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An Effective Field Theory for the Calculation of Matrix Elements Involving Heavy Quarks

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To measure matrix elements involving heavy quarks on the lattice, including f_B and the B meson decay constant, the dependence on the heavy quark mass must first be extracted analytically. We present the resulting continuum effective field theory action and illustrate its utility by calculating the one-loop renormalization of an arbitrary heavy-light bilinear. We also discuss the limits of the approximation.

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For several matrix elements of interest, including those which determine $f_{\rm B}$ and the B parameter, the large rest energy of the b quark is not deposited into lighter hadrons. For these matrix elements it ought to be possible to find an effective field theory in which all the dependence on the large rest mass of the heavy quark has been removed analytically. The effective field theory will be an expansion in the heavy quark's spatial momentum or kinetic energy over its rest energy, termed the $1/m$ expansion. Not only is this effective field theory a useful analytical tool for extracting dependences on the heavy quark mass, for several matrix elements all the remaining scales in the problem are small enough that they can be numerically calculated on the lattice.

Indeed, doing calculations at the zeroth order in this expansion has already been used as a basis for deriving heavy quark potentials[1]. The approximation to the heavy quark propagator was written down in position space for an external gauge field. In this rather singular limit, whatever the momentum the heavy quark is created with, it doesn't move at all. In position space the propagator is proportional to $\delta^3(\mathbf{y}-\mathbf{x})$. In an external gauge field it comes multiplied by phase which is just the Wilson line. Thus it looks somewhat like a static source and the approximation has been called the static approximation. However, the reader should not be mislead by the name as the color of the the heavy quark is not fixed, momentum is conserved at vertices involving the heavy quark, and $1/m$ corrrections are possible to include perturbatively.

This approximation for the propagator has also been used to calculate the logarithmic corrections to $f_{\rm B}$ and the B parameter[2] already obtained in the full theory[3]. The discretized version of the position space propagator for numerical simulations[4] has been used by two lattice gauge theory groups to obtain $f_{\rm B}$[5], and the relationship between the time component of the axial current defined on the lattice in this appoximation to its continuum counterparts has been studied previously[6].

All of these calculations, including those in reference [2] where the language of effective field theories is used, have taken as a starting point the position space expression for the propagator in an external gauge field and gone to considerable effort to avoid doing perturbation theory in momentum space. We write down the effective field theory action from which the propagator can be obtained, take this as our starting point, and confront the minor technicalities of working with the theory in momentum space. Calculations in this framework are substantially simpler than previous calculations. Actually the propagator previously used differs from the propagator we obtain in that it was for a four-component field that tried to describe the propagation of both heavy quarks and heavy anti-quarks and had a trivial but annoying remaining dependence on the heavy quark mass. We phrase the effective field theory in terms of a two-component field which is more natural and the dependence on the heavy quark mass is eliminated.

While we believe we have clarified the static effective field theory, we are dealing mostly with issues already sorted out by Caswell and Lepage in non-relativistic QED[7], an effective field theory in which they have done calculations of stunning accuracy. Their starting point is an action valid to first order in the $1/m$ expansion. The ultraviolet behavior of the propagator in the non-relativistic effective field theory is completely different from the propagator in the static effective field theory, just as both of these propagators behave completely differently from the propagator in the full theory. There are differences in the infrared behavior of the two theories as well; the rather singular nature of the static approximation makes the static effective theory invalid as a starting point for some problems, including some where the non-relativistic effective field theory is an acceptable first approximation.

It should also be noted that it is just conjectured, not demonstrated, that these are approximately renormalizable effective field theories[8]; no proof analagous to the one given for the four-Fermi effective theory of the weak interactions[9] has been given, even for a limited class of Green's functions. On the other hand, it has been shown in the full theory that when external momenta are restricted to values where the $1/m$ expansion should be applicable, that the regions of loop integration which contribute after renormalization are regions where one can use the non-relativistic approximation for the heavy quark[10]. We will not worry more about this important subject here.

In what follows we describe the static effective field theory and illustrate it's use and the technicalities encountered by doing some one-loop calculations, including a calculation which reproduces the logarithmic corrections to the matrix element determining f_B[2][3]. Our method allows us to obtain the full order α_S contributions to the effective theory-full theory matching of an arbitrary heavy-light bilinear. It reproduces corrections for the time component of the axial current which can be extracted from work applying this theory to the lattice[6]. We conclude by giving an example of a process which illustrates the limits of the static approximation.

