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The Effect of Topological Defects on Phase Transitions in the Early Universe.

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Abstract

In this paper we attempt to develop an analytic description of a phase transition which leads to the formation of topological defects. In detail we consider the effect of 'Nielsen-Olesen' string like solutions on the phase transition. The statistical properties of such strings are derived. Neglecting

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interactions our results are in agreement with those of Mitchell and Turok [1]. We however, also show how the results are modified by including self-interactions. We discover the temperature at which strings are formed and show how the Ginzburg length arises in our description.

1 Introduction

It has previously been demonstrated by Weinberg and others [2], that at sufficiently high temperatures, the full gauge symmetry of a spontaneously broken simple gauge group which could describe the interactions of particles can be restored. The big bang model of the universe suggests that it was once very hot and dense. At these high temperatures we would therefore expect the full symmetry to be manifest. As the universe expanded and cooled it would undergo a series of phase transitions during which this symmetry would be broken. At this time topological defects may be formed [3], such as strings. These strings have been suggested as providing the seed mass about which galaxies and clusters of galaxies might form (see Vilenkin [6] and references there in).

Up to now calculations to investigate the nature of these phase transitions have been based on the mean field approach. In this paper we investigate how the presence of topological defects affects the phase transition. We also investigate the affect of finite temperature on the profile of a cosmic string, and derive the flat space equilibrium distribution of strings. Our results are in good agreement with theoretical calculations based on a rather different approach [1], and on computer simulations [7,17].

The paper is divided into five parts. In Section 2 we consider the simplest theory that undergoes a phase transition, that of a real scalar field theory. At the phase transition domain walls can form. We discuss qualitatively how we expect their formation to effect the phase transition. In Section 3 we derive the partition function describing the equilibrium properties of a U(1) gauge theory. We consider the effects of string formation on the phase transition in Section 4, and show that the dominant contribution to the partition function at temperatures well below the critical temperature comes from the constant field configuration (i.e the mean field approximation is good in this regime). However, as the temperature is increased, the high density of states (entropy) available for the topological defects balances the energy required to form them, and they then make the dominant contribution to the partition function. In this section we also derive the statistical properties of strings and the effect of temperature on the width and energy per unit length of a string. The final section is devoted to a summary and discussion of our results and their cosmological significance.

2 The partition function for a real scalar field theory

We shall start our discussion of phase transitions by considering the simplest theory that would possess one; that of a real scalar field theory with Lagrange density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + \frac{1}{2}m_0^2\phi^2 - \frac{\lambda}{4!}\phi^4 \quad (1)$$

On taking $m_0^2 > 0$, $\mathcal{L}$ possess a double well potential which breaks the $\phi \rightarrow -\phi$ symmetry of the theory.

The partition function of this theory in thermal equilibrium at temperature T has the path integral representation in terms of Euclidean fields

$$Z \propto \int \mathcal{D}\phi exp[-I_\beta(\phi)] \quad (2)$$

where [10]

$$I_\beta(\phi) = \int_0^\beta d\tau \int d^3x[-\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m_0^2\phi^2 + \frac{\lambda}{4!}\phi^4] \quad (3)$$

The sum over configurations of $\phi(\tau, \underline{x})$ is restricted to fields periodic in τ with period β and we shall take the signature of our Euclidean space to be -4.

The standard approach to evaluating this partition function is to adopt the mean field approximation. One restricts the path integral to periodic configurations whose Euclidean space-time average is specified in advance to be:

$$\frac{1}{\beta v}\int_0^\beta d\tau \int d^3x\phi(\tau, \underline{x}) = \overline{\phi} \quad (4)$$

say (where v is the spatial volume of the system).The resulting expression [4]:

$$Z(\overline{\phi}) = \int \mathcal{D}\phi[\delta(\overline{\phi} - \frac{1}{\beta v}\int_0^\beta d\tau \int d^3x\phi(\tau, \underline{x}))]exp(-I_\beta[\phi]) \quad (5)$$

has the following interpretation. Let us couple the field $\phi(\tau, \underline{x})$ to a constant source j. The effect of this source is to enable the thermal average $< \phi >$ to take any value we wish (almost), ($< \phi >$ will be constant by virtue of the translational invariance of j for large v). If we now write Z as

$$Z(\overline{\phi}) = e^{-\beta v V(\overline{\phi})} \tag{6}$$

then $V(\overline{\phi})$ is the Helmholtz energy density (effective potential) of the system when j is chosen so that $< \phi >= \overline{\phi}$.

With $V(\overline{\phi})$ satisfying

$$\frac{\partial V}{\partial \overline{\phi}} = 0 \tag{7}$$

in the absence of external sources, the thermal average $< \phi >$ for the original system is the value of $\overline{\phi}$ for which V is minimized. In principle it is straight forward to compute $< \phi >$ by performing a saddle point expansion for $Z(\overline{\phi})$ [5]. This is equivalent to performing a loop expansion for $V(\overline{\phi})$, the generating function for zero momentum Green functions. The existence of a phase transition is already present at the one-loop level and we shall restrict ourselves to this alone.

The calculation is so well known, we shall only quote the result that, at large T, $V(\overline{\phi})$ takes the form:

$$V(\overline{\phi}) = -\frac{1}{2}m_0^2(1 - \frac{T^2}{T_c^2})\overline{\phi}^2 + \frac{\lambda}{4!}\overline{\phi}^4 + ... \tag{8}$$

where

$$T_c^2 = \frac{24m_0^2}{\lambda} \tag{9}$$

and one loop renormalisation has been implemented. Higher order terms are suppressed either by a factor $\frac{\lambda T}{m} \sim \sqrt{\lambda}$, (near T_c) or by a factor of λ [2].

There is a possible problem in that V of (8) is not concave, as the free energy must be [5]. We can ignore this since our interpretation of V in the remainder of this section will be rather different from that above. However, taking (8) at face value, we see that, as T increases from zero to T_c, so $< \phi >$, satisfying

$$< \phi >^2= \frac{6m_0^2}{\lambda}(1 - \frac{T^2}{T_c^2}) \tag{10}$$

decreases smoothly to zero, implying a second order phase transition.

Convexity apart, the picture implied by the effective potential is that the thermal average decreases uniformly across all of space. This is very unlikely to be true. The classical equation of motion derived from (1),

$$(\Box - m_0^2)\phi + \frac{\lambda}{6}\phi^3 = 0 \tag{11}$$

permits static domain wall solutions of the form

$$\phi_w(x,y,z) = \eta tanh(\frac{m_0 z}{\sqrt{2}}) \tag{12}$$

[11], [where we have taken a wall in the x-y plane as an example]. The field ϕ flips value across this wall from $-\eta$ to η, where $\eta^2 = \frac{6m_0^2}{\lambda}$. A much more likely scenario for the phase transition is that, as the temperature is increased, more and more domain wall area will be formed until the whole of space is filled, at which point the symmetric phase $< \phi >= 0$ is achieved. That is, the effective potential description corresponds to the averaging of a much more complicated structure. Until we are very close to the phase transition, however, we would expect the effective potential averaging to be reliable since the domains will be large (see later).

To evaluate the effect of this domain wall formation on the temperature and nature of the phase transition some care has to be taken. The thickness of a domain wall at zero temperature is $\zeta = O(m_0^{-1})$ and its surface tension $\sigma = O(\frac{m_0^3}{\lambda})$. Calculations that rely on holding these fixed at finite temperature [12] will give the wrong answer. The long range correlations that are associated with a phase transition arise because the effective scalar mass $m_{eff} = (\frac{\partial^2 V}{\partial \phi^2}\mid_{\phi=\bar{\phi}})^{\frac{1}{2}}$ vanishes at $T = T_c$. The effect of non-zero temperature (to $O(\lambda)$) on a domain wall will be to replace m_0 in ζ and σ by m_{eff}. In this way the surface tension of a single domain wall vanishes at $T = T_c$, enabling the creation of domain walls at no energetic cost.

There are two ways of seeing this from the functional representation of Z. One of the methods is outlined below, the other in Appendix A.

Since $\phi(\tau, \underline{x})$ is periodic in τ it permits the Taylor expansion:

$$\phi(\tau, \underline{x}) = \sum_n \phi_n(\underline{x}) e^{\frac{i2\pi n\tau}{\beta}}, \quad \phi_n^* = \phi_{-n} \tag{13}$$

in terms of a denumerable set of three-dimensional fields. The action I_β of (3) then takes the form

$$I_\beta[\phi] = \beta \overline{I_\beta}[\{\phi_n\}] \tag{14}$$

where $\overline{I}_\beta$ can be decomposed into the contributions from ϕ_0 and $\phi_n (n \neq 0)$ [termed ϕ'] as

$$\overline{I}_\beta[\phi_0, \phi'] = H_0[\phi_0] + H_0[\phi'] + H_I[\phi_0, \phi'] \tag{15}$$

$$H_0[\phi_0] = \int d^3x [\frac{1}{2}(\underline{\nabla}\phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda}{4!}\phi_0^4] \tag{16}$$

$$H_0[\phi'] = \sum_{n\neq 0} H_0[\phi_n] = \sum_{n\neq 0} \int d^3x \frac{1}{2}[(\nabla\phi_n^*)(\nabla\phi_n) - (m_0^2 + (\frac{2\pi n}{\beta})^2)\phi_n^*\phi_n] \tag{17}$$

$$H_I[\phi_0, \phi'] = \frac{\lambda}{4!} \sum_{p+q+r+s=0} \int d^3x \phi_p\phi_q\phi_r\phi_s \quad (excluding \ \ \phi_0^4)$$

$$= \frac{\lambda}{4} \sum_{n\neq 0} \int d^3x \phi_0^2 \phi_n^* \phi_n + terms\ containing\ n \neq 2\ \ \phi' \tag{18}$$

[2] From (17) we see that the masses of the $\phi_n (n \neq 0)$ modes are large at high temperatures. We refer to ϕ_0, ϕ' as light and heavy modes respectively. Z now becomes :

$$Z \propto \int \mathcal{D}\phi e^{-I_\beta} = \int \mathcal{D}\phi_0 \mathcal{D}\phi' e^{-\beta \overline{I}[\phi_0, \phi']} \tag{19}$$

$$= \int \mathcal{D}\phi_0 e^{-\beta(H[\phi_0] + v[\phi_0])} \tag{20}$$

on integrating out the heavy modes, where

$$e^{-\beta v[\phi_0]} \propto \int \mathcal{D}\phi' e^{-\beta H_0[\phi'] - \beta H_I[\phi_0, \phi']} \tag{21}$$

The 'effective' potential $v[\phi_0]$ for the three-dimensional field ϕ_0 contains **temperature dependent** parameters. Most importantly, at high temperatures and to first order in λ, $v[\phi_0]$ is the spatial integral of a local density. This can be seen by expanding (21) to first order in λ:

$$e^{-\beta v[\phi_0]} \propto \int \mathcal{D}\phi' e^{-\beta H_0[\phi']}(1 - \beta H_I[\phi_0, \phi'])$$

[2] to $O(\lambda)$ these extra terms will not contribute to Z

$$\propto 1 - \frac{\beta\lambda}{4}\int d^3x\phi_0^2\sum_{n\neq 0}^{\infty}\int \mathcal{D}\phi_n\mathcal{D}\phi_n^* e^{-\beta H_0[\phi_n]}\phi_n^*(x)\phi_n(x) + ... \tag{22}$$

$$= 1 - \beta\int d^3x\phi_0^2(\underline{x})\frac{\Delta m^2}{2} + ... \tag{23}$$

where we have dropped terms down by at least a factor of $\frac{\lambda T}{m}$ or λ. After performing the zero temperature renormalisation of the mass, we have [2]:

$$\Delta m^2 = \frac{\lambda}{2\beta}\sum_{n\neq 0}\int d^3p\frac{1}{\underline{p}^2 + m_0^2 + (\frac{2\pi n}{\beta})^2} \tag{24}$$

$$\approx \frac{\lambda T^2}{24} = m_0^2\frac{T^2}{T_c^2} \tag{25}$$

Exponentiating (23) gives

$$v[\phi_0] = \int d^3x\frac{1}{2}\Delta m^2\phi_0^2(\underline{x}) + O(\frac{\lambda T}{m}, \lambda) \tag{26}$$

Diagrammatically the light-mode mass increment Δm^2 has the representation shown in Figure 1, where the solid line denotes ϕ_0, the light mode, and the dashed line ϕ', the heavy modes.

Inserting the first term of $v[\phi_0]$ in (26) gives Z as

$$Z \propto \int \mathcal{D}\phi_0 e^{-\beta\bar{I}[\phi_0]} \tag{27}$$

the vacuum functional for the three-dimensional field with 'action'

$$\bar{I}[\phi_0] = \int d^3x[\frac{1}{2}(\underline{\nabla}\phi_0)^2 - \frac{1}{2}m_0^2(1 - \frac{T^2}{T_c^2})\phi_0^2 + \frac{\lambda}{4!}\phi_0^4] \tag{28}$$

Equivalently in terms of the one-loop effective potential $V(\phi)$ of (8)

$$\bar{I}[\phi_0] = \int d^3x[\frac{1}{2}(\underline{\nabla}\phi_0)^2 + V(\phi_0)] \tag{29}$$

Thus, as well as its definition for constant ϕ, $V(\phi)$ plays the role of an 'effective' potential for the non constant three-dimensional light mode $\phi_0(\underline{x})$. From this viewpoint, it is the vanishing of the scalar mass in the effective three-dimensional theory that triggers the long range correlations characterising a phase transition.

Now let us consider the nature of the phase transition. The dominant contributions to the partition function (27), will come from solutions to the semi-classical equation

$$\frac{\delta \bar{I}}{\delta \phi_0} = 0. \tag{30}$$

As well as the constant solution $\bar{\phi}^2 = \frac{6m_0^2}{\lambda}[1 - \frac{T^2}{T_c^2}]$, there are domain wall solutions of the form (12), in which m_0 has been replaced by the effective scalar mass m(T), where $m^2(T) = m_0^2(1 - \frac{T^2}{T_c^2})$. Away from the critical temperature the solution $\phi = constant$, being the minimum energy solution, makes the dominant contribution to the partition function and the mean field approach is a good approximation. As we approach the critical temperature however, the energy required to produce a section of domain wall becomes smaller and smaller. Eventually because of the large number of different configurations of domain walls of a given size it may be possible for their entropy to counterbalance the Boltzmann coefficient and they may come to dominate the partition function. They may then drive the system into undergoing a phase transition at a temperature slightly less than T_c.

We shall not bother to evaluate the effect of domain walls on the phase transition in any more detail. The reason for this is that theories that produce domain walls at a phase transition in the early universe can be ruled out as inconsistent with present day observations [11]. For example they would produce large anisotropies in the microwave background. Instead we will perform our analysis for a more complicated, but cosmologically more interesting theory, that which would lead to the formation of cosmic strings.

3 The Partition Function for a U(1) Scalar Gauge Theory

The simplest theory to possess vortex solutions is scalar QED, with Lagrange density:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2} \mid (\partial_\mu + ieA_\mu)\phi \mid^2 + \frac{1}{2}m_0^2 \mid \phi \mid^2 - \frac{\lambda}{4!} \mid \phi \mid^4 \tag{31}$$

where ϕ is a complex scalar field. The partition function for this theory takes the form:

$$Z \propto \int \mathcal{D}\phi \mathcal{D}\phi^* \mathcal{D}A(detM)exp(-I_\beta[\phi, A]) \tag{32}$$

where $detM$ describes the gauge fixing, and

$$I_\beta[\phi, A] = -\int_0^\beta d\tau \int d^3x \mathcal{L}_E[\phi, A] \tag{33}$$

with $\mathcal{L}_E$ the Euclidean form of the Lagrangian (31).

As in Section 2, the thermodynamic free energy for $< \phi >= \overline{\phi}$, $V(\phi)$ is obtained by fixing the spatial average of $| \phi(\underline{x}) |$ to $\overline{\phi}$ in (32). $V(\phi)$ is necessarily a gauge-variant quantity, since it is the generator of zero momentum 1PI Green functions. However physical conclusions drawn from it should be gauge invariant . It is most convenient to calculate $V(\phi)$ in the covariant gauge, for which, at high temperatures and to $O(\lambda, e^2)$, [2],

$$V(\phi) = -\frac{1}{2}m_0^2 | \phi |^2 (1 - \frac{T^2}{T_c^2}) + \frac{\lambda}{4!} | \phi |^4 \tag{34}$$

T_c^2, which now includes gauge fixed loops, takes the form $T_c^2 = \frac{m_0^2}{\frac{\lambda}{18} + \frac{e^2}{4}}$. To evaluate the partition function we shall adopt the same procedure as in Section 2. An alternative approach is outlined in Appendix A. We first decompose the scalar ϕ and vector A_μ fields into light $(\phi_0, A_{0\mu})$ and heavy (ϕ', A'_μ) modes. As before we can write Z as:

$$Z \propto \int \mathcal{D}\phi_0 \mathcal{D}A_0 exp(-\beta \overline{I}[\phi_0, A_0]) \tag{35}$$

where

$$exp(-\beta \overline{I}[\phi_0, A_0]) = \int \mathcal{D}\phi' \mathcal{D}A'(detM)exp(-I_\beta[\phi, A]) \tag{36}$$

Obtaining the mass corrections by keeping terