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FACTORIZATION AT SMALL x*

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We show how to modify the Lipatov equation to treat short distance cross sections in the factorization theorem.

1 INTRODUCTION

In this paper we will present a solution to the problem of combining the ordinary hard scattering formalism and the perturbative reggeon of Lipatov[1]. The formalism we will describe is designed so that a single formula can be applied both at small x and at large x (and hence in the region of intermediate x). It is set up so that one can systematically use the results of higher order calculations of hard scattering cross sections and of the (Gribov-Lipatov)-Altarelli-Parisi kernel. Thus we are able to discuss nonleading logarithms, and indeed non-logarithmic terms. Our aims and results are therefore more general than those of the work described at this workshop by Catani[2].

We will not attempt to discuss the saturation effects that gained so much attention at this meeting. At sufficiently small x they dominate the physics, even though they are higher twist. But in this paper we will concentrate on the issue of how to make more accurate calculations when saturation effects are not important. There is a wide kinematic region when these effects are ignorable, but where x is sufficiently small that the higher order perturbative corrections to the standard hard scattering formalism are large.

In Sects. 2 and 3 we will briefly review the standard factorization formula and the Lipatov equation. This will serve to establish our notation. Then, in Sect. 4, we will present our modified Lipatov equation. In Sect. 5 we will show how to solve it in closed form, and in Sect. 6 we will present the results of numerical calculations. Sect. 7 contains our conclusions.

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2 FACTORIZATION

In this section we will review the usual hard scattering formalism[3]. For the sake of simplicity we will consider a case such as deep inelastic scattering or the inclusive photoproduction of heavy quarks or jets, where there is one initial-state hadron. We characterize the dominant scale of virtuality or transverse momentum in the hard scattering by Q^2. In deep inelastic scattering this is the virtuality of the exchanged electroweak boson, while in the photoproduction of a heavy quark it is the mass squared of the heavy quark.

The factorization theorem states that when Q is large, the cross section may be written

$$\begin{aligned}\sigma(s,Q^2) &= \sum_i \int_0^1 \mathrm{d}\xi\, \hat{\sigma}_i(\xi s, Q^2; \mu^2)\, f_i(\xi;\mu^2) \\ &= \hat{\sigma}\otimes f.\end{aligned} \tag{1}$$

Here f_i is the distribution of partons of type i in the initial-state hadron, and $\hat{\sigma}_i(\hat{s}, Q^2)$ is the coefficient function, or short-distance cross section, with a parton target of type i.

Both f_i and $\hat{\sigma}_i$ quantities are defined with an auxiliary scale μ, and the μ dependence is given by the (Gribov-Lipatov)-Altarelli-Parisi equation:

$$\frac{\mathrm{d}f_i(x)}{\mathrm{d}\ln\mu^2} = \sum_j \int_x^1 \mathrm{d}\xi\, \gamma_{\mathrm{AP}}^{ij}(\xi/x, \alpha_s) f_j(\xi). \tag{2}$$

This scale should be set to be of order Q, to avoid large logarithms $\ln(Q^2/\mu^2)$ in higher order perturbative corrections.

The rules for calculating the short-distance cross section are well-known[3]. They are to calculate the cross section at the parton level and then to make subtractions to remove the collinear region. If one sets μ to be of order Q, then the subtractions force the internal transverse momenta of graphs to be of order Q.

A great advantage of this formalism is that one can make systematic calculations for higher order corrections, in powers of α_s, for both $\hat{\sigma}$ and for the Altarelli-Parisi kernel γ_{AP}. However, if one considers the case that the ratio s/Q^2 is large, then these higher order corrections contain large logarithms of $\hat{s}/Q^2$ and ξ/x. This ruins the accuracy of the calculations, and creates the set of problems associated with small-x physics. (We have defined x to be the minimum parton momentum fraction in eq. (2); it is proportional to Q^2/s.)

