

# Fermi National Accelerator Laboratory

FERMILAB-PUB-89/231-T

UCLA/89/TEP/52

November 89

## Anyonic Superconductivity

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## ABSTRACT

The renormalized Chern-Simons term at finite density is shown to vanish when the renormalized coefficient at zero density takes values  $\frac{Ne^2}{2\pi}$ . We thus demonstrate that a system of anyons at zero temperature is a superfluid. This result is shown to hold to all orders in perturbation theory by generalizing a non-renormalization theorem of the zero density case. Superfluidity (and thus superconductivity for charged anyons) is lost at finite temperature. The mass of the resulting "pseudo-Goldstone" mode is derived.

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<sup>†</sup> Work supported by the "Julian Schwinger Post-Doctoral Fellowship"



It was first suggested by Laughlin<sup>[1,2]</sup> that anyons<sup>[3]</sup> (particles with fractional statistics in two spatial dimensions) may exhibit superconductivity. In particular, the work of Fetter, Hanna and Laughlin,<sup>[4]</sup> which was expanded upon by Chen, Halperin, Wilczek, and Witten,<sup>[5]</sup> showed that in the random phase approximation a free gas of anyons with statistics parameter  $\gamma = \pi(1 - \frac{1}{N})$  where  $N$  is a large integer, has a massless pole in the current-current correlation at zero temperature and thus exhibits superfluidity. This then implies that a charged gas of anyons would be superconducting at zero temperature. The main limitation of these results is that it was not known to what extent the random phase approximation is valid and, in particular, whether these results would survive improvements in this approximation. It was also not known how nonzero temperature affects the results.

Banks and Lykken<sup>[6]</sup> studied a field theoretic realization of an anyonic system in which charged fermions in 2+1 dimensions are coupled to ordinary photons plus an additional “statistics” gauge field possessing a Chern-Simons (CS) term. They argued that superconductivity (at zero temperature) occurs if and only if the renormalized CS term vanishes – i.e. if the quantum corrections to the bare CS term precisely cancel it.

In this letter we follow the approach of ref. [6] and calculate the renormalized CS coefficient. The system we analyze is with nonzero anyon density.

We present the results of a detailed calculation<sup>[7]</sup> showing that the renormalized Chern-Simons term at finite density vanishes if and only if the zero density renormalized Chern-Simons coefficient  $2\pi\theta_R/e^2$  is a positive integer,  $N$ . This establishes that (even if the density is so high that the system behaves relativistically) this system of anyons is a superfluid at zero temperature.

We then show that this result extends to all orders in perturbation theory and thus does not depend on the mean field approximation, nor on the large  $N$  limit.<sup>[8]</sup> We do this by showing that the nonrenormalization theorem of Coleman and Hill<sup>[9]</sup> can be extended to the Chern-Simons theory at finite density. We

also show that at any finite temperature the Chern-Simons term does *not* cancel. Superfluidity (and thus superconductivity in the charged anyon gas) is lost at finite temperature. We derive an expression for the mass of the resulting “pseudo-Goldstone” mode.

We consider a single two-component massive fermion field coupled to a fictitious (“statistics”) U(1) gauge field  $A_\mu$ , which has a Chern-Simons term but no Maxwell term.\* The relationship of the Chern-Simons Lagrangian to anyons has been studied by many authors.<sup>[10-14]</sup> The Euclidean path integral expression for the  $T = 0$  partition function of the Chern-Simons theory at finite chemical potential  $\mu$  is given by:

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_\mu \exp(-S_E) \quad (1)$$

with

$$S_E = \int d^3x \left( \bar{\psi}(\not{\partial} - m)\psi + i\frac{\theta}{2}\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - \mu\psi^\dagger\psi \right) \quad (2)$$

We shall work throughout with a nonnegative chemical potential  $\mu$ . We choose to work in Coulomb gauge ( $\partial_i A^i = 0$ ). We proceed by integrating out the  $A_0$  field, which simply gives the Gauss law constraint  $\delta(B - \frac{e}{\theta}\psi^\dagger\psi)$ . This delta function now allows us to do the integral over  $A_1$  and  $A_2$  by setting

$$A_i = \frac{e}{\theta}\epsilon_{ij}\frac{\partial_j}{\nabla^2}\psi^\dagger\psi \quad (3)$$

