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THE COLOR STRUCTURE OF GLUON EMISSION

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Abstract

The color structure of an arbitrary QCD diagram is studied. A basis for the color form factors of a generic amplitude is suggested, in which all the coherence effects are automatically and simply accounted for to the leading order in N_c. Some applications of this formalism are given for tree level amplitudes describing processes with one or two pairs of fermions and radiation of gluons and photons. The different coherence properties of the abelian and non-abelian radiation are exhibited with some analytic and explicit examples, leading to the appearance of string-effect-like correlations for *hard* non-abelian radiation.

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1 Introduction

Processes with many hard partons in the final state play a key role in high energy collisions. They allow for tests of perturbative QCD and they provide large backgrounds for more exotic phenomena [1]. The capability of performing precise predictions for these processes is severely limited by the complexity of the calculations involved: the large number of Feynman diagrams contributing to the amplitudes and the complex algebraic structure of the vertices call for *ad-hoc* techniques to treat the problem [2-10]. In particular, the organization of the helicity and the color structure of the amplitudes turns out to be fundamental.

Recently a criterion was proposed to organize the color structure of amplitudes with many gluons [6,7] and with gluons and a quark-antiquark pair [5,9]. In this paper I describe a general way to efficiently organize the color structure of a generic QCD amplitude, containing an arbitrary number of quarks and gluons plus possibly arbitrary color singlet sources. As a by-product I will get exact expressions for some sets of partonic amplitudes relevant in hadronic collisions, e^+e^- interactions and deep inelastic scattering. These expressions will explicitly exhibit interesting phenomena of color coherence, generalizing results known in the case of soft radiation.

This paper is organized as follows. In Section 2 I describe the construction of the color structures in which to expand an arbitrary QCD amplitude. I will first treat the tree-level case and then the case with loops, proving in an algebraical way the gauge invariance of the decomposition in terms of color-form-factors. In Section 3 I apply these results to the case of quark-gluon scattering, using a set of exact helicity-amplitudes to exhibit the properties of color coherence of the non-abelian radiation in relation to the case of abelian radiation. A non-trivial example of antenna-type patterns of emission for non-abelian radiation is given in Section 4, where processes with gluons and two quark pairs are described. Once again the relation with the emission of photons is established. As an example of the versatility of the technique, in Section 5 I will extend these results to processes with color-singlet sources, as e^+e^- scattering and Deep Inelastic Scattering (DIS). Throughout the paper all the particle's momenta are taken as outgoing. This fact must be taken into account when using the given amplitudes in physical processes.

2 The Color Form Factors

In this Section I explicitly construct a set of color-form-factors which can be used to decompose any QCD amplitude in a gauge-invariant manner. The construction is very much in the spirit of 't Hooft's $1/N$ expansion [11].

To start with, let me consider the color structure of an amplitude with quarks only, at tree level. I will take all the particles as outgoing, and will assign indices $i_1,\ldots,i_m$ to the quarks and indices $j_1,\ldots,j_m$ to the antiquarks. It is understood that the quark i_k is continuously connected through a fermionic line to the antiquark j_k, for each $1\le k\le m$. Helicity will be conserved along this quark line, as well as flavor. We can furthermore assume all quarks to be different, the case with identical quarks being similar but more confusing. It is easy to verify that the color functions accompanying each diagram contributing to this scattering process can be decomposed in terms of the following color structures:

$$D(\{\alpha\}) = \frac{1}{N^p}\,\delta_{i_1\alpha_1}\delta_{i_2\alpha_2}\ldots\delta_{i_m\alpha_m} \tag{2.1}$$

where $\{\alpha\}=(\alpha_1,\ldots,\alpha_m)$ is a permutation of $(j_1,\ldots,j_m)$ and p is the number of antiquark indices that are kept fixed in the permutation. In other words, p is the number of $\delta_{i_kj_k}$ appearing in the product, with the caveat that when *all* of the delta functions connect quarks that belong to the same fermionic line then $p=m-1$. Each color structure $D(\{\alpha\})$ defines the color flow pattern inside the diagram. Whenever the color flows continuously along a quark line, from the beginning to the end, this configuration is suppressed by a factor N^{-1}. In figure 1 the case $m=2$ is shown, with the possible color factors given by:

$$\delta_{i_1j_2}\delta_{i_2j_1} \quad , \quad \frac{1}{N}\delta_{i_1j_1}\delta_{i_2j_2} \tag{2.2}$$