3 LIPATOV'S EQUATION

The Lipatov equation[1] is applicable to cross sections in the limit of high s. If we consider hadron-hadron scattering, then graphs have the form of factors for each hadron connected by a generalized ladder (Fig. 1). The ladder satisfies a Bethe-Salpeter-like equation; this is the Lipatov equation. By making suitable kinematic restrictions, the equation may be applied in hard scattering processes, like minijet production (with $E_\perp \ll \sqrt{s}$). Note that the

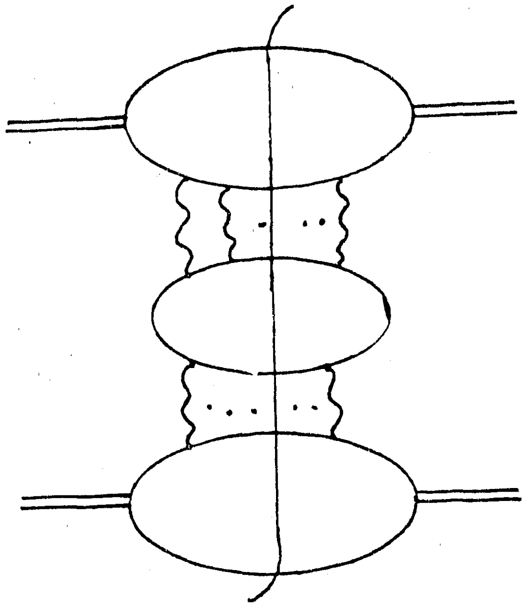

Fig. 1. Ladders for perturbative Reggeon.

ladder form is not valid for the contributions of individual graphs, but one must use Ward identities in a rather subtle way to obtain the ladder form[4].

Once one has the ladder structure, the total ladder is obtained by summing over rungs: $1 + L + L^2 + L^3 + ...$, where L represents one rung. The multiplication of the Ls is in the sense of a convolution of the loop momenta that connect the individual rungs. The sum is a geometrical series, $1/(1 - L)$, and satisfies the following equation:

$$\frac{1}{1-L} = 1 + \frac{1}{1-L} \times L. \tag{3}$$

An explicit formula is most easily obtained by performing a Mellin transform in the center-of-mass energy:

$$\tilde{\sigma}(j) = \int \mathrm{d}s\, s^{-j-1} \sigma(s). \tag{4}$$

Then the equation is

$$X(j, k_\perp) = \text{lowest order} + \frac{\alpha_s N_c}{\pi(j-1)} \int \mathrm{d}k_\perp'^2 \left[\frac{X(j, k'_\perp) - X(j, k_\perp)}{|k_\perp'^2 - k_\perp^2|} + \frac{X(j, k_\perp)}{\sqrt{k_\perp'^4 + 4k_\perp^4}} \right], \tag{5}$$

where X represents the sum of the ladders, possibly convoluted with the 'impact factor'[5] associated with one of the hadrons.

The kernel of the equation has a pole at $j = 1$; this is the result of the exchange of the spin-1 gluon. The effect of gluon exchange in individual graphs is to give cross sections that at high energy are constant up to logarithms of s. After solving the equation we get cross sections that grow like a power of s.

The Lipatov equation is valid when $s/Q^2 \gg 1$. It makes no separation between hard and soft physics. Indeed a more exact version of the equation would have the fixed coupling α_s

replaced by a running coupling $\alpha_s(k'_\perp)$. It is this lack of separation between hard and soft physics that we wish to overcome in this paper. The problem is that the solution of the equation as it stands inevitably brings in the region of low transverse momentum, where the running coupling is large and where one must therefore treat nonperturbative effects.

Note also that at large enough s, saturation effects become important. This is inevitable, since the equation leads to cross sections that grow like a power of s. The saturation effects referred to in the introduction are needed to restore the Froissart bound. We will assume that s is not so large that saturation is important.

4 MODIFIED LIPATOV EQUATION

The results we are in the process of deriving apply to the short-distance cross section at the parton level and to the Altarelli-Parisi kernel. Thus we retain the familiar structure of a short-distance cross section convoluted with parton distribution functions that evolve according to the Altarelli-Parisi equation. Since the higher order terms in the perturbation expansion for such quantities have subtractions that remove the collinear region, we must modify the Lipatov equation when we apply it to these quantities, by incorporating corresponding subtractions. Our resulting modified Lipatov equation only involves transverse momenta of order the scale Q of the hard scattering. Thus we are freed from the need to consider the infrared region of nonperturbative effects. In effect our equation resums the large logarithms that appear in the short-distance cross section when Q^2/s is small and in the Altarelli-Parisi kernel when x/ξ is small.

