale these variables to the interval $[0,1]$, by defining $\hat{\nu}_i = \operatorname{Im}\nu_i/\operatorname{Im}\tau$, we end up with an integral over the imaginary part of the modular parameter (the integral over the real part of the modular parameter merely enforces the level-matching condition on the closed-string states in the infinite string tension limit) of the form

$$\int^{\infty} d\operatorname{Im}\tau\,(\operatorname{Im}\tau)^{n_\nu-3} e^{-\lambda s \operatorname{Im}\tau} = \frac{\Gamma(n_\nu-2)}{(\lambda s)^{n_\nu-2}} \tag{5.2}$$

where n_ν is the number of surviving *unpinched* ν variables. (There are two additional powers of $(\operatorname{Im}\tau)^{-1}$ from the modular measure, and one from the partition function in four dimensions.) This produces an additional $n_\nu - 2$ factors of λ^{-1}. From this we see that to get a surviving field theory contribution each integral over the pinched variables *must* contribute a factor of λ^{-1} so as to completely cancel the overall λ^{n-2}.

There are two contributions that deserve special consideration, when there is either one surviving insertion on the world-sheet ($n_\nu = 1$), or two surviving insertions ($n_\nu = 2$). In both of these cases, the integrals are divergent. The former case corresponds to a tree of gluons coupled to a massless loop via a dilaton. Naively, these contributions will survive in the limit, but a more careful analysis [22] using an appropriate regulator shows that they do in fact drop out. For our purposes here, we will be content to note that this contribution comes from an isolated region in the space of all ν_i, and thus could be thrown away by hand; and we shall discard them in the following discussion. The second set, with two surviving insertions, are more interesting; these graphs are logarithmically divergent, and correspond to the usual UV divergences of a cut-off field theory when the cut-off is taken to infinity. They can be handled either with the physical cut-off provided by the string, or via dimensional regularization [37]; in either case, the would-be divergence is absorbed as usual into the renormalization of the gauge coupling from its infinite-energy value to its value at a physical scale. In the case where the pinch effectively isolates the loop on an external leg (that is, all variables but one are pinched together), the on-shell Polyakov amplitude suffers from the 0/0 ambiguity mentioned in the previous section; with the use of the Minahan prescription, one can treat this contribution on the same footing as the other $n_\nu = 2$ contributions. These issues have no effect on any of the arguments in this paper, and so we shall treat these contributions in the same manner as the remaining contributions with three or more surviving insertions; the latter contain only on-shell infrared divergences (which can also be handled with dimensional regularization, though we shall not do so explicitly).

For the purposes of this section, it will be convenient to use the surviving remnant of conformal invariance on the torus to fix the coordinate ν_n of the last leg to be the modular parameter τ. This does not change any of the counting arguments in the previous paragraph, since an explicit power of $\mathrm{Im}\,\tau$ will appear in the integrand to compensate for the missing integral. Let us begin with the contribution where none of the νs have been pinched together, and determine which color structures survive in the gauge-theory limit. As we shall show later, the contributions with pinches do not give rise to any additional trace structures.

To understand the structure of the surviving contributions, we should therefore consider the expansion of the partition function and the Green functions in powers of $\hat{q}^{1/2} = e^{-\pi\,\mathrm{Im}\,\tau}$. The integer powers of $\hat{q}^{1/2}$ correspond to mass[-squared] levels of the string in units of $4\pi/\lambda$. Terms with negative powers of $\hat{q}^{1/2}$ would correspond to tachyonic divergences (but as we shall see, there aren't any); terms with positive integer powers of $\hat{q}^{1/2}$ would disappear in the infinite-tension limit, because they would fail to produce inverse powers of λ in front of the integral. Only terms with no surviving integer powers of $\hat{q}^{1/2}$ will give rise to massless-particle contributions in the gauge theory;

and such contributions can only arise in certain sectors of the models. In the case of the particular model we are using, there is only one sector with massless states — the Neveu-Schwarz sector or W_0 sector in the notation of KLT [26].

In any term in the expansion of the integrand, the powers of $\hat{q}^{1/2}$ coming from the left-movers are accompanied by powers of $e^{i\pi \operatorname{Re}\tau}$, while the powers of $\hat{q}^{1/2}$ coming from the right movers are accompanied by powers of $e^{-i\pi \operatorname{Re}\tau}$. Thus unless the left-movers supply the same number of powers of $\hat{q}^{1/2}$ in any given term as do the right-movers, there will be a surviving factor of $e^{\pm i\pi \operatorname{Re}\tau \times \text{ integer}}$, and the $\operatorname{Re}\tau$ integral will kill the term. That is, the $\operatorname{Re}\tau$ integral (which in the field theory limit varies between $-1/2$ and $+1/2$) enforces the level-matching condition of the string, and allows us to consider the expansions of the left- and right-movers separately. For the purposes of determining the structure of the color decomposition, the expansions of the right-movers are in fact largely unimportant.

The left-mover contributions to the partition function from the Neveu-Schwarz or W_0 sector have the form

$$\mathcal{Z}_L\begin{bmatrix} W_0 \\ \vec{\beta} \end{bmatrix}(\tau) \to \hat{q}^{-1}e^{-2\pi i \operatorname{Re}\tau}(1+24\hat{q}e^{2\pi i \operatorname{Re}\tau}) \\ \times \left(1 - 2\hat{q}^{1/2}e^{\pi i \operatorname{Re}\tau}\sum_{i=1}^{22}\cos 2\pi\beta_{Li} + 4\hat{q}e^{2\pi i \operatorname{Re}\tau}\sum_{i<j=1}^{22}\cos 2\pi\beta_{Li}\cos 2\pi\beta_{Lj} \right). \tag{5.3}$$

Since the leading right-mover contribution is at order $\hat{q}^{-1/2}$, there is no corresponding $\hat{q}^{-1}$ term in the expansion of the right-movers, and the leading term will be killed by the $\operatorname{Re}\tau$ phase integral. The $\hat{q}^{-1/2}$ term would give rise to a tachyon contribution; but it is killed by the generalized GSO projection in the particular model we are considering. For the overall $\hat{q}^0$ pieces, the ones in which we are really interested, we must also consider the expansion of the left-mover Green functions,

$$G_F\begin{bmatrix} 1/2 \\ \beta \end{bmatrix}(\nu) \to -i\,\operatorname{sign}(\hat{\nu})\Big(e^{\pi i\,\operatorname{sign}(\hat{\nu})\operatorname{Re}\nu}e^{-\pi|\hat{\nu}|\operatorname{Im}\tau} - \hat{q}^{1/2}e^{\pi i \operatorname{Re}\tau}e^{-\pi i\,\operatorname{sign}(\hat{\nu})\operatorname{Re}\nu}e^{\pi|\hat{\nu}|\operatorname{Im}\tau}e^{2\pi i\,\operatorname{sign}(\hat{\nu})\beta} \\ -\hat{q}e^{2\pi i\operatorname{Re}\tau}e^{-\pi i\,\operatorname{sign}(\hat{\nu})\operatorname{Re}\nu}e^{\pi|\hat{\nu}|\operatorname{Im}\tau}e^{4\pi i\,\operatorname{sign}(\hat{\nu})\beta}\Big)$$

$$S_F\begin{bmatrix} 1/2 \\ \beta \end{bmatrix} \to -2\hat{q}^{1/2}e^{\pi i\operatorname{Re}\tau}\sin(2\pi\beta) - 2\hat{q}e^{2\pi i\operatorname{Re}\tau}\sin(4\pi\beta). \tag{5.4}$$

In expanding the product of the Green functions and the partition function, there are in principle three possible ways of obtaining the two powers of $\hat{q}^{1/2}$ needed to cancel off the leading $\hat{q}^{-1}$ from the partition function: (a) a power of $\hat{q}$ from the partition function, combined with the leading ($\hat{q}^0$) power from the Green functions (b) a power of $\hat{q}$ or two powers of $\hat{q}^{1/2}$ from the Green functions or (c) one power of $\hat{q}^{1/2}$ from the Green functions, and one from the expansion of the partition function.

Terms which contain the self-contraction Green functions $S_F[{}^{1/2}_{\beta}]$ must have at least one power of $\hat{q}^{1/2}$ coming from the Green functions, simply because the leading behavior of a self contraction Green function *is* $\hat{q}^{1/2}$. In terms not containing any self-contraction Green functions, the leading term in the expansion of the left-mover Green function $G_F[{}^{1/2}_{\beta}]$ contains a decaying exponential in $|\hat{\nu}_i|\,\mathrm{Im}\,\tau$, which would in turn imply a vanishing contribution in the field theory limit were this term taken alone. On the other hand, the coefficients of the $\hat{q}^{1/2}$ and $\hat{q}$ contain growing exponentials in the $|\hat{\nu}_i|\,\mathrm{Im}\,\tau$ that can cancel the decaying ones present in the leading-order term. Thus the only terms that will survive are those with an appropriate combination of leading-order and higher-order terms from the Green functions. In particular, at least one of the powers of $\hat{q}^{1/2}$ *must* come from the Green functions even if the term does not contain self-contraction Green functions, and so alternative (a) is not viable.

This result for terms without self-contraction Green functions can also be obtained by considering the phases $\exp\left[\pm\pi i\,\mathrm{sign}(\hat{\nu}_{ij})\,\mathrm{Re}\,\nu_{ij}\right]$, where in correspondence with the notation for the ν, we define $\hat{\nu}_{ij} = \hat{\nu}_i - \hat{\nu}_j$. Since each ν_i appears exactly twice in a left-mover term (the vertex operator at position ν_i contains two world sheet left-mover fermions), this means that for each ν_i the phases either cancel to give a factor of unity or else add to give a factor of $e^{\pm 2\pi i\,\mathrm{Re}\,\nu_i}$. Integrating the latter factor over $\mathrm{Re}\,\nu_i$ then would lead to a vanishing result. This implies that the only non-vanishing contributions are those where the phases completely cancel. This can only occur when appropriate combinations are taken of the leading and higher order terms (in $\hat{q}$) from the Green functions again leading to the conclusion that at least one power of $\hat{q}^{1/2}$ must come from the Green functions.

For the left-movers of the model in Appendix I, we may group the world sheet 'time' boundary conditions, which control the generalized GSO projection into triplets $(W + 0\cdot W_1, W + 1\cdot W_1, W + 2\cdot W_1)$. Each of the time-boundary conditions in any given triplet shares the same coefficient $C^{W_0}_{\vec{\beta}}$, since W_1 has a zero in the first right-mover position or "spin-component" (which denotes the world-sheet fermion carrying the space-time index) and the coefficient $C^{W_0}_{\vec{\beta}} = -\cos 2\pi\beta_{\uparrow} = (-1)^{n_0+n_2+n_3+n_4+1}$ where $\vec{\beta} \equiv \sum_{i=0}^{4} n_i W_i$. In our model, the complex exponentials $e^{-2\pi i\beta_G}$ and $e^{-4\pi i\beta_G}$ are simply the cube roots of either 1 or -1, and so will vanish when summed over all time-boundary conditions. Only terms where the factors of $e^{\pm 2\pi i\beta_G}$ completely cancel can survive. This tells us that terms where the $\hat{q}$ comes from the third term in the expansion of a single Green function cannot contribute, because these would not give rise to an appropriate 'interference'.

We are thus finally left with two options: (a) one power of $\hat{q}^{1/2}$ comes from the partition function, and one from a Green function, or (b) each power of $\hat{q}^{1/2}$ comes from a different Green function, with opposite phases for the complex exponentials of β_G. In the first case, the sum in the expansion of the partition function will leave only the sum over those oscillators that correspond to the gauge group of interest; there are N_c of these for an $SU(N_c)$ model. This will yield an explicit factor of N_c when the powers of $\hat{q}^{1/2}$ go according to the first option.

As an example, consider a left-mover term from the three-point function

$$\mathcal{Z}_L\begin{bmatrix} W_0 \\ \vec{\beta} \end{bmatrix}(\tau) G_F\begin{bmatrix} 1/2 \\ \beta \end{bmatrix}(-\nu_{12}) G_F\begin{bmatrix} 1/2 \\ \beta \end{bmatrix}(-\nu_{23}) G_F\begin{bmatrix} 1/2 \\ \beta \end{bmatrix}(\nu_{13}) \,. \tag{5.5}$$

Taking the ordering $\hat{\nu}_1 \leq \hat{\nu}_2 \leq \hat{\nu}_3$ as $\hat{q} \to 0$ ($\mathrm{Im}\,\tau \to \infty$), this expression becomes

$$\begin{aligned} &\hat{q}^{-1} e^{-2\pi i\,\mathrm{Re}\,\tau}(1 + 24\hat{q}e^{2\pi i\,\mathrm{Re}\,\tau}) \left(1 - \hat{q}^{1/2} e^{\pi i\,\mathrm{Re}\,\tau}\Big(N_c(e^{2\pi i\beta_G} + e^{-2\pi i\beta_G}) + \sum_{i=N_c+1}^{\mathrm{len}\,W_{0L}} \cos 2\pi\beta_{Li}\Big) \right) \\ &\times (-i)\left(e^{\pi i\,\mathrm{Re}\,\nu_{21}} e^{-\pi|\hat{\nu}_{21}|\,\mathrm{Im}\,\tau} - \hat{q}^{1/2} e^{\pi i\,\mathrm{Re}\,\tau} e^{-\pi i\,\mathrm{Re}\,\nu_{21}} e^{\pi|\hat{\nu}_{21}|\,\mathrm{Im}\,\tau} e^{2\pi i\beta_G}\right) \\ &\times (-i)\left(e^{\pi i\,\mathrm{Re}\,\nu_{32}} e^{-\pi|\hat{\nu}_{32}|\,\mathrm{Im}\,\tau} - \hat{q}^{1/2} e^{\pi i\,\mathrm{Re}\,\tau} e^{-\pi i\,\mathrm{Re}\,\nu_{32}} e^{\pi|\hat{\nu}_{32}|\,\mathrm{Im}\,\tau} e^{2\pi i\beta_G}\right) \\ &\times (+i)\left(e^{\pi i\,\mathrm{Re}\,\nu_{31}} e^{-\pi|\hat{\nu}_{31}|\,\mathrm{Im}\,\tau} - \hat{q}^{1/2} e^{\pi i\,\mathrm{Re}\,\tau} e^{-\pi i\,\mathrm{Re}\,\nu_{31}} e^{\pi|\hat{\nu}_{31}|\,\mathrm{Im}\,\tau} e^{-2\pi i\beta_G}\right) \end{aligned} \tag{5.6}$$

where we have dropped the $\mathcal{O}(\hat{q})$ terms in both the world-sheet fermionic contributions to the partition function and Green functions following the earlier discussion about the sum over boundary conditions. Expanding this out, we must only keep those terms where all factors of $e^{\pm\pi i\,\mathrm{Re}\,\nu_i}$ and $e^{\pm\pi\hat{\nu}_i}$ cancel. This leaves us with the simple result $-iN_c$.

In general, when we extract terms proportional to $\hat{q}^{1/2}$ from a product of Green functions, we will end up with a factor of the form

$$\exp\Big(|\hat{\nu}_{kl}| - \sum |\hat{\nu}_{ij}|\,\mathrm{Im}\,\tau\Big) \,. \tag{5.7}$$

As mentioned previously, in order to avoid an eventual exponential suppression in $\mathrm{Im}\,\tau$, the sum must add up to exactly cancel the leading term. This will happen only if each $\hat{\nu}_i$ appears once with a positive and once with a negative sign after expressing the absolute values in terms of the $\hat{\nu}_i$s directly. After fixing $\nu_n = \mathrm{Im}\,\tau$, that is $\hat{\nu}_n = 1$, we may divide the integration over the remaining $\hat{\nu}_i$ into different regions, where in each region these variables have a definite ordering, for example $\hat{\nu}_1 \leq \hat{\nu}_2 \leq \cdots \hat{\nu}_{n-1} \leq \hat{\nu}_n$. In this particular case, if we consider the first option, where all Green functions contribute to the exponential in equation (5.7), then because $\hat{\nu}_1$ is smaller than all other $\hat{\nu}_i$, and because $\hat{\nu}_n$ is larger than the others, we can only obtain a cancellation if the leading term is $\hat{\nu}_{n,1}$, and all other $\hat{\nu}$'s appear only in the combinations $\hat{\nu}_{i+1,i}$. That is, in arguments of the Green functions, the ν's must appear as a cyclicly ordered set,

$$G_F\begin{bmatrix} \alpha_G \\ \beta_G \end{bmatrix}(\nu_{1,n}) G_F\begin{bmatrix} \alpha_G \\ \beta_G \end{bmatrix}(-\nu_{12}) G_F\begin{bmatrix} \alpha_G \\ \beta_G \end{bmatrix}(-\nu_{23}) \cdots G_F\begin{bmatrix} \alpha_G \\ \beta_G \end{bmatrix}(-\nu_{n-1,n}) \tag{5.8}$$

or

$$G_F\begin{bmatrix} \alpha_G \\ \beta_G \end{bmatrix}(-\nu_{1,n}) G_F\begin{bmatrix} \alpha_G \\ \beta_G \end{bmatrix}(\nu_{12}) G_F\begin{bmatrix} \alpha_G \\ \beta_G \end{bmatrix}(\nu_{23}) \cdots G_F\begin{bmatrix} \alpha_G \\ \beta_G \end{bmatrix}(\nu_{n-1,n}) \,. \tag{5.9}$$

These two are associated with the trace structures $\mathrm{Tr}(T^{a_1}\cdots T^{a_n})$ and $\mathrm{Tr}(T^{a_n}\cdots T^{a_1})$. For any permutation of this ordering of the ν_i, we obtain contributions to the coefficients of the appropriate permutations of the trace structure.

In general, if we want a single power of $\hat{q}^{1/2}$, only a product of Green functions whose arguments form an ***indivisible cyclic set*** will survive in the infinite-tension limit. Given a set of n variables $\{\nu_1,\ldots,\nu_n\}$, a cyclic set is the set of differences ν_{ij} of the form $\nu_{\sigma(j),\sigma(j+1)}$ for some permutation σ, and where j and $j+1$ are taken mod n. A cyclic set is indivisible if the underlying set of variables cannot be partitioned so that the cyclic set is the union of the cyclic sets generated by the different partitions. For example, given four variables $\nu_1,\ldots,\nu_4$, then $\{\nu_{12},\nu_{23},\nu_{34},\nu_{41}\}$ is an indivisible cyclic set; $\{\nu_{12},\nu_{34}\}$ is a divisible cyclic set; and $\{\nu_{12},\nu_{13},\nu_{14}\}$ is not a cyclic set at all. A m-times divisible set is one that can be partitioned into m indivisible cyclic sets.