This leads to the following effective 4-fermi theory:

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left[-\int d^3x \left( \bar{\psi}(\not{\partial} - m - \mu\gamma^0)\psi - \frac{ie^2}{\theta}(\bar{\psi}\gamma^i\psi)\frac{\epsilon_{ij}\partial^j}{\nabla^2}(\bar{\psi}\gamma^0\psi) \right)\right] \quad (4)$$

We use the gamma matrices  $\gamma_1 = \sigma_1$ ,  $\gamma_2 = \sigma_2$ , and  $\gamma_0 = \sigma_3$  where  $\sigma_i$  are the Pauli spin matrices. Notice that the effect of the chemical potential is simply to

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\* In such a pure Chern-Simons theory, a Maxwell term is, in fact, generated at one loop. This implies that the results of our calculation should hold even if a bare Maxwell term is present. We believe that our calculation can be extended to this case.

replace  $\partial_0$  by  $\partial_0 - \mu$ . We thus define  $\bar{\partial}_\nu$  to be equal to  $\partial_\nu$ , unless  $\nu = 0$  in which case  $\bar{\partial}_0 = \partial_0 - \mu$ .

We begin by studying the fermion propagator  $S(x, y)$  for this theory. The bare fermion propagator  $S_0(x, y)$  is simply  $1/(\bar{\partial} - m)$ . When including perturbative corrections to this propagator we notice that there are tadpole contributions to  $S$ . These are shown diagrammatically in Figure 1. These tadpoles are nonvanishing since  $\langle J_0 \rangle = \rho_0$  is nonzero when  $\mu$  is nonzero. Our first key observation is that we can compute the entire contribution of these tadpoles as a function of the mean density  $\rho_0$ . Each tadpole contributes an amount

$$i \frac{e^2}{\theta} \gamma^i \epsilon_{ij} \frac{\partial_j}{\nabla^2} \rho_0 \quad (5)$$

to the fermion propagator. This can be written as  $ie\gamma^i \mathcal{A}_i$  where

$$\mathcal{A}_i = \frac{e}{\theta} \epsilon_{ij} \frac{\partial_j}{\nabla^2} \rho_0 \quad (6)$$

Notice that  $\mathcal{A}_i$  is precisely the gauge potential one would obtain from a constant fictitious magnetic field  $\mathcal{B} = \frac{e}{\theta} \rho_0$ . The full tadpole contribution is found by summing the geometric series of Figure 1 which results in the expression

$$S_T = (S_0^{-1} - ie\gamma^i \mathcal{A}_i)^{-1} = [\gamma^\mu (\partial_\mu - ie\mathcal{A}_\mu) - m - \mu\gamma^0]^{-1} \quad (7)$$

where  $\mathcal{A}_0 = 0$ . Thus the tadpole-corrected propagator  $S_T$  is the Green's function for a free fermion in a constant magnetic field  $\mathcal{B} = \frac{e}{\theta} \rho_0$  and with chemical potential  $\mu$ .

The next step is to find the fermion propagator  $S_T$  under this circumstance. We have done this by two methods, using the Euclidean version of Schwinger's proper time method<sup>[18]</sup> and by directly solving the Green's function equation. In this letter we concentrate on the latter method. We choose an asymmetric gauge

in which  $\mathcal{A}_y = \mathcal{B}x$ ,  $\mathcal{A}_z = 0$ ,  $\mathcal{A}_0 = 0$  which is consistent with Coulomb gauge. To find  $S_F$  we want to invert the operator  $\bar{\mathcal{D}} - m$  where  $\bar{D}_\mu = \bar{\partial}_\mu - ie\mathcal{A}_\mu$ . Note that

$$[\bar{\mathcal{D}} - m]^{-1} = (\bar{\mathcal{D}} + m)[(\bar{\mathcal{D}} - m)(\bar{\mathcal{D}} + m)]^{-1} = (\bar{\mathcal{D}} + m)[\bar{D}^2 - m^2 + e\sigma_3\mathcal{B}]^{-1} \quad (8)$$