The static approximation is the zeroth order approximation in the expansion in p^μ/m, where m is the heavy quark mass, and we have removed $(m, \mathbf{0})$ from the momentum of the heavy quark. So the approximation is that the heavy quark is nearly at rest and nearly on shell. Whenever an operator creates a heavy quark we will have removed $(m, \mathbf{0})$ from what we call the momentum inserted at the operator, and whenever an operator annihilates a heavy quark we will have added this in to what we call the inserted momentum. This removes the last trivial dependence in the static effective field theory on the heavy quark mass. The only remaining dependence comes from matching the static theory to the full theory at the heavy quark mass. It is important to note that for the matching to work, not only must the external momenta be in the regime of validity of the static appoximation, the amputated Green's function has to be sandwiched between spinors describing a heavy quark with this momentum.

The static effective field theory Lagrangian in Minkowski space is

$$\mathcal{L}_{\mathrm{M}} = b^{\dagger}\left(i\partial_0 + gA_0\right)b. \tag{1}$$

The b field annihilates heavy quarks and its hermitian conjugate creates them. A completely independent field would have to be introduced to describe processes involving heavy anti-quarks. The b field is a two-component field, so there is a supressed two-by-two identity in the action as well as in the free propagator. We will denote this propagator with a double line. Incoming particles will always appear on the right and outgoing on the left so that in problems with heavy quarks and heavy anti-quarks, you can distinguish them by whether the arrow on the line points from incoming to outgoing or from outgoing to incoming, respectively. The free propagator in Minkowski space is

$$\frac{i}{p_0 + i\epsilon}. \tag{2}$$

The $i\epsilon$ prescription is there so that the heavy quark propagates forward in time. In position space the propagator from x to y is

$$\theta(y_0 - x_0)\delta^3(\mathbf{y} - \mathbf{x}). \tag{3}$$

The $i\epsilon$ prescription is necessary even in Euclidean space. If the first order term in the $1/m$ expansion had been kept, the prescription wouldn't be necessary in Euclidean space since the pole in the propagator would have been pushed off to $-i\mathbf{p}^2/2m$. While our answers will depend on the fact that they have an $i\epsilon$ prescription as opposed to a $-i\epsilon$ prescription, they will not depend on ϵ as $\epsilon \to 0$.

Although the matching between the full theory and the effective theory is best done in Minkowski space, our perturbative calculations are done in Euclidean space so we will quote all our conventions there. The Euclidean static effective field theory is obtained by the following procedure: First the comparison between the effective theory and the full theory is made in Minkowski space. Then a Wick rotation to the Euclidean static effective field theory is done. Finally, for lattice applications, a discretized version of the theory is chosen.

With our conventions, the heavy quark part of the Lagrangian in Euclidean space is

$$\mathcal{L}_{\mathrm{E}} = b^{\dagger}\left(i\partial_0 + gA_0\right)b. \tag{4}$$

In the Euclidean funtional integral, $b^{\dagger}$ and b are independent fields. The propagator in momentum space is

$$\frac{1}{p_0 + i\epsilon}. \tag{5}$$

The Feynman rule for heavy quark-gauge field interaction is a gauge group generator times $-g$. Only the zeroth component of the gauge field participates, and the matrix in spin space is just the identity.

Here we are interested in comparing operators defined in the effective theory and the full theory. When comparing operators, it is important to calculate an unambiguous and gauge invariant quantity especially if one is interested in more than the leading logarithms in the heavy quark mass. We will take a matrix element of the heavy-light bilinear between an incoming light quark and an outgoing heavy quark. For calculational convenience, we will set the light quark mass to zero. The light quark propagator in our conventions is then $1/\not{p}$ and all four γ_μ are hermitian.

Our mass shell point will have the incoming and outgoing momenta set to zero. The infrared divergences at this point are eliminated by giving the gluon a mass, λ, which is acceptable because all of our diagrams are QED-like. All λ dependence, as well as light quark mass dependence were we to have included it, should drop out of the difference of the matrix elements defined in the two different theories. A more satisfying procedure than introducing a gluon mass would be to demonstrate the local cancellation in the difference of the low momentum parts of the loop integration.

Before we illustrate a loop calculation we should do the tree level matching of an arbitrary bilinear in the full and effective theories. Consider the operator in the full theory $\bar{b}\Gamma q$ where q is the light quark field. Take its matrix element between an incoming light quark with a spinor u and an outgoing heavy quark with a spinor u'. The spinor u' is a four component spinor in the full theory. It is normalized to satisfy $\overline{u}'u = 2m$. At zeroth order in the $1/m$ expansion, it is

$$u' = \sqrt{2m}\begin{pmatrix} U' \\ 0 \end{pmatrix}, \tag{6}$$

where U' is the non-relativistic spinor normalized to satisfy $U'^{\dagger}U' = 1$. Parametrize the arbitrary matrix Γ in the operator by two by two blocks,

$$\Gamma = \begin{pmatrix} \alpha \, \beta \\ \gamma \, \delta \end{pmatrix}. \tag{7}$$

The tree level matrix element of this operator is then $\overline{u}'\Gamma u$ which simplifies to

$$\sqrt{2m}U'^{\dagger}(\alpha\,\beta)u \tag{8}$$

at zeroth order in the $1/m$ expansion. Clearly the only operator in the static effective field theory that has the same matrix element for arbitrary U' and u is

$$\sqrt{2m}\, b^{\dagger}(\alpha\,\beta)q. \tag{9}$$

What we need to do an order α_S comparison of the matrix elements of the full theory and effective theory operators is the vertex corrections in the full theory and the effective theory, the wave function renormalization of a heavy quark in the full theory, and the wave function renormalization of a quark treated in the static approximation. The light quark is treated the same way in the effective theory and the full theory so it's wave function renormalization drops out of the difference of the matrix elements of the operators.