In fact, left-mover fermionic Green functions always produce (divisible) cyclic sets, and each indivisible cyclic set is associated with a single trace. In the full string theory, there is no limit (up to the number of Green functions) to the number of indivisible cyclic sets that can appear in any term; but in the infinite-tension limit, as pointed out above, each indivisible cyclic set also carries a power of $\hat{q}^{1/2}$, and we can have no more than two if we want to obtain a non-vanishing contribution. Thus, the only surviving trace structures in the gauge-theory limit are those with one or two non-trivial traces, in agreement with the open string intuition.

The self-contractions are associated with traces of a single matrix; but since the expansion of S_F starts with $q^{1/2}$, these traces behave in exactly the same way as longer traces.

Furthermore, the terms with a single non-trivial trace, as we have seen, pick up one power of $\hat{q}^{1/2}$ from a set of fermionic Green functions, and another from the partition function, the latter being accompanied by an explicit power of the number of colors. This gives us exactly the trace structure $\mathrm{Gr}_{n;1}$. The terms with two non-trivial traces give us the remaining trace structures $\mathrm{Gr}_{n;j}$. The remaining pieces of the integrand know nothing about the number of colors, and so the kinematic factors that multiply these group theory coefficients are 'universal' — independent of the particular gauge group we are considering. Using the symmetry properties of the amplitudes, we can restrict the set of $\mathrm{Gr}_{n;j}$ to those with $j\leq\lfloor n/2\rfloor+1$.

So much for the contributions with no pinches. What happens if we take account of pinches? In principle, pinches of cyclic set of Green could lead to extra traces without extra powers of $\hat{q}^{1/2}$, in turn leading to terms with more than two traces in the field theory limit. However, as we argue below, such terms will not survive.

Let us consider first pinching together some subset of the variables $\{\nu_{p_j}\}_{j=1}^{p}$, leaving the remaining ones $(\{\nu_{u_j}\}_{j=1}^{n-p})$ unpinched. (The argument generalizes in a straightforward way to the case where we pinch distinct sets of variables at different locations on the world-sheet.) It is convenient to make the following change of variables,

$$\begin{aligned}
\rho &= \nu_{p_1} \\
\eta &= \nu_{p_1,p_2} \\
\omega_j &= \nu_{p_j,p_{j+1}}/\eta \qquad j=2,3,\ldots,p-1 \qquad (\omega_1 = 1).
\end{aligned} \tag{5.10}$$

The η coordinate is the size of a (small) disk on the torus at location ρ, which contains the points $\{\nu_{p_j}\}$, while the ω_j are the relative locations of the vertex operators within the disk. Under this change of variables the measure becomes

$$\int \prod_{j=1}^{p} d^2\nu_{p_j} = \int d^2\rho d^2\eta \prod_{j=2}^{p} d^2\omega_j \, |\eta|^{2p-4} \, . \tag{5.11}$$

Recall that we are interested only in terms which produce p powers of λ^{-1}. Since each integration can produce at most one power of λ^{-1}, and there are only $p-1$ of the ω_i, we can restrict our attention to those terms for which the η integration near $\eta \simeq 0$ also produces a pole in λ. Such a pole can arise only from terms of the form $|\eta|^{-2-\lambda s_{ij}/\pi}$. Terms with mismatched powers of η and $\bar{\eta}$ will be killed by the integration over the phase of η, while terms with a higher negative power of $|\eta|$ correspond to the would-be propagation of the fictitious tachyon and will cancel by virtue of world-sheet supersymmetry.

In the $\eta \to 0$ limit, the Green functions behave as follows,

$$\begin{aligned}
G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu_{p_j,u_l}) &\to G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\rho-\nu_{u_l}) + \mathcal{O}(\eta) \\
G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu_{p_1,p_2}) &\to \frac{1}{\eta}G_F^{\rm tree}(1) - 2\hat{q}^{1/2}e^{\pi i\,{\rm Re}\,\tau}\sin(2\pi\beta_G) + \mathcal{O}(\eta) \\
G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu_{p_j,p_l}) &\to \frac{1}{\eta}G_F^{\rm tree}(\Sigma_{k=j}^{l-1}\omega_k) - 2\hat{q}^{1/2}e^{\pi i\,{\rm Re}\,\tau}\sin(2\pi\beta_G) + \mathcal{O}(\eta)
\end{aligned} \tag{5.12}$$

with analogous formulæ for the right-movers. These $\mathcal{O}(\eta^0)$ terms do not vanish in general because the β_G take on values which are a multiple of $1/3$.

It is now a straightforward matter of counting powers of η and independent ωs in order to determine which terms might survive in the gauge theory limit, and which will not. Let us begin with an example where we pinch a subset of ν_{ij} that does not itself contain a cyclic set. The left-mover terms associated with the color trace ${\rm Tr}(T^{a_1}T^{a_2}T^{a_3})\,{\rm Tr}(T^{a_4}T^{a_5})\,{\rm Tr}(T^{a_6}T^{a_7})$ is

$$\Big[G_F(-\nu_{12})G_F(-\nu_{23})G_F(\nu_{13})\Big]\Big[G_F(-\nu_{45})G_F(\nu_{45})\Big]\Big[G_F(-\nu_{67})G_F(\nu_{67})\Big] \tag{5.13}$$

where we have simplified the notation by dropping the boundary conditions associated with the fermionic Green functions. The pinch $\nu_1 - \nu_2 \to 0$ results in a contribution of the form

$$-\frac{1}{2\pi\eta}\Big[G_F(-\nu_{23})G_F(\nu_{23})\Big]\Big[G_F(-\nu_{45})G_F(\nu_{45})\Big]\Big[G_F(-\nu_{67})G_F(\nu_{67})\Big] \tag{5.14}$$

where $\eta \equiv \nu_{12}$. The fact that we have a single pole in η means that we obtain (assuming the right-movers also have a single pole in $\bar{\eta}$) the required λ^{-1} from performing the η integral. This pinch, of course, has not increased the number of potential trace structures in the field theory limit since even after the pinch, each cycle still contributes at least a power of $\hat{q}^{1/2}$.

In general, at most $p-1$ Green functions can involve two ν_{p_j} — otherwise the arguments of the Green functions *would* contain a cyclic set — so we will end up with η^{-1} or higher powers, depending on the precise arguments of the Green functions in the term. If exactly $p-1$ Green functions involve two ν_{p_j}, then assuming the right-movers supply the necessary poles a surviving contribution will emerge from this term. Such a term will have the [left-mover] structure

$$G_F\left(\nu_{\mathrm{u}_1,\mathrm{p}_1}\right)\prod_{j=2}^{p} G_F\left(\nu_{\mathrm{p}_{j-1},\mathrm{p}_j}\right)\; G_F\left(\nu_{\mathrm{p}_j,\mathrm{u}_2}\right) \tag{5.15}$$

of which the leading pinch part will be

$$\frac{1}{\eta^{p-2+1}}\prod_{j=2}^{p}\frac{1}{2\pi\omega_j}\, G_F\left(\nu_{\mathrm{u}_1,\mathrm{p}_1}\right) G_F\left(\nu_{\mathrm{p}_j,\mathrm{u}_1}\right)\ . \tag{5.16}$$

The overall power of η, after combining with the powers from the measure, will be η^{-1}, which will lead to a factor of λ^{-1} (assuming the right-movers also supply a factor of $1/\bar{\eta}$); and each Green function will give rise a pole in an independent combination of ωs, so that the ω integrations will each give one factor of λ^{-1} (again assuming the right-movers supply the appropriate single poles in the $\bar{\omega}$s), for an over-all factor of $\lambda^{-(p-1)}$, as needed. In this case, however, the charge matrices associated with the pinched variables will be in the same trace as the charge matrices associated with ρ and those unpinched νs which complete a cyclic set; and in general an independent trace is not pinched off.

If we do attempt to pinch off a set of ν's which by themselves yield a cyclic set for a subset of arguments ν_{ij} of the Green functions, then we will find that either the η integration, or at least one of the ω integrations, will fail to give rise to a required pole, or that we lose a power of $\mathrm{Im}\,\tau$ at the end of the expansions. In each of these three possibilities, we lose a power of λ^{-1}, and the term will die in the gauge theory limit.

As an example, consider the left-mover term in the six-point amplitude associated with the trace structure $\mathrm{Tr}(T^{a_1}T^{a_2})\,\mathrm{Tr}(T^{a_3}T^{a_4})\,\mathrm{Tr}(T^{a_5}T^{a_6})$,

$$\Big[G_F(-\nu_{12})G_F(\nu_{12})\Big]\Big[G_F(-\nu_{34})G_F(\nu_{34})\Big]\Big[G_F(-\nu_{56})G_F(\nu_{56})\Big]\ . \tag{5.17}$$

If we don't pinch some of the ν_i, this term will not survive in the field theory limit, because it contains three traces. Let us now show that even if we pinch a a complete cyclic set of ν_{ij}, this term will not survive in the field-theory limit. Taking the pinch $\nu_1 - \nu_2 \to 0$ and keeping terms through $\mathcal{O}(\eta^{-1})$ yields

$$\begin{aligned}&\Big(-\frac{1}{2\pi\eta} - 2\hat{q}^{1/2}e^{\pi i\,\mathrm{Re}\,\tau}\sin(2\pi\beta_G)\Big)\Big(\frac{1}{2\pi\eta} - 2\hat{q}^{1/2}e^{\pi i\,\mathrm{Re}\,\tau}\sin(2\pi\beta_G)\Big)G_F(-\nu_{34})G_F(\nu_{34})G_F(-\nu_{56})G_F(\nu_{56})\\ &= \Big(-\frac{1}{4\pi^2\eta^2}\Big)G_F(-\nu_{34})G_F(\nu_{34})G_F(-\nu_{56})G_F(\nu_{56})\end{aligned} \tag{5.18}$$

where $\eta \equiv \nu_1$. Note that the single pole in η disappears (the two contributions cancel). As discussed above, unless there is a single pole in η we cannot obtain sufficient powers of λ^{-1}, and this term will disappear in the field theory limit. Pinching additional variables beyond the complete cycle does not change matters: consider the same term (5.17) but with ν_1, ν_2 and ν_3 pinched together. Including the power of η that comes from the measure (5.11) we obtain in this pinch limit through $\mathcal{O}(\eta)$

$$\begin{aligned}&-\eta\frac{1}{2\pi\eta}\,\frac{1}{2\pi\eta}G_F(\nu_4-\rho)G_F(\rho-\nu_4)G_F(-\nu_{56})G_F(\nu_{56})\\ &= -\frac{1}{4\pi^2\eta}G_F(\nu_4-\rho)G_F(\rho-\nu_4)G_F(-\nu_{56})G_F(\nu_{56})\end{aligned} \tag{5.19}$$

where $\rho \equiv \nu_1$. Because of the extra η from the measure, we are left with one inverse power of η, so the integral over η might lead to a power of λ^{-1}. However, the left-mover contribution is independent of $\omega_2 \equiv \nu_{23}/\eta$. In particular, the left-movers term cannot have a single pole in ω_2 so that it is not possible to extract a λ^{-1} from the ω_2 integral. This would leave us at least one factor of λ^{-1} short of completely canceling the overall power of λ^4 since every integral must contribute one power of λ^{-1}.

More generally, if the arguments of the Green functions involving the pinched variables form exactly a cyclic set (with no open 'edges'),

$$G_F\left(\nu_{\mathrm{p}_p,\mathrm{p}_1}\right)\prod_{j=2}^{p}G_F\left(\nu_{\mathrm{p}_{j-1},\mathrm{p}_j}\right) \tag{5.20}$$

then the powers of η will be wrong: we will end up with a leading term of η^{-2}, which will eventually be killed by world-sheet supersymmetry as it corresponds to the fictitious F_1-formalism tachyon. The coefficient of the subleading pole η^{-1} in the expansion of the left-movers will vanish since it is proportional to $\sum_{\mathrm{cycle}}\nu_{ij} = 0$. It is also possible to obtain a subleading pole by expanding the right-mover contributions, because the bosonic Green function and its derivative contain non-analytic contributions; they depend on the ν's in addition to the $\overline{\nu}$'s. However, the non-analytic piece of $\dot{G}_B$ has the form

$$\frac{\eta\omega_i}{\mathrm{Im}\,\tau} \tag{5.21}$$

so that even if we obtain a single pole in η and $\bar{\eta}$ by expanding to include such factors, we will lose over-all powers of $\mathrm{Im}\,\tau$, because of the $\mathrm{Im}\,\tau$ in the denominator. The loss of a power of $\mathrm{Im}\,\tau$ will translate into a loss of a power of λ^{-1}, and such a contribution will again die in the field-theory limit.

On the other hand, if there is a proper subset of the pinched variables that is a cyclic set, the term will be

$$
\begin{aligned}
&G_F\left(\nu_{\mathrm{u}_1,\mathrm{p}_{l+1}}\right) G_F\left(\nu_{\mathrm{p}_1,\mathrm{p}_l}\right) \prod_{j=2}^{l} G_F\left(\nu_{\mathrm{p}_{j-1},\mathrm{p}_j}\right) \prod_{j=l+2}^{p} G_F\left(\nu_{\mathrm{p}_{j-1},\mathrm{p}_j}\right) G_F\left(\nu_{\mathrm{p}_p,\mathrm{u}_2}\right) \\
&\quad \to \frac{1}{\eta^{p-1}} \frac{1}{2\pi\left(1+\sum_{j=2}^{l-1}\omega_j\right)} \prod_{j=2}^{l-1} \frac{1}{2\pi\omega_j} \prod_{j=l+1}^{p-1} \frac{1}{2\pi\omega_j} G_F\left(\nu_{\mathrm{u}_1}-\rho\right) G_F\left(\rho-\nu_{\mathrm{u}_2}\right) .
\end{aligned}
\tag{5.22}
$$

Because the arguments of the Green functions contain a cyclic set, they will not be linearly independent. Since the total number of Green functions is fixed, this in turn means that the integrand is independent of some ωs (or linear combinations of them) — in the expansion above, independent of ω_l. The integration over such ωs will fail to produce a power of λ^{-1}; but since any given integration can produce at most a single power of λ^{-1}, we will fail to get a sufficient number of poles. [Note that even an integral — for example $\int d^2\omega |\omega|^{-2+\lambda s}|1-\omega|^{-2+\lambda s'}$ — which produces powers of λ^{-1} from different parts of the region of integration, will give only a single overall power of λ^{-1}, because the different contributions just add.]

In summary, only single-trace or double-trace structures survive in the infinite-tension limit; and single trace trace terms are always accompanied by an explicit power of the number of colors, so that the gauge-theory one-loop amplitude can be written in the form

$$
\mathcal{A}_n = \sum_{j=1}^{\lfloor n/2\rfloor+1} \sum_{\sigma\in S_n/S_{n;j}} \mathrm{Gr}_{n;j}(\sigma) A_{n;j}(\sigma) . \tag{5.23}
$$

6. Decoupling Equations

Amplitudes with any number of external $U(1)$ gauge bosons will vanish in a gauge theory with no matter content. We can use this simple observation to derive a variety of decoupling equations for loop amplitudes, similar to equation (2.6) for tree-level amplitudes. Unlike the tree-level case, however, there is a larger variety of equations at loop level. This comes about because there are many different trace structures which contribute, and in addition, substituting additional photons for gauge bosons beyond the first can lead to new equations.

In this section, we give the explicit form of the decoupling equations for the four-, five-, and six-point amplitudes; we present the general form in the next section.

The arguments of the previous section show that the gauge-theory four-point amplitude can be written as follows,

$$
\begin{aligned}
\mathcal{A}_4 &= \sum_{j=1}^{3} \sum_{\sigma\in S_4/S_{4;j}} \mathrm{Gr}_{4;j}(\sigma(1)\cdots\sigma(4)) A_{4;j}(\sigma(1),\cdots,\sigma(4)) \\
&= \sum_{\sigma\in S_4/Z_4} N_c\, \mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}) A_{4;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4)) \\
&\quad + \sum_{\sigma\in S_4/Z_3} \mathrm{Tr}(T^{a_{\sigma(1)}})\,\mathrm{Tr}(T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}) A_{4;2}(\sigma(1),\sigma(2),\sigma(3),\sigma(4)) \\
&\quad + \sum_{\sigma\in S_4/Z_2^3} \mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}})\,\mathrm{Tr}(T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}) A_{4;3}(\sigma(1),\sigma(2),\sigma(3),\sigma(4))\ .
\end{aligned}
\tag{6.1}
$$

In the second term of this equation, Z_3 refers to the cyclic permutations of the matrices inside the second trace, while the Z_2^3 refers to the exchange of two matrices inside each of the traces as well as the exchange of the two traces.

If we take the fourth leg to be a photon, T^4 becomes a matrix proportional to the identity matrix, and we obtain

$$
\begin{aligned}
&\sum_{\sigma\in S_4/Z_4} \mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}) A_{4;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4)) \\
&\qquad + \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}) A_{4;2}(4,1,2,3) + \mathrm{Tr}(T^{a_1}T^{a_3}T^{a_2}) A_{4;2}(4,1,3,2) = 0\ .
\end{aligned}
\tag{6.2}
$$

Since

$$
\begin{aligned}
&\sum_{\sigma\in S_4/Z_4} \mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}) A_{4;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4)) \\
&\quad = \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}) \sum_{\sigma\in Z_3} A_{4;1}(\sigma(1),\sigma(2),\sigma(3),4) + \mathrm{Tr}(T^{a_1}T^{a_3}T^{a_2}) \sum_{\sigma\in Z_3} A_{4;1}(\sigma(1),\sigma(3),\sigma(2),4)
\end{aligned}
\tag{6.3}
$$

and since the coefficients of the two independent trace structures in equation (6.2) must vanish independently we have the decoupling equation

$$
\sum_{\sigma\in Z_3} A_{4;1}(\sigma(1),\sigma(2),\sigma(3),4) + A_{4;2}(4,1,2,3) = 0\ . \tag{6.4}
$$

If we substitute photons for both the third and fourth legs we find an independent equation,

$$
\begin{aligned}
&\sum_{\sigma\in S_4/Z_4} A_{4;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4)) \\
&\qquad + \Big(A_{4;2}(4,1,2,3) + A_{4;2}(4,1,3,2) + A_{4;2}(3,1,2,4) + A_{4;2}(3,1,4,2)\Big) \\
&\qquad + A_{4;3}(1,2,3,4) = 0\ .
\end{aligned}
\tag{6.5}
$$

The one-photon equation (6.2) allows us to eliminate the partial amplitude $A_{4;2}$ when choosing to sum over all the colors in $U(N)$ rather than merely those in $SU(N)$, something that is convenient to do, as we shall see in later sections. (Were we to sum only over $SU(N)$, $A_{4;2}$ would not appear in a color-summed object since its trace coefficient vanishes when all external legs lie within the $SU(N)$ algebra.) Substituting this equation into the two-photon equation yields a constraint on the other two partial amplitudes,

$$A_{4;3}(1,2,3,4) = \sum_{\sigma\in S_4/Z_4} A_{4;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4)) \,. \tag{6.6}$$

A computation of $A_{4;1}$ is thus sufficient to determine the entire one-loop four-point amplitude. An explicit calculation will be presented elsewhere.