Our procedure is to recognize first that ordinary factorization continues to apply. The proof[3] needs generalization from what is actually done in the literature. Next we treat the problem that higher order corrections have large logarithms of kinematic ratios — logarithms of $\xi s/Q^2$ in eq. (1) and of ξ/x in eq. (2) — and that these imply that low order perturbation theory is inaccurate. Then in the short distance cross section $\hat{\sigma}$ and in the Altarelli-Parisi kernel, we find a generalized factorization scheme for the contributions that dominate when $\xi s/Q^2$ or ξ/x gets large. Our modified Lipatov equation applies to these further factorizations. Finally we add correction terms so that we have a formula that is also applicable when Q^2/s is not small. (In perturbation theory, the correction terms have no singularities near $j = 1$.)

The equation, and therefore the final result for the cross section, are written in terms of quantities that have no small x logarithms, and that we may therefore usefully expand in powers of α_s.

The basic idea for our equation can be obtained by considering how to obtain factorization from two-particle-reducible graphs for deep inelastic scattering: The graphs can be considered as an irreducible part, S, associated with the hadron, an irreducible part, H, associated with the virtual photon, and a ladder connecting them. If we treat the ladders as a sum over rungs, then we have (Fig. 2)

$$H \frac{1}{1-R} S. \tag{6}$$

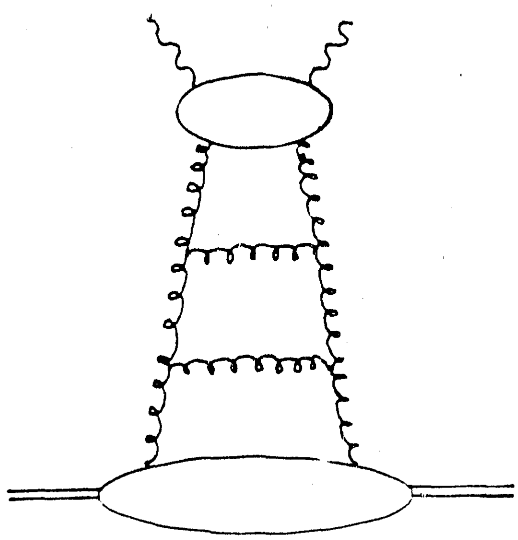

Fig. 2. Ladder graph for deep inelastic scattering, with lowest order subgraphs for the rungs.

To obtain the hard scattering, we consider the parton level cross section, which is $H/(1-R)$. The subtractions amount to a projection that kills small momenta, and the short-distance cross section can be written as

$$\hat{\sigma} = H\,\frac{1}{1-(1-A)R}. \tag{7}$$

Here, A is an operator that projects out the asymptote when the transverse momentum coming from the R factor to its right (below in the figures) is much less than the transverse momentum to its left. The scale μ arises in the definition of this projection.

Note that in QCD, the above picture of a (generalized) ladder for the dominant contributions to the deep inelastic scattering is only valid in light-cone gauge. In a general gauge, the ladders must be modified by the exchange of extra longitudinal gluons.

As an example, consider the one-rung case (Fig. 3). If we have

$$HR = \int \mathrm{d}^2k'_\perp\, H(Q,k'_\perp)\, R(k'_\perp,k_\perp), \tag{8}$$

then the projected quantity is

$$H(1-A)R = \int \mathrm{d}^2k'_\perp\, R(k'_\perp,k_\perp)\,[H(Q,k'_\perp) - H(Q,0)\,\theta(k'_\perp < \mu)]\,. \tag{9}$$

A separate calculation[6] can be made to relate μ to the more standard scale $\mu_{\overline{\mathrm{MS}}}$, and we find that at small x, $\mu = \mu_{\overline{\mathrm{MS}}}$.

The equation for the ladders at small x is obtained by a further projection to separate out the small x. We thus obtain a more complicated factorization than the standard one, which is still true. Similar ideas and results apply to the Altarelli-Parisi kernel. The modified

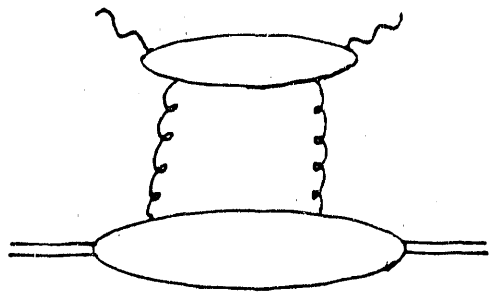

Fig. 3. One rung ladder.

