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The Rigidity of Cosmic Strings.

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Abstract

We re-examine the derivation of the effective action for a string and show that the curvature term enters with the same sign as the Nambu term. We then analyse the equations of motion to show explicitly that strings are rigid. We discuss the cosmological implications for the cosmic string scenario of galaxy formation.

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In this letter we seek to address a misconception that has arisen in recent years in the study of cosmic strings. Cosmic strings are topologically stable defects that may have been produced during a phase transition in the early universe[1]. These strings play a central role in one scenario of galaxy formation[2]. To test this scenario it is important to understand how a cosmic string network evolves. For this, it is necessary to know the effective action for a string, i.e. the action that effectively describes the motion of the string.

The standard approach to deriving this action neglects the finite width of the string, the effective action is therefore the action for a two dimensional worldsheet, i.e. the Nambu Goto action. How good an approximation to the true effective action is this? Most of the time, the neglected terms in the action are small, of order the square of the ratio of the string width to its radius of curvature. However, the Nambu action predicts structures such as cusps or kinks at which the neglected terms are not negligible. These structures are **extremely** important when considering gravitational radiation from strings; indeed linearized calculations predict that a large fraction of the energy loss from a loop typically occurs via radiation from such structures[3]. Recent calculations have put very stringent bounds on the cosmic string scenario of galaxy formation, based on the amount of gravitational radiation loops emit[4]. In the light of this it is obviously important to examine the effective action of the string in more detail in order to ascertain whether cusp or kink formation is favoured or suppressed.

As we shall see, previous articles purporting to do this[5] all contained flawed arguments concerning the sign of the correction terms. This is critical in deciding whether cusp (kink) formation is favoured or suppressed. We shall see that, contrary to previous claims, cosmic strings are rigid and therefore cusp formation is suppressed. This turns out to make the bounds on the cosmic string scenario even more stringent than those so far suggested.

The letter is organized into three parts. Firstly we will briefly review the derivation of the effective action, we show that it contains only the Nambu and extrinsic curvature terms and does not contain the extra 'twist' term suggested by Maeda and Turok[5]. We then argue that the stability of the straight static string implies that strings are rigid. To confirm this, in the second section we derive the equations of motion associated with the action (without resorting to any special gauge) and consider the growth of small perturbations to an infinitely long straight string trajectory. In the final section we will consider the implications of the rigidity of cosmic strings for the string scenario of galaxy formation.

1 The Effective Action for Cosmic Strings

To build an effective action, the four-dimensional integral in the field theoretic action must be reduced to a two-dimensional worldsheet integral by projecting the structure of the string vortex onto its core. There is a possible difficulty in doing this in that, although each field configuration uniquely defines a worldsheet, the converse is not true. The different field configurations corresponding to the same worldsheet differ by the number of massive excitations of the fields of the vortex. We shall look only at the lowest energy solutions and thereby neglect these types of string excitations. We will take as our string model the Nielsen-Olesen U(1) vortex[6] in flat spacetime. It is however interesting to note that most of the qualitative arguments we give are independent of the model.

The Nielsen-Olesen string[6] is a vortex solution to the U(1) abelian Higgs model:

$$\mathcal{L}[\phi, A_a] = D_a\phi^\dagger D^a\phi - \frac{1}{4}\tilde{F}_{ab}\tilde{F}^{ab} - \frac{\lambda_0}{4}(\phi^\dagger\phi - \eta_0^2)^2 \tag{1}$$

(where $D_a = \nabla_a + ieA_a$ is the usual gauge covariant derivative, and $\tilde{F}_{ab}$ the field strength associated with A_a). In this paper we shall take lower case latin indices to run from 0 to 3 and our metric convention is $(+,-,-,-)$. The Nielsen-Olesen vortex solution[6] takes the form

$$\phi_0 = \eta_0\chi_0(r)e^{i\theta}, \qquad A_a = \frac{1}{e}(P_0(r) - 1)\nabla_a\theta = \frac{1}{e}(P_a - \nabla_a\theta) \tag{2}$$

in cylindrical polar coordinates. (In fact the total physical content of this model is contained in the fields $|\phi| = \eta\chi$, and P_a.)