Equation (6.6) also reveals that the partial amplitude $A_{4;3}$ also has more symmetry than expected from the arguments of section 4. Its right-hand side is invariant under exchange of any two legs; and thus the left-hand side must be as well.

The equations obtained by substituting three or four photons for gluons in the four-point amplitude do not yield any new information; the three-photon-one-gluon amplitude vanishes identically, while the four-photon case gives a linear combination of the previous equations.

Once again using the representation (5.23), we can write the five-point amplitude in the following form,

$$\begin{aligned}\mathcal{A}_5 &= \sum_{j=1}^{3}\sum_{\sigma\in S_5/S_{5;j}} \mathrm{Gr}_{5;j}(\sigma(1)\cdots\sigma(5))A_{5;j}(\sigma(1),\cdots,\sigma(5))\\ &= \sum_{\sigma\in S_5/Z_5} N_c\,\mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}T^{a_{\sigma(5)}})A_{5;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5))\\ &\quad+\sum_{\sigma\in S_5/Z_4} \mathrm{Tr}(T^{a_{\sigma(1)}})\,\mathrm{Tr}(T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}T^{a_{\sigma(5)}})A_{5;2}(\sigma(1)\sigma(2),\sigma(3),\sigma(4),\sigma(5))\\ &\quad+\sum_{\sigma\in S_5/Z_2\times Z_3} \mathrm{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}})\,\mathrm{Tr}(T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}T^{a_{\sigma(5)}})A_{5;3}(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5))\,.\end{aligned} \tag{6.7}$$

Taking the fifth leg to be a photon, and setting the coefficient of $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$ to zero, we obtain the first decoupling equation,

$$\sum_{\sigma\in Z_4} A_{5;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4),5) + A_{5,2}(5;1,2,3,4) = 0\,. \tag{6.8}$$

A new feature of the five-point amplitude is the emergence of additional constraints from other trace structures, still considering the one-photon substitution; the coefficient of $\mathrm{Tr}(T^{a_1}T^{a_2})\,\mathrm{Tr}(T^{a_3}T^{a_4})$ must vanish, which means that

$$A_{5;3}(1,2,3,4,5) + A_{5;3}(1,2,4,3,5) + A_{5;3}(3,4,1,2,5) + A_{5;3}(3,4,2,1,5) = 0\,. \tag{6.9}$$

If we substitute two photons for gluons, only one equation emerges,

$$\sum_{\sigma\in COP_4^{\{123\}}} A_{5;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4),5) + \sum_{\sigma\in Z_3}\Big(A_{5;2}(5,\sigma(1),\sigma(2),\sigma(3),4) + A_{5;2}(4,\sigma(1),\sigma(2),\sigma(3),5)\Big) + A_{5;3}(4,5,1,2,3) = 0 \tag{6.10}$$

where $COP_n^{\{\{a_j\}\}}$ ('cyclicly ordered permutations') denotes the subset of S_n that leaves the ordering of the a_j unchanged up to a cyclic transformation. We can now solve these equations to eliminate the partial amplitudes $A_{5;2}$ and $A_{5;3}$. Using the one-photon single trace equation (6.8) to substitute for $A_{5;2}$ in the two-photon equation (6.10) we obtain

$$A_{5;3}(4,5,1,2,3) = \sum_{\sigma\in COP_4^{\{123\}}} A_{5;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4),5)\,. \tag{6.11}$$

From this equation along with the one-photon double trace equation (6.9) we obtain also the constraint

$$\sum_{\sigma\in S_5/Z_5} A_{5;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5)) = 0 \tag{6.12}$$

which is trivially satisfied because of the reflection identity (4.17). Just as in the four-point case, additional equations obtained by substituting three or more photon legs are not independent.

The infinite tension limit of the six-point amplitude has the form

$$\begin{aligned}\mathcal{A}_6 = &\sum_{\sigma\in S_6/Z_6} N_c \operatorname{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}T^{a_{\sigma(5)}}T^{a_{\sigma(6)}}) A_{6;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5)\sigma(6))\\
&+\sum_{\sigma\in S_6/Z_5} \operatorname{Tr}(T^{a_{\sigma(1)}})\operatorname{Tr}(T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}T^{a_{\sigma(5)}}T^{a_{\sigma(6)}}) A_{6;2}(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5),\sigma(6))\\
&+\sum_{\sigma\in S_6/Z_2\times Z_4} \operatorname{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}})\operatorname{Tr}(T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}T^{a_{\sigma(5)}}T^{a_{\sigma(6)}}) A_{6;3}(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5),\sigma(6))\\
&+\sum_{\sigma\in S_6/Z_3^2\times Z_2} \operatorname{Tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}})\operatorname{Tr}(T^{a_{\sigma(4)}}T^{a_{\sigma(5)}}T^{a_{\sigma(6)}}) A_{6;4}(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5),\sigma(6))\,.\end{aligned} \tag{6.13}$$

Following the same approach as in the four- and five-point amplitudes leads to the one-photon equations,

$$\sum_{\sigma\in Z_5} A_{6;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5),6) + A_{6;2}(6,1,2,3,4,5) = 0 \tag{6.14}$$

from the coefficient of $\operatorname{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5})$ and

$$\Big(A_{6;3}(1,2,3,4,5,6) + A_{6;3}(1,2,3,4,6,5) + A_{6;3}(1,2,3,6,4,5)\Big) + \Big(A_{6;4}(1,2,6,3,4,5) + A_{6;4}(2,1,6,3,4,5)\Big) = 0 \tag{6.15}$$

from the coefficient of $\mathrm{Tr}(T^{a_1}T^{a_2})\,\mathrm{Tr}(T^{a_3}T^{a_4}T^{a_5})$.

The two-photon equation arising from the $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$ term is

$$\begin{aligned}
0 = &\sum_{\sigma\in COP_5^{\{1234\}}} A_{6;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5),6) \\
&+ \sum_{\sigma\in Z_4} \Big(A_{6;2}(6,\sigma(1),\sigma(2),\sigma(3),\sigma(4),5) \\
&\qquad\qquad + A_{6;2}(5,\sigma(1),\sigma(2),\sigma(3),\sigma(4),6) \Big) \\
&+ A_{6;3}(5,6,1,2,3,4)
\end{aligned} \tag{6.16}$$

while from the coefficent of $\mathrm{Tr}(T^{a_1}T^{a_2})\,\mathrm{Tr}(T^{a_3}T^{a_4})$ we obtain

$$\begin{aligned}
&\sum_{\sigma\in S_4/Z_4} \Big(A_{6;3}(1,2,\sigma(3),\sigma(4),\sigma(5),\sigma(6)) + A_{6;3}(3,4,\sigma(1),\sigma(2),\sigma(5),\sigma(6)) \Big) \\
&\quad + A_{6;4}(1,2,5,3,4,6) + A_{6;4}(2,1,5,3,4,6) + A_{6;4}(1,2,5,4,3,6) + A_{6;4}(2,1,5,4,3,6) \\
&\quad + A_{6;4}(1,2,6,3,4,5) + A_{6;4}(2,1,6,3,4,5) + A_{6;4}(1,2,6,4,3,5) + A_{6;4}(2,1,6,4,3,5) = 0\,.
\end{aligned} \tag{6.17}$$

The latter equation contains no new information, however, as it is simply a sum of equations of the form (6.15).

Finally, by taking the fourth, fifth, and sixth legs to be photons we obtain from the coefficient of $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3})$ the decoupling equation

$$\begin{aligned}
&\sum_{\sigma\in COP_5^{\{123\}}} A_{6;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5),6) \\
&\quad + \sum_{\sigma\in COP_4^{\{123\}}} \Big(A_{6;2}(6,\sigma(1),\sigma(2),\sigma(3),\sigma(4),5) + A_{6;2}(5,\sigma(1),\sigma(2),\sigma(3),\sigma(4),6) \\
&\qquad\qquad + A_{6;2}(4,\sigma(1),\sigma(2),\sigma(3),\sigma(5),\sigma(6)) \Big) \\
&\quad + \sum_{\sigma\in Z_3} \Big(A_{6;3}(5,6,\sigma(1),\sigma(2),\sigma(3),4) + A_{6;3}(4,5,\sigma(1),\sigma(2),\sigma(3),6) \\
&\qquad\qquad + A_{6;3}(4,6,\sigma(1),\sigma(2),\sigma(3),5) \Big) \\
&\quad + \Big(A_{6;4}(1,2,3,4,5,6) + A_{6;4}(1,2,3,4,6,5) \Big) = 0\,.
\end{aligned} \tag{6.18}$$

No new equations emerge from terms with two traces after the substitution of three photon legs, and no new equations emerge from substituting more than three photons for gluons.

We can again solve for the second partial amplitude, $A_{6;2}$; substituting into the two-photon single-trace equation (6.14) yields

$$A_{6;3}(5,6,1,2,3,4) = \sum_{\sigma\in COP_5^{\{1234\}}} A_{6;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5),6) \tag{6.19}$$

and substituting this into the three-photon equation (6.18) yields

$$A_{6;4}(1,2,3,4,5,6) + A_{6;4}(1,2,3,4,6,5) = -\sum_{\sigma\in COP_5^{\{123\}}} A_{6;1}(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5),6)\,. \quad (6.20)$$

Note that in all the decoupling equations, $A_{6;4}$ appears only as a sum of two terms, so that none of the decoupling equations will separate the two. In the six-point case, we see that the one-photon double-trace equation (6.15) is automatically satisfied for solutions of equations (6.19) and (6.20). This is in contrast to the situation in the five-point case, where the one-photon double-trace equation does lead to an additional constraint.

7. General Decoupling Equations

In considering the n-point gluon amplitude, we can derive decoupling equations by substituting one or more photons for gluons (that is, replacing an $SU(N)$ generator by the identity matrix), and then looking at the coefficients of the various independent trace terms.

Let us look first at the equations obtained from substituting a single photon for a gluon. We take this gluon to be the last one; the coefficient of $\mathrm{Tr}(T^{a_1}T^{a_2}\cdots T^{a_{n-1}})$ yields the equation

$$\sum_{\sigma\in Z_{n-1}\{1,\ldots,n-1\}} A_{n;1}(\sigma(1),\ldots,\sigma(n-1),n) + A_{n;2}(n,1,\ldots,n-1) = 0\,. \quad (7.1)$$

where $Z_n\{s_1,\ldots,s_n\}$ refers to the set of cyclic permutations of n objects acting on the positions occupied by the symbols $s_1,\ldots,s_n$. This equation allows us to eliminate $A_{n;2}$ in all other equations. (This partial amplitude does not appear in the $SU(N)$ amplitude, because it is the coefficient of a group-theory structure which contains the trace of a single generator. It may however appear in formulæ for cross-sections calculated by summing over $U(N)$ instead of $SU(N)$; this equation would then be required to show that the two forms are equivalent. It is not necessarily desirable to eliminate $A_{n;2}$, since formulæ which contain it may well be more compact than those which eliminate it in favor of a sum of $A_{n;1}$'s.) Extracting the coefficient of $\mathrm{Tr}(T^{a_1}\cdots T^{a_{j-1}})\,\mathrm{Tr}(T^{a_j}\cdots T^{a_{n-1}})$, we find the decoupling equation ($3\le j\le \lfloor n/2\rfloor$)

$$\begin{aligned}&\sum_{\sigma\in Z_{n-j}\{j,\ldots,n-1\}} A_{n;j}(1,\ldots,j-1,\sigma(j),\ldots,\sigma(n-1),n)\\ &\quad+\sum_{\sigma\in Z_{j-1}\{1,\ldots,j-1\}} A_{n;j+1}(\sigma(1),\ldots,\sigma(j-1),n,j,\ldots,n-1) = 0\,.\end{aligned} \quad (7.2)$$

If n is odd, we obtain one more equation, involving only $A_{n;\lfloor n/2\rfloor+1}$:

$$\begin{aligned}&\sum_{\sigma\in Z_{(n-1)/2}\{\frac{n+1}{2},\ldots,n-1\}} A_{n;(n+1)/2}\left(1,\ldots,\frac{n-1}{2},\sigma\left(\frac{n+1}{2}\right),\ldots,\sigma(n-1),n\right)\\ &\quad+\sum_{\sigma\in Z_{(n-1)/2}\{1,\ldots,\frac{n-1}{2}\}} A_{n;(n+1)/2}\left(\frac{n+1}{2},\ldots,n-1,\sigma(1),\ldots,\sigma\left(\frac{n-1}{2}\right),n\right) = 0\,.\end{aligned} \quad (7.3)$$

In presenting the remaining equations, we will find it convenient to introduce the following notation for:

1. A set of *ordered permutations*, $OP_n^{(x_1\ldots x_m)}\{s_1,\ldots,s_n\}$, that is the set of permutations of n objects, acting on the symbols $s_1,\ldots,s_n$, while keeping the order of $x_1,\ldots,x_m$ fixed. (The x_j are a subset of the s_j.) This set has $n!/m!$ elements. For example*,

$$OP_4^{(1\,2)}\{1,2,3,4\} = \{(1\,2\,3\,4),(1\,3\,2\,4),(1\,3\,4\,2),(3\,1\,2\,4),(3\,1\,4\,2),(3\,4\,1\,2),\\ (1\,2\,4\,3),(1\,4\,2\,3),(1\,4\,3\,2),(4\,1\,2\,3),(4\,1\,3\,2),(4\,3\,1\,2)\}\,. \tag{7.4}$$

2. A set of *cyclicly ordered permutations*, $COP_n^{(x_1\ldots x_m)}\{s_1,\ldots,s_n\}$, that is the set of permutations of n objects, acting on the symbols $s_1,\ldots,s_n$, while keeping the order of $x_1,\ldots,x_m$ fixed up to a cyclic permutation. (The x_j are again a subset of the s_j. Note that this definition is a generalization of the usage in the previous section, where $s_1,\ldots,s_n$ were implicitly $1,\ldots,n$.) This set has $n!/(m-1)!$ elements. For example,

$$COP_4^{(1\,2\,3)}\{1,2,3,4\} = \{(1\,2\,3\,4),(1\,2\,4\,3),(1\,4\,2\,3),(4\,1\,2\,3)\\ (2\,3\,1\,4),(2\,3\,4\,1),(2\,4\,3\,1),(4\,2\,3\,1)\\ (3\,1\,2\,4),(3\,1\,4\,2),(3\,4\,1\,2),(4\,3\,1\,2)\}\,. \tag{7.5}$$

3. The set of distinct partitions of the n symbols $s_1,\ldots,s_n$ into subsets of length $m-1$ and $n-m+1$, $P\binom{n}{m}\{s_1,\ldots,s_n\}$. The elements of P are expressed as permutations, with the elements mapped into the first partition listed first, in increasing order, followed by the elements mapped into the second partition, again in increasing order. This set has $n!/\left((m-1)!(n-m+1)!\right)$ elements, unless n is even and $m=n/2+1$, when it has half that many elements (because we must then divide out by the Z_2 symmetry exchanging the two partitions). For example,

$$P\binom{6}{4} = \{(1\,2\,3\,4\,5\,6),(1\,2\,4\,3\,5\,6),(1\,3\,4\,2\,5\,6),(1\,2\,5\,3\,4\,6),(1\,3\,5\,2\,4\,6),\\ (1\,4\,5\,2\,3\,6),(1\,2\,6\,3\,4\,5),(1\,3\,6\,2\,4\,5),(1\,4\,6\,2\,3\,5),(1\,5\,6\,2\,3\,4)\}\,. \tag{7.6}$$

We can re-express any element of a *COP* in terms of an element of the corresponding *OP*, followed by a cyclic permutation,

$$COP_n^{(x_1\ldots x_m)}\{s_1,\ldots,s_n\} = Z_m\{x_1,\ldots,x_m\}\circ OP_n^{(x_1\ldots x_m)}\{s_1,\ldots,s_n\}\,. \tag{7.7}$$

In particular, note that

$$COP_n^{(s_1\ldots s_n)}\{s_1,\ldots,s_n\} = Z_n\{s_1,\ldots,s_n\}\,. \tag{7.8}$$

* We use a standard notation for permutations, the image of its action on $1,\ldots,n$; $(3\,1\,2\,5\,4)[\{a\,b\,c\,d\,e\}] = \{c\,a\,b\,e\,d\}$.