These two color factors arise from the following decomposition of the color function multiplying the diagram:

$$\sum_{a=1}^{N^2-1}(\lambda^a)_{i_1j_1}(\lambda^a)_{i_2j_2} \;=\; \delta_{i_1j_2}\delta_{i_2j_1} \;-\; \frac{1}{N}\delta_{i_1j_1}\delta_{i_2j_2}, \tag{2.3}$$

where I have chosen the following normalization of the λ matrices:

$$[\lambda^a,\lambda^b] \;=\; i\sqrt{2}f^{abc}\lambda^c \qquad , \qquad tr(\lambda^a\lambda^b) \;=\; \delta^{ab}. \tag{2.4}$$

If the gauge group were $U(N)$ instead of $SU(N)$ only the first color structure in Figure 1 would appear. The second structure corresponds to the subtraction of the trace of the $U(N)$ gauge field. I will call this trace the '$U(N)$-photon'. In Figure 1 the propagation of the $U(N)$-photon is represented by the slim closed ellipse between the two quark lines, and one can interpret the negative power of N appearing in equation (2.1) as being the number of $U(N)$-photons propagating in the color-flow pattern determined by $D(\{\alpha\})$.

Given a helicity configuration for the external states the matrix element for m quark-pair scattering can then be expressed as:

$$A_m \;=\; \sum_{\{\alpha\}} D(\{\alpha\})\,\tilde{A}_{\{\alpha\}}(p,h), \tag{2.5}$$

where the sum is over the permutations of $(j_1, \ldots, j_m)$ and $\tilde{A}_{\{\alpha\}}(p,h)$ are $m!$ functions of the $2m$ momenta and helicities. We will call these functions ***sub-amplitudes.*** To the leading order in N only the terms in the sum with $\alpha_k \neq j_k$ will contribute, and the sum over colors of the amplitude squared will be the sum of the squares of the functions $\tilde{A}_{\{\alpha\}}$, the interferences being suppressed by negative powers of N:

$$\sum_{col} |A_m|^2 = N^m \widehat{\sum_{\{\alpha\}}} |\tilde{A}_{\{\alpha\}}(p,h)|^2. \tag{2.6}$$

The hat on the sum sign constrains the sum to extend over permutations $(\alpha_1, \ldots, \alpha_m)$ with $\alpha_k \neq j_k$ for all k's.

This construction admits a straightforward generalization to the case when gluons are radiated. Suppose we have a process with m quark pairs and n gluons as external states. Then it is easy to check that the color structure of each Feynman diagram can be decomposed in terms of the following color functions, in all the possible permutations of the gluon color indices:

$$\frac{1}{N^p} (\lambda^1 \ldots \lambda^{n_1})_{i_1\alpha_1} (\lambda^{n_1+1} \ldots \lambda^{n_2})_{i_2\alpha_2} \ldots (\lambda^{n_{m-1}+1} \ldots \lambda^{n})_{i_m\alpha_m} \tag{2.7}$$

The power p is determined as before, and a product of zero λ matrices has to be interpreted as a Kroneker delta.

To give an example, in the case of two quark pairs and two gluons the possible color structures are the following:

$$(\lambda^a\lambda^b)_{i_1j_2}\,\delta_{i_2j_1} \quad , \quad (\lambda^a)_{i_1j_2}(\lambda^b)_{i_2j_1} \quad , \quad \delta_{i_1j_2}(\lambda^a\lambda^b)_{i_2j_1}, \tag{2.8}$$

$$\tfrac{1}{N}(\lambda^a\lambda^b)_{i_1j_1}\,\delta_{i_2j_2} \quad , \quad \tfrac{1}{N}(\lambda^a)_{i_1j_1}(\lambda^b)_{i_2j_2} \quad , \quad \tfrac{1}{N}\delta_{i_1j_1}\,(\lambda^a\lambda^b)_{i_2j_2}, \tag{2.9}$$

where a and b represent the color indices of the two gluons, and the six additional color structures with a and b interchanged have been omitted. A graphical representation of these color factors is given in Figure 2. As another example, in the case of one quark-pair and n gluons the color factors are the $n!$ permutations of the following expression:

$$(\lambda^{a_1}\lambda^{a_2} \ldots \lambda^{a_n})_{ij}. \tag{2.10}$$

The structure of processes with only gluons was studied in detail in reference [6,7].