The method we use in deriving the effective action[5,15] involves taking a vortex field configuration which is a solution to the field equations of motion, integrating out over directions orthogonal to the worldsheet, to leave a two-dimensional integral over the worldsheet coordinates. In order to do this an Ansatz solution built from the Nielsen-Olesen solution is chosen and the action is expanded around this non-stationary point:

$$S = S_0 + \int d^4x\sqrt{-g}\left[\left(\frac{\delta S}{\delta\phi}\right)_0 \delta\phi + \left(\frac{\delta S}{\delta A_a}\right)_0 \delta A_a\right]. \tag{3}$$

We then integrate out over the orthogonal directions obtaining a worldsheet integral: the effective action. To perform the projection, a worldsheet based coordinate system is required. This is readily formed firstly by extending the worldsheet coordinates σ, τ to be constant on the orthogonal planes to the worldsheet, then, choosing a pair of unit normal fields $\{n^a, m^a\}$ to the worldsheet, we define the perpendicular cartesian coordinates ξ, η by requiring $\left(\frac{\partial}{\partial\xi}\right)_0^a = n^a$, $\left(\frac{\partial}{\partial\eta}\right)_0^a = m^a$. We choose the Ansatz solution to be the Nielsen-Olesen vortex solution, where r and θ are now the cylindrical polar coordinates associated with ξ, η. Since the Ansatz field depends only on these perpendicular directions, the calculation of S_0 is purely geometric, involving an expansion of the volume element, $\sqrt{-g}$, around the worldsheet. The calculation of the second term, the field theoretic part, in principle involves solving for the perturbations around the Ansatz configuration.

For the first part, calculating $\sqrt{-g}$, we simply Taylor expand around the worldsheet:

$$\begin{aligned}\sqrt{-g} &= \sqrt{-g}|_0 + \xi\partial_n\sqrt{-g}|_0 + \eta\partial_m\sqrt{-g}|_0 + \tfrac{\xi^2}{2}\partial_n\partial_n\sqrt{-g}|_0 + \tfrac{\eta^2}{2}\partial_m\partial_m\sqrt{-g}|_0 \\ &\quad +\xi\eta\partial_n\partial_m\sqrt{-g}|_0 + \text{higher order terms.}\end{aligned}$$

Upon integration, terms linear in ξ and η disappear, therefore we only require $\partial_n\partial_n\sqrt{-g}|_0$ and $\partial_m\partial_m\sqrt{-g}|_0$. To find these, we need the covariant derivatives of the normals in terms of worldsheet quantities. These are[7]:

$$\begin{aligned}\nabla_a n_b &= K_{1ab} - \beta_a m_b \\ \nabla_a m_b &= K_{2ab} + \beta_a n_b\ .\end{aligned}$$

The K_{iab} are the extrinsic curvatures. These represent how the worldsheet curves in spacetime and are defined in terms of the derivatives[7] of the normals:

$$\begin{aligned}K_{1ab} &= h^c_{(a}h^d_{b)}\nabla_c n_d \\ K_{2ab} &= h^c_{(a}h^d_{b)}\nabla_c m_d\ ,\end{aligned}$$

where $h_{ab} = g_{ab} + n_a n_b + m_a m_b$ is the first fundamental form, or projection tensor onto the worldsheet.

$$\beta_a = m_b\nabla_a\, n^b = -n_b\nabla_a\, m^b$$

is the normal fundamental form[7] of the worldsheet, and measures the rotation of m^a and n^a in the orthogonal planes as one moves around the worldsheet. Note however that β_a is a gauge dependent object, depending on the choice of the normal fields.

With this information, and noting that for a scalar, $\partial_n \equiv \mathcal{L}_n$, the Lie derivative, we see that

$$\begin{aligned}\partial_n\sqrt{-g} &= \mathcal{L}_n\sqrt{-g} \\ &= \tfrac{1}{2}\sqrt{-g}\; g^{ab}\mathcal{L}_n g_{ab} \\ &= \sqrt{-g}\; g^{ab}\nabla_{(a}n_{b)} \\ &= \sqrt{-g}K_1 \qquad (4)\end{aligned}$$

and also

$$\begin{aligned}\partial_n K_1 &= n^a\nabla_a\nabla_b n^b = n^a\nabla_b\nabla_a n^b \\ &= -(\nabla_b n_a)(\nabla^a n^b) \\ &= -K_{1ab}K_1^{ab}\end{aligned}$$

with similar expressions for m^a.