If we substitute two photons for two gluons, say the ultimate and penultimate ones, and examine the coefficients of the various trace terms, we obtain the following equations,

$$
\begin{aligned}
&\sum_{\sigma\in COP^{(1\ldots n-2)}_{n-1}\{1,\ldots,n-1\}} A_{n;1}(\sigma(1),\ldots,\sigma(n-1),n)\\
&+\sum_{\sigma\in Z_{n-2}\{1,\ldots,n-2\}} A_{n;2}\,(n-1;\sigma(1),\ldots,\sigma(n-2),n)+A_{n;2}\,(n;\sigma(1),\ldots,\sigma(n-2),n-1) \qquad (7.9a)\\
&+\quad A_{n;3}(n-1,n;1,\ldots,n-2)=0
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{\sigma\in COP^{(j\ldots n-2)}_{n-j}\{j,\ldots,n-1\}} A_{n;j}(1,\ldots,j-1;\sigma(j),\ldots,\sigma(n-1),n)\\
&+\sum_{\sigma\in Z_{j-1}\{1,\ldots,j-1\}\times Z_{n-j-1}\{j,\ldots,n-2\}} [A_{n;j+1}(\sigma(1),\ldots,\sigma(j-1),n-1;\sigma(j),\ldots,\sigma(n-2),n)\\
&\qquad +A_{n;j+1}(\sigma(1),\ldots,\sigma(j-1),n;\sigma(j),\ldots,\sigma(n-2),n-1)] \qquad (7.9b)\\
&+\sum_{\sigma\in COP^{(1\ldots j-1)}_{j}\{1,\ldots,j-1,n-1\}} A_{n;j+2}(\sigma(1),\ldots,\sigma(j-1),\sigma(n-1),n;j,\ldots,n-2)=0, \qquad (3\le j\le \lfloor n/2\rfloor-1)
\end{aligned}
$$

where here we have indicated the separation into the first $j-1$ and the remaining $n-j+1$ arguments of the $A_{n;j}$ with a semicolon. The first of these two equations allows us to solve for $A_{n;3}$. In general, the remaining $A_{n;j}$ will always appear as sums in the decoupling equations, and one cannot solve for them. The six-point amplitude, discussed in the previous section, provides an explicit example of this point (in equation (6.20)). If we substitute for $A_{n;2}$ using equation (7.1), we find

$$A_{n;3}(n-1,n;1,\ldots,n-2)=\sum_{\sigma\in COP^{(1\ldots n-2)}_{n-1}\{1,\ldots,n-1\}} A_{n;1}(\sigma(1),\ldots,\sigma(n-1),n)\,. \qquad (7.10)$$

The equations of the type (7.9b) are not independent of equation (7.2). To see this, note that in addition to the decomposition (7.7) given above, we can also decompose the *COP*'s in the second equation as follows,

$$COP^{(j\ldots n-2)}_{n-j}\{j,\ldots,n-1\}=Z_{n-j}\{j,\ldots,n-1\}\circ Z_{n-j-1}\{j,\ldots,n-2\}\,. \qquad (7.11)$$

We can then rewrite the left-hand side of equation (7.9b),

$$\begin{aligned}
&\sum_{\substack{z\in Z_{n-j}\{j,\ldots,n-1\}\\ \sigma\in Z_{n-j-1}\{j,\ldots,n-2\}}} A_{n;j}(1,\ldots,j-1;z(\sigma(j)),\ldots,z(\sigma(n-2)),z(n-1),n)\\
&+\sum_{\substack{z\in Z_{j-1}\{1,\ldots,j-1\}\\ \sigma\in Z_{n-j-1}\{j,\ldots,n-2\}}} A_{n;j+1}(z(1),\ldots,z(j-1),n;\sigma(j),\ldots,\sigma(n-2),n-1)\\
&+\sum_{\substack{z\in Z_{n-j-1}\{j,\ldots,n-2\}\\ \sigma\in Z_{j-1}\{1,\ldots,j-1\}}} A_{n;j+1}(\sigma(1),\ldots,\sigma(j-1),n-1;\sigma(j),\ldots,\sigma(n-2),n)\\
&+\sum_{\substack{z\in Z_{j}\{1,\ldots,j-1,n-1\}\\ \sigma\in Z_{j-1}\{1,\ldots,j-1\}}} A_{n;j+2}(z(\sigma(1)),\ldots,z(\sigma(j-1)),z(n-1),n;j,\ldots,n-2)\\
&=\sum_{\sigma\in Z_{n-j-1}\{j,\ldots,n-2\}}\Bigg[\sum_{z\in Z_{n-j}\{j,\ldots,n-1\}} A_{n;j}(1,\ldots,j-1;z(\sigma(j)),\ldots,z(\sigma(n-2)),z(n-1),n)\\
&\qquad\qquad+\sum_{z\in Z_{j-1}\{1,\ldots,j-1\}} A_{n;j+1}(z(1),\ldots,z(j-1),n;\sigma(j),\ldots,\sigma(n-2),n-1)\Bigg]\\
&+\sum_{\sigma\in Z_{j-1}\{1,\ldots,j-1\}}\Bigg[\sum_{z\in Z_{n-j-1}\{j,\ldots,n-2\}} A_{n;j+1}(\sigma(1),\ldots,\sigma(j-1),n-1;\sigma(j),\ldots,\sigma(n-2),n)\\
&\qquad\qquad+\sum_{z\in Z_{j}\{1,\ldots,j\}} A_{n;j+2}(z(\sigma(1)),\ldots,z(\sigma(j-1)),z(n-1),n;j,\ldots,n-2)\Bigg]
\end{aligned}\tag{7.12}$$

which vanishes upon substitution of equation (7.2) in the two different terms.

This is a general feature of the decoupling equations generated by substitution of more than one photon leg: only those which link coefficients of color factors with different numbers of non-trivial traces — in the one-loop case, linking coefficients of color factors with two non-trivial traces to the coefficient of the color factor with a lone non-trivial trace — are independent of previous equations.

Thus we may need to substitute up to $\lfloor n/2\rfloor$ photons for gluons in order to obtain the full set of decoupling equations (because this is the minimum number of identity matrices that can convert a double trace into a single trace), but further substitutions will yield only linear combinations of existing equations. (As we have seen in the previous section, not all the equations obtained for $\lfloor n/2\rfloor$ photons are necessarily independent.) The independent equations, besides equations (7.1–7.3) are obtained by substituting the identity for $T^{a_{n-m+2}},\ldots,T^{a_n}$, and extracting the coefficient of the resulting single-trace term,

$$\sum_{\sigma\in COP_{n-1}^{(1\ldots n-m+1)}\{1,\ldots,n-1\}} A_{n;1}(\sigma(1),\ldots,\sigma(n-1),n)$$
$$+\sum_{j=2}^{m} \sum_{\substack{p\in P\binom{m-1}{j}\{n-m+2,\ldots,n\} \\ \rho\in S_{j-1}/Z_{j-1}\{p(1),\ldots,p(j-1)\} \\ \sigma\in COP_{n-j}^{(1\ldots n-m+1)}\{1,\ldots,n-m+1,p(j),\ldots,p(n-1)\}}}$$
$$A_{n;j}(\rho(p(1)),\ldots,\rho(p(j-1)),\sigma(1),\ldots,\sigma(n-m+1),\sigma(p(j)),\ldots,\sigma(p(n-1)),p(n))$$
$$=0 \tag{7.13}$$

where $3\leq m\leq\lfloor n/2\rfloor+1$. For $m=3$ we obtain once again equation (7.9a). If we substitute for $A_{n;2}$, using equation (7.1), we obtain

$$-\sum_{\sigma\in COP_{n-1}^{(1\ldots n-m+1)}\{1,\ldots,n-1\}} A_{n;1}(\sigma(1),\ldots,\sigma(n-1),n)$$
$$+\sum_{j=3}^{m} \sum_{\substack{p\in P\binom{m-1}{j}\{n-m+2,\ldots,n\} \\ \rho\in S_{j-1}/Z_{j-1}\{p(1),\ldots,p(j-1)\} \\ \sigma\in COP_{n-j}^{(1\ldots n-m+1)}\{1,\ldots,n-m+1,p(j),\ldots,p(n-1)\}}}$$
$$A_{n;j}(\rho(p(1)),\ldots,\rho(p(j-1)),\sigma(1),\ldots,\sigma(n-m+1),\sigma(p(j)),\ldots,\sigma(p(n-1)),p(n))$$
$$=0\,. \tag{7.14}$$

If we continue to substitute using equations for smaller m, we find that although we cannot solve for $A_{n;m>3}$, we can eliminate them from the equations for larger m. Thus we obtain a simple form of the remaining decoupling equations,

$$\begin{aligned}\sum_{\sigma\in S_{m-1}/Z_{m-1}\{n-m+2,\ldots,n\}} A_{n;m}\left(\sigma(n-m+2),\ldots,\sigma(n),1,\ldots,n-m+1\right) = \\ (-1)^{m+1}\sum_{\sigma\in COP_{n-1}^{(1\ldots n-m+1)}\{1,\ldots,n-1\}} A_{n;1}\left(\sigma(1),\ldots,\sigma(n-1),n\right)\end{aligned} \tag{7.15}$$

where $3\leq m\leq\lfloor n/2\rfloor+1$. For n odd, the independent equations can be combined to yield another equation involving only the $A_{n;1}$,

$$\sum_{\sigma\in COP_{n-1}^{(1\ldots(n-1)/2)}\{1,\ldots,n-1\}} A_{n;1}(\sigma(1),\ldots,\sigma(n-1),n)+\sum_{\sigma\in COP_{n-1}^{((n+1)/2\ldots n-1)}\{1,\ldots,n-1\}} A_{n;1}(\sigma(1),\ldots,\sigma(n-1),n)=0\,. \tag{7.16}$$

The decoupling equations discussed in the section and in the previous section can provide a valuable consistency check on explicit calculations of the partial amplitudes, or else can be used to reduce the amount of work, for example eliminating the need to calculate the $A_{n;3}$ independently.

8. The Color-Summed Cross Section

Ultimately, we are interested in calculating a differential cross-section, summed over final colors, and averaged over initial colors. At tree level, we may use [6,5] the color decomposition (2.2) to write the color-summed differential cross section as

$$\begin{aligned}\sum_{\text{colors}} \mathcal{A}_n^* \mathcal{A}_n &= g^{2n-4} \sum_{\sigma,\rho \in S_n/Z_n} \operatorname{Tr}\left(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}\right)^* \operatorname{Tr}\left(T^{a_{\rho(1)}} \cdots T^{a_{\rho(n)}}\right) \\ &\qquad \times A_n^*(\sigma(1),\ldots,\sigma(n)) A_n(\rho(1),\ldots,\rho(n)) \\ &= g^{2n-4} \sum_{\sigma,\rho \in S_n/Z_n} \operatorname{Tr}\left(T^{a_{\sigma(n)}} \cdots T^{a_{\sigma(1)}}\right) \operatorname{Tr}\left(T^{a_{\rho(1)}} \cdots T^{a_{\rho(n)}}\right) \; A_n^*(\sigma) A_n(\rho)\end{aligned} \tag{8.1}$$

where summation over all a_i is understood, and where we have introduced the abbreviation $A_n(\sigma) = A_n(\sigma(1),\ldots,\sigma(n))$.

Because the extra $U(1)$ gauge boson decouples, one can sum over $U(N)$ instead of $SU(N)$; this is easier, because the $U(N)$ Fierz identities,

$$\operatorname{Tr}(T^a X) \operatorname{Tr}(T^a Y) = \operatorname{Tr}(XY) \tag{8.2a}$$

$$\operatorname{Tr}(T^a X T^a Y) = \operatorname{Tr}(X) \operatorname{Tr}(Y) \tag{8.2b}$$

are simpler than their $SU(N)$ counterparts. The decoupling equations, discussed in the previous sections, can be used to show explicitly the identity of the $U(N)$-summed cross section with the $SU(N)$-summed one. Using these identities, and the form of the Casimir $T^a T^a = N_c$ (recall that we use the normalization $\operatorname{Tr}(T^a T^b) = \delta^{ab}$), it is easy to see that

$$\operatorname{Tr}\left(T^{a_n} \cdots T^{a_1}\right) \operatorname{Tr}\left(T^{a_1} \cdots T^{a_n}\right) = N_c^n \tag{8.3}$$

while for any $\sigma \neq 1$ (we denote the identity permutation by 1),

$$\operatorname{Tr}\left(T^{a_{\sigma(n)}} \cdots T^{a_{\sigma(1)}}\right) \operatorname{Tr}\left(T^{a_1} \cdots T^{a_n}\right) \leq N_c^{n-2} \tag{8.4}$$

because in this case one takes at least one detour through the Fierz identities, and every such detour costs a power of N_c^2.

Thus

$$\begin{aligned}\sum_{\text{colors}} \mathcal{A}_n^* \mathcal{A}_n = g^{2n-4} \Bigg[& N_c^n \sum_{\sigma \in S_n/Z_n} |A_n(\sigma)|^2 \\ & + \sum_{\substack{\sigma,\rho \in S_n/Z_n \\ \sigma \neq \rho}} \operatorname{Tr}\left(T^{a_{\sigma(n)}} \cdots T^{a_{\sigma(1)}}\right) \operatorname{Tr}\left(T^{a_{\rho(1)}} \cdots T^{a_{\rho(n)}}\right) \; A_n^*(\sigma) A_n(\rho) \Bigg]\end{aligned} \tag{8.5}$$

where the terms in the second sum are of $\mathcal{O}(N_c^{n-2})$ or smaller. There is a subset of the $\mathcal{O}(N_c^{n-2})$ terms, that may be distinguished from the rest, because they are also absolute values squared. To extract them, separate out a set generated by cyclic permutations on $n-1$ legs,

$$\sum_{\text{colors}} \mathcal{A}_n^* \mathcal{A}_n =$$
$$g^{2n-4} \left[N_c^n \sum_{\sigma \in S_n/Z_n} |A_n(\sigma)|^2 \right.$$
$$+ \sum_{\substack{\sigma \in S_n/Z_n \\ \rho \in Z_{n-1}, \rho \neq 1}} \operatorname{Tr}\left(T^{a_{\sigma(n)}} \ldots T^{a_{\sigma(1)}}\right) \operatorname{Tr}\left(T^{a_{\rho\cdot\sigma(1)}} \ldots T^{a_{\rho\cdot\sigma(n)}}\right) A_n^*(\sigma) A_n(\rho \cdot \sigma) \tag{8.6}$$
$$\left. + \sum_{\substack{\sigma, \rho \in S_n/Z_n \\ \rho \notin Z_{n-1}}} \operatorname{Tr}\left(T^{a_{\sigma(n)}} \ldots T^{a_{\sigma(1)}}\right) \operatorname{Tr}\left(T^{a_{\rho\cdot\sigma(1)}} \ldots T^{a_{\rho\cdot\sigma(n)}}\right) A_n^*(\sigma) A_n(\rho \cdot \sigma) \right]$$

(in the second and third terms, Z_{n-1} should be understood as a subset of S_n, so that each of its elements acts trivially on the n-th symbol in its argument, that is $\rho(n)$ is understood to be n).

Now, for $\rho \in Z_{n-1}$, $\rho \neq 1$ (ρ shifting the the elements j units to the left),

$$\begin{aligned}
\operatorname{Tr}\left(T^{a_n} \ldots T^{a_1}\right) \operatorname{Tr}\left(T^{a_{\rho(1)}} \ldots T^{a_{\rho(n-1)}} T^{a_n}\right) &= \operatorname{Tr}\left(T^{a_{n-1}} \ldots T^{a_1} T^{a_{\rho(1)}} \ldots T^{a_{\rho(n-1)}}\right) \\
&= \operatorname{Tr}\left(T^{a_{n-1}} \ldots T^{a_1} T^{a_{j+1}} \ldots T^{a_{n-1}} T^{a_1} \ldots T^{a_j}\right) \\
&= \operatorname{Tr}\left(T^{a_{n-1}} \ldots T^{a_2} T^{a_2} \ldots T^{a_j}\right) \operatorname{Tr}\left(T^{a_{j+1}} \ldots T^{a_{n-1}}\right) \\
&= \operatorname{Tr}\left(T^{a_{j+1}} \ldots T^{a_{n-2}} T^{a_{n-1}} \ldots T^{a_2} T^{a_2} \ldots T^{a_j}\right) \\
&= N_c^{(n-2-j)+(j-1)+1} \\
&= N_c^{n-2}
\end{aligned} \tag{8.7}$$

so that

$$\sum_{\text{colors}} \mathcal{A}_n^* \mathcal{A}_n = g^{2n-4} \left[N_c^n \sum_{\sigma \in S_n/Z_n} |A_n(\sigma)|^2 \right.$$
$$+ N_c^{n-2} \sum_{\substack{\sigma \in S_n/Z_n \\ \rho \in Z_{n-1}, \rho \neq 1}} A_n^*(\sigma) A_n(\rho \cdot \sigma)$$
$$\left. + \sum_{\substack{\sigma, \rho \in S_n/Z_n \\ \rho \notin Z_{n-1}}} \operatorname{Tr}\left(T^{a_{\sigma(n)}} \ldots T^{a_{\sigma(1)}}\right) \operatorname{Tr}\left(T^{a_{\rho\cdot\sigma(1)}} \ldots T^{a_{\rho\cdot\sigma(n)}}\right) A_n^*(\sigma) A_n(\rho \cdot \sigma) \right] . \tag{8.8}$$

Using the decoupling equation (2.6), we can rewrite the second term so that it has the same form as the first one, and thus

$$\sum_{\text{colors}} \mathcal{A}_n^* \mathcal{A}_n = g^{2n-4} \Bigg[N_c^{n-2} \left(N_c^2 - 1\right) \sum_{\sigma \in S_n / Z_n} |A_n(\sigma)|^2$$

$$+ \sum_{\substack{\sigma, \rho \in S_n / Z_n \\ \rho \notin Z_{n-1}}} \mathrm{Tr}\left(T^{a_{\sigma(n)}} \ldots T^{a_{\sigma(1)}}\right) \mathrm{Tr}\left(T^{a_{\rho \cdot \sigma(1)}} \ldots T^{a_{\rho \cdot \sigma(n)}}\right) \; A_n^*(\sigma) A_n(\rho \cdot \sigma) \Bigg] \; . \tag{8.9}$$

For the four-point case, the first term is in fact the entire story; the second term vanishes because of the decoupling equation, and so we obtain

$$\sum_{\text{colors}} \mathcal{A}_4^* \mathcal{A}_4 = g^4 N_c^2 \left(N_c^2 - 1\right) \sum_{\sigma \in S_4 / Z_4} |A_4(\sigma)|^2 \; . \tag{8.10}$$

In the five-point case, using both the decoupling equation (2.6) and the reflection identity (2.3), explicit calculation allows us to put the second term of equation (8.9) in the form

$$2 N_c \left(N_c^2 - 1\right) \sum_{\sigma \in S_5 / Z_5} A_5^*(\sigma) A_5(r \cdot \sigma) \tag{8.11}$$

where $r = (2\,4\,1\,3\,5)$. Note that r^2 is equivalent to a reflection: $r^2 = (4\,3\,2\,1\,5)$. We could redefine the summation variable $\sigma \to r\sigma$, since the latter variable also sweeps over S_5/Z_5; averaging between these two forms, we obtain

$$\begin{aligned} &\frac{1}{2} \times 2 N_c \left(N_c^2 - 1\right) \sum_{\sigma \in S_5 / Z_5} \left[A_5^*(\sigma) A_5(r \cdot \sigma) + A_5^*(r \cdot \sigma) A_5(r^2 \cdot \sigma) \right] \\ &= N_c \left(N_c^2 - 1\right) \sum_{\sigma \in S_5 / Z_5} \left[A_5^*(\sigma) A_5(r \cdot \sigma) - A_5^*(r \cdot \sigma) A_5(\sigma) \right] \end{aligned} \tag{8.12}$$

where we have used the reflection identity (2.3). The full amplitude squared is of course real, and the first term in equation (8.9) is manifestly real, since it is a sum of absolute values squared. Thus only the real part of equation (8.12) would survive. But the latter expression is imaginary, and so it drops out, and for the five-point cross section as well, we find that the first term in equation (8.9) is exact,

$$\sum_{\text{colors}} \mathcal{A}_5^* \mathcal{A}_5 = g^6 N_c^3 \left(N_c^2 - 1\right) \sum_{\sigma \in S_5 / Z_5} |A_5(\sigma)|^2 \; . \tag{8.13}$$

All $\mathcal{O}(N_c^{n-2})$ terms arise from a single detour through the Fierz identities in the process of collapsing the product of the two traces. Consider for example,

$$\begin{aligned} &\mathrm{Tr}(T^{a_7} T^{a_6} T^{a_5} T^{a_4} T^{a_3} T^{a_2} T^{a_1}) \, \mathrm{Tr}(T^{a_1} T^{a_4} T^{a_5} T^{a_6} T^{a_2} T^{a_3} T^{a_7}) \\ &= \mathrm{Tr}(T^{a_6} T^{a_5} T^{a_4} T^{a_3} T^{a_2} T^{a_1} T^{a_1} T^{a_4} T^{a_5} T^{a_6} T^{a_2} T^{a_3}) \end{aligned} \tag{8.14}$$

where the subset of consecutive elements 2, 3, 4, 5, 6 had been shifted cyclicly by two units in the second trace on the first line. Because of this shift, a detour through the Fierz identities is required. Using the Fierz identity (8.2b) we break the last trace back into two traces

$$N_c^2 \operatorname{Tr}(T^{a_6}T^{a_5}T^{a_4}) \operatorname{Tr}(T^{a_4}T^{a_5}T^{a_6}) \,. \tag{8.15}$$

Observe that an additional cyclic shift of the elements of the second trace on the right-hand-side has no effect on the value of the product after summing over the remaining indices; this means that we could have performed a cyclic shift of these elements in the second trace on the first line in eq. (8.14) without changing the of the color sum of this term. (A further use of the Fierz identity (8.2a) reduces this term to N_c^5.)

