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Scattering amplitudes in hot gauge theories

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Abstract

A consistent approach to the perturbative calculation of scattering amplitudes in hot gauge theories is developed. As an example, the damping rate for a heavy fermion is computed.

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Understanding the collision of heavy nuclei at ultra-relativistic energies requires insight into the properties of QCD at a temperature T, both in and out of equilibrium.[1] Attention has focused recently on the properties near equilibrium for "hot" gauge theories: QCD, in its chirally symmetric, deconfined phase, or for QED, at temperatures much greater than the fermion mass. Particularly confusing is the infrared limit: for instance, on its mass shell the damping rate for the gluon appears to be gauge dependent, and, in certain gauges, of the wrong sign,[2] neither of which should be true for the mass shell of a physical excitation.

In this Letter I argue that for quantities like damping rates, in hot gauge theories an infinite subset of diagrams of higher order in the loop expansion contribute to the same order in the coupling constant g as the lowest order result. I generalize the methods of ref. 3 to develop an efficient technique for resumming all effects to leading order in g. Doing so, I expect that the unphysical properties found previously[2] reflect nothing more than incomplete calculations.

The essential step in computing processes that involve soft quanta in hot gauge theories is the recognition that one must differentiate between effects that are of order $\sim gT$, and those that are g times terms $\sim gT$.

To start with, it is necessary to compute the terms $\sim gT$ for every propagator and vertex. By this I mean: let an external momentum for a given diagram be P^μ ($P^\mu = (p_0, \bar{p}), p \equiv |\bar{p}|$), and analytically continue the diagram from euclidean p_0 to real energies, $\omega = ip_0$. The terms $\sim gT$ are those such that when the components of all external momenta are soft, ω and $p \sim gT$, in magnitude the diagram is some power of gT, with no extra factors of g left over.

The self-energy terms $\sim gT$ have been computed for fermion and gauge fields by Klimov and Weldon.[4] These calculations illustrate something generic to all terms $\sim gT$, including vertex renormalizations:[5] after the discrete sum over the loop k_0 is done, in the remaining integral over spatial $\bar{k}$, terms $\sim gT$ only arise from hard $k \sim T$, and not from soft $k \sim gT$. At hard k, however, loop corrections are never greater $\sim g^2$. Consequently, any term $\sim gT$ is simply computed by using the bare propagators and vertices to one-loop order; even then, only a small part of the one-loop graph contributes, when k is hard.

What is difficult to compute are quantities that are g times a power of $\sim gT$, since typically they receive contributions from both hard and soft loop momenta. For soft momenta, by definition terms $\sim gT$ are as large as the bare propagators and vertices. Thus when a term $\sim gT$ is inserted into a diagram, such as through a self-energy or vertex correction, in magnitude the result is as large as the diagram without the insertion, although according to the loop expansion it is nominally of higher order. Thus to include all effects of leading order in g, the complete renormalized propagators and vertices to $\sim gT$ must be used, sewn together by the Schwinger-Dyson equations.

The necessity of including terms $\sim gT$ for soft quanta was recognized by Kalashnikov and Klimov,[2] by Gross, the author, and Yaffe,[1] by Heinz, Kajantie, and Toimela,[2] and again in ref. 3. Unlike these discussions, where only some terms $\sim gT$ were considered, I emphasize that the effects of every term $\sim gT$ — including not only self-energy but vertex renormalizations — must be incorporated.

This is unlike low temperatures, such as in cold QED, when the fermion mass $m \gg T$. Then any infrared divergence in a diagram is cut off by m, and loop effects are uniformly small, $\leq g^2$.

Even for hot theories, the euclidean green's functions are far simpler than its scattering amplitudes. For the former, since the momenta p_0 is a multiple of πT, one isolates the infrared divergences as coming from boson lines with $p_0 = 0$. For scattering amplitudes, after analytic continuation the energy $ip_0 = \omega$ of either fermion or boson lines is arbitrary, and the only scale that cuts off infrared divergences in loop diagrams is radiatively induced, $\sim gT$, which is small relative to T.