Thus

$$\int d^4x\sqrt{-g}\mathcal{L}_0 = -\mu\int d^2\sigma\sqrt{-\gamma}[1+\frac{\alpha_0}{2\mu}\sum_{i=1}^{2}(K_i^2 - K_{iab}^2)]$$

where $\mu = -\int d\xi d\eta \mathcal{L}(\chi_0, P_{0b})$ is the energy density per unit length of the Nielsen-Olesen vortex, and

$$\alpha_0 = -\int d\xi d\eta \xi^2 \mathcal{L}(\chi_0, P_{0b}) = -\int d\xi d\eta\ \eta^2\mathcal{L}(\chi_0, P_{0b})$$

is some positive constant of order μr_s^2. However, the Gauss-Codazzi equations[7] for a two-dimensional surface embedded in a flat space imply $\sum_{i=1}^{2} K_i^2 - K_{iab}^2 = {}^2R$, the intrinsic curvature of the worldsheet, which by the Gauss-Bonnet theorem[7] gives a topological invariant upon integration over the two-dimensional worldsheet. Thus the geometric contribution to the effective action reduces to

$$S_g = -\mu\int d^2\sigma\sqrt{-\gamma}$$

i.e. the Nambu action.

The field theoretic contribution to the action arises because the vortex field does not satisfy the field equations of motion for a general curved worldsheet. In order to calculate the field theoretic contribution we need to find the first order correction to the fields from the equations of motion. To simplify the analysis we rescale the coordinates by setting

$$x = \sqrt{\lambda_0}\eta_0\ \xi\ ,\ y = \sqrt{\lambda_0}\eta_0\ \eta\ ,\ s = \sqrt{\lambda_0}\eta_0\ \sigma\ ,\ t = \sqrt{\lambda_0}\eta_0\ \tau\ .$$

Then setting $\gamma = \lambda_0/2e^2$, the lagrangian and equations of motion in terms of χ and P_a become

$$\begin{aligned} \frac{\mathcal{L}}{\lambda_0\eta_0^4} = (\nabla_a\chi)^2 + P_a^2\chi^2 - \frac{\gamma}{2}F_{ab}^2 - \tfrac{1}{4}(\chi^2 &- 1)^2 \\ \nabla^a\nabla_a\chi - P_a^2\chi + \tfrac{1}{2}\chi(\chi^2-1) &= 0 \\ \nabla^a F_{ab} + \frac{1}{\gamma}P_b\chi^2 &= 0\ . \end{aligned} \tag{5}$$

Let χ, P_b be the true solutions to these field equations. The vortex Ansatz (χ_0, P_{0b}) satisfies

$$\begin{aligned} -\chi_0'' - \frac{\chi_0'}{r} + \frac{P_0^2\chi_0}{r^2} + \tfrac{1}{2}\chi_0(\chi_0^2 - 1) &= 0 \\ -P_0'' + \frac{P_0'}{r} + \frac{\chi_0^2 P_0}{\alpha} &= 0 \end{aligned} \tag{6}$$

where $r = \sqrt{x^2+y^2}$ and primes denote differentation with respect to r. We write the true solution as a perturbation of the vortex solution, i.e.

$$\begin{aligned} \chi &= \chi_0 + \delta\chi \\ P_b &= P_{0b} + \delta P_b \, . \end{aligned}$$

From (5,6), after linearizing in the perturbations, one finds the following equations of motion

$$\begin{aligned} (\nabla^a\nabla_a - P_{0a}^2 + \tfrac{1}{2}(3\chi_0^2 - 1))\delta\chi &- 2\chi_0 P_{0a}\delta P^a \\ &= -\nabla^a\nabla_a\chi_0 + P_{0a}^2\chi_0 - \tfrac{1}{2}\chi_0(\chi_0^2-1) \\ &= \frac{1}{\sqrt{-g}}(\sqrt{-g})_1'\chi_0' \end{aligned} \tag{7}$$

and

$$\begin{aligned} \nabla_a\delta F^{ab} + \frac{1}{\gamma}\chi_0^2\delta P^b + \frac{2}{\gamma}\chi_0 P_0^b\delta\chi &= -\nabla_a F_0^{ab} - \frac{1}{\gamma}P_0^b\chi_0^2 \\ &= -\frac{1}{\sqrt{-g}}(\partial_a\sqrt{-g})_1 F_0^{ab} \end{aligned} \tag{8}$$

where $(\ )_1$ indicates the first order part of the expression in parentheses. We can thus read these off from eqn (4) as

$$\begin{aligned} (\partial_b\sqrt{-g})_1 &= \sqrt{-\gamma}(\delta_b^x\tilde{K}_1 + \delta_b^y\tilde{K}_2) \\ &= \sqrt{-\gamma}\,\delta_b^\theta\, r(-\tilde{K}_1\sin\theta + \tilde{K}_2\cos\theta) + \sqrt{-\gamma}\,\delta_b^r(\tilde{K}_1\cos\theta + \tilde{K}_2\sin\theta) \end{aligned} \tag{9}$$

where $\tilde{K}_i = \sqrt{\lambda_0}\eta_0 K_i$ are the rescaled extrinsic curvatures.