This example illustrates the general structure of all $\mathcal{O}(N_c^{n-2})$ contributions; these come from a combination of two cyclic permutations. The first cyclic permutation forces the Fierz detour while the second cyclic permutation comes from the cyclic symmetry of the second trace in the double trace term of the Fierz detour.

In order to express the general structure of the $\mathcal{O}(N_c^{n-2})$ terms succinctly, it is convenient to introduce some more notation. Let $z_0 \in Z_m\{s_1, \ldots, s_m\}$ be the cyclic permutation that shifts elements by one unit to the left; we then write $Z_m = \{1, z_0, z_0^2, \ldots, z_0^{m-1}\}$. We want to introduce a set which contains an additional cyclic shift of all the elements to the left of the resulting position of s_1, for each $z \in Z_m$:

$$\bigcup_{l=1}^{m} Z_{m-l}\{s_{l+1}, \ldots, s_m\} \circ z_0^l \,. \tag{8.16}$$

(Z_0 and Z_1 are understood to contain only the identity element.) In Appendix IV, we show that the union of such sets for $s_1 = 1, \ldots, n-2$ furnishes a complete list of additional elements of S_n/Z_n that contribute terms of $\mathcal{O}(N_c^{n-2})$. The existing constraints on the sum in equation (8.9) however indicate that we should remove some elements, and thus we define the set

$$\mathbb{Z}_k\{s_1, \ldots, s_k\} = \left[\bigcup_{j=1}^{k-1} \bigcup_{m=j+1}^{k} \bigcup_{l=1}^{m-j+1} Z_{m-j-l+1}\{s_{j+l-1}, \ldots, s_m\} \circ z_0 \left[\begin{smallmatrix} m \\ j \end{smallmatrix}\right]^l\right] - Z_k\{s_1, \ldots, s_k\} \tag{8.17}$$

where $z_0 \left[\begin{smallmatrix} m \\ j \end{smallmatrix}\right]$ is the unit left generator of $Z_{m-j+1}\{s_j, \ldots, s_m\}$. The set $\mathbb{Z}_k$ ('double-Z sub k') contains

$$\begin{aligned} &\left[\sum_{j=1}^{k-1} \sum_{m=j+1}^{k} \sum_{l=1}^{m-j} (m-j-l+1)\right] - k + 1 \\ &= \binom{k+2}{4} - k + 1 \\ &= \frac{1}{24}(k-1)(k-2)(k^2+5k+12) \end{aligned} \tag{8.18}$$

elements. For example,

$$
\begin{aligned}
\mathbb{Z}_4 = \{&(3\,4\,2\,1),(4\,2\,3\,1),(4\,3\,1\,2),(2\,3\,1\,4),(3\,2\,1\,4),(3\,1\,2\,4),\\
&(1\,3\,4\,2),(1\,4\,3\,2),(1\,4\,2\,3),(2\,1\,3\,4),(1\,3\,2\,4),(1\,2\,4\,3)\}\,.
\end{aligned}
\tag{8.19}
$$

With the set $\mathbb{Z}_{n-1}$ in hand, we can write

$$
\begin{aligned}
\sum_{\text{colors}} \mathcal{A}_n^* \mathcal{A}_n = g^{2n-4} \Bigg[& N_c^{n-2}\left(N_c^2-1\right) \sum_{\sigma\in S_n/Z_n} |A_n(\sigma)|^2 \\
& + N_c^{n-2} \sum_{\substack{\sigma\in S_n/Z_n\\ \rho\in \mathbb{Z}_{n-1}}} A_n^*(\sigma) A_n(\rho\cdot\sigma) \\
& + \sum_{\substack{\sigma,\rho\in S_n/Z_n\\ \rho\notin \mathbb{Z}_{n-1},\rho\notin \mathbb{Z}_{n-1}}} \mathrm{Tr}\left(T^{a_{\sigma(n)}}\cdots T^{a_{\sigma(1)}}\right)\mathrm{Tr}\left(T^{a_{\rho\cdot\sigma(1)}}\cdots T^{a_{\rho\cdot\sigma(n)}}\right)\; A_n^*(\sigma)A_n(\rho\cdot\sigma)\Bigg]
\end{aligned}
\tag{8.20}
$$

where $\mathbb{Z}_{n-1}$ is also understood to be embedded in S_n. The third term in the sum is $\mathcal{O}(N_c^{n-4})$. For large n, the use of $\mathbb{Z}_{n-1}$ in computing the next-to-leading color terms is probably preferrable to that of the $(1,2)$ basis of Kleiss and Kuijf [16] because the number of terms in it grows only polynomially rather than factorially.

9. Corrections to the Cross Section

In order to calculate the next-to-leading corrections to the differential cross section, we must compute the interference of tree and one-loop amplitudes

$$
\begin{aligned}
\sum_{\text{colors}} \mathcal{A}_n^* \mathcal{A}_n &= \sum_{\text{colors}} \left(\mathcal{A}_n^{\text{tree}\,*} + \mathcal{A}_n^{1-\text{loop}\,*} + \mathcal{O}(g^{n+2})\right)\,\left(\mathcal{A}_n^{\text{tree}} + \mathcal{A}_n^{1-\text{loop}} + \mathcal{O}(g^{n+2})\right)\\
&= \sum_{\text{colors}} \mathcal{A}_n^{\text{tree}\,*}\mathcal{A}_n^{\text{tree}} + 2\,\mathrm{Re}\left[\mathcal{A}_n^{\text{tree}\,*}\mathcal{A}_n^{1-\text{loop}}\right] + \mathcal{O}(g^{2n})\,.
\end{aligned}
\tag{9.1}
$$

(Throughout the discussion here, the gauge coupling g should be understood to depend on the renormalization scale in the usual manner, though we shall not display that dependence explicitly.) As in the previous section, we will compute the next-to-leading correction to the cross section by summing over $U(N)$ rather than $SU(N)$. By virtue of the decoupling equations, this yields the same answer as summing over $SU(N)$; but the calculation is simpler this way. In performing the color sums, we will encounter two types of terms in the next-to-leading corrections, the first with two non-trivial traces (and an explicit power of the number of colors), and the second with three non-trivial traces. The latter terms are again suppressed by at least one power of N_c^2 compared to the former, since we will use up an additional pair of charge matrices in reducing the three traces to one trace.

Thus the next-to-leading order terms can be written as follows

$$
\begin{aligned}
\sum_{\text{colors}} [\mathcal{A}_n^* \mathcal{A}_n]_{\text{NLO}} = & \\
2g^{2n-2}\,\text{Re} \sum_{\sigma\in S_n/Z_n} \sum_{j=1}^{\lfloor n/2\rfloor+1} \sum_{\rho\in S_n/S_{n;j}} & \text{Tr}(T^{a_{\sigma(n)}}\ldots T^{a_{\sigma(1)}})\,\text{Tr}(T^{a_{\rho(1)}}\ldots T^{a_{\rho(j-1)}})\,\text{Tr}(T^{a_{\rho(j)}}\ldots T^{a_{\rho(n)}}) \\
& \times A_n^{\text{tree}\,*}(\sigma) A_{n;j}(\rho) \\
= 2g^{2n-2} \Bigg[\text{Re} \sum_{\sigma,\rho\in S_n/Z_n} & N_c\,\text{Tr}(T^{a_{\sigma(n)}}\ldots T^{a_{\sigma(1)}})\,\text{Tr}(T^{a_{\rho(1)}}\ldots T^{a_{\rho(n)}})\; A_n^{\text{tree}\,*}(\sigma) A_{n;1}(\rho) \\
+ \text{Re} \sum_{\sigma\in S_n/Z_n} \sum_{j=2}^{\lfloor n/2\rfloor+1} \sum_{\rho\in S_n/S_{n;j}} & \text{Tr}(T^{a_{\sigma(n)}}\ldots T^{a_{\sigma(1)}})\,\text{Tr}(T^{a_{\rho(1)}}\ldots T^{a_{\rho(j-1)}})\,\text{Tr}(T^{a_{\rho(j)}}\ldots T^{a_{\rho(n)}}) \\
& \times A_n^{\text{tree}\,*}(\sigma) A_{n;j}(\rho) \Bigg] \\
= 2g^{2n-2} \Bigg[N_c^{n+1}\,\text{Re} \sum_{\sigma\in S_n/Z_n} & A_n^{\text{tree}\,*}(\sigma) A_{n;1}(\sigma) \\
+ \text{Re} \sum_{\substack{\sigma,\rho\in S_n/Z_n \\ \sigma\neq 1}} & N_c\,\text{Tr}(T^{a_{\sigma\cdot\rho(n)}}\ldots T^{a_{\sigma\cdot\rho(1)}})\,\text{Tr}(T^{a_{\rho(1)}}\ldots T^{a_{\rho(n)}})\; A_n^{\text{tree}\,*}(\sigma) A_{n;1}(\rho) \\
+ \text{Re} \sum_{\sigma\in S_n/Z_n} \sum_{j=2}^{\lfloor n/2\rfloor+1} \sum_{\rho\in S_n/S_{n;j}} & \text{Tr}(T^{a_{\sigma(n)}}\ldots T^{a_{\sigma(1)}})\,\text{Tr}(T^{a_{\rho(1)}}\ldots T^{a_{\rho(j-1)}})\,\text{Tr}(T^{a_{\rho(j)}}\ldots T^{a_{\rho(n)}}) \\
& \times A_n^{\text{tree}\,*}(\sigma) A_{n;j}(\rho) \Bigg] \; .
\end{aligned}
\tag{9.2}
$$

As in the discussion of the tree-level cross section, we can identify a subset of the next-to-leading color terms which have the same structure as the leading color terms, so that

$$
\begin{aligned}
&\sum_{\text{colors}} [\mathcal{A}_n^* \mathcal{A}_n]_{\text{NLO}} = \\
&\quad 2g^{2n-2} \Bigg[N_c^{n-1} \left(N_c^2 - 1\right) \mathrm{Re} \sum_{\sigma \in S_n / Z_n} A_n^{\text{tree}\,*}(\sigma) A_{n;1}(\sigma) \\
&\qquad + \mathrm{Re} \sum_{\substack{\sigma, \rho \in S_n / Z_n \\ \sigma \notin Z_{n-1}}} N_c \, \mathrm{Tr}\left(T^{a_{\sigma\cdot\rho(n)}} \cdots T^{a_{\sigma\cdot\rho(1)}}\right) \mathrm{Tr}\left(T^{a_{\rho(1)}} \cdots T^{a_{\rho(n)}}\right) A_n^{\text{tree}\,*}(\sigma) A_{n;1}(\rho) \\
&\qquad + \mathrm{Re} \sum_{\sigma \in S_n / Z_n} \sum_{j=2}^{\lfloor n/2 \rfloor + 1} \sum_{\rho \in S_n / S_{n;j}} \mathrm{Tr}\left(T^{a_{\sigma(n)}} \cdots T^{a_{\sigma(1)}}\right) \mathrm{Tr}\left(T^{a_{\rho(1)}} \cdots T^{a_{\rho(j-1)}}\right) \mathrm{Tr}\left(T^{a_{\rho(j)}} \cdots T^{a_{\rho(n)}}\right) \\
&\qquad\qquad \times A_n^{\text{tree}\,*}(\sigma) A_{n;j}(\rho) \Bigg] \; .
\end{aligned} \tag{9.3}
$$

(Note that only the tree-level decoupling equations are required to obtain this result.) The last two sums inside the brackets are of order $\mathcal{O}(N_c^{n-1})$.

The $j = 2$ terms in the last sum can be rewritten as follows,

$$
\begin{aligned}
&\mathrm{Re} \sum_{\sigma \in S_n / Z_n} \sum_{\rho \in S_n / S_{n;2}} \mathrm{Tr}\left(T^{a_{\sigma(n)}} \cdots T^{a_{\sigma(1)}}\right) \mathrm{Tr}\left(T^{a_{\rho(1)}}\right) \mathrm{Tr}\left(T^{a_{\rho(2)}} \cdots T^{a_{\rho(n)}}\right) A_n^{\text{tree}\,*}(\sigma) A_{n;j}(\rho) \\
&= \mathrm{Re} \sum_{\rho \in S_n / S_{n;2}} \sum_{\substack{\sigma \in S_{n-1} / Z_{n-1}\{1, \ldots, \widehat{\rho(1)}, \ldots, n\} \\ z \in Z_{n-1}\{\sigma(1), \ldots, \widehat{\rho(1)}, \ldots, \sigma(n)\}}} \mathrm{Tr}\left(T^{a_{\sigma(n)}} \cdots \widehat{T^{a_{\rho(1)}}} \cdots T^{a_{\sigma(1)}}\right) \mathrm{Tr}\left(T^{a_{\rho(2)}} \cdots T^{a_{\rho(n)}}\right) \\
&\qquad\qquad \times A_n^{\text{tree}\,*}(z \cdot \sigma(1), \ldots, \widehat{\rho(1)}, \ldots, z \cdot \sigma(n), \rho(1)) A_{n;j}(\rho(1), \ldots, \rho(n-1), \rho(n))
\end{aligned} \tag{9.4}
$$

since the color sum is independent of z. (The hat $\widehat{\ }$ denotes an omitted element.) Using the tree-level decoupling equation, we then find that this sum vanishes identically. (Note that the trace of a single generator does *not* vanish identically, because of the $U(1)$ generator, but as expected this contribution drops out.)

In the case of the four-point amplitude, the complete symmetry of $A_{4;3}$, noted in section 6, means that the $j = 3$ terms in equation (9.3) have the following form,

$$\begin{aligned}
&\mathrm{Re} \sum_{\sigma \in S_4/Z_4} \sum_{\rho \in S_4/S_{4;3}} \mathrm{Tr}(T^{a_{\sigma(4)}} \cdots T^{a_{\sigma(1)}})\, \mathrm{Tr}(T^{a_{\rho(1)}} \cdots T^{a_{\rho(2)}})\, \mathrm{Tr}(T^{a_{\rho(3)}} \cdots T^{a_{\rho(4)}})\, A_4^{\mathrm{tree}\,*}(\sigma) A_{4;3}(\rho) \\
&= \mathrm{Re} \Bigg[A_{4;3}(1,2,3,4) \\
&\qquad \times \sum_{\sigma \in S_4/Z_4} \sum_{\rho \in S_4/S_{4;3}} \mathrm{Tr}(T^{a_{\sigma(4)}} \cdots T^{a_{\sigma(1)}})\, \mathrm{Tr}(T^{a_{\rho(1)}} \cdots T^{a_{\rho(2)}})\, \mathrm{Tr}(T^{a_{\rho(3)}} \cdots T^{a_{\rho(4)}}) A_4^{\mathrm{tree}\,*}(\sigma) \Bigg] .
\end{aligned} \tag{9.5}$$

The sum over ρ leads to a completely symmetric tensor in the a_j, so that the color sum is independent of σ; as a result, we obtain a result proportional to

$$\mathrm{Re} \left[A_{4;3}(1,2,3,4) \sum_{\sigma \in S_4/Z_4} A_4^{\mathrm{tree}\,*}(\sigma) \right] \tag{9.6}$$

which vanishes using the tree-level decoupling equation.

The second term in equation (9.3) also vanishes for the four-point amplitude, and so the next-to-leading correction in this case has the simple form

$$\sum_{\mathrm{colors}} [A_4^* A_4]_{\mathrm{NLO}} = 2g^6 N_c^3 \left(N_c^2 - 1\right) \mathrm{Re} \sum_{\sigma \in S_4/Z_4} A_4^{\mathrm{tree}\,*}(\sigma) A_{4;1}(\sigma) \,. \tag{9.7}$$

The calculation of $A_{4;1}$ will be described in detail in a future paper.

For the five-point function, we can follow the same steps as in the tree cross section, and rewrite the second term as

$$\begin{aligned}
&2N_c^2 \left(N_c^2 - 1\right) \mathrm{Re} \sum_{\rho \in S_5/Z_5} A_5^{\mathrm{tree}\,*}(r \cdot \rho) A_{5;1}(\rho) \\
&= N_c^2 \left(N_c^2 - 1\right) \mathrm{Re} \sum_{\rho \in S_5/Z_5} \left[A_5^{\mathrm{tree}\,*}(r \cdot \rho) A_{5;1}(\rho) - A_5^{\mathrm{tree}\,*}(\rho) A_{5;1}(r \cdot \rho) \right]
\end{aligned} \tag{9.8}$$

where $r = (2\,4\,1\,3\,5)$ has the same definition as in section 8.

Explicit calculation shows that the $j = 3$ terms in the five-point cross section can be written using a subset of S_5/Z_5,

$$H_5 = \{(1\,2\,3\,4\,5), (3\,4\,1\,2\,5), (3\,1\,2\,4\,5), (2\,1\,3\,4\,5), (3\,2\,1\,4\,5), (3\,4\,2\,1\,5)\} \tag{9.9}$$

as follows,

$$\begin{aligned}
&\mathrm{Re} \sum_{\sigma \in S_5/Z_5} \sum_{\rho \in S_5/S_{5;3}} \mathrm{Tr}(T^{a_{\sigma(5)}} \cdots T^{a_{\sigma(1)}})\, \mathrm{Tr}(T^{a_{\rho(1)}} T^{a_{\rho(2)}})\, \mathrm{Tr}(T^{a_{\rho(3)}} T^{a_{\rho(4)}} T^{a_{\rho(5)}})\, A_5^{\mathrm{tree}\,*}(\sigma) A_{5;3}(\rho) \\
&= N_c^2 \left(N_c^2 - 1\right) \mathrm{Re} \sum_{h \in H_5} \sum_{\rho \in S_5/S_{5;3}} A_5^{\mathrm{tree}\,*}(h \cdot \rho) A_{5;3}(\rho) \\
&= 2N_c^2 \left(N_c^2 - 1\right) \mathrm{Re} \sum_{h \in H_5} \sum_{p \in P\binom{5}{3}} A_5^{\mathrm{tree}\,*}(h \cdot p) A_{5;3}(p)
\end{aligned} \tag{9.10}$$

where the last form of the equation uses the fact that for each partition in $P\binom{5}{3}$, there are two elements of $S_5/S_{5;3}$, and these are related by the reflection identity (4.17). This set of partitions is

$$\begin{aligned} P\binom{5}{3} &= P\binom{5}{3}\{1,\ldots,5\} \\ &= \{(1\,2\,3\,4\,5),(1\,3\,2\,4\,5),(1\,4\,2\,3\,5),(1\,5\,2\,3\,4),(2\,3\,1\,4\,5), \\ &\qquad (2\,4\,1\,3\,5),(2\,5\,1\,3\,4),(3\,4\,1\,2\,5),(3\,5\,1\,2\,4),(4\,5\,1\,2\,3)\}\,. \end{aligned} \tag{9.11}$$

Putting all of the terms together, we can write

$$\begin{aligned} \sum_{\text{colors}} [\mathcal{A}_5^*\mathcal{A}_5]_{\text{NLO}} = 2g^8 \Bigg[& N_c^4\left(N_c^2-1\right)\text{Re}\sum_{\sigma\in S_5/Z_5} A_5^{\text{tree}\,*}(\sigma)A_{5;1}(\sigma) \\ & + N_c^2\left(N_c^2-1\right)\text{Re}\sum_{\rho\in S_5/Z_5}\left[A_5^{\text{tree}\,*}(r\cdot\rho)A_{5;1}(\rho) - A_5^{\text{tree}\,*}(\rho)A_{5;1}(r\cdot\rho)\right] \\ & + 2N_c^2\left(N_c^2-1\right)\text{Re}\sum_{h\in H_5}\sum_{p\in P\binom{5}{3}} A_5^{\text{tree}\,*}(h\cdot p)A_{5;3}(p)\Bigg]\,. \end{aligned} \tag{9.12}$$

One could also use the decoupling equations to eliminate $A_{5;3}$ in favor of $A_{5;1}$; in this case, the third term inside the parentheses could be written as

$$\begin{aligned} & 2N_c^2\left(N_c^2-1\right)\text{Re}\sum_{\sigma\in S_5/Z_5} A_5^{\text{tree}\,*}(\sigma) \\ & \times\Big[3A_{5;1}(\sigma\cdot(1\,2\,3\,4\,5)) + A_{5;1}(\sigma\cdot(4\,1\,2\,3\,5)) + A_{5;1}(\sigma\cdot(1\,2\,4\,3\,5)) + A_{5;1}(\sigma\cdot(3\,1\,2\,4\,5)) \\ & \quad + A_{5;1}(\sigma\cdot(3\,4\,1\,2\,5)) - A_{5;1}(\sigma\cdot(4\,3\,1\,2\,5)) - A_{5;1}(\sigma\cdot(1\,3\,2\,4\,5)) - 3A_{5;1}(\sigma\cdot(4\,1\,3\,2\,5)) \\ & \quad - A_{5;1}(\sigma\cdot(1\,3\,4\,2\,5)) - A_{5;1}(\sigma\cdot(2\,4\,1\,3\,5)) + A_{5;1}(\sigma\cdot(1\,4\,2\,3\,5)) - 3A_{5;1}(\sigma\cdot(1\,4\,3\,2\,5))\Big]\,. \end{aligned} \tag{9.13}$$

The general structure of the subleading color pieces for the $j = 1$ part of equation (9.3) is similar to that at tree level. For the $j = 2$, these terms were shown to vanish above. To display these terms explicitly for $3 \le j \le \lfloor n/2\rfloor + 1$, we must define another bit of notation, for *cyclicly consecutive partitions*, or $CCP\binom{n}{j}$, the subset of $P\binom{n}{j}$ that maps the set $1\ldots j-1$ into a set of elements that are consecutive up to a cyclic transformation (of $1\ldots n$) [so that 1 is considered the successor of n]. Because of the definition of the full set of partitions, the set $j\ldots n$ will then also be mapped into a cyclicly consecutive set of elements. For example,

$$CCP\binom{5}{3} = \{(1\,2\,3\,4\,5),(2\,3\,1\,4\,5),(3\,4\,1\,2\,5),(4\,5\,1\,2\,3),(1\,5\,2\,3\,4)\}\,. \tag{9.14}$$

Note that for n even, and $j = n/2+1$, this set has already been reduced by the Z_2 symmetry exchanging the two traces, because of the way $P\binom{n}{j}$ is defined. The number of elements in $CCP\binom{n}{j}$ is n, except for n even and $j = n/2+1$, when it is $n/2$.

We show in Appendix V that

$$\mathrm{Tr}(T^{a_n}\cdots T^{a_1})\,\mathrm{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(j-1)}})\,\mathrm{Tr}(T^{a_{\sigma(j)}}\cdots T^{a_{\sigma(n)}}) = N_c^{n-1} \tag{9.15}$$

if and only if $\sigma \in CCP\binom{n}{j}$, which is the condition that no detours are taken through the Fierz identities. (For other elements of $S_n/S_{n;j}$, the color-summed traces will be N_c^{n-4} or smaller.) With these sets, along with $\mathbb{Z}_n$, defined in the previous section, we can write the next-to-leading correction to the cross-section for general n as follows,

$$\begin{aligned}
\sum_{\text{colors}} [\mathcal{A}_n^* \mathcal{A}_n]_{\text{NLO}} = 2g^{2n-2} \Bigg[& N_c^{n-1}\,(N_c^2-1)\,\mathrm{Re}\sum_{\sigma\in S_n/Z_n} A_n^{\text{tree}\,*}(\sigma) A_{n;1}(\sigma) \\
& + N_c^{n-1}\,\mathrm{Re}\sum_{\substack{\rho\in S_n/\mathbb{Z}_n\\ \sigma\in\mathbb{Z}_{n-1}}} A_n^{\text{tree}\,*}(\sigma\cdot\rho) A_{n;1}(\rho) \\
& + N_c^{n-1}\,\mathrm{Re}\sum_{\sigma\in S_n/Z_n}\ \sum_{j=3}^{\lfloor n/2\rfloor+1}\ \sum_{\rho\in CCP\binom{n}{j}} A_n^{\text{tree}\,*}(\sigma) A_{n;j}(\rho\cdot\sigma) \\
& + \mathrm{Re}\sum_{\substack{\rho\in S_n/\mathbb{Z}_n\\ \sigma\notin\mathbb{Z}_{n-1},\,\sigma\notin\mathbb{Z}_{n-1}}} N_c\,\mathrm{Tr}(T^{a_{\sigma\cdot\rho(n)}}\cdots T^{a_{\sigma\cdot\rho(1)}})\,\mathrm{Tr}(T^{a_{\rho(1)}}\cdots T^{a_{\rho(n)}})\; A_n^{\text{tree}\,*}(\sigma\cdot\rho) A_{n;1}(\rho) \\
& + \mathrm{Re}\sum_{\sigma\in S_n/Z_n}\ \sum_{j=3}^{\lfloor n/2\rfloor+1}\ \sum_{\rho\in S_n/S_{n;j}-CCP\binom{n}{j}} \\
& \qquad \mathrm{Tr}(T^{a_{\sigma(n)}}\cdots T^{a_{\sigma(1)}})\,\mathrm{Tr}(T^{a_{\rho\cdot\sigma(1)}}\cdots T^{a_{\rho\cdot\sigma(j-1)}})\,\mathrm{Tr}(T^{a_{\rho\cdot\sigma(j)}}\cdots T^{a_{\rho\cdot\sigma(n)}}) \\
& \qquad \times A_n^{\text{tree}\,*}(\sigma) A_{n;j}(\rho\cdot\sigma) \Bigg]
\end{aligned} \tag{9.16}$$

where the last two terms are of $\mathcal{O}(N_c^{n-3})$.

As we have seen, for processes involving a small number of external gluons, the color-summed cross section through next-to-leading order in α_s has a simple form. For a larger number of external gluons, we may note that although the number of colors in QCD is not that large, the expansion presented here is in $1/N_c^2$ (rather than just $1/N_c$), and thus a restriction to the leading term can provide a useful approximation. (For six or fewer final-state gluons, this approximation is quite good at tree level [38].) Of course, one may also wish to calculate the exact tree or next-to-leading matrix element; in this case, the expansion in $1/N_c^2$ provides a useful way of organizing a numerical calculation. For example, it is possible in a Monte Carlo calculation of an integrated cross section to perform the summation over the series in a probabilistic way, evaluating the leading terms (which are numerically more significant but cheaper to evaluate) more often, and the higher-order terms (which are numerically less significant but more expensive to evaluate) less often. The differing frequencies of evaluation will be compensated by different weight factors for the different terms, with the frequencies (for example) adjusted so that each term in the color expansion contributes roughly an equal amount to the total error in the Monte Carlo evaluation. (An adaptive algorithm such as VEGAS [39] could be used to select the frequencies and calculate the appropriate weights.)

10. Summary

We have presented a color decomposition for one-loop amplitudes in an $SU(N)$ gauge theory with no matter content, analogous to the trace decomposition of tree-level amplitudes. The full on-shell amplitude $\mathcal{A}_n$ can be written as a sum over double traces times partial amplitudes $A_{n;j}$,

$$\mathcal{A}_n^{1-loop} = g^n \sum_{j=1}^{\lfloor n/2 \rfloor+1} \sum_{\sigma \in S_n/S_{n;j}} \mathrm{Gr}_{n;j}\left(\sigma(1),\ldots,\sigma(n)\right) A_{n;j}\left(\sigma(1),\ldots,\sigma(n)\right) \tag{10.1}$$

where $\mathrm{Gr}_{n;j}$ denotes a double trace structure,

$$\begin{aligned} \mathrm{Gr}_{n;1}\left(1,\ldots,n\right) &= \mathrm{Tr}(1)\,\mathrm{Tr}(T^{a_1}\ldots T^{a_n}) \\ &= N_c\,\mathrm{Tr}(T^{a_1}\ldots T^{a_n}) \\ \mathrm{Gr}_{n;j}\left(1,\ldots,n\right) &= \mathrm{Tr}(T^{a_1}\ldots T^{a_{j-1}})\,\mathrm{Tr}(T^{a_j}\ldots T^{a_n}) \end{aligned} \tag{10.2}$$

and $S_{n;j}$ is the subset of the permutation group S_n that leaves the trace structure $\mathrm{Gr}_{n;j}$ invariant.

The one-loop $A_{n;j}$ have properties analogous to those of their tree-level counterparts: they are gauge invariant, and satisfy a symmetry equation,

$$\forall \sigma \in S_{n;j}, \qquad A_{n;j}(\sigma(1),\ldots,\sigma(n)) = A_{n;j}(1,\ldots,n) \tag{10.3}$$

and a reflection identity,

$$A_{n;j}(R_{n;j}(1,\ldots,n)) = (-1)^n A_{n;j}(1,\ldots,n) \tag{10.4}$$

where

$$R_{n;j}(i_1,\ldots,i_n) = (i_{j-1},\ldots,i_1,i_n,\ldots,i_j)\,. \tag{10.5}$$

In addition, they satisfy the decoupling equations

$$\sum_{\sigma \in Z_{n-1}\{1,\ldots,n-1\}} A_{n;1}(\sigma(1),\ldots,\sigma(n-1),n) + A_{n;2}(n,1,\ldots,n-1) = 0 \tag{10.6a}$$

$$\begin{aligned} &\sum_{\sigma \in Z_{n-j}\{j,\ldots,n-1\}} A_{n;j}(1,\ldots,j-1,\sigma(j),\ldots,\sigma(n-1),n) \\ &+ \sum_{\sigma \in Z_{j-1}\{1,\ldots,j-1\}} A_{n;j+1}(\sigma(1),\ldots,\sigma(j-1),n,j,\ldots,n-1) = 0 \end{aligned} \qquad (3 \le j \le \lfloor n/2 \rfloor) \tag{10.6b}$$

$$\sum_{\sigma\in Z_{(n-1)/2}\{\frac{n+1}{2},\ldots,n-1\}} A_{n;(n+1)/2}\left(1,\ldots,\frac{n-1}{2},\sigma\left(\frac{n+1}{2}\right),\ldots,\sigma(n-1),n\right)$$
$$+\sum_{\sigma\in Z_{(n-1)/2}\{1,\ldots,\frac{n-1}{2}\}} A_{n;(n+1)/2}\left(\frac{n+1}{2},\ldots,n-1,\sigma(1),\ldots,\sigma\left(\frac{n-1}{2}\right),n\right)=0 \qquad (n \text{ odd}) \qquad (10.6c)$$

$$\sum_{\sigma\in S_{j-1}/Z_{j-1}\{n-j+2,\ldots,n\}} A_{n;j}\left(\sigma(n-j+2),\ldots,\sigma(n),1,\ldots,n-j+1\right)=$$
$$(-1)^{j+1}\sum_{\sigma\in COP_{n-1}^{\{1,\ldots,n-j+1\}}\{1,\ldots,n-1\}} A_{n;1}\left(\sigma(1),\ldots,\sigma(n-1),n\right)\,. \qquad (3\leq j\leq \lfloor n/2\rfloor+1)(10.6d)$$

Examples of these equations for the four- through six-point amplitudes can be found in section 6. We have discussed the structure of the color-summed cross-section through next-to-leading order in α_s in sections 8 and 9.

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Appendix I. String Model

To construct an appropriate four-dimensional string model, we follow the fermionic formulation of Kawai, Lewellen, and Tye (KLT) [26]. We discuss the construction of a heterotic string theory containing an $SU(9)$ pure gauge theory in its infinite-tension limit. There is no particular significance to nine colors; it just happens to be an easy model to construct and analyze. It is not difficult to construct such models with other gauge groups, including $SU(3)$.

The four-dimensional model at hand is specified by the five "basis" vectors,

$$\begin{aligned}
W_0 &= \left(\tfrac{1}{2}^{22}\middle|\,\tfrac{1}{2}^{10}\right)\\
W_1 &= \left(\tfrac{1}{3}^{18}0^4\middle|\,0^{10}\right)\\
W_2 &= \left(0^9\tfrac{1}{2}^9\tfrac{1}{2}000\middle|\,\tfrac{1}{2}\left(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\right)\left(\tfrac{1}{2}00\right)\left(\tfrac{1}{2}00\right)\right)\\
W_3 &= \left(0^9\tfrac{1}{2}^9 0\tfrac{1}{2}00\middle|\,\tfrac{1}{2}\left(0\tfrac{1}{2}0\right)\left(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\right)\left(0\tfrac{1}{2}0\right)\right)\\
W_4 &= \left(0^9\tfrac{1}{2}^9 00\tfrac{1}{2}0\middle|\,\tfrac{1}{2}\left(00\tfrac{1}{2}\right)\left(00\tfrac{1}{2}\right)\left(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\right)\right)
\end{aligned} \qquad (I.1)$$

where l^n signifies n contiguous components with value l. The triplet grouping for internal right-movers arises from requiring world sheet supersymmetry [26]. The gauge group we will be interested in corresponds to the first nine left-mover oscillators, while the space-time index for vectors is carried by the first right-mover complex fermion. It is straightforward to show that this model satisfies the KLT constraint equations presented in ref. [26] and hence is modular invariant.

We do not present a complete analysis here; it is straightforward though tedious to verify the properties of the model. (A more detailed discussion was presented in earlier work [23].) Here we are interested in the spectrum of massless particles, since only these will survive in the infinite-tension limit. Sectors containing massless particles must have both left and right vacuum energies that are zero or negative. There are seventeen such sectors; sixteen of these are easily eliminated, as exciting a gauge oscillator in those sectors would necessarily yield a massive state. These sectors also contain no tachyons. The remaining sector is the Neveu-Schwarz (W_0) sector, which is the one containing the graviton and the gauge bosons. The coefficients of various terms in the generalized GSO projection in the Neveu-Schwarz sector are given by $C^{W_0}_{\vec{\beta}} = -\cos 2\pi\beta_\dagger$ where $\beta_\dagger$ is the boundary condition of the first right-mover. The generalized GSO projections in this model imply that the only massless particles which carry gauge charge of the $SU(9) \times SU(9)$ subgroup are the gauge bosons themselves. (The above properties hold independent of the choice of KLT 'structure constants', so long as they are consistent in the sense of ref. [26].)

As a result, this string model yields a tachyon-free pure gauge theory in the infinite-tension limit.

Appendix II. Integration By Parts and Banishment of Double Derivatives

In this appendix, we argue that it is always possible to remove the double derivatives of the bosonic Green functions ($\ddot{G}_B$) from all terms in the amplitude (4.4) (without generating any $\dot{G}_F$ terms) by appropriate integrations by parts. The idea is to integrate by parts until the double derivative is shifted to a Green function which depends on a $\overline{\nu}_i$ which appears only once in the term of interest. A further integration by parts with respect to this $\overline{\nu}_i$ then eliminates all $\ddot{G}_B$'s.

This strategy might founder only if we cannot find a $\overline{\nu}$ appearing only once in the arguments of the different Green functions, that is, if the $\overline{\nu}$'s which are in the argument of a $\ddot{G}_B$ are part of a cyclic set (in the sense of section 5), so that repeated integrations by parts only shifts the extra derivative around; in a cycle each $\overline{\nu}$ appears twice.

However, in the general N-point amplitude (4.4) elements of a cyclic set cannot appear in the arguments of any $\ddot{G}_B$'s, because of the multi-linearity requirement on the polarization vectors. For example, consider a potential cycle of the form $\ddot{G}_B(\overline{\nu}_{12})\dot{G}_B(\overline{\nu}_{23})\cdots\dot{G}_B(\overline{\nu}_{N1})$ we would have $\varepsilon_1 \cdot \varepsilon_2$

associated with $\ddot{G}_B(\overline{\nu}_{12})$, an ε_3 associated with $\dot{G}_B(\overline{\nu}_{23})$ and so on until we come to $\dot{G}_B(\overline{\nu}_{N1})$ which would again require an ε_1; this is not allowed from the multilinearity of the polarizations. This argument extends to the appearance of more than one $\ddot{G}_B$ with arguments in a cyclic set.

We can also show that such cycles containing $\ddot{G}_B$'s cannot be created by integrating by parts a $\ddot{G}_B$ external to the cycle. Consider, for example, the term

$$(\dot{G}_B(\overline{\nu}_{12})\dot{G}_B(\overline{\nu}_{23})\cdots\dot{G}_B(\overline{\nu}_{N1}))\ddot{G}_B(\overline{\nu}_{1M}). \tag{II.1}$$

One might worry that an integration by parts with respect to $\overline{\nu}_1$ could introduce a $\ddot{G}_B$ into the cycle. However, this term cannot appear in the amplitude (4.4) because once again the multilinearity of the amplitude prevents it; a term of this form would require ε_1 to appear twice. This argument can be extended in a straightforward way to show that the process of integrating by parts does not create any $\ddot{G}_B$'s or $\dot{G}_F$'s in cycles.