For the gauge field, to $\sim gT$ the longitudinal and transverse self-energies are[4,5]

$$\Pi_\ell = -3m_g^2\left(1 - \frac{ip_0}{2p}L(ip_0,p)\right) \ , \ \Pi_t = \frac{3m_g^2 p_0^2}{2p^2}\left(1 - \left(1 + \frac{p_0^2}{p^2}\right)\frac{ip_0}{2p}L(ip_0,p)\right) \ , \quad (1)$$

with $L(ip_0, p) \equiv log((ip_0 + p)/(ip_0 - p))$, and $m_g \sim gT$ is the gauge field "mass"; $m_g^2 = (N + N_f/2)(gT)^2/9$ for $SU(N)$ with N_f isodoublet fermions.

The self-energies to $\sim gT$ are independent of gauge for either fermion or gauge fields.[4,5] This is crucial to the consistency of the present approach, since the renormalized propagators to $\sim gT$ determines the mass shell conditions that enter in going beyond leading order, to $\sim g(gT)$, *etc.*.

For gauge fields, the form of the renormalized propagator does depend on the choice of gauge. In Coulomb gauge, $\partial_i A^i = 0$, the renormalized gauge propagator is $\Delta_{00} = \Delta_\ell$, $\Delta_{0i} = 0$, and $\Delta_{ij} = (\delta^{ij} - p^i p^j/p^2)\Delta_t$, with $\Delta_\ell = 1/(p^2 - \Pi_\ell)$, $\Delta_t =$

$1/(p_0^2 + p^2 - \Pi_t)$.

It would be onerous to compute with the renormalized propagators formed from eq. (1) in the usual fashion, by performing the discrete sum over p_0 in loop integrals. To be able to efficiently calculate with these renormalized propagators, I fourier transform with respect to p_0, and work in a coordinate representation for the euclidean time, τ.[3,5] This produces a spectral representation for the Δ's:

$$\Delta_{\ell,t}(\tau,p) = T \sum_{\substack{n=-\infty \\ p_0=2\pi nT}}^{+\infty} e^{-ip_0\tau}\Delta_{\ell,t} = \int_0^\infty d\omega\, \Big((1+n(\omega))\, e^{-\omega\tau} + n(\omega)e^{+\omega\tau} \Big)\, \rho_{\ell,t}(\omega,p) \ , \tag{2.a}$$

$n(\omega) = 1/(exp(\omega/T) - 1)$, and

$$\rho_{\ell,t}(\omega,p) = \rho^{res}_{\ell,t}(\omega_{\ell,t}(p),p)\delta(\omega - \omega_{\ell,t}(p)) + \rho^{disc}_{\ell,t}(\omega,p)\theta(p-\omega) \ , \tag{2.b}$$

$\theta(x) = 0,1$ for $x <,> 0$ The spectral densities $\rho_{\ell,t}$ are determined by the behavior of $\Delta_{\ell,t}$ for $ip_0 = \omega + i0^+$: each Δ has a single pole above the light cone, lying on the mass shell $\omega = \omega_{\ell,t}(p)$, with residue $\rho^{res}_{\ell,t}$. Due to Landau damping there is a discontinuinuity below the light cone, for $0 \le \omega \le p$, which determines ρ^{disc}.