We now see that the equation for the perturbations contain inhomogeneous terms which are proportional to the extrinsic curvatures. Hence we expect the inhomogeneous solutions to be of the form*

$$\begin{aligned} \delta P_\theta &= \pi(r)(\tilde{K}_1\cos\theta + \tilde{K}_2\sin\theta) \\ \delta P_r &= \psi(r)(-\tilde{K}_2\cos\theta + \tilde{K}_1\sin\theta) \\ \delta\chi &= \xi(r)(\tilde{K}_1\cos\theta + \tilde{K}_2\sin\theta), \end{aligned}$$

where $\pi(r), \xi(r)$ and $\psi(r)$ are particular inhomogeneous solutions†. Inserting this information back into the action (using (7), (8) and (9) to find $\left(\frac{\delta S}{\delta\phi}\right)_0$ and $\left(\frac{\delta S}{\delta A_b}\right)_0$) one finds

$$\begin{aligned} S_f &= -\lambda_0\eta_0^4\pi\int d^2\sigma\sqrt{-\gamma}[K_1^2+K_2^2]\Big[\int dr(r\xi\chi_0' + \frac{\alpha\pi P_0'}{r} + \alpha P_0'\psi)\Big] \\ &= -\gamma_0\int d^2\sigma\sqrt{-\gamma}[K_1^2+K_2^2] \ . \end{aligned}$$

*We are assuming that the derivatives in the tangential directions on the worldsheet are suppressed by a factor of K relative to those in the normal directions.

†Including non-zero homogeneous solutions corresponds to including massive excitations of the string.

The most important feature of this correction term is its sign, which can be argued on stability grounds. Firstly, the stability‡ of the Nielsen-Olesen vortex solution actually guarantees that the solution X, P_b exists. Secondly, we know that because the Nielsen-Olesen vortex is stable[8] and static, it sits at a local energy minimum. What we have here is the peculiarity that the true solution, which is neighbouring to the Nielsen-Olesen vortex, actually has higher energy (as measured in our coordinate system) than the vortex Ansatz. The true solution, whilst being a stationary point of the action, is not static and therefore does not sit at a local energy minimum. In fact, the equations of motion for δX, δP_b (the differences between the true and the Ansatz solutions) are the equations for a perturbation of the Nielsen-Olesen vortex with a driving term. Thus the perturbed solution (i.e. our true solution) has higher energy than the Nielsen-Olesen vortex. Since the time derivatives of the fields are small compared to the spatial derivatives, the energy is approximately equivalent to minus the action, and thus the sign of the correction term, S_f, is negative. Therefore the total action is

$$\begin{aligned} S &= S_g + S_f \\ &= -\mu \int d^2\sigma \sqrt{-\gamma}\left[1 + \frac{\gamma_0}{\mu}(K_1^2 + K_2^2)\right] , \end{aligned}$$

where $\varepsilon = \frac{\gamma_0}{\mu}$ is a positive constant. The flaw in the arguments of previous authors[5] that lead them to conclude incorrectly that the sign of S_f was positive, was to assume that the true solution sat at a local energy minimum. This is only true for the case of static solutions, non-static solutions, of which the true solution to the equation of motion is an example, sit at saddle points.

2 The Equation of Motion for, and the Stability of, Strings

Having derived the effective action for the string

$$S = -\mu \int d^2\sigma \sqrt{-g}\left[1 + \varepsilon(K_1^2 + K_2^2)\right] \tag{10}$$

we shall now determine the equation of motion for the string. To do this we will continue to use the Gauss-Codazzi formalism.

Let $X^a(\sigma^A)$ (where $\sigma^A = \{\tau, \sigma\}$, uppercase latin indices take the values 0 and 1) be the coordinates of the string in four-dimensions. The induced metric on the two-dimensional worldsheet is given by[7]

$$\gamma_{AB} = \frac{\partial X^a}{\partial \sigma^A}\frac{\partial X^b}{\partial \sigma^B} g_{ab} \tag{11}$$

‡Bogomol'nyi[8] showed the stability of straight Nielsen-Olesen string to radial perturbations, his proof can be easily generalized to show stability to z-dependent perturbations.