[We can in fact avoid worrying about the appearance of $\dot{G}_F$'s, for if a θ_{i3} integration has brought down a fermionic Green function, the corresponding θ_{i4} integration cannot bring down a bosonic Green function, since a θ_{i3} and θ_{i4} with the same external index always multiply a given bosonic Green function, whether $\dot{G}_B$ or $\ddot{G}_B$. Rather, the corresponding θ_{i4} must bring down another fermionic Green function, which will contain a $\overline{\nu}_i$ in its argument. Thus a $\overline{\nu}$ appearing in the argument of any initial $\ddot{G}_B$ will not appear in the argument of a G_F. Furthermore, when we integrate by parts, in those cases when we do not eliminate all double derivatives, we shift it to another bosonic Green function in whose argument one of the $\overline{\nu}$'s is the same as one in the argument of the previous $\ddot{G}_B$ (and thus cannot appear in the argument to a fermionic G_F), and whose other argument is associated with a $\theta_{i3}\theta_{i4}$ pair (and for that reason cannot appear in the argument to a fermionic Green function). Thus at no stage would integration by parts create a $\dot{G}_F$.]

Integration by parts can thus always be used to remove all $\ddot{G}_B$'s from the amplitude (4.4) so that the only Green functions appearing in the amplitude are G_F's, G_B's and $\dot{G}_B$'s.

Appendix III. Notation and Normalizations

We define theta functions for general twisted boundary conditions by

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu\,|\,\tau) = \sum_{n\in Z} e^{\pi i(n+\alpha-1/2)^2\tau} e^{2\pi i(n+\alpha-1/2)(\nu-\beta-1/2)} \tag{III.1}$$

Then

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu\,|\,\tau) = e^{\pi i\alpha^2\tau} e^{2\pi i\alpha(\nu-\beta-1/2)}\vartheta_1(\nu+\alpha\tau-\beta|\tau) \tag{III.2}$$

where $\vartheta_1 = \vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ is the conventional first Jacobi theta function.

We remind the reader of the definition of the Dedekind η function,

$$\begin{aligned}\eta(\tau) &= e^{\pi i\tau/12}\prod_{n=1}^{\infty}\left(1-e^{2\pi i n\tau}\right)\\ &= \sqrt[3]{\vartheta'\begin{bmatrix}0\\0\end{bmatrix}(0\,|\,\tau)\Big/2\pi}\end{aligned} \tag{III.3}$$

where the prime indicates differentiation with respect to the first argument.

The bosonic partition function is

$$\mathcal{Z}_B(\tau) = \left(\eta(\tau)\overline{\eta}(\tau)\sqrt{\operatorname{Im}\tau}\right)^{2-D} \tag{III.4}$$

where D is the number of spacetime dimensions.

We define $\mathcal{Z}_1\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right](\tau)$ to be the partition function for a single left-moving complex fermion with $\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]$ boundary conditions,

$$\mathcal{Z}_1\begin{bmatrix}\alpha\\\beta\end{bmatrix}(\tau) = \operatorname{Tr}\left[e^{2\pi i\hat{H}_\alpha\tau}e^{2\pi i(\frac{1}{2}-\beta)\hat{N}_\alpha}\right] = \frac{e^{-2\pi i(1/2-\alpha)(1/2+\beta)}\vartheta\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right](0\,|\,\tau)}{\eta(\tau)} \tag{III.5}$$

where the phase is present in order to be consistent with the KLT definition [26]. It is really irrelevant, and could be absorbed into the definitions of the summation coefficients $C^{\vec{\alpha}}_{\vec{\beta}}$.

Putting the pieces together, the complete partition function for the set of fermions with $\left[\begin{smallmatrix}\vec{\alpha}\\\vec{\beta}\end{smallmatrix}\right]$ boundary conditions is

$$\begin{aligned}\mathcal{Z}\begin{bmatrix}\vec{\alpha}\\\vec{\beta}\end{bmatrix}(\tau) &= \mathcal{Z}_B(\tau)\prod_{i=1}^{\operatorname{len}\alpha_L}\mathcal{Z}_1\begin{bmatrix}\alpha_{Li}\\\beta_{Li}\end{bmatrix}(\tau)\prod_{i=1}^{\operatorname{len}\alpha_R}\overline{\mathcal{Z}_1}\begin{bmatrix}\alpha_{Ri}\\\beta_{Ri}\end{bmatrix}(\tau)\\ &= \left(\eta(\tau)\overline{\eta}(\tau)\sqrt{\operatorname{Im}\tau}\right)^{2-D}\\ &\quad\prod_{i=1}^{\operatorname{len}\alpha_L}\frac{e^{-2\pi i(1/2-\alpha_{Li})(1/2+\beta_{Li})}\vartheta\left[\begin{smallmatrix}\alpha_{Li}\\\beta_{Li}\end{smallmatrix}\right](0\,|\,\tau)}{\eta(\tau)}\prod_{i=1}^{\operatorname{len}\alpha_R}\frac{e^{2\pi i(1/2-\alpha_{Ri})(1/2+\beta_{Ri})}\overline{\vartheta}\left[\begin{smallmatrix}\alpha_{Ri}\\\beta_{Ri}\end{smallmatrix}\right](0\,|\,\tau)}{\overline{\eta}(\tau)}\,.\end{aligned} \tag{III.6}$$

The bosonic correlation function, $G_B(\nu)$, is defined via

$$\langle X^\mu(\nu_1,\overline{\nu}_1)\,X^\nu(\nu_2,\overline{\nu}_2)\rangle_\tau = \eta^{\mu\nu}G_B(\nu=\nu_1-\nu_2)\,. \tag{III.7}$$

It can be expressed in terms of theta functions,

$$G_B(\nu) = -\frac{1}{\pi}\ln\left|2\pi e^{-\pi(\operatorname{Im}\nu)^2/\operatorname{Im}\tau}\frac{\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](\nu\,|\,\tau)}{\vartheta'\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](0\,|\,\tau)}\right|\,. \tag{III.8}$$

A dotted variable, for our purposes, will always be taken to signify differentiation with respect to $\overline{\nu}$,

$$\dot{X} = \partial_{\overline{\nu}} X \ . \tag{III.9}$$

In a slight abuse of notation, we write the correlation function for right-movers as $\dot{G}_B(\overline{\nu})$ although in fact it is equal to $\overline{\dot{G}_B(\nu)}$.

We thus have

$$\dot{G}_B(\overline{\nu}) = \frac{i\,\mathrm{Im}\,\nu}{\mathrm{Im}\,\tau} - \frac{1}{2\pi}\frac{\vartheta'\left[{0 \atop 0}\right](\overline{\nu}\,|\,-\overline{\tau})}{\vartheta\left[{0 \atop 0}\right](\overline{\nu}\,|\,-\overline{\tau})} \ . \tag{III.10}$$

The fermionic particle correlation function $G_F\left[{\alpha \atop \beta}\right](\nu)$ and anti-particle correlation function $\hat{G}_F\left[{\alpha \atop \beta}\right](\nu)$ are defined as follows (excluding the case $\alpha = \beta = 0$):

$$\begin{aligned} \left\langle \Psi^{i\dagger}(\nu_1)\,\Psi^{j}(\nu_2)\right\rangle^{\alpha}_{\beta;\tau} &= \delta^{ij} G_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu = \nu_1 - \nu_2) \\ \left\langle \Psi^{i}(\nu_1)\,\Psi^{j\dagger}(\nu_2)\right\rangle^{\alpha}_{\beta;\tau} &= \delta^{ij} \hat{G}_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu = \nu_1 - \nu_2) \end{aligned} \tag{III.11}$$

where here the expectation value is understood to exclude a factor of the partition function.

These correlation functions can also be expressed in terms of theta functions,

$$\begin{aligned} G_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu) &= \frac{\vartheta\left[{\alpha \atop \beta}\right](\nu\,|\,\tau)\vartheta'\left[{0 \atop 0}\right](0\,|\,\tau)}{2\pi\vartheta\left[{0 \atop 0}\right](\nu\,|\,\tau)\vartheta\left[{\alpha \atop \beta}\right](0\,|\,\tau)} \\ \hat{G}_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu) &= G_F\begin{bmatrix}1-\alpha\\ 1-\beta\end{bmatrix}(\nu) = -G_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(-\nu) \end{aligned} \tag{III.12}$$

where the last equality derives from a theta function identity.

As $\nu \to 0$ the various Green's functions have the following behavior:

$$\begin{aligned} \exp\left(G_B(\nu)\right) &\to |\nu|^{-1/\pi} \times \text{constant} \\ \dot{G}_B(\overline{\nu}) &\to -\frac{1}{2\pi\overline{\nu}} + \mathcal{O}(\overline{\nu}) \\ G_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu) &\to \frac{1}{2\pi\nu} + \frac{\vartheta'\left[{\alpha \atop \beta}\right](0)}{2\pi\vartheta\left[{\alpha \atop \beta}\right](0)} + \mathcal{O}(\nu) \end{aligned} \tag{III.13}$$

The self-contractions are obtained by taking $\nu \to 0$ but with the pole piece subtracted [40] so that

$$S_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{\vartheta'\left[{\alpha \atop \beta}\right](0)}{2\pi\vartheta\left[{\alpha \atop \beta}\right](0)} \ . \tag{III.14}$$

The vertex operator for emission of a gauge boson, in the F_1 picture for the right-movers, is

$$V^{a}(\varepsilon, k; \nu, \overline{\nu}) = -\sqrt{2}g\sqrt{\lambda}T^{a}{}_{i}{}^{j} : \Psi^{i\dagger}(\nu)\Psi_j(\nu)\,\varepsilon\cdot\left(\partial_{\overline{\nu}}\overline{X}(\overline{\nu}) + i\sqrt{\lambda}\overline{\Psi}(\overline{\nu})\,k\cdot\overline{\Psi}(\overline{\nu})\right)e^{i\sqrt{\lambda}k\cdot(X(\nu,\overline{\nu}))} : \tag{III.15}$$

or, using Grassman variables to put it into an exponential form,

$$V^a(\varepsilon, k; \nu, \overline{\nu}) = \sqrt{2}g\sqrt{\lambda}T^a{}_i{}^j : \int d\theta_1\, d\theta_2\, d\theta_3\, d\theta_4$$
$$\exp\Big(i\sqrt{\lambda}k \cdot X(\nu,\overline{\nu}) + \theta_1\Psi^{i\dagger}(\nu) + \theta_2\Psi_j(\nu)$$
$$+\theta_3\theta_4\varepsilon \cdot \dot{\overline{X}}(\overline{\nu}) + i\sqrt{\lambda}\theta_3 k \cdot \overline{\Psi}(\overline{\nu}) + \theta_4\varepsilon \cdot \overline{\Psi}(\overline{\nu})\Big) : . \tag{III.16}$$

The N-point amplitude is then given by

$$\mathcal{A}_N = \frac{1}{2(16\pi^2)}\frac{1}{\lambda^2}\int \frac{d^2\tau}{(\operatorname{Im}\tau)^2}(\operatorname{Im}\tau)\int d^2\nu_1 \cdots \int d^2\nu_{N-1}$$
$$\mathcal{Z}_B(\tau)\sum_{\vec{\alpha},\vec{\beta}} C^{\vec{\alpha}}_{\vec{\beta}}\mathcal{Z}_F\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}(\tau)\ \langle V^{a_1}(\nu_1)\cdots V^{a_N}(\nu_N)\rangle^{\vec{\alpha}}_{\vec{\beta};\tau}\ . \tag{III.17}$$

Evaluating the correlation functions gives (in Minkowski space)

$$\frac{1}{2(16\pi^2)}\lambda^{n/2-2}(\sqrt{2}g)^n\, T^{a_1}{}_{m_1}{}^{\widehat{m}_1}\cdots T^{a_n}{}_{m_n}{}^{\widehat{m}_n}$$
$$\int\frac{d^2\tau}{(\operatorname{Im}\tau)^2}\int\left(\prod_{i=1}^n d\theta_{i1}\, d\theta_{i2}\, d\theta_{i3}\, d\theta_{i4}\right)\int\left(\prod_{i=1}^n d^2\nu_i\right)\sum_{\vec{\alpha},\vec{\beta}} C^{\vec{\alpha}}_{\vec{\beta}}\mathcal{Z}^{\vec{\alpha}}_{\vec{\beta}}(\tau)$$
$$\prod_{i=1}^n \exp\left[-\theta_{i1}\theta_{i2}\delta^{m_i}{}_{\widehat{m}_i} S_F\begin{bmatrix}\alpha_{m_i}\\ \beta_{\widehat{m}_j}\end{bmatrix}\right]$$
$$\prod_{i<j}\exp\Bigg[\lambda k_i \cdot k_j\, G_B(\nu_i-\nu_j)$$
$$-\theta_{i1}\theta_{j2}\delta^{m_i}{}_{\widehat{m}_j} G_F\begin{bmatrix}\alpha_{m_i}\\ \beta_{\widehat{m}_j}\end{bmatrix}(\nu_i-\nu_j) - \theta_{i2}\theta_{j1}\delta^{m_j}{}_{\widehat{m}_i}\hat{G}_F\begin{bmatrix}\alpha_{m_j}\\ \beta_{\widehat{m}_i}\end{bmatrix}(\nu_i-\nu_j)$$
$$-\theta_{i3}\theta_{j3}\lambda k_i \cdot k_j\, G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i-\overline{\nu}_j)$$
$$+i\sqrt{\lambda}(\theta_{i3}\theta_{j4}k_i \cdot \varepsilon_j + \theta_{i4}\theta_{j3}k_j \cdot \varepsilon_i)\, G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i-\overline{\nu}_j)$$
$$-i\sqrt{\lambda}(\theta_{i3}\theta_{i4}k_j \cdot \varepsilon_i - \theta_{j3}\theta_{j4}k_i \cdot \varepsilon_j)\, \dot{G}_B(\overline{\nu}_i-\overline{\nu}_j)$$
$$+\theta_{i4}\theta_{j4}\varepsilon_i \cdot \varepsilon_j\, G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i-\overline{\nu}_j)$$
$$+\theta_{i3}\theta_{i4}\theta_{j3}\theta_{j4}\varepsilon_i \cdot \varepsilon_j\, \ddot{G}_B(\overline{\nu}_i-\overline{\nu}_j)\Bigg]\ . \tag{III.18}$$

This formula is valid in all sectors except Ramond–Ramond, where the fermionic zero mode demands special treatment. However, that sector enters only into parity-violating amplitudes, and so is not relevant to any of the calculations in this paper. The normalization of the amplitude has been calculated by Polchinski [41] and Sakai and Tanii [42].

Appendix IV. Structure of Two-Trace Next-to-Leading Color Terms

We wish to show that any relative permutation which gives rise to a term of $\mathcal{O}(N_c^{n-2})$ from the sum over color indices of a product of two traces, is either an element of $Z_{n-1}\{1,\ldots,n-1\}$ or of $\mathbb{Z}_{n-1}\{1,\ldots,n-1\}$, where the latter set was defined in equation (8.17).

Consider the product

$$C = \mathrm{Tr}\,(T^{a_n}\cdots T^{a_1})\;\mathrm{Tr}\left(T^{a_{\rho(1)}}\cdots T^{a_{\rho(n-1)}}{}^{i}T^{a_n}\right) = \mathrm{Tr}\,(T^{a_{n-1}}\cdots T^{a_1}T^{a_{\rho(1)}}\cdots T^{a_{\rho(n-1)}}) \qquad \text{(IV.1)}$$

summed over all a_i. Assume only that $\rho \neq 1$. Then, without loss of generality, we may take

$$\begin{aligned} \rho(j) &= j \qquad \text{for } j = 1,\ldots,f-1 \qquad (f \text{ may be } 1)\\ \rho(j) &= j \qquad \text{for } j = l+1,\ldots,n-1 \qquad (l \text{ may be } n-1) \end{aligned} \qquad \text{(IV.2)}$$

where $l > f$. Define $t = \rho(f)$ and $s = \rho^{-1}(f)$ (note that $f < t \leq l$ and $f < s \leq l$); using the Fierz identities, we can re-write C as

$$\begin{aligned} &N_c^{n-2+f-l}\,\mathrm{Tr}\,(T^{a_l}\cdots T^{a_f}T^{a_{\rho(f)}}\cdots T^{a_{\rho(l)}})\\ &= N_c^{n-2+f-l}\,\mathrm{Tr}\,(T^{a_l}\cdots T^{a_{t+1}}T^{a_{\rho(f+1)}}\cdots T^{a_{\rho(l)}})\;\mathrm{Tr}\,(T^{a_{t-1}}\cdots T^{a_f})\\ &= N_c^{n-2+f-l}\,\mathrm{Tr}\,(T^{a_l}\cdots T^{a_{t+1}}T^{a_{\rho(f+1)}}\cdots T^{a_{\rho(s-1)}}T^{a_{t-1}}\cdots T^{a_{f+1}}T^{a_{\rho(s+1)}}\cdots T^{a_{\rho(l)}})\,. \end{aligned} \qquad \text{(IV.3)}$$

The maximal power of N_c that can emerge from the trace in this case is $1+(l-t)+(t-f-1) = l-f$, so that C (as we already know) cannot be greater than N_c^{n-2}. This means that we cannot afford any more detours through the Fierz identities, so all the products of matrices inside the trace must collapse to Casimir operators; thus, we must have

$$\begin{aligned} \rho(f+1) &= t+1\\ \rho(f+2) &= t+2\\ &\;\vdots\\ \rho(f+l-t) &= l \end{aligned} \qquad \text{(IV.4)}$$

and

$$\begin{aligned} \rho(s+1) &= f+1\\ \rho(s+2) &= f+2\\ &\;\vdots\\ \rho(l) &= f+l-s \end{aligned} \qquad \text{(IV.5)}$$

and, in addition, we must have $t-f \geq l-s+1$ and

$$\begin{aligned} \rho(s-1) &= t-1\\ \rho(s-2) &= t-2\\ &\;\vdots\\ \rho(s-(t-f-l+s-1)) &= t-(t-f-l+s-1)\,. \end{aligned} \qquad \text{(IV.6)}$$

From equations (IV.5) and (IV.6) it is clear, that by performing a cyclic transformation on the image of elements $s-(t-f-l+s-1),\ldots,s,\ldots,l$ under ρ, one can bring them into an ordered sequence. After such a cyclic transformation, the image of ρ has the form

$$s-(t-f-l+s-1),\ldots,s,\ldots,l,f,f+1,\ldots,f+l-t \qquad \text{(IV.7)}$$

because of equation (IV.4). A further cyclic transformation on these elements will then bring them all back into an ordered sequence. From the definition of $\mathbb{Z}_k$, however, one can see that this sequence of cyclic transformations is exactly the inverse of an element of $\mathbb{Z}_{n-1}\cup Z_{n-1}$. The reader may verify that the boundary cases (for example, $t=l$) are included correctly in this description.