The transverse pole represents the usual transverse excitations of a gauge field, renormalized by temperature to lie above the light cone by $\sim m_g$: as $p \to 0$, $\omega_t(p) \approx m_g$, $\rho^{res}_t \approx 1/(2m_g)$; as $p \to \infty$, $\omega_t(p) \approx p + 3m_g^2/(4p)$, $\rho^{res}_t \approx 1/(2p)$. The longitudinal pole behaves like a massive excitation about zero momentum, $\omega_\ell(p) \approx m_g$ and $\rho^{res}_\ell \approx (-1/p^2)1/(2m_g)$. As a collective mode, the residue of the longitudinal mode is only significant for soft momenta $\le m_g$: for $p \gg m_g$, the longitudinal mass shell is exponentially close to the light cone, and its residue exponentially small: $\omega_\ell(p) \approx p(1+2x_\ell)$, $\rho^{res}_\ell \approx (-1/p^2)x_\ell(4p/(3m_g^2))$, with $x_\ell \equiv exp(-2p^2/(3m_g^2) - 2)$.

Previous attention has focused on these quasi-particle excitations in the Δ's,[2] but the most important contribution to damping rates is from the discontinuities in the spectral densities. For the damping rate of a heavy fermion, all that is needed is the limit about zero energy:

$$\rho^{disc}_\ell(\omega,p) \underset{\omega\to 0}{\approx} \left(-\frac{1}{p^2}\right)\frac{3}{2}\,\frac{m_g^2\omega p}{(p^2+3m_g^2)^2} \ , \ \rho^{disc}_t(\omega,p) \underset{\omega\to 0}{\approx} \frac{3}{4}\,\frac{m_g^2\omega p}{\left(p^6 + \left(3\pi m_g^2\omega/4\right)^2\right)} \ . \tag{3}$$

Since the self-energies to $\sim gT$ are independent of gauge, to this order the renormalized propagator satisfies properties expected for a physical field. The transverse spectral density is never negative, $\rho_t \geq 0$; from the equal time commutation relations, it satisfies a sum rule, $\int_0^\infty d\omega(2\omega)\rho_t(\omega,p) = 1$, which was checked numerically.[5]

Unlike the transverse density, the longitudinal density is never positive, $\rho_\ell \leq 0$; indeed, it is only smooth about zero momentum if one pulls out an overall factor of $1/p^2$, as in eq. (3). From the example of the coupling of an (abelian) gauge field to an external, conserved current,[5] it can be shown that $\rho_\ell \leq 0$ ensures that ρ_ℓ contributes to physical quantities with positive weight; despite the $1/p^2$ in ρ_ℓ, its effects are infrared finite. This happens because while Δ_t couples to the space-like part of the current, Δ_ℓ couples, essentially, to the time-like part. Also, ρ_ℓ does not satisfy a sum rule like that for ρ_t.

The treatment of the renormalized fermion propagator to $\sim gT$ is similar. For massless fermions, Klimov and Weldon[4] noticed that at positive energy ($\omega > 0$), the renormalized fermion propagator to $\sim gT$ has not one, but two branches above the light cone. One branch is standard: it has "mass" $\sim gT$, a residue that is ~ 1 for all p, and chirality equal to helicity. The second branch is a collective mode — also above the light cone by $\sim gT$, along this branch chirality is equal to *minus* the helicity, while its residue decreases exponentially for $p \gg gT$.

It is direct to treat the case of fermions with non-zero mass, at least if $m \sim gT$.[5] To $\sim gT$, there are still two branches in the fermion propagator, but as m increases from zero, while the mode that had flipped chirality/helicity at $m = 0$ becomes lighter, its residue descreases. By the time that $m \sim T$, the residue of the collective mode is $\leq g^2$, and its effects are negligible. For any mass, the spectral density in the fermion propagator $\sim \gamma^0$ satisfies a sum rule like that for ρ_t.

At temperatures $T \approx 100 - 300MeV$ in the quark-gluon plasma, the up and down quarks are essentially massless, with the strange quark mass $\sim T$. Thus the propagation of up and down quarks is strongly renormalized over soft momenta, and exhibits a collective mode with flipped chirality/helicity; for strange quarks, loop effects are $\leq g^2$, and the collective mode can be ignored.

The great difficulty in computing in hot gauge theories is the necessity of computing terms $\sim gT$ for the vertices. The vertex corrections $\sim eT$ have been computed for hot QED:[5] while independent of gauge, when the external legs are soft its form is involved, and not merely some function of the external momentum times γ^μ.