The color structure in equation (2.7) has a very simple physical interpretation. In fact it corresponds to the emission of the gluons off the color-flow lines defined by the functions $D(\{\alpha\})$. Each function $D(\{\alpha\})$ defines a net of color flows, as shown in figure 1 for the case $m = 2$. Each of these color-flow lines, specified by a pair of indices $(i_k, j_{k'})$, acts as a sort of antenna, that radiates gluons with an associated color factor $(\lambda \ldots \lambda)_{i_k j_{k'}}$ (see Figure 2). This color factor is the one appearing in the QED-type diagrams, *i.e.* diagrams in which all the gauge bosons are emitted from the fermionic line and no three- or four-vector vertices are present. Equation (2.7) shows that even graphs with non-abelian vertices can be decomposed as sums of QED-like diagrams. The full amplitude is then given by:

$$A_{m,n} = \sum \Lambda(\{n_i\},\{\alpha\})\, \bar{A}_{\{n_i\},\{\alpha\}}(p,h). \tag{2.11}$$

$\Lambda(\{n_i\},\{\alpha\})$ are the color factors appearing in equation (2.7): they depend upon the partition and permutation $\{n_i\}$ of the gluon indices and upon the antenna pattern determined by the permutation of indices $\{\alpha\}$. If some of the external states are in a given color configuration, for example in a color singlet, the amplitude can be easily obtained by contracting equation (2.11) with the proper projector. The sub-amplitudes $\bar{A}(p,h)$ multiplying a given color factor are functions of the momenta and helicities of the external particles. These sub-amplitudes are obtained by summing contributions from various different Feynman diagrams.

The color-form-factors given in equations (2.1) and (2.7) will also describe loop amplitudes, as can be seen by applying repeatedly to each diagram the Lie-Algebra identity $f^{abc}\lambda^c = -i/\sqrt{2}[\lambda^a,\lambda^b]$ and the identity (2.3). In this case, though, for each partition $\{n_i\}$ and permutation $\{\alpha\}$ there exist different possible powers of N multiplying the form-factor. This is because additional powers of N are introduced by the traces over the colors of the particles entering the loops. We can then introduce the following color structures:

$$\Lambda^q(\{n_i\},\{\alpha\}) = N^q\,(\lambda^1\dots\lambda^{n_1})_{i_1\alpha_1}(\lambda^{n_1+1}\dots\lambda^{n_2})_{i_2\alpha_2}\dots(\lambda^{n_{m-1}+1}\dots\lambda^{n})_{i_m\alpha_m} \tag{2.12}$$

Now the power q can be also a positive integer. The decomposition of each diagram in terms of these color structures can be performed studying the possible color-flow patterns inside the given diagram. Every loop amplitude can then be uniquely written in the following form:

$$A_{m,n} = \sum_q \sum_{\{n_i\},\{\alpha\}} \Lambda^q(\{n_i\},\{\alpha\})\, \bar{A}^{(q)}_{\{n_i\},\{\alpha\}}(p,h). \tag{2.13}$$

The sum is over the integers q giving rise to a non-zero contribution.

Each sub-amplitude $\bar{A}^{(q)}_{\{n_i\},\{\alpha\}}(p,h)$ is invariant under gauge transformations of the gluon polarizations $\epsilon^i_\mu \to \epsilon^i_\mu + \beta p^i_\mu$. To prove this one does not need Ward identities, being sufficient the orthogonality, to the leading order in N, of the color factors. I will now prove this fact.

Let $\delta\bar{A}^{(q)}_{\{n_i\},\{\alpha\}}(p,h)$ be the gauge variation of a given sub-amplitude . Suppose $\bar{q}$ is the largest q for which some non-zero sub-amplitude exists, and let $\{\bar{n}_i\},\{\bar{\alpha}\}$ be a given partition and a given permutation of quark and gluon indices. Then the following identity follows:

$$0 = \sum_{col} \Lambda^{(\bar{q})}_{\{\bar{n}_i\},\{\bar{\alpha}\}}(p,h)\delta A_{m,n} = N^{2\bar{q}+n+m}\,\delta\bar{A}^{(\bar{q})}_{\{\bar{n}_i\},\{\bar{\alpha}\}}(p,h) + \mathcal{O}(1/N^2). \tag{2.14}$$

This shows that all the sub-amplitudes labelled by $\bar{q}$ are gauge-invariant, since gauge-invariance does not depend on N and variations of $\mathcal{O}(1/N^2)$ cannot cancel the leading piece. One can then proceed to the next non-trivial q and repeat the analysis, continuing until all the values of q have been considered.

This gauge invariance is particularly useful for the calculation of the sub-amplitudes, since different gauges can be chosen for different sub-sets of gauge invariant diagrams[1]. This property was key in the calculation of the amplitudes for the six-parton processes carried out in [6,7].