Appendix V. Structure of Three-Trace Next-to-Leading Color Terms

We wish to show that if $\rho\in S_n/S_{n;j}$, then

$$C\equiv \mathrm{Tr}\left(T^{a_n}\cdots T^{a_1}\right)\,\mathrm{Tr}\left(T^{a_{\rho(1)}}\cdots T^{a_{\rho(j-1)}}\right)\,\mathrm{Tr}\left(T^{a_{\rho(j)}}\cdots T^{a_{\rho(n)}}\right)=N_c^{n-1} \qquad \text{(V.1)}$$

if and only if $\rho\in CCP\binom{n}{j}$, where the set is defined just before equation (9.14). We will assume that $j>1$ and that $n-j+1\geq j-1$.

Define $t=\rho(1)$. If $\rho\in CCP\binom{n}{j}$, then using the Fierz identities, we find for $t+j-2\leq n$ that

$$\begin{aligned}
C &= \mathrm{Tr}\left(T^{a_n}\cdots T^{a_1}\right)\,\mathrm{Tr}\left(T^{a_t}\cdots T^{a_{t+j-2}}\right)\,\mathrm{Tr}\left(T^{a_{t+j-1}}\cdots T^{a_n}T^{a_1}\cdots T^{a_{t-1}}\right)\\
&= \mathrm{Tr}\left(T^{a_1}\cdots T^{a_{t-1}}T^{a_{t+j-1}}\cdots T^{a_{n-1}}T^{a_{n-1}}\cdots T^{a_1}\right)\,\mathrm{Tr}\left(T^{a_t}\cdots T^{a_{t+j-2}}\right)\\
&= \mathrm{Tr}\left(T^{a_1}\cdots T^{a_{t-1}}T^{a_{t+j-1}}\cdots T^{a_{n-1}}T^{a_{n-1}}\cdots T^{a_{t+j-1}}T^{a_t}\cdots T^{a_{t+j-3}}T^{a_{t+j-3}}\cdots T^{a_1}\right)\\
&= N_c^{(j-2)+(t-1)+(n-t-j+1)+1}\\
&= N_c^{n-1}\,.
\end{aligned} \qquad \text{(V.2)}$$

The evaluation in the case $t+j-2>n$ is similar, and the result is identical.

To prove the 'only if' part of the assertion, consider C in the general case. We continue to label $\rho(1)$ by t, and define $u=\rho(j)$; then

$$\begin{aligned}
C &= \mathrm{Tr}\left(T^{a_n}\cdots T^{a_{t+1}}T^{a_{\rho(2)}}\cdots T^{a_{\rho(j-1)}}T^{a_{t-1}}\cdots T^{a_1}\right)\,\mathrm{Tr}\left(T^{a_{\rho(j)}}\cdots T^{a_{\rho(n)}}\right)\\
&= \mathrm{Tr}\left(T^{a_n}\cdots T^{a_{u+1}}T^{a_{\rho(j+1)}}\cdots T^{a_{\rho(n)}}T^{a_{u-1}}\cdots T^{a_{t+1}}T^{a_{\rho(2)}}\cdots T^{a_{\rho(j-1)}}T^{a_{t-1}}\cdots T^{a_1}\right)
\end{aligned} \qquad \text{(V.3)}$$

if $t<u$, with a similar expression if $t>u$. Because there are only $2n-4$ matrices inside the trace, the maximal power of N_c that we can obtain after summing over the remaining a_i is N_c^{n-1}, and that value we will obtain only if we avoid a detour through the Fierz identities, that is if the string of matrices collapses to a product of Casimir operators. This can happen only if for some b

$$\begin{aligned} \rho(j+1) &= u+1 \\ \rho(j+2) &= u+2 \\ &\vdots \\ \rho(j+b) &= u+b \bmod n \end{aligned} \qquad \text{and} \qquad \begin{aligned} \rho(n) &= u-1 \\ \rho(n-1) &= u-2 \\ &\vdots \\ \rho(j+b+1) &= u-(n-j-b) \end{aligned} \tag{V.4}$$

and for some b'

$$\begin{aligned} \rho(2) &= t+1 \\ \rho(3) &= t+2 \\ &\vdots \\ \rho(b') &= t+b'-1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \rho(j-1) &= t-1 \\ \rho(j-2) &= t-2 \\ &\vdots \\ \rho(b'+1) &= t-(j-b'-1) \bmod n\,. \end{aligned} \tag{V.5}$$

Equations (V.4) tell us that the elements $j,\ldots,n$ must be mapped into a cyclic permutation of a set of elements that are consecutive $\bmod n$, that is up to a cyclic transformation on all n indices. Equations (V.5) tell us that the same thing is true of the elements $1,\ldots,j-1$. Using the freedom to make cyclic transformations on the indices inside each trace, inherent in picking a ρ out of a set equivalent under $S_{n;j}$, we can then express ρ as an element of $CCP\binom{n}{j}$.

Appendix VI. The Color Decomposition from Feynman Diagrams

We give here an outline of a derivation of the color decomposition of pure gauge theory amplitudes from a conventional diagrammatic point of view. Every Feynman diagram in a $SU(N)$ gauge theory with no matter fields is built out of five components: a three-gluon vertex, V_3; a four-gluon vertex, V_4; a ghost-ghost-gluon vertex, $V_{\rm gh}$; a gluon propagator, $D_{\rm gl}$; and a ghost propagator, $D_{\rm gh}$. The color structure of these objects is expressed in terms of the Kronecker delta on Lie algebra indices and in terms of the group structure constants. These structure constants are related to the generators via

$$f^{abc} = -\frac{i}{\sqrt{2}}\,{\rm Tr}\left(\left[T^a,T^b\right]T^c\right) \tag{VI.1}$$

where the appearance of $\sqrt{2}$ is due to our normalization of the generators.

The first observation is that because all structure constants involving an additional $U(1)$ field will vanish, it will decouple from physical quantities; we can thus add a $U(1)$ gauge field (and corresponding ghost), and perform all our sums in $U(N)$ rather than $SU(N)$. We can now rewrite the vertices as sums of coefficients times traces of generators:

$$
\begin{aligned}
V_3 &= igf^{a_1a_2a_3}K_3^{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)\\
&= \frac{g}{\sqrt{2}}\left(\mathrm{Tr}\left(T^{a_1}T^{a_2}T^{a_3}\right) - \mathrm{Tr}\left(T^{a_1}T^{a_3}T^{a_2}\right)\right) K_3\left(k_1^{\mu_1},k_2^{\mu_2},k_3^{\mu_3}\right)\\
V_4 &= -ig^2\left[f^{ba_1a_2}f^{ba_3a_4}K_{4a}^{\mu_1\mu_2\mu_3\mu_4} + f^{ba_1a_3}f^{ba_4a_2}K_{4b}^{\mu_1\mu_2\mu_3\mu_4} + f^{ba_1a_4}f^{ba_2a_3}K_{4c}^{\mu_1\mu_2\mu_3\mu_4}\right]\\
&= \frac{ig^2}{2}\Big[\left(\mathrm{Tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right) + \mathrm{Tr}\left(T^{a_4}T^{a_3}T^{a_2}T^{a_1}\right)\right)\left(K_{4a}^{\mu_1\mu_2\mu_3\mu_4} - K_{4c}^{\mu_1\mu_2\mu_3\mu_4}\right)\\
&\quad + \left(\mathrm{Tr}\left(T^{a_1}T^{a_3}T^{a_4}T^{a_2}\right) + \mathrm{Tr}\left(T^{a_1}T^{a_2}T^{a_4}T^{a_3}\right)\right)\left(K_{4b}^{\mu_1\mu_2\mu_3\mu_4} - K_{4a}^{\mu_1\mu_2\mu_3\mu_4}\right)\\
&\quad + \left(\mathrm{Tr}\left(T^{a_1}T^{a_4}T^{a_2}T^{a_3}\right) + \mathrm{Tr}\left(T^{a_1}T^{a_3}T^{a_2}T^{a_4}\right)\right)\left(K_{4c}^{\mu_1\mu_2\mu_3\mu_4} - K_{4b}^{\mu_1\mu_2\mu_3\mu_4}\right)\Big]\\
V_{\mathrm{gh}} &= -igf^{a_1a_2a_3}K_{\mathrm{gh}}^{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)\\
&= -\frac{g}{\sqrt{2}}\left(\mathrm{Tr}\left(T^{a_1}T^{a_2}T^{a_3}\right) - \mathrm{Tr}\left(T^{a_1}T^{a_3}T^{a_2}\right)\right) K_3\left(k_1^{\mu_1},k_2^{\mu_2},k_3^{\mu_3}\right)
\end{aligned}
\tag{VI.2}
$$

where the K_i are kinematic factors depending upon the momenta flowing into the loop and the corresponding Lorentz indices.

The propagators are proportional to $\delta^{a_1a_2}$, so that in constructing [off-shell] tree diagrams, they will simply contract indices. Expanding a given diagram using equation (VI.2), and performing the contractions term-by-term will lead to products of traces such as

$$
\mathrm{Tr}\left(T^{a_1}T^{a_2}T^{a_I}\right)\,\mathrm{Tr}\left(T^{a_I}T^{a_3}T^{a_4}\right)\,. \tag{VI.3}
$$

So long as we are not creating a loop, the contracted indices will lie in different traces, and using the $U(N)$ Fierz identities, we can reduce this to

$$
\mathrm{Tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right), \tag{VI.4}
$$

so that any off-shell tree diagram* can be written as a sum of kinematic coefficients times single traces of products of generators,

$$
\sum \mathrm{Tr}\left(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}\right) K(\sigma)\,. \tag{VI.5}
$$

If we now close a loop, we will be contracting the indices of two generators with the same loop, and there are two possibilities, depending on whether these generators sit next to each other, or are separated by additional generators:

$$
\begin{aligned}
&\mathrm{Tr}\left(T^{a_1}\cdots T^{a_m}T^{a_I}T^{a_I}T^{a_{m+1}}\cdots T^{a_n}\right)\\
&\mathrm{Tr}\left(T^{a_1}\cdots T^{a_m}T^{a_I}T^{a_2}\cdots T^{a_3}T^{a_I}T^{a_{m+1}}\cdots T^{a_n}\right)\,.
\end{aligned}
\tag{VI.6}
$$

---

* The reader should be careful to note that this statement depends on the off-shell continuation of the theory. It is *not* invariant under arbitrary field redefinitions (which correspond to different off-shell continuations); however it is true for the conventional choice of field variables used in Feynman diagrams.

In the first case, the contraction will produce a Casimir operator, so that we will be left with a single non-trivial trace, and an explicit power of the number of colors,

$$N_c \, \mathrm{Tr}\,(T^{a_1} \cdots T^{a_n}) \tag{VI.7}$$

while in the second case, the other Fierz identity tells us that we will get two non-trivial traces,

$$\mathrm{Tr}\,(T^{a_1} \cdots T^{a_m} T^{a_{m+1}} \cdots T^{a_n}) \, \mathrm{Tr}\,(T^{a_2} \cdots T^{a_3}) \tag{VI.8}$$

in agreement with the string approach. While the heterotic string approach may be less familiar, it provides a direct and efficient method for explicit computations of the partial amplitudes.

Beyond one loop, one can continue to split traces, or to generate additional powers of N_c, keeping the number of traces fixed. These actions generated terms with $T \leq L + 1$ traces, and an explicit coefficient of N_c^{L+1-T}. It is also possible to combine previously split traces, generating terms with fewer than $L + 1$ traces, and an explicit coefficient less than the maximal one by some number of powers of N_c^2. One must keep in mind that any given term may disappear if the corresponding kinematic coefficients conspire appropriately; this diagrammatic derivation only tells us the set of allowed trace structures.

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Figure Captions

Fig. 1. The planar diagram contribution to the one-loop five-gluon amplitude in an open string theory.

Fig. 2. A non-planar diagram contribution to the one-loop five-gluon amplitude in an open string theory.

Fig. 3. A generic higher-loop contribution to the five-gluon amplitude in an open string theory.

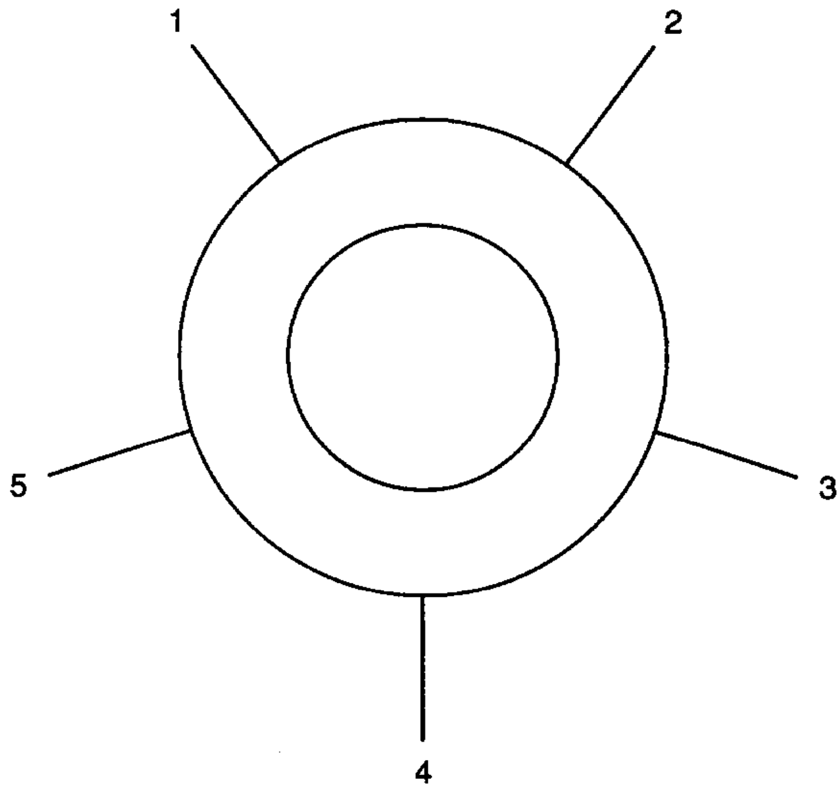

Fig. 1

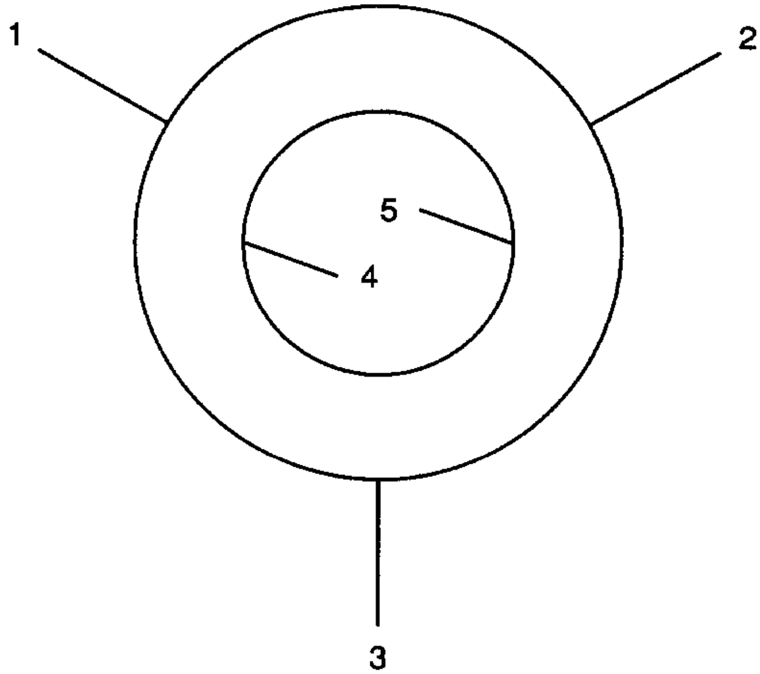

Fig. 2

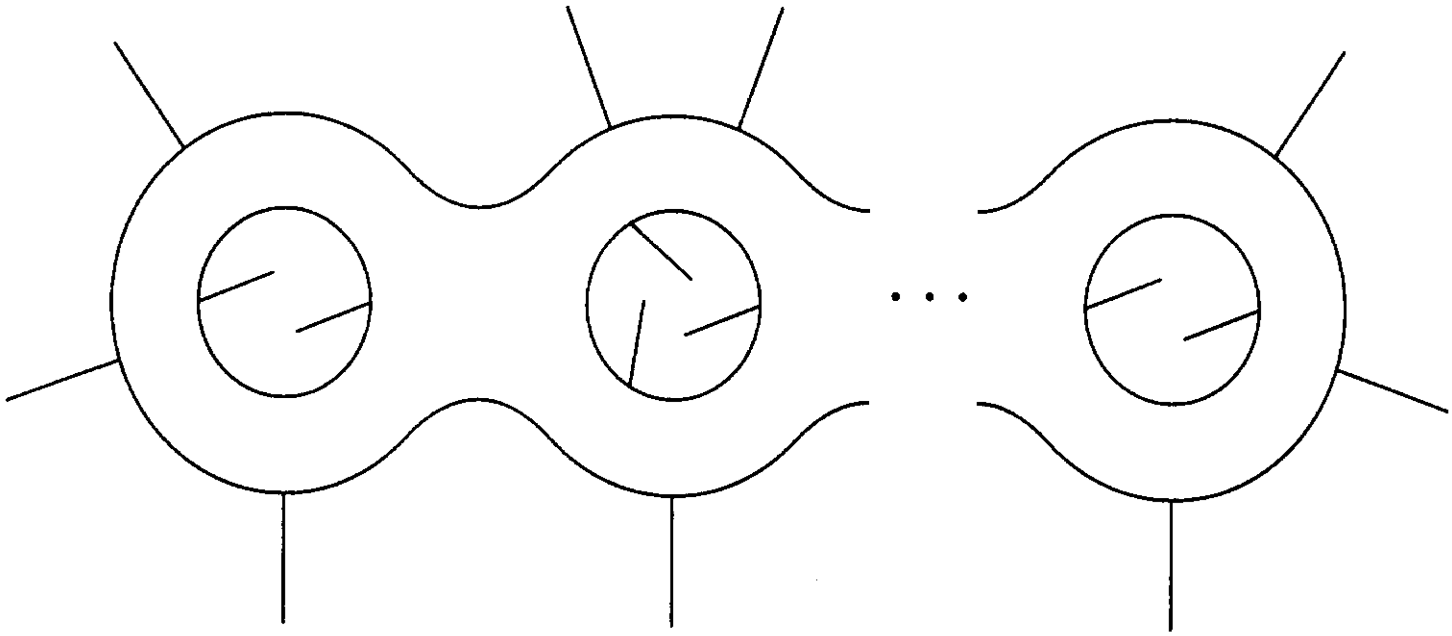

Fig. 3