Lipatov equation for the hard scattering cross section has the form:

$$UX(j,k_\perp) = U(j,k_\perp) + \frac{\alpha_s N_c}{j-1} UX \otimes \bar{L}. \tag{10}$$

Here $\bar{L}$ is our modified Lipatov kernel; we have explicitly separated out its single factor of $1/(j-1)$. The impact factor U has no singularity near $j = 1$. We obtain the complete short distance cross section for scattering off a gluon by adding a remainder term:

$$\hat{\sigma} = UX + H_{\rm rem}, \tag{11}$$

where $H_{\rm rem}$ represents the cross section with subtractions not only to cancel the collinear regions but also to cancel all the leading large $\hat{s}$ behavior. It also has no singularity near $j = 1$.

For hard scattering off a quark, we must convolute UX with a bottom impact factor B associated with the quark, and in addition make a collinear subtraction:

$$UX(1-A)\frac{1}{j-1}B = \int {\rm d}k_\perp{}^2 [UX(j,k_\perp) - UX(j,0)] \frac{1}{j-1} B(k_\perp). \tag{12}$$

The factor of $1/(j-1)$ is associated with the lowest order exchange of a gluon between the upper impact factor U and the bottom impact factor B. There will also be a remainder term.

The precise form of our modified equation, with the lowest order kernel is

$$X(j,k_\perp) = \text{lowest order} + \frac{\alpha_s N_c}{\pi(j-1)} \int {\rm d}k'_\perp{}^2 \left\{ \frac{X(j,k'_\perp) - X(j,k_\perp)}{|k'_\perp{}^2 - k_\perp{}^2|} + \frac{X(j,k_\perp)}{\sqrt{k'_\perp{}^4 + 4k_\perp{}^4}} \right.$$
$$\left. - X(j,0) \left[\frac{\theta(k'_\perp < \mu) - \theta(k_\perp < \mu)}{|k'_\perp{}^2 - k_\perp{}^2|} + \frac{\theta(k_\perp < \mu)}{\sqrt{k'_\perp{}^4 + 4k_\perp{}^4}} \right] \right\}. \tag{13}$$

The original Lipatov equation allows $k'_\perp \ll k_\perp$: this region gives the $j = 1$ pole of the Altarelli-Parisi kernel. We have already taken care of this region by the ordinary factorization theorem, and therefore it is subtracted off.

5 SOLUTION TO THE MODIFIED EQUATION

We can solve the equation in closed form by a Mellin transform on $k_\perp$. This transform diagonalizes the original Lipatov equation, but extra work is needed to treat the subtraction term in the modified equation. We define

$$\overline{UX}(j,\gamma) = \gamma \int \frac{\mathrm{d}k_\perp{}^2}{k_\perp{}^2} \left(\frac{k_\perp{}^2}{\mu}\right)^\gamma UX(j,k_\perp). \tag{14}$$

Here we have convoluted X with the 'impact factor' for the photon-gluon fusion process. Then the solution to eq. (13) is

$$\overline{UX}(j,\gamma) = \frac{(j-1)\overline{U}(j,\gamma) - \frac{\alpha_s}{\pi}N_c\chi(\gamma)\overline{U}(j,\gamma_c(j))}{j-1-\frac{\alpha_s}{\pi}N_c\chi(\gamma)}. \tag{15}$$

Here the function $\chi(\gamma)$ is the same as the one in the solution of the ordinary Lipatov equation: it is the eigenvalue of the kernel, and is given by

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma). \tag{16}$$

It has a minimum value $4\ln 2$ at $\gamma = \frac{1}{2}$, and has singularities when γ is 0 and 1. The function $\gamma_c(j)$ is the functional inverse of $\chi(\gamma)$. That is, it satisfies the equation

$$j = 1 + \frac{\alpha_s N_c}{\pi}\chi(\gamma_c(j)). \tag{17}$$

It is defined for real values of j greater than

$$j_L \equiv 1 + \frac{\alpha_s N_c}{\pi} 4\ln 2. \tag{18}$$

The short-distance cross section is then

$$\begin{aligned} \hat{\sigma}(\hat{s}) &= \int \frac{\mathrm{d}j}{2\pi i}\, \hat{s}^{j-1}\, UX(j,0) + H_{\rm rem} \\ &= \int \frac{\mathrm{d}j}{2\pi i}\, \hat{s}^{j-1}\, \overline{U}(j,\gamma_c(j)) + H_{\rm rem}, \end{aligned} \tag{19}$$

where the remainder term goes to zero like a power of $\hat{s}$ as $\hat{s} \to \infty$. In this limit,

$$\hat{\sigma}(\hat{s}) \propto \hat{s}^{j_L - 1} \times \frac{1}{\ln^{3/2}\hat{s}}. \tag{20}$$