The two-dimensional Levi-Civita connection[7] may then be expressed as:

$$\begin{aligned}\Gamma^A_{BC} &= \frac{1}{2}\gamma^{AE}(\gamma_{EB,C} + \gamma_{EC,B} - \gamma_{BC,E}) \\ &= \gamma^{AE} X^a_{,E} X_{a,BC}\end{aligned} \tag{12}$$

which implies that

$$\begin{aligned}D_A D_B X^a &= \partial_A \partial_B X^a - \Gamma^C_{AB} \partial_C X^a \\ &= n^a K_{1AB} + m^a K_{2AB}\end{aligned} \tag{13}$$

and hence

$$\Box X^a \Box X_a = -K_1^2 - K_2^2 \tag{14}$$

We can therefore rewrite the action as

$$S = -\mu \int d^2\sigma \sqrt{-\gamma}\left[1 - \varepsilon(\Box X^a)^2\right] \tag{15}$$

To find the equations of motion, we simply vary the action with respect to X^a, remembering that γ_{AB}, and hence D_A, depend upon X^a. This calculation is somewhat tedious and so we have reserved the details for the appendix and only quote the result here:

$$\begin{aligned}2\varepsilon\Box^2 X^a + \Box X^a - 2\varepsilon\Box^2 X_b\, D_C X^b\, D^C X^a + 4\varepsilon D_C \Box X_b\, D_A D^C X^b\, D^A X^a \\ +4\varepsilon\Box X_b\, D_A D_C X^b\, D^A D^C X^a + 2\varepsilon D_C \Box X_b\, D^C X^b\, \Box X^a + \varepsilon(\Box X^b)^2\, \Box X^a = 0\end{aligned} \tag{16}$$

The equation of motion for the pure Nambu action ($\varepsilon = 0$) can readily be seen to be $\Box X^a = 0$. It is therefore easy to see that if $X^a(\sigma^A)$ is a Nambu trajectory, and contains no **singular points**[§], then it is also a solution of the generalized action. Thus the flat worldsheet corresponding to an infinitely long straight string (the Nielsen-Olesen vortex) is also a solution of these generalized equations of motion. We will now consider the stability of such a worldsheet.

In order to investigate the stability of an infinitely long straight string we will consider perturbing it by a small amount. Such a solution is known to be a stable field configuration and ensuring stability when considering our effective action will enable us to check the sign of the correction term. We take the unperturbed solution to be $X^a_0 = (\tau, \sigma, 0, 0)$ and the perturbed solution to be $X^a = X^a_0 + \delta X^a$. Varying the equations of motion, noting that $\Box X^a = 0$, we find

$$\begin{aligned}2\varepsilon\Box(\delta(\Box X^b))[\delta^a_b - D^C X^a D_C X_b] + \delta(\Box X^a) + 4\varepsilon D^C(\delta(\Box X^b)) D_A D_C X_b D^A X^a \\ +4\varepsilon(\delta(\Box X^b)) D_A D_C X_b D^A D^C X^a = 0\end{aligned} \tag{17}$$

[§]If the trajectory does contain singular points in the induced metric, for example points at which $\gamma = 0$, then these trajectories must be examined with care as additional derivatives of γ_{AB} and $\Box X^a$ appear in the full equation of motion

Using equation (A.6), we see that

$$\delta(\Box X^a) = -2D^C D^D X^a D_C \delta X^b D_D X_b + \Box\delta X^a - D_C X^a D^C X_b \Box\delta X^b,$$

which is perpendicular to the worldsheet.

Taking the parallel component of equation (17) to the worldsheet gives

$$\begin{aligned} D_A D^C X_b D_C[\delta\Box X^b] &= 0 \\ \Rightarrow \qquad D_C X_b \Box[\delta\Box X^b] &= D_A[D_C X_b D^A(\delta\Box X^b)] \\ &= -D_A[D^A D_C X_b \delta\Box X^b] \\ &= -R_C^D D_D X_b \delta\Box X^b \\ &= 0. \end{aligned}$$

Thus the perturbation equations now reduce to

$$\begin{aligned} &(2\varepsilon\Box + 1)\delta(\Box X^a) + 4\varepsilon\delta(\Box X^b) D_A D_C X_b D^A D^C X^a \\ &= (2\varepsilon\Box + 1)\delta(\Box X^a) + 4\varepsilon n_i^a K_{iAC} K_j^{AC} n_{jb} \delta(\Box X^b) = 0 \end{aligned} \tag{18}$$

For the case of a static straight string, we may use the specific form of X^a to conclude that the perturbation equations to first order in δX are:

$$(2\varepsilon\Box + 1)\Box\delta X^\alpha = 0 \tag{19}$$

where $\alpha = 2,3$ are the perpendicular components. δX^0 and δX^1 components can be arbitrary functions of σ and τ because $\delta(\Box X^0) \equiv \delta(\Box X^1) \equiv 0$ for arbitrary δX^0 and δX^1. These variations correspond to reprametrizations of the string coordinates.