We thus first invert the operator  $[\bar{D}^2 - m^2 + e\sigma_3\mathcal{B}]$  by finding its eigenvalues and eigenfunctions. The result is given by

$$\frac{1}{[\bar{D}^2 - m^2 + e\sigma_3\mathcal{B}]} = - \sum_{n=0}^{\infty} \int \frac{d\omega}{2\pi} \int \frac{dP_y}{2\pi} \left\{ \left[ \frac{1}{2} \left( \frac{1}{d_{n+1}} + \frac{1}{d_n} \right) + \frac{1}{2} \left( \frac{1}{d_{n+1}} - \frac{1}{d_n} \right) \sigma_3 \right] e^{-i\omega(t-t')} e^{-iP_y(y-y')} \Psi_n \left( x - \frac{P_y}{e\mathcal{B}} \right) \Psi_n^* \left( x' - \frac{P_y}{e\mathcal{B}} \right) \right\}, \quad (9)$$

where  $\Psi_n$  is the  $n^{\text{th}}$  normalized eigenfunction of the a harmonic oscillator with frequency  $e\mathcal{B}$  and where  $-d_n$  and  $-d_{(n+1)}$  are the eigenvalues given by  $d_n = [(\omega - i\mu)^2 + 2ne\mathcal{B} + m^2]$ .

Now that we have the fermion propagator, we calculate  $\rho_0$  as a function of  $\mu$  and  $\mathcal{B}$ . We will argue below that the lowest order calculation presented here is in fact an exact result.

$$\begin{aligned} \rho_0 &= \langle \psi^\dagger(x)\psi(x) \rangle = -T\tau[\gamma_0 S_T(x, x)] \\ &= \frac{-ie\mathcal{B}}{2\pi} \sum_{n=0}^{\infty} \int \frac{d\omega}{2\pi} \left[ (\omega - i\mu) \left( \frac{1}{d_{n+1}} + \frac{1}{d_n} \right) + im \left( \frac{1}{d_{n+1}} - \frac{1}{d_n} \right) \right] \\ &= \frac{e\mathcal{B}}{4\pi} \sum_{n=0}^{\infty} \left[ \Theta(\mu - \sqrt{2ne\mathcal{B} + m^2}) + \Theta(\mu - \sqrt{2(n+1)e\mathcal{B} + m^2}) \right] \\ &\quad - \frac{e\mathcal{B}}{4\pi} \frac{m}{|m|} \Theta(|m| - \mu) \\ &= \frac{e\mathcal{B}}{2\pi} \left[ \text{Int} \left( \frac{\mu^2 - m^2}{2e\mathcal{B}} \right) + \frac{1}{2} \right] \Theta(\mu - |m|) - \frac{e\mathcal{B}}{4\pi} \frac{m}{|m|} \Theta(|m| - \mu), \end{aligned} \quad (10)$$

where  $\text{Int}$  stands for the integer part of its argument and  $\mu$  and  $\mathcal{B}$  are taken to be positive. Notice that when  $\frac{\mu^2 - m^2}{2e\mathcal{B}}$  is itself an integer, the value of the density  $\rho_0$  is ambiguous. Notice also that the density  $\rho_0$  is nonzero at  $\mu = 0$ . This is due

to the spectral asymmetry of our parity non-invariant Hamiltonian and to our definition of  $\rho_0$  as  $\langle \psi^\dagger \psi \rangle$ . One thus has to define the physical density

$$\rho_{ph} = \rho(\mu) - \rho(\mu = 0) = \frac{eB}{2\pi} \left( \text{Int} \left( \frac{\mu^2 - m^2}{2eB} \right) + 1 \right) \quad (11)$$