From now on I will give results only for the tree-level case. As in the case of the amplitude with only quarks, to the leading order in N the sum over colors of the square of the amplitude with quarks and gluons is given by the sum of the squares of the sub-amplitudes :

$$\sum_{col} |A_{m,n}|^2 = g^{2(n+m-1)} N^{m+n} \widehat{\sum_{\{n_i\}\{\alpha\}}} |\tilde{A}_{\{n_i\},\{\alpha\}}(p,h)|^2. \tag{2.15}$$

Once again the 'hat' restricts the sum to the permutations $\{\alpha\}$ with $\alpha_k \neq j_k$ for all k's. This remarkable property is unique to this color decomposition. Linear combinations of the color factors Λ, for example, would still give rise to gauge-invariant sub-amplitudes , but would not enjoy the property (2.15). Therefore we can think of the prescription described in this note as of a systematic and simple procedure to isolate the leading-N contribution to a given process. All of the coherence effects are exactly accounted for, to the leading order in N, by equation (2.15): color coherence, azimuthal correlations, interference between final-state and initial-state radiation, etc. In particular, no distinction is actually made between final- and initial-state radiation, both being treated on the same ground.

3 Color Coherence in qg Scattering

Color coherence effects were first observed in e^+e^- collisions [13], where the soft radiation emitted from a final state containing a hard q-$\bar{q}$ pair and a hard gluon is suppressed in the region between the two quark-jets. This effect is known as the *string effect*, and has been described by the Lund string-model [14] and by Marchesini and Webber [15], in the context of Monte-Carlo simulations, and by the authors of reference [16] in the context of perturbative QCD. The possibility of observing color-coherence effects in hadronic collisions would be of fundamental importance, because it would provide a way to separate quark and gluon jets[2] and would significantly reduce the background to important processes [18]. An analysis of color coherence effects in hadronic collisions was recently carried out in reference [19,20]. The actual detectability of these effects has still to be thoroughly investigated, but available results [21] seem to suggest

[1]After completing this work, the paper in reference [12] was brought to my attention. There the issue of finding a gauge-invariant decomposition of a QCD amplitude in terms of color-form-factors was also analyzed. The choice of QED-type form-factors was discussed and recipies for the decomposition of higher loop diagrams were given.

[2]For a review of the properties of soft QCD radiation in jet physics see reference [17].

that, at least at current energies, the phenomenon is washed out by the fluctuations. One important issue that will have to be answered in the future is if the coherence effects become strong enough with higher energies to overcome the smearing operated by the fluctuations.

For processes with purely hadronic final states the preferred channel to look for color coherence is $qg \to qg$ plus gluons (q represents either a quark or an anti-quark). This process was studied in the case of single gluon emission in [19,20,21]. Using our formalism we can gain additional insight on the way the coherence effect arises and we can gain some analytic control on the multi-gluon emission case.

Using results contained in [9] we can explicitly write the matrix elements of the most helicity violating (and non-vanishing) amplitudes, *i.e.* those with all but one of the gluons carrying the same helicity:

$$A_{1,n}(h_q, h_g) = i\, g^n \frac{\{pg\}^3\{p'g\}}{\{pp'\}} \sum_{\{1,2,\dots,n\}} (\lambda^{a_1}\dots\lambda^{a_n})_{ij} \frac{1}{\{p1\}\{12\}\dots\{np'\}}. \tag{3.1}$$

Here the index g represents the gluon with helicity different from all the others; p and p' are the momenta of the quarks, p representing the quark with the same helicity as g. The symbol $\{ij\}$, with i and j being the momenta of the external particles carrying the respective indices, satisfies[3]:

$$|\{ij\}|^2 = (i+j)^2 = 2\, i\cdot j \tag{3.2}$$

The phase of $\{ij\}$ depends upon the polarization of the gluon g. In the notation of reference [3]:

$$\{ij\}_{h_g=+} = \langle ij\rangle \equiv \bar\psi(p_i)(\frac{1+\gamma_5}{2})\psi(p_j) \quad , \quad \{ij\}_{h_g=-} = [ij] \equiv \langle ji\rangle^* \tag{3.3}$$

$\psi(p)$ is a spinor satisfying the massless Dirac equation with momentum p: $p\cdot\gamma\,\psi(p) = 0$. For $n = 2,3$ the amplitude represented in equation (3.1) is the only independent amplitude necessary to completely describe the process. For $n \geq 4$ there are contributions from different helicity configurations where at least two gluons have the same polarization for each polarization. The matrix elements for these helicity amplitudes will be very complicated but, in analogy with what shown by Maxwell [22] in the case of the Parke and Taylor formula [23] for amplitudes with gluons only, their overall contribution can be approximated quite well in most of the phase space once we know (3.1). I will then use the amplitude described by (3.1) to model the behaviour of the full amplitude, summed over all the helicity configurations.

The first important observation to make is the following: if we put the color factors $(\lambda^{a_1}\dots\lambda^{a_n})_{ij}$ equal to δ_{ij} for each permutation of 1 through n, then equation (3.1) gives rise to the QED result for the amplitude with one quark-pair and n photons. This can be easily

[3]Since I have taken all the particles as outgoing, in case the energy components of the two momenta i and j had a different sign an extra minus sign should be introduced in the following relation.

proved diagrammatically by observing that diagrams with non-abelian gluon vertices entering the graph expansion for the sub-amplitudes cancel in pairs when we perform the sum over permutations. In this way the only diagrams left are the QED-type diagrams, with the common color structure given by δ_{ij}. This result is independent of the helicity configuration, and for the helicities considered above we then obtain:

$$A^{\gamma}_{1,n}(h_q, h_\gamma) = i\, e^n \delta_{ij} \frac{\{p\gamma\}^3\{p'\gamma\}}{\{pp'\}} \sum_{\{1,2,\ldots,n\}} \frac{1}{\{p1\}\{12\}\ldots\{np'\}}. \tag{3.4}$$

Now γ is the momentum of the photon with helicity different from the others. The second observation is that the following remarkable identity holds:

$$\sum_{\{1,2,\ldots,n\}} \frac{\{pp'\}}{\{p1\}\{12\}\ldots\{np'\}} = \prod_{i=1}^{n} \frac{\{pp'\}}{\{pi\}\{ip'\}}. \tag{3.5}$$

Equation (3.5) can be proved by iteratively using the Fierz identity:

$$\{pp'\}\{qq'\} = \{pq'\}\{qp'\} + \{pq\}\{p'q'\}. \tag{3.6}$$

Equation (3.5) can be thought of as a sort of 'square root' of the eikonal identity. It allows us to put equation (3.4) into the eikonalized form:

$$A^{\gamma}_{1,n}(h_q, h_\gamma) = i\, e^n \delta_{ij} \frac{\{p\gamma\}^3\{p'\gamma\}}{\{pp'\}^2} \prod_{i=1}^{n} \frac{\{pp'\}}{\{pi\}\{ip'\}}. \tag{3.7}$$

Equations (3.1), (3.4) and (3.7) offer a nice example of the difference between the properties of the non-abelian radiation as opposed to the abelian radiation. Let us take, in fact, the square of these three expressions, summed over the colors of the quarks and of the gluons (when present):

$$\sum_{col} |A_{1,n}(h_q, h_g)|^2 = g^{2n} N^{n+1} \frac{(pg)^3(p'g)}{(pp')} \sum_{\{1,2,\ldots,n\}} \frac{1}{(p1)(12)\ldots(np')} + \frac{1}{N^2}(interf.), \tag{3.8}$$

$$\sum_{col} |A^{\gamma}_{1,n}(h_q, h_\gamma)|^2 = e^{2n} N \frac{(p\gamma)^3(p'\gamma)}{(pp')} \sum_{\{1,2,\ldots,n\}} \frac{1}{(p1)(12)\ldots(np')} + (interf.), \tag{3.9}$$

$$\sum_{col} |A^{\gamma}_{1,n}(h_q, h_\gamma)|^2 = e^{2n} N \frac{(p\gamma)^3(p'\gamma)}{(pp')^2} \prod_{i=1}^{n} \frac{(pp')}{(pi)(ip')}. \tag{3.10}$$

Here $(ij) = 2\, i \cdot j$. Equations (3.9) and (3.10) are identical, thanks to the eikonal identity, but I wrote them in the two different ways to establish a connection with the expression for the gluon emission. Equation(3.10) shows that the photon emission is incoherent: the photons only know about their source, *i.e.* the quark line, but they do not know about each other. Up to the overall factor in front, the probability for the emission of n photons is just the product of the probabilities for the independent emission of each of them. [4]

[4] This result, which is exact for this specific helicity configuration, also holds for any other helicity configuration in the limit of soft-photon emission. The reason why it cannot hold for an arbitrary helicity configuration is that in general the amplitude will have poles of the kind $1/(p+k+k')^2$, k and k' being arbitrary photon momenta.

On the contrary, if we now look at equation (3.8) we see that the gluon emission is not incoherent: gluons know of each other's presence, and the full probability is not a product of probabilities. The interference terms coming from the product of different permutations are suppressed by a factor of $1/N^2$; this suppression originates from the interferences of the color factors. For the photon emission, vice versa, we can see from equation (3.9) that the interferences among different permutations are not suppressed and they conspire to cancel the coherence apparent into the sum of squares, giving rise to the factorized expression given in (3.10). The difference between equation (3.8) and equation (3.10) is the essence of the string-effect.

4 $q\bar{q}q\bar{q}$ plus gluons

In general the factorization of the color structure exhibited in equation (2.7) does not imply a similar factorization of the kinematical part of the amplitude. In other words, the sub-amplitude that multiplies a given color factor does not factorize into products of terms that only depend upon the kinematical variables (helicities and momenta) of the particles belonging to the same antenna. One remarkable non-trivial exception to this general feature is given by the amplitude for a process with two quark pairs and an arbitrary number of like-helicity gluons (all the particles are outgoing). Up to an overall factor that only depends upon the helicity configuration each sub-amplitude factorizes into the product of two terms only depending upon the momenta of the gluons emitted by one or the other of the two antennas:

$$\begin{aligned} A(h_p, h_q, h_g) \quad &= \; i\, g^{n+2} A_0(h_p, h_q, h_g) \;\cdot \\ \sum \quad & \frac{\{p\bar{q}\}}{\{pa_1\}\{a_1a_2\}\cdots\{a_k\bar{q}\}} \frac{\{q\bar{p}\}}{\{qb_1\}\{b_1b_2\}\cdots\{b_{k'}\bar{p}\}} (\lambda^{a_1}\cdots\lambda^{a_k})_{i_1j_2}(\lambda^{b_1}\cdots\lambda^{b_{k'}})_{i_2j_1} \\ -\frac{1}{N} \quad & \frac{\{p\bar{p}\}}{\{pa_1\}\{a_1a_2\}\cdots\{a_k\bar{p}\}} \frac{\{q\bar{q}\}}{\{qb_1\}\{b_1b_2\}\cdots\{b_{k'}\bar{q}\}} (\lambda^{a_1}\cdots\lambda^{a_k})_{i_1j_1}(\lambda^{b_1}\cdots\lambda^{b_{k'}})_{i_2j_2}. \end{aligned} \tag{4.1}$$

This equation can be proved in the spirit of reference [9]: the helicity structure of the amplitude uniquely determines the pole structure and the residues of these poles, through unitarity. Equation(4.1) is the only Lorentz invariant amplitude that gives rise to the right poles and the right residues. Alternatively, one can use recursive relations that were given recently [10], connecting amplitudes with $n+1$ gluons to amplitudes with n gluons.

The symbols $\{ij\}$ have the same meaning as before. The indices p and q ($\bar{p}$ and $\bar{q}$) refer to the quarks (antiquarks) and the indices a_α, b_β refer to the gluons. The arguments h represent

(h_p, h_q, h_g)	$a_0(h_p, h_q, h_g)$
$(+,+,+)$	$\langle\bar{p}\bar{q}\rangle^2[p\bar{p}][q\bar{q}]$
$(+,+,-)$	$[pq]^2\langle p\bar{p}\rangle\langle q\bar{q}\rangle$
$(+,-,+)$	$\langle\bar{p}q\rangle^2[p\bar{p}][q\bar{q}]$
$(+,-,-)$	$[p\bar{q}]^2\langle p\bar{p}\rangle\langle q\bar{q}\rangle$

Table 1: The universal functions $a_0(h_p, h_q, h_g)$. The meaning of angle- and square-brakets is given in equation (3.3).

the helicities of the two quarks and of the gluons; the helicities of the anti-quarks are fixed by helicity conservation along the fermion lines. The sum is over all the partitions of the n gluons ($k + k' = n, k = 0, 1, \ldots, n$) and over the permutations of the gluon indices. When $k = (0, n)$ the product of zero λ matrices becomes a Kroneker delta and one of the two kinematical factors is equal to one. The overall factor A_0 can be written as follows:

$$A_0(h_p, h_q, h_g) = \frac{a_0(h_p, h_q, h_g)}{(p + p')^2(q + q')^2}. \tag{4.2}$$

The functions a_0 are given in Table 1, where helicity configurations obtained by permuting the quark helicities have been omitted. The functions a_0 are universal, in the sense that they only depend upon the spin-1/2 nature of the quarks. As we will see later, they also enter in processes like deep inelastic scattering or e^+e^- annihilation.

To the leading order in N, the amplitude squared summed over colors is furthermore given by:

$$\sum_{col} |A(h_p, h_q, h_g)|^2 = g^{2n+4} N^{n+2} |A_0(h_p, h_q, h_g)|^2$$
$$\sum \frac{(p\bar{q})}{(pa_1)(a_1a_2)\cdots(a_k\bar{q})} \frac{(q\bar{p})}{(qb_1)(b_1b_2)\cdots(b_{k'}\bar{p})}. \tag{4.3}$$

If the quarks are identical we must add the contribution from the crossed channel $p \leftrightarrow q$.

As in the case with one quark pair, we can here compare the properties of photon radiation with those of gluon radiation. A reasoning similar to the one used in the previous section allows us to write the amplitude for the emission of n like-helicity photons off two quark-pairs:

$$A(h_p, h_q, h_g) = i\, g^2 e^n A_0(h_p, h_q, h_g) \cdot$$

$$\sum \frac{\{p\bar{p}\}}{\{pa_1\}\{a_1a_2\}\cdots\{a_k\bar{p}\}} \frac{\{q\bar{q}\}}{\{qb_1\}\{b_1b_2\}\cdots\{b_{k'}\bar{q}\}}(\delta_{i_1j_2}\delta_{i_2j_1} - \frac{1}{N}\delta_{i_1j_1}\delta_{i_2j_2}). \quad (4.4)$$

Only the contribution from gluon exchange is shown. The effect of photon exchange between the two quark-pairs can be easily added. A repeated use of the Fierz identity, equation (3.6), leads then to the following form of equation (4.4):

$$A(h_p, h_q, h_g) = i\, g^2 e^n A_0(h_p, h_q, h_g) \cdot$$

$$\prod_{i=1}^n \left(\frac{\{p\bar{p}\}}{\{pi\}\{i\bar{p}\}} + \frac{\{q\bar{q}\}}{\{qi\}\{i\bar{q}\}}\right)(\delta_{i_1j_2}\delta_{i_2j_1} - \frac{1}{N}\delta_{i_1j_1}\delta_{i_2j_2}). \quad (4.5)$$

This expression shows that photons are emitted independently. Once again we expect this result to hold for an arbitrary helicity configuration in the soft-photon limit.

If we substitute the color factor in equation(4.5) with the Abelian one, $\delta_{i_1j_1}\delta_{i_2j_2}$, and if we put g=e, then we obtain the amplitude for the process $e^+e^-\mu^+\mu^-$ *photons*, as given in reference [24].

An incisive study of soft gluon emission in $q\bar{q} \to q\bar{q}$ processes was recently given in reference [25].

5 e^+e^- and DIS

It is easy to derive expressions analogous to (4.1) and (4.3) for the process $l\bar{l}q\bar{q}$ *gluons*, where $l\bar{l}$ is a lepton-antilepton pair (for example e^+e^- or $e^-\bar{\nu}$). Again the gluons have all the same helicity:

$$A^l(h_l, h_q, h_g) = i\, g^n \sum_{V=\gamma,Z,W} A^l_V(h_l, h_q, h_g) \sum_{\{1,2,\ldots,n\}} (\lambda^{a_1}\ldots\lambda^{a_n})_{ij} \frac{1}{\{q1\}\{12\}\ldots\{nq'\}}, \quad (5.1)$$

$$\sum_{col} |A^l(h_l, h_q, h_g)|^2 = g^{2n} N^{n+1} | \sum_{V=\gamma,Z,W} A^l_V(h_l, h_q, h_g)|^2 \sum_{\{1,2,\ldots,n\}} \frac{1}{(q1)(12)\ldots(nq')}. \quad (5.2)$$

q and q' are the quark momenta and 1 through n are the gluon momenta. The contributions from photon,W and Z exchange are explicitly exhibited. The functions $A_V^l(h_l, h_q, h_g)$ are given by:

$$A_V(h_l, h_q, h_g) = \frac{Q_V(h_l)Q_V(h_q)}{(q+q')^2(s-M_V^2)} a_0(h_l, h_q, h_g) \tag{5.3}$$

$Q_V(h_l)$ $(Q_V(h_q))$ is the charge corresponding to the interaction of a lepton (quark) of helicity h_l (h_q) with the vector V. Furthermore $s = (p+p')^2$, with p and p' being the lepton momenta, and M_V^2 is the mass squared of the vector boson V. The universal functions $a_0(h_l, h_q, h_g)$ coincide with those given in Table 1. Equation (5.2) was also obtained independently in reference [10].

For e^+e^- scattering the effect of photon radiation (both from the initial and the final state) can be easily incorporated into equation (5.1) by using equation (4.5). Here I will display directly the result for the square of the amplitude with n gluons and m photons, to the leading order in N:

$$\begin{aligned} \sum_{col} |A^{e^+e^-}(h_l, h_q, h_g)|^2 \quad &= \quad e^{2m} g^{2n} N^{n+1} \mid \sum_{V=\gamma,Z} A_V^l(h_l, h_q, h_g)|^2 \\ & \Pi_{i=1}^m \quad |(\frac{\{p\bar{p}\}}{\{pk_i\}\{k_i\bar{p}\}} + \frac{\{q\bar{q}\}}{\{qk_i\}\{k_i\bar{q}\}})|^2 \sum_{\{1,2,\ldots,n\}} \frac{1}{(q1)(12)\ldots(nq')} . \end{aligned} \tag{5.4}$$

The k_i's are the momenta of the photons. Once again this result is only exact if all the gluons *and* the photons have the same helicity, but this is the behaviour of all the other helicity configurations in the case of soft emission. The difference between equation (5.2) and equation (5.4) is the source of the string effect in e^+e^- collisions.