The solution can be written in terms of untransformed quantities:

$$\hat{\sigma}(\hat{s}) = \int \frac{\mathrm{d}j}{2\pi i}\, \hat{s}^{j-1} \int_{4M^2}^{\infty} \mathrm{d}\bar{s}\, \bar{s}^{-j} \int_0^\infty \frac{\mathrm{d}k_\perp{}^2}{k_\perp{}^2} \left(\frac{k_\perp{}^2}{\mu^2}\right)^{\gamma_c(j)} \gamma_c(j)\, U(\bar{s},k_\perp) + H_{\rm rem}. \tag{21}$$

A similar equation can be used for the Altarelli-Parisi kernel. The solution for the moment in the gluon-gluon case is

$$\begin{aligned}\gamma_{gg}(j,\alpha_s) &= \gamma_c(j)\left[1+\frac{\alpha_s}{\pi}\int_0^\infty \frac{dk_\perp{}^2}{k_\perp{}^2}\left(\frac{k_\perp{}^2}{\mu^2}\right)^{\gamma_c(j)}\overline{\gamma}_1(j,k_\perp)+O(\alpha_s^2)\right] \\ &\quad + \text{remainder} \\ &\approx \gamma_c(j).\end{aligned} \tag{22}$$

The leading singularities of the remainder and of $\overline{\gamma}_1$ are to the left of $j=1$. The last line of eq. (22) reproduces a result of Jaroszewicz[7]. Our formula has the possibility of including higher order and nonleading logarithm corrections.

If we take the limit as the strong coupling α_s goes to zero, we find that

$$\gamma_c(j) \rightarrow \frac{\alpha_s}{\pi}\frac{N_c}{j-1}, \tag{23}$$

which enables us to reproduce the leading singularity of the lowest order Altarelli-Parisi kernel.

6 CALCULATIONS

The above ladder factor is supposed to be convoluted with an impact factor $U(k_\perp, \hat{s}, M)$. In our present work, we are considering the case of heavy quark production, so we have replaced the scale Q by the heavy quark mass, M. The impact factor is essentially an off-shell cross section. It is projected onto transverse polarization for the gluon. It is subtracted at higher order, so that it never has a singularity above or near $j=1$. It has been calculated by Catani, Ciafaloni and Marchesini[2], as described by Catani at this workshop. Ellis and Ross[8] have also calculated this impact factor, but only at $j=1$.

We have used the solution to our modified Lipatov equation to make a numerical calculation of the short distance cross section for photon-gluon fusion to heavy quarks. We have compared it with the Born approximation. At low $\hat{s}$ the two calculations give similar results, since the higher order corrections are smaller by a power of α_s and have no extra logarithms. At large $\hat{s}$, the Born approximation falls off rapidly with energy, as is well known, but the full solution to our equation gives a cross section that rises like a power of $\hat{s}$. So we have also compared it with its asymptote:

$$\hat{\sigma}_{\text{asy}} \propto \frac{\hat{s}^{j_L-1}}{\ln^{3/2}\hat{s}}\int_{4M^2}^\infty \hat{s}^{-j_L}\int \frac{dk_\perp{}^2}{k_\perp{}^2}\left(\frac{k_\perp{}^2}{\mu^2}\right)^{1/2} I(k_\perp, \hat{s}, M). \tag{24}$$

It turns out that $\hat{s}$ has to be several orders of magnitude above threshold before asymptopia is reached.

7 OUTLOOK

We have an equation for the short-distance cross section and for the Altarelli-Parisi kernel. This resums the $j = 1$ singularities that are responsible for the logarithms as $x \to 0$ or $\hat{s} \to \infty$. It enables calculations to be done over the whole range of x.

We are developing methods for calculating higher order corrections. The consequence should be much more accurate perturbative predictions in kinematic regions important for collider physics: e.g., deep inelastic scattering at HERA, and heavy flavor production at HERA and at hadron colliders. The calculations are considerably complicated by the presence of many soft and similar singularities in individual Feynman graphs. These cancel in the final answer.[9]

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