for  $\mu > m > 0$ . The final result for  $\rho_{ph}$  when  $\mu > m > 0$  is drawn in Fig. 2. We now see the correspondence of  $\rho_{ph}$  to the density of fermions in Landau levels. Notice that the filled Landau levels are described by the horizontal sections (constant  $\rho_{ph}$ ) in Fig 2. These occur when  $2\pi\rho_{ph}/eB$  is an integer  $N$ . This implies that  $2\pi\rho_0/eB = N - \frac{1}{2}$  (for  $m > 0$ ). Using the definition of  $B$ , i.e.  $B = \frac{e}{\delta}\rho_0$ , we see that a filled Landau level occurs when  $\theta/2\alpha = N - \frac{1}{2}$  with  $\alpha = e^2/4\pi$ . This is precisely the condition that the renormalized Chern-Simons coefficient  $\theta_R$  at zero density is  $2\alpha$  times an integer (see ref.[9]).\* Furthermore an unfilled level corresponds to the vertical parts of Fig. 2. for which  $\frac{\mu^2 - m^2}{2eB}$  is an integer.

We proceed now to evaluate the one-loop contribution to the CS term in the effective action, which is given by the parity odd part  $\Pi_{odd}(k^2 = 0)$  of the vacuum polarization

$$\Pi_{\mu\nu}(k) = \Pi^e_{\mu\nu}(k^2) + \epsilon_{\mu\nu\lambda} k^\lambda \Pi_{odd}(k^2) \quad (12)$$

where  $\Pi^e$  is symmetric under interchange of  $\mu$  and  $\nu$ . Note that gauge invariance requires the odd part to have the above form even at finite density where Lorentz invariance is lost. The one loop expression for  $\Pi_{\mu\nu}$  is given by

$$\Pi_{\mu\nu}(x, y) = -e^2 T\tau [\gamma_\mu S_T(x, y) \gamma_\nu S_T(y, x)] \quad (13)$$

To evaluate  $\Pi_{odd}$  we extract the term in eqn. (13) which is proportional to  $\epsilon_{ij}k_0$ . The expression for  $\Pi_{odd}(0)$  using both Schwinger's method<sup>[7]</sup> and the method pre-

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\* We shall see below that this is the case for the regularization scheme which we chose. The point is that the zero density one loop corrections to the Chern-Simons coefficient are regularization dependent as is shown in ref. [10]. It is only the renormalized  $\theta$  which is related to the statistics parameter of the anyons.

sented above is the following:

$$\Pi_{odd} = e^2 \frac{e\mathcal{B}}{\pi} \sum_{n=0}^{\infty} \int \frac{d\omega}{2\pi} \left[ \frac{-m}{d_n d_{n+1}} + 2ie\mathcal{B} \frac{(\omega - i\mu)[(\omega - i\mu)^2 + m^2]}{d_n^2 d_{n+1}^2} \right] \quad (14)$$

We now compare this expression for  $\Pi_{odd}$  to the previous expression for  $\rho_0$ . The term proportional to  $m$  is identical to the similar term in  $\rho_0$  (divided by  $\frac{\mathcal{B}}{e}$ ). The second term in the integrand can be rewritten as

$$2ie\mathcal{B} \left[ \frac{(\omega - i\mu)}{(2e\mathcal{B})^2} \left( \frac{1}{d_{n+1}} + \frac{1}{d_n} \right) - \frac{2n}{2e\mathcal{B}} \frac{(\omega - i\mu)}{[(\omega - i\mu)^2 + M^2(n)]^2} \right] \quad (15)$$

where  $M^2(n) = m^2 + 2en\mathcal{B}$ . The first term of eqn. (15) is identical to the remaining term in  $\rho_0$ . For the second term of eqn. (15) we perform the  $\omega$  integration which results in an expression which is proportional to  $\delta(\mu - M(n))$ . This term vanishes whenever  $\frac{\mu^2 - m^2}{2e\mathcal{B}} \neq \text{integer}$ . Therefore for any  $\mu$  such that  $\frac{\mu^2 - m^2}{2e\mathcal{B}} \neq \text{integer}$  we get the following relation between  $\Pi_{odd}(0)$  and  $\rho_0$ :

$$\Pi_{odd}(k^2 = 0) = \frac{e^2 \rho_0}{e\mathcal{B}} = \theta \quad (16)$$

where we have used the tadpole relation between  $\rho_0$  and  $\mathcal{B}$ . We see that the one loop correction to the Chern-Simons term precisely cancels the bare Chern-Simons term *provided*  $\frac{\mu^2 - m^2}{2e\mathcal{B}}$  is not an integer. As discussed previously and as is seen from Figure 1 this occurs precisely when we have any number  $N$  of filled Landau levels, in which case  $\frac{\theta}{2\alpha} = N - \frac{1}{2}$  (for  $m > 0$ ). If we now apply the above equation and eqn.(10) to the case  $\mu = 0$  we see that  $\Pi_{odd}(k^2 = 0, \mu = 0) = -\alpha$ . This renormalizes the *zero density* Chern-Simons term from  $\theta$  to  $\theta_R = \theta + \alpha$ . Thus at *nonzero*  $\mu$  the condition for having  $N$  filled Landau levels becomes  $\frac{\theta_R}{2\alpha} = N$ . We thus conclude that to first order in the tadpole-corrected perturbation calculation a CS coefficient whose renormalized value at zero density is  $\frac{Ne^2}{2\pi}$  is, at any finite density, *renormalized to zero*. Note that this value of  $\theta_R$  corresponds precisely to anyons with a statistics parameter  $\gamma = \pi(1 - \frac{1}{N})$ .

The above result for  $\Pi_{odd}$  could have been anticipated without an explicit calculation once the result for  $\rho_0$  was calculated. The reason for this is that at any order, there is a general relation between those diagrams which contribute to  $\Pi_{odd}(k=0)$  and those which contribute to  $\rho$ . Here  $\Pi_{odd}(0)$  is defined by summing one-particle-irreducible diagrams to any order in tadpole-corrected perturbation theory. To see this imagine taking  $\delta/\delta\mathcal{B}$  (for fixed  $\mu$ ) of any diagram which contributes to  $\rho$ . This has the effect of removing a tadpole insertion and replacing it by  $(1/e)\epsilon^{ij}\gamma_j\frac{\partial}{\partial x_i}$ . One easily sees that the resulting diagram is one-particle-irreducible (in terms of tadpole-corrected lines) and contributes precisely to the odd part of the  $\Pi(0)/e$ , i.e.  $\Pi_{odd}(0)$ . We obtain the general relation:

$$\left.\frac{\delta\rho}{\delta e\mathcal{B}}\right|_{\mu} = \frac{1}{e^2}\Pi_{odd}(0) \quad (17)$$

Thus, having calculated  $\rho_0$  we can simply differentiate with respect to  $\mathcal{B}$  at fixed  $\mu$ . From Figure 2 we see that for filled Landau levels  $\rho_0$  is simply proportional to  $\mathcal{B}$ , and thus  $\Pi_{odd}(0)$  is simply equal to  $e\rho_0/\mathcal{B}$  which agrees with our previous result.

We will now show that  $\rho$ ,  $\Pi_{odd}(0)$ , and thus eqn. (16), are unaffected by higher order radiative corrections in tadpole-corrected perturbation theory. The nonrenormalization theorem can be proven either by a topological argument,<sup>[16]</sup> or by a direct extension of the Coleman-Hill theorem<sup>[8]</sup> to the finite density case. Briefly stated, (for details, see [7]) the topological argument begins by noting that  $\rho$  measures the spectral asymmetry of the  $\mu$  and  $m \neq 0$  Dirac operator in a gauge field background. Extending the analysis of Boyanovsky and Blankenbcler<sup>[17]</sup> to finite density, one sees explicitly that, provided  $\frac{\mu^2 - m^2}{2e\mathcal{B}} \neq \text{integer}$ ,  $\frac{\rho}{e\mathcal{B}}$  is proportional to the index of the background field Dirac operator. Thus fluctuations in the gauge field do not affect the value of  $\rho$ . Since we showed above that  $\Pi_{odd}(0)$  is determined by  $\rho$  to all orders we see that the topological argument above covers the nonrenormalization of  $\Pi_{odd}(0)$  as well.