There are instances in which vertex and (some) self-energy corrections can be neglected. For a heavy fermion F of mass M, $M \geq T/g$, self-energy and vertex corrections are $\leq g^2T/M$, so that to order $\sim gT$, they can be dropped; the only effects of $\sim gT$ that need to be included are for the gauge field. (A similar simplification occurs for any field with hard momentum.)

To leading order in g^2, the imaginary part of F's self–energy is[6]

$$Im\,\Sigma_F \approx -\frac{g^2}{2}C_F T\pi \int \frac{d^3k}{(2\pi)^3}\int_0^\infty \frac{d\omega}{\omega}\sum_{i=\ell,t}\rho_i(\omega,k)G_i(\delta(E_p^F-E_{p-k}^F-\omega)+\delta(E_p^F-E_{p-k}^F+\omega)) \,. \tag{4}$$

I assume that F is on its mass shell ($p^\mu = (iE_p^F,\bar{p}), E_p^F = \sqrt{p^2+M^2}$), and used Coulomb gauge; C_F is the Casimir, and $G_\ell \approx \gamma^0 + 1$, $G_t \approx 2(-\gamma^0+1)$. There are other terms to $\sim g^2$ not indicated in eq. (4), but these do not contribute to eqs. (5) — (7); these results are independent of the choice of gauge.[5]

Start with the case at rest, $p=0$. The only way to satisfy momentum conservation is through the last term in eq. (4), $\omega = E_k^F - M \approx k^2/(2M)$. For $k \sim m_g$, $\omega \approx m_g^2/M \ll k$, so only the ρ^{disc}'s of eq. (3) enter. Then

$$Im\,\Sigma_F \underset{p=0}{\approx} (\gamma^0-1)\,\frac{3g^2C_FT}{16\pi}\,\frac{m_g^2}{\mu_{cutoff}^2} + (\gamma^0+1)\,\frac{g^2C_FT}{16\pi} + \ldots\,; \tag{5.a}$$

an infrared cutoff μ_{cutoff} was introduced. The first term is due to ρ_t^{disc}, and contributes to wave–function renormalization for F,

$$Z_F \underset{p=0}{\approx} 1 + i\,\frac{3g^2C_FT}{16\pi M}\,\frac{m_g^2}{\mu_{cutoff}^2}\,. \tag{5.b}$$

There are other, gauge-dependent terms in Z_F; this represents the most infrared singular, imaginary term. Since Z_F is not a physical quantity, the presence of μ_{cutoff} is of no concern. For instance, to one-loop order in cold QED, terms $\sim g^2T/\mu_{cutoff}$ appear in Z_F. The higher power of $1/\mu_{cutoff}$ in eq. (5.b) arises because, in including all effects of $\sim gT$ for the gauge field, one is implicitly including a subset of effects to two-loop, and higher, order: eq. (5.b) is $Z_F - 1 \approx g^4T^2/\mu_{cutoff}^2(T/M)$.

The second term in eq. (5.a) is due exclusively to ρ_ℓ^{disc}, and alters the mass shell for F; the pole in the renormalized propagator, $-i\,\not{p} + M - \Sigma_F$, is no longer at

$\omega_{mass} = M$, but at

$$\omega_{mass} \underset{p=0}{\approx} M - i\,\frac{g^2 C_F}{8\pi}\,T\,. \tag{5.c}$$

The sign of $Im\ \omega_{mass}$ is such that the pole is off the physical sheet, as it should be, for a causal, stable theory. As only ρ_ℓ contributes to eq. (5.c), that $-Im\ \omega_{mass}$ has positive sign is another example (like that of an external current,[5] mentioned before) of why ρ_ℓ is negative.