Now let us concentrate on equation (19). If we consider a localized perturbation of the string we can fourier decompose δX^α as

$$\delta X^\alpha = \int dk e^{ik\sigma} \delta\tilde{X}^\alpha(\tau, k)$$

Doing a frequency analysis of $\delta\tilde{X}^\alpha(\tau,k)$ as $\delta\tilde{X}^\alpha(\tau,k) = \delta\hat{X}^\alpha(\omega,k)e^{i\omega\tau}$, we see from (19) that

$$(-2\varepsilon(\omega^2 - k^2) + 1)(\omega^2 - k^2)\delta X^\alpha(\omega, k) = 0.$$

Now, since $k^2 > 0$, we see that for $\varepsilon \geq 0$, $\omega^2 \geq 0$, which implies that a rigid infinitely long straight string is stable to perturbations. If, however, we had chosen $\varepsilon < 0$, then there would have been unstable modes for which the perturbations grow exponentially. These would be perturbations for which $k < \frac{1}{\sqrt{2|\varepsilon|}}$, i.e. for which the wavelength of the perturbation is larger than $\sqrt{2|\varepsilon|}$. However, as $|\varepsilon|$ is of order r_s^2, this would say that any perturbation of wavelength longer than the scaled width of the string are be unstable. Since our approximation breaks down at order

r_s^2, this would mean that, in the regime in which our approximation are valid, all perturbations were unstable - clearly in contradiction with the facts!

Thus we see that to order ε a straight infinitely long string is only stable if strings are rigid. One might now worry that higher order terms in the expansion of the action with respect to ε might change this result, especially since ε multiplied the highest derivatives in (19). Fortunately this does not turn out to be the case. Remembering that the only possible higher order terms in the action are products of extrinsic curvatures, it is easy to see that the differential equation satisfied by the perturbation will never be of degree higher than four. Thus our conclusion, that strings are rigid for small angle bending, holds; going to higher order will not change this.

In fact, we can generalize this argument quite straightforwardly to include non-singular Nambu trajectories in the following way. Consider $N_{ij} = K_{iAB}K_j^{AB}$. Clearly this is a symmetric matrix in i, j and so can be diagonalised by suitable choice of normals. Now, dropping the i-subscript on K_{AB} for notational convenience, and choosing worldsheet coordinates in which K_{AB} is diagonal, we see that

$$K_A^A = 0 \Rightarrow K_{00}\gamma^{00} + K_{11}\gamma^{11} = 0$$

which further implies

$$\begin{aligned} K_{AB}K^{AB} &= K_{00}^2\gamma^{00^2} + K_{11}^2\gamma^{11^2} + 2K_{00}K_{11}\gamma^{01^2} \\ &= K_{00}^2\frac{\gamma^{00}}{\gamma^{11}}\left(\gamma^{00}\gamma^{11} - 2\gamma^{01^2}\right) \\ &\geq 0. \end{aligned}$$

Thus N_{ij} is a positive semi-definite symmetric matrix. Therefore in our perturbation equations, $M_b^a = -n_i^a N_{ij} n_{jb}$ is a positive semi-definite matrix, since both the normals and $\delta\Box X^b$ are spacelike. Therefore, writing the perturbation equations as

$$\left(\delta_b^a\Box + \frac{1}{2\varepsilon}\delta_b^a - 2M_b^a\right)\delta\Box X^b = 0 \qquad (20)$$

we see that provided the eigenvalues of M_b^a are smaller than ε^{-1}, the perturbations are stable. Since the eigenvalues of M_b^a are of the order of the square of the extrinsic curvatures, and ε is of order r_s^2, we see that provided the curvature of the worldsheet remains small compared with the radius of the string, the Nambu solutions are stable. However, for large curvatures (such as those near a cusp) the dominant term is now M_b^a, and we have unstable modes arising. This means that a cusp is not a stable structure, and is rounded off at the order of the string width.

3 Discussion

Let us now discuss the cosmological consequences of the rigidity of cosmic strings. We derived the equation of motion for the rigid string (16) and saw that all Nambu trajectories that do not contain singular points are solutions to the equation of motion. However, as the equation of motion for a rigid string is quartic, a far richer set of solutions than just the nonsingular Nambu solutions exist.