The nonrenormalization of  $\rho$  and  $\Pi_{odd}(0)$  is also seen by extending the arguments of Coleman and Hill.<sup>[9]</sup> Consider the Euclidean  $n$ -photon effective vertex, at



finite density, given by summing all graphs consisting of a single *tadpole-corrected* fermion loop with  $n$  external photons attached. We denote this by:

$$\Gamma_{\mu_1 \dots \mu_n}^{(n)}(k_1 \dots k_n) \quad (18)$$

All diagrams in tadpole-corrected perturbation theory which contribute to  $\rho$  or  $\Pi_{odd}(0)$  can be constructed from the  $\Gamma^{(n)}$ 's, by sewing together photon lines. To prove our nonrenormalization theorem for  $\rho$  and  $\Pi_{odd}(0)$ , it suffices to show that, for  $k_1, k_2 \rightarrow 0$ :

$$\begin{aligned} \Gamma_{\dots}^{(n)}(k_1 \dots) &= \mathcal{O}(k_1), & n > 1 \\ \Gamma_{\dots}^{(n)}(k_1, k_2, \dots) &= \mathcal{O}(k_1 k_2), & n > 2 \end{aligned} \quad (19)$$

By gauge invariance and the argument of ref. [9], these relations are true provided that  $k \rightarrow 0$  is in the region of analyticity of the  $\Gamma^{(n)}$ .

We prove the nonrenormalization theorem therefore by demonstrating the analyticity of the  $\Gamma^{(n)}$  as  $k^2 \rightarrow 0$  in the Euclidean region. This is obvious for the zero density system, since the physical (Minkowski) threshold for fermion-antifermion pairs begins at  $k^2 = 4m^2$ . At finite density, however, one must also worry about the production of fermion-hole pairs. In our case, since the  $\Gamma^{(n)}$  are defined in tadpole-corrected perturbation theory, this corresponds to a (Minkowski) photon being absorbed by a fermion in a Landau level, causing a transition to an unoccupied state. The Landau levels allow continuous values of momentum but are discretely spaced in energy, with spacing  $e\mathcal{B}/m$ . Therefore, when we have  $N$  completely filled Landau levels, physical singularities are absent for (Minkowski)  $k_0 < e\mathcal{B}/m$ . Thus as we approach  $k^2 \rightarrow 0$  from the Euclidean region the  $\Gamma^{(n)}$  are analytic, and the nonrenormalization theorem holds precisely for  $\theta = Ne^2/2\pi$ .

Next we analyze the finite temperature behavior of the anyonic system, namely, that of the partition function given in eqn. (2). The evaluation of the fermion propagator follows the same lines as for the zero temperature case

apart from replacing the integral over  $\omega$  with the sum over discrete Matsubara frequencies  $\omega_l = \frac{2\pi}{\beta}(l + \frac{1}{2})$ . Inserting the resulting propagator into (10) and using standard contour integrals<sup>[7]</sup> to replace the summations we get the following expression for the density at finite temperature:

$$\rho_o = \frac{e\mathcal{B}}{4\pi} \left\{ \sum_{n=0}^{\infty} \left[ \text{tgh} \left[ \frac{\beta}{2}(\mu + M(n)) \right] + \text{tgh} \left[ \frac{\beta}{2}(\mu - M(n)) \right] \right] + \frac{m}{|m|} \text{tgh} \left[ \frac{\beta}{2}(\mu - |m|) \right] \right\} \quad (20)$$

In the limit  $\beta \rightarrow \infty$  this expression reduces to the one given in eqn.(10).