6 Conclusions

In this paper I analyzed the color structure of general QCD processes, with and without color-singlet sources. A general procedure to decompose any amplitude in terms of color-form-factors led to the identification of gauge-invariant sub-amplitudes in terms of which the amplitude can be written. This procedure applies to tree-level processes as well as to loop amplitudes.

This prescription also enables to isolate the leading-N contribution to a given process and explicitly exhibits the coherence properties of the radiation of colored particles. I showed this in a few examples, where use of exact tree-level matrix elements was made. These results suggest that string-effect-type correlations can be found not just by looking at the soft radiation, but for hard radiation as well.

Throughout the paper I only studied cases with massless external states. The decomposition in terms of color-form-factors is independent of this assumption and certainly goes through for amplitudes with massive quarks or massive color-singlet vectors. In this case, alas, simple analytic expressions are not known for any class of amplitudes. On general grounds, though, the qualitative properties of color coherence are still present [21]. Further developments in this area will hopefully lead to the ability of tagging jets.

Finally, it would be very important to be able to introduce into the explicit formulas given in this paper the effects of the virtual corrections. This is necessary to eliminate the divergencies associated with the soft emission and would allow to make more quantitative predictions of the coherence effects, offering an alternative to Monte-Carlo simulations based on branching processes.

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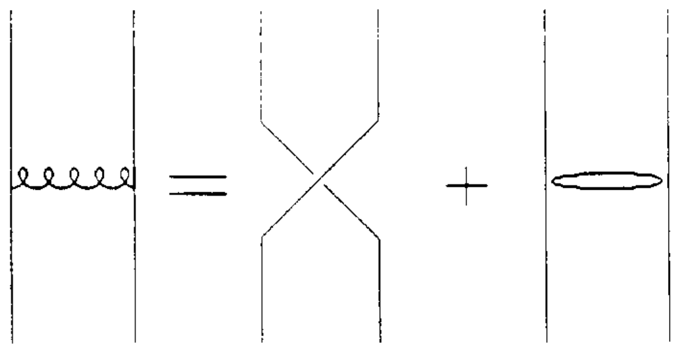

$$\delta_{i_1 j_2}\delta_{i_2 j_1} \qquad -\frac{1}{N}\,\delta_{i_1 j_1}\delta_{i_2 j_2}$$

Figure 1: The color factors for two quark pair scattering.

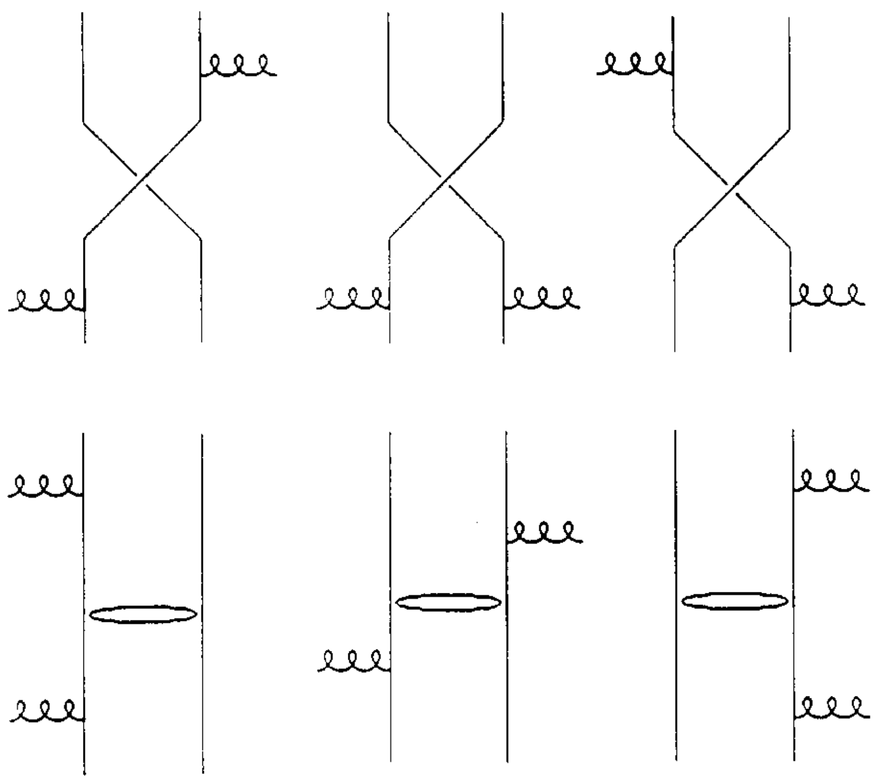

Figure 2: The color structures for the scattering of two quark pairs and two gluons.