For understanding the cosmological evolution of a network of open strings[¶] it might be marginally plausible to neglect these new solutions. After all, the lowest energy solutions for a string of fixed length will be straight. For closed strings however, periodic boundary conditions prevent the extrinsic curvature from being zero everywhere. We therefore expect the Nambu trajectories to be modified. Obviously the effect of rigidity is going to be greatest when the mean value of $K_{ab}K^{ab}$ is large, i.e. for small loops, and we would anticipate that the rigid string trajectories for these loops would be significantly different from Nambu trajectories. In fact the rigid string equations of motion permit a static circular solution[9] of radius of order $\sqrt{\varepsilon}$. This particular new, non-Nambu solution, is however cosmologically uninteresting. This is because it is unstable to radial perturbations[10] due to its large extrinsic curvature; a fact that can be seen from equation (21). If you believe the simple model of string evolution proposed by Albrecht and Turok[4] you would also expect the number of small loops present in the early universe at a given time to be less than that predicted neglecting rigidity.

For large loops on large scale the mean value of $K_{ab}K^{ab}$ will be small and therefore on large scales the Nambu approximation to the true string trajectories should be reasonable. Even for large loops however, there may be regions where the string trajectory departs significantly from that predicted by the Nambu approximation. For example, if the action for a string trajectory infinitesimally close to a cusp or kink is calculated it is found to diverge[9]. This is suggestive that a string strongly resist forming cusps, but is not conclusive. Our perturbative analysis supports this by showing that a cusp is unstable and indicates that it will typically be rounded off on scales of order the string width.

A powerful test of the cosmic string scenario comes from the quantity of gravitational radiation produced by strings. The rounding off of cusps may have significant effects on the amount of gravitational radiation produced. Calculations of the power, P, emitted by cosmic strings predict[3]:

$$P = \Gamma G \mu^2,$$

where G is the gravitational coupling constant and Γ is a constant that depends on the loop trajectories. For loop trajectories with cusps or kinks Γ is typically of order 50-150. In 1984, Turok[11] evaluated the radiation from trajectories without cusps or kinks and found a Γ of order 10. The regularity of the timing of the millisecond pulsar[4] imposes strong limits on the energy density in gravitational radiation present today. The predicted contribution to this energy density produced by strings is

[¶] A model of monopoles connected by string in which the monopole mass is of the same order of magnitude as the string tension would be we approximated by open strings[14] proportional to $\left(\frac{G\mu}{\Gamma}\right)$. The factor of 10 on going from loops without cusps/kinks to those with could significantly (although a factor of 10 is probably an over estimate of the effect of rigidity) change the bound on $G\mu$, so much so that it becomes to small for strings to provide a successful scenario of galaxy formation.

Further improvements to our calculation would involve including gravity in our effective action. This would be a worthwhile exercise as it might enable the effect of backreaction on the amount of gravitational radiation emitted by loops to be estimated. Again it would be expected that the inclusion of this would further suppress cusp/kink formation and thereby reduce the rate of emission of gravitational radiation. This would mean that the energy density in radiation might not be redshifted enough to be consistent with the low levels currently observed. It is obviously important to assess how great the suppression would be in order to understand whether the cosmic string scenario with $G\mu = 10^{-6}$ is ruled out. It should be pointed that there are effects that could reduce the strength of the bound on $G\mu$, for example if the string emits significant amounts of other types of radiation.

There are perhaps other factors that may, for the present, prevent us from ruling out the cosmic string scenario of galaxy formation. The most significant of these is the uncertainty in the normalization factor ν of the loop distribution function. This has been determined from numerical simulations of the evolution of a string network by two groups of researchers[4,11] and they obtain very different answers. This difference is sufficient that one group would rule the string scenario out while the other, even with $\Gamma = 10$, would not. Although it merits further work to establish the degree of suppression of Γ due to rigidity, it is probably more important to first understand the discrepancy in the value of ν obtained by the two groups.

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Appendix

Here we find the equations of motion associated with the action

$$S = -\mu \int d^2\sigma \sqrt{-\gamma}[1 - \varepsilon(\Box X^a)^2]. \tag{A.1}$$

In varying the action with respect to X^a, we must remember that both the metric, γ_{AB}, and the connection, Γ^C_{AB} depend on X^a. For the metric we have:

$$\delta\gamma_{AB} = \delta X^a_{,A} X_{a,B} + X^a_{,A}\delta X_{a,B}$$

$$= \delta X^a_{;A} X_{a;B} + X^a_{;A}\delta X_{a;B} \tag{A.2}$$

and for the connection we have:

$$\begin{aligned}
\delta\Gamma^C_{AB} &= \frac{1}{2}\delta[\gamma^{CD}(\gamma_{DA,B}+\gamma_{DB,A}-\gamma_{AB,D})] \\
&= -\frac{1}{2}\gamma^{CE}\gamma^{DF}\delta\gamma_{EF}(\gamma_{DA,B}+\gamma_{DB,A}-\gamma_{AB,D}) \\
&\quad +\frac{1}{2}\gamma^{CD}(\delta\gamma_{DA,B}+\delta\gamma_{DB,A}-\delta\gamma_{AB,D}) \\
&= -\gamma^{CE}\Gamma^F_{AB}[X^b_{,E}\delta X_{b,F}+\delta X^b_{,E}X_{b,F}]+\gamma^{CD}[\delta X^b_{,AB}X_{b,D}+X^b_{,AB}\delta X_{b,D}] \\
&= X_{b,D}\gamma^{CD}D_AD_B\delta X^b+\gamma^{CD}\delta X_{b,D}D_AD_BX^b
\end{aligned} \tag{A.3}$$

Therefore, varying S with respect to X^a gives

$$\begin{aligned}
\delta S &= -\mu\int d^2\sigma\delta\sqrt{-\gamma}[1-\epsilon(\Box X^a)^2]-2\epsilon\sqrt{-\gamma}(\Box X_a)\delta(\Box X^a) \\
&= \mu\int d^2\sigma\frac{1}{2}\left(\sqrt{-\gamma}\gamma^{AB}\delta\gamma_{AB}[1-\epsilon(\Box X^a)^2]\right) \\
&\qquad -2\epsilon\sqrt{-\gamma}\Box X_a\delta[\gamma^{AB}D_AD_BX^a] \\
&= \delta S_1+\delta S_2
\end{aligned} \tag{A.4}$$

The first term in this expression gives, upon integration by parts,

$$\delta S_1 = \mu\int d^2\sigma\sqrt{-\gamma}\delta X^aD_A\left(D^AX_a[1-\epsilon(\Box X^b)^2]\right) \tag{A.5}$$

The second term contains three pieces from the variation

$$\begin{aligned}
\delta[\gamma^{AB}D_AD_BX^a] &= -\gamma^{AC}\gamma^{BD}\delta\gamma_{CD}D_AD_BX^a+\Box\delta X^a-\gamma^{AB}\delta\Gamma^C_{AB}X^a_{\ ,C} \\
&= -2D^CD^DX^aD_C\delta X^bD_DX_b+\Box\delta X^a \\
&\quad -D_CX^aD^CX_b\Box\delta X^b-D_CX^aD^C\delta X^b\Box X_b
\end{aligned} \tag{A.6}$$

Substituting from (A.2) and (A.3) and integrating by parts as necessary gives

$$\begin{aligned}
\delta S_2 &= \mu\int d^2\sigma\sqrt{-\gamma}\delta X^a\Big\{4\epsilon D_C\left(\Box X_bD_AD^CX^bD^AX_a\right)+2\epsilon\Box^2X_a \\
&\qquad -2\epsilon\Box\left(\Box X_bD_CX^bD^CX_a\right)+2\epsilon D_C\left(\Box X_bD^CX^b\Box X_a\right)\Big\}.
\end{aligned} \tag{A.7}$$

Thus, imposing $\frac{\delta S}{\delta X^a}=0$, (A.5) and (A.7) give as the equations of motion for the string:

$$\begin{aligned}
&- D_A\left(D^AX_a[1-\epsilon(\Box X^b)^2]\right)-4\epsilon D_C\left(\Box X_bD_AD^CX^bD^AX_a\right)-2\epsilon\Box^2X_a \\
&+ 2\epsilon\Box\left(\Box X_bD_CX^bD^CX_a\right)-2\epsilon D_C\left(\Box X_bD^CX^b\Box X_a\right)=0,
\end{aligned} \tag{A.8}$$

which, upon expansion of the terms and cancellation, gives

$$\begin{aligned}
&2\epsilon\Box^2X^a+\Box X^a-2\epsilon\Box^2X_b\,D_CX^b\,D^CX^a+4\epsilon D_C\Box X_b\,D_AD^CX^b\,D^AX^a \\
&+4\epsilon\Box X_b\,D_AD_CX^b\,D^AD^CX^a+2\epsilon D_C\Box X_b\,D^CX^b\,\Box X^a+\epsilon(\Box X^b)^2\,\Box X^a=0
\end{aligned} \tag{A.9}$$

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