We now compute the renormalized CS term at finite temperature. The simplest way to do this is to use eqn.(17) which is valid at finite temperature. We see immediately that  $\rho$  is no longer proportional to  $\mathcal{B}$  and, in fact  $\frac{\rho}{\mathcal{B}}$  is a monotonic function of  $\mathcal{B}$ . Equation (16) is thus never valid, and the renormalized CS term is nonzero for any finite temperature. Instead, for fully filled levels, the renormalized CS term is given by  $\theta_R(\mu, T) = -e\mathcal{B} \frac{d(\rho/\mathcal{B})}{d\mathcal{B}}$ . Alternately we could repeat the steps that led to eqn.(14). We now get the same expression with the sum over  $\omega_l$  replacing integration over  $\omega$ . Recall that the cancellation of the bare and one loop CS term was a result of the vanishing of the second term in eqn. (15) for the filled Landau levels -i.e.,  $\frac{\mu^2 - m^2}{2e\mathcal{B}} \neq \text{integer}$ . At finite temperature this term does not vanish.

We can now compute the mass of the resulting ‘‘pseudo-Goldstone’’ mode which is proportional to the renormalized CS coefficient. Using either of the above methods we find

$$\theta_R(\mu, T) = -\frac{\alpha e\mathcal{B}}{2} \beta \sum_{n=0}^{\infty} \frac{n}{M(n)} \left[ \text{tgh}^2 \left[ \frac{\beta}{2}(\mu + M(n)) \right] - \text{tgh}^2 \left[ \frac{\beta}{2}(\mu - M(n)) \right] \right] \quad (21)$$

For small temperature and large mass  $\beta m \gg 1$ ,  $|\mu^2 - m^2| \ll m^2$  this reduces to

$$|\theta_R(\mu, T)| = 2\alpha e\mathcal{B} \beta \sum_{n=0}^{\infty} \frac{n}{M(n)} e^{-\beta|\mu - M(n)|} \quad (22)$$

In this limit the only significant terms are those with  $2ne\mathcal{B} \sim |\mu^2 - m^2| \ll$

$m^2$  so that  $\theta_R(\mu, T) \sim \alpha\beta\left(\frac{\mu^2 - m^2}{m}\right) \left(e^{-\beta\frac{eB}{m}\delta} + e^{-\beta\frac{eB}{m}(1-\delta)}\right)$  where  $\delta = \frac{\mu^2 - m^2}{2eB} - \text{Int}\left(\frac{\mu^2 - m^2}{2eB}\right)$ . Since the mass of the “pseudo-Goldstone” mode is given by the the absolute value of the renormalized CS coefficient, this last expression is the mass.

Acknowledgements: Nathan Weiss wishes to thank the Natural Sciences and Engineering Research Council of Canada for their support of this research. Jacob Sonnenschein wishes to thank the “Julian Schwinger Post-Doctoral Fellowship”. We also wish to thank I. Affleck, J. M. Cornwall, B. Hill, H.Quinn and G. Semenoff for their comments and suggestions.

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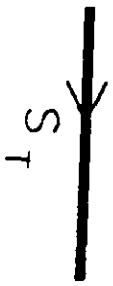
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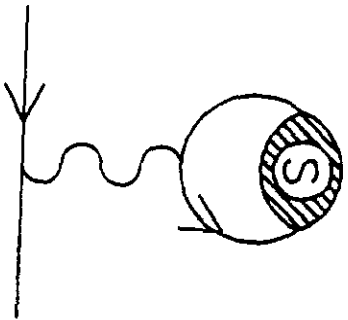
## FIGURE CAPTIONS

Figure 1. Tadpole contributions to the fermion propagator. The bold line indicates tadpole-corrected fermion line.

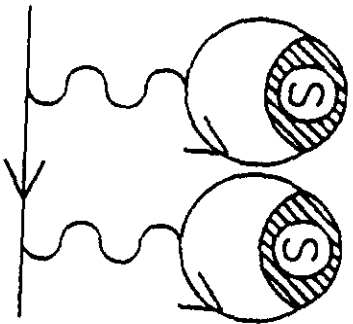
Figure 2. The fermion density as a function of the chemical potential  $\mu$  and the magnetic field  $\mathcal{B}$ .



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