In the full theory, the one loop correction to the vertex is $(g^2/12\pi^2)\Gamma$ times

$$-1 - \ln\frac{\lambda^2}{m^2} - \frac{1}{2}HG + \frac{1}{4}H^2\ln\frac{\mu^2}{m^2} + \frac{3}{4}H^2 - HH' \tag{10}$$

The only subtlety in getting this result is that to zeroth order in the $1/m$ expansion, when γ_0 is next to $\overline{u}'$, we used that the spinor u' describes a heavy quark nearly at rest and replaced γ_0 by the identity. We have used modified minimal subtraction to eliminate the $1/\epsilon - \gamma_E + \ln 4\pi$ that comes with the logarithm of μ^2, and we have had to introduce H defined by $H\Gamma = \gamma_\mu\Gamma\gamma_\mu$ and G defined by $G\Gamma = \gamma_0\Gamma\gamma_0$. H' is the derivative with respect to d of H in d dimensions. In a particular case of interest, $\Gamma = \gamma_0\gamma_5$, and with the extension of the gamma matrix algebra that γ_5 commutes with γ_μ when $4 < \mu \leq d$ we find $H = 2$ and $H' = G = -1$. Eqn. (10) simplifies to

$$\ln\frac{\mu^2}{\lambda^2} + 5. \tag{11}$$

If we had taken γ_5 to anticommute with all γ_μ we would have $H' = 1$ and the constant in (11) would be 1.

The integral for the corresponding effective field theory graph, figure 1, is the tree graph times

$$\frac{4}{3}g^2\int\frac{d^4l}{(2\pi)^4}\frac{1}{l_0 + p_0 + i\epsilon}\frac{1}{\not{l} + \not{k}}\gamma_0\frac{1}{l^2 + \lambda^2}. \tag{12}$$

Except for the fact that only the time component of the gauge field interacts with the heavy quark and the unusual form of the heavy quark propagator, the factors are familiar. The loop momentum is l, and the momentum of the incoming light quark, k, and the outgoing heavy quark, p, we promptly set to zero (remember $(m, 0)$ has been removed from p).

This integral is particularly easy. After setting the momentum to zero and rationalizing the light quark propagator, we have $\not{l}$ in the numerator. The vector part of $\not{l}$ is odd leaving us with $l_0\gamma_0$. Thus we don't have to worry about the pole prescription and the integral simplifies to a standard covariant dimensionally regularized integral. The result is $\sqrt{2m}(g^2/12\pi^2)(\alpha\,\beta)$ times

$$\ln\frac{\mu^2}{\lambda^2}+1. \tag{13}$$

The result doesn't depend on the matrix at the vertex, in contrast to the result in the full theory. However, notice that the coefficient of the logarithm of λ^2 in (10) is the same for any Γ and will drop out of the difference with equation (13). This is a non-trivial check on the validity of the effective theory.

In the full theory, the one loop self energy graph gives a contribution to the wave function renormalization of the heavy quark of $g^2/12\pi^2$ times

$$-4+\ln\frac{m^2}{\mu^2}+2\ln\frac{m^2}{\lambda^2}. \tag{14}$$

The integral for the corresponding effective field theory graph, figure 2, is

$$\frac{4}{3}g^2\int\frac{d^4l}{(2\pi)^4}\frac{1}{l_0+p_0+i\epsilon}\frac{1}{l^2+\lambda^2}. \tag{15}$$

The evaluation of this integral will illustrate how to deal with the non-covariant poles. The trick will be to isolate the pole in an integrand which one subtracts and adds to the original integrand. In the difference, the singularity cancels. The integral added back in is chosen to be easily doable.

A preliminary step whose utility will become clear momentarily is to symmetrize the integral under $l_0 \to -l_0$. The symmetrized integral is

$$\frac{4}{3}g^2\int\frac{d^4l}{(2\pi)^4}\frac{-(p_0+i\epsilon)}{l_0^2-(p_0+i\epsilon)^2}\frac{1}{l^2+\lambda^2}. \tag{16}$$

The integrand now has poles at $\pm(p_0+i\epsilon)$. Replace the gauge field propagator by

$$\frac{1}{l^2+\lambda^2}-\frac{1}{p_0^2+\mathbf{l}^2+\lambda^2}+\frac{1}{p_0^2+\mathbf{l}^2+\lambda^2}. \tag{17}$$

The difference has zeroes at $l_0 = \pm(p_0 + i\epsilon)$ so in the difference we simplify naively, cancel the poles and take the derivative with respect to p_0 at zero to obtain wave function renormalization. The result is $g^2/12\pi^2$ times

$$2 \ln \frac{\mu^2}{\lambda^2}. \tag{18}$$

The poles are isolated in the integral we have added back in. This integral is simple because it factors into the product of an l_0 integral and a $d-1$-dimensional **l** integral. Notice that this integral would be poorly defined if we had done the pole isolation procedure without first symmetrizing. One does the l_0 integral by contour integration and shortly discovers that the result is even under $p_0 \to -p_0$ (contrary to the integral's naive appearance) and so it doesn't contribute to wave function renormalization. Equation (18) is the entire answer. Again, notice that it's dependence on λ is the same as in the full theory, equation (14). While it is not necessary for the order α_S renormalization of a heavy-light bilinear, we note that the same techniques show that the graph of figure 2 gives no contribution to mass renormalization in dimensional regularization.

The ratio of the matrix elements of the operator in the full theory to the operator in the effective theory is given by (10) minus (13) plus one half of (14) minus one half of (18). In the case of most interest, $\Gamma = \gamma_0\gamma_5$, it is

$$1 + \frac{g^2}{12\pi^2}\left(2 - \frac{3}{2}\ln\frac{\mu^2}{m^2}\right). \tag{19}$$

We have not resummed the logarithms to extend the range of validity of this comparison as the non-logarithmic corrections which we have obtained would be only one of the comparable non-leading corrections in the extended range of validity.

Boucaud, Lin and Pène studied the renormalization of the time component of the axial current in the static approximation for application to the lattice[6]. They did a position space calculation of the correlator of the time component of the axial current with itself. If we use the other extension of the gamma matrix algebra noted below equation (11), the constant in (19) is changed to -2 and then agrees with the result extacted from equations (2.10) and (2.30) of their work.

We conclude by examining a process where the static effective field theory breaks down. Consider the scattering of two heavy quarks at one loop. There are two graphs, one where the gluon lines don't cross and one where they do. Concentrate on the one where they don't cross (the other graph is well behaved). The bad behavior we are about to illustrate occurs at any mass shell point, so for simplicity take it to be all momenta zero. The integral for the graph contains

$$\int \frac{d^4 l}{(2\pi)^4} \frac{1}{l_0 + i\epsilon} \frac{1}{-l_0 + i\epsilon} \frac{1}{(l^2+\lambda^2)^2}. \tag{20}$$

Already one can see the problem as the $i\epsilon$ prescription has placed the poles so they pinch the l_0 integration. To isolate this we replace the gluon propagators by

$$\frac{1}{(l^2+\lambda^2)^2} - \frac{1}{(\mathbf{l}^2+\lambda^2)^2} + \frac{1}{(\mathbf{l}^2+\lambda^2)^2}. \tag{21}$$

In the difference, we can simplify without worrying about the pole prescription. The term added back in factors into the product of an l_0 integration and a $d-1$ dimensional $\mathbf{l}$ integral. The result of the l_0 integration is a pole in ϵ. If we were using the less singular non-relativistic effective field theory, instead of having ϵ in the denominator of the result, we would have $\mathbf{l}^2/2m$, which is integrable.

In general, for processes involving two (or more) heavy quarks in a state, the results of the static approximation can be invalid and care must be used in choosing the quantities to be computed. For example, in a heavy quark-antiquark system, the potential is the quantity that can be computed using the static approximation[1]. The sign that a quantity is sensitive to the approximation is ϵ dependence as $\epsilon \to 0$. Physically, the problem is that the contribution of intermediate states with two heavy quarks whose total energy is near their combined rest energy is incorrect because the energy difference is neglected in the static approximation. In these circumstances one cannot start at zeroth order in the $1/m$ expansion and evaluate $1/m$ corrections perturbatively. One must include the kinetic energy in the heavy quark propagator and use the non-relativistic effective field theory of Caswell and Lepage[7].

To summarize, we have calculated the full order α_S contributions to the renormalization of an arbitrary heavy-light bilinear. We have clarified the basis of the static effective field theory and illustrated the simplicity of momentum space calculations using the static approximation once a prescription for defining and isolating the poles is supplied. In this framework, perturbation theory beyond one loop and calculations of $1/m$ corrections will be systematic and tractable. Similar simplification is possible in the calculation of the corrections to the value of f_B measured on the lattice[6][11].

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Figure Captions

Fig. 1: Effective Theory Vertex Correction

Fig. 2: Effective Theory Self-Energy Correction