Consider now $p \neq 0$; I assume that $p \sim T$, so $m_g \ll p \ll M$. Both terms in eq. (4) contribute, $\omega = \pm\left(E_p^F - E_{p-k}^F\right) \approx \pm pk\cos(\theta)/M$. The dominant contribution in eq. (4) is from the transverse density, when $\theta \approx \pi/2$, and $\omega \approx 0$, eq. (3):

$$Im\ \Sigma_F \underset{p\sim T}{\approx} (\gamma^0 - 1)\,\frac{g^2 C_F T}{4\pi}\,\frac{M}{p}\,log\left(\frac{m_g}{\mu_{phys}}\right)\,. \tag{6.a}$$

An infrared divergence appears, which I cutoff at μ_{phys}. Eq. (6.a) contributes to wave-function renormalization,

$$Z_F \underset{p\sim T}{\approx} 1 + i\frac{g^2 C_F T}{4\pi p}\,log\left(\frac{m_g}{\mu_{phys}}\right)\,, \tag{6.b}$$

and moves the pole for F off the physical sheet, to:

$$\omega_{mass} \underset{p\sim T}{\approx} M + \frac{p^2}{2M} - i\,\frac{p}{M}\left(\frac{g^2 C_F T}{8\pi}\,log\left(\frac{m_g}{\mu_{phys}}\right)\right)\,. \tag{6.c}$$

Despite appearances, $Im\ \Sigma_F$ behaves smoothly about zero momentum. Eqs. (6) cannot be extended from $p \sim T$ to $p = 0$, since then the approximate form of $\omega \approx \pm pk\cos(\theta)/M$ no longer holds in eq. (4).

As is common to theories with infrared divergences, μ_{cutoff} is a parameter which drops out of any measurable quantity. In contrast, μ_{phys} contributes to the damping rate of eq. (6.c), and so is a parameter of physical siginificance.

For non–abelian gauge theories, it is clear what determines μ_{phys}. The logarithmic divergence in eq. (6) arises from the behavior of $\rho_t^{disc}(\omega, p)$ for $\omega \ll p \ll m_g$, eq. (3). In this limit, ρ_t^{disc} is so much more singular than ρ_ℓ^{disc} because for $\omega = 0$, $p \to 0$, $\Pi_\ell \sim (gT)^2$, while to $\sim gT$, Π_t vanishes. This is special to keeping terms $\sim gT$ — even for $\omega = 0$, there are terms in $\Pi_t \approx g(gT)p$.[7] These will cutoff the logarithmic divergence in eq. (6), and fix $\mu_{phys} = \mu_0 g^2 T$, with $\mu_0 \sim 1$. Thus for QCD,

$$log\left(\frac{m_g}{\mu_{phys}}\right) \approx log\left(\frac{1}{g}\right) + \cdots \tag{7}$$

For hot QED with scalar matter fields, one similarly expects $\mu_{phys} \sim e^2 T$. For hot QED with only fermion matter fields, presumably to any order in e, $\Pi_t \approx p^2$ for $\omega = 0, p \to 0$, and it is not apparent what sets the scale for μ_{phys} — it cannot be larger than $e^2 T$, but it could well be smaller.

In hot QED, the damping rates for light and heavy fermions are similar:[5] at zero momentum, $-Im\ \omega_{mass}$ is $\sim +e^2 T$, while for $p \neq 0$, it is $\sim +e^2 T\ log(eT/\mu_{phys})$; both terms are independent of gauge. A power counting analysis[5] indicates the same qualitative behavior for the damping rates in hot QCD.

After this manuscript was completed, G. Baym and C. Pethick informed me that they and H. Monien have also investigated a resummation of terms $\sim gT$.[8] Although they did not resum all terms of $\sim gT$, they did include the most important — those for Landau damping. They find that 1/viscosity is $\sim g^4\ log(1/g)$: this $\sim log(1/g)$ differs from that in the self-energy, eq. (6), as it arises from $\sim log(T/(gT))$, instead of the $\sim log((gT)/(gT^2))$ of eq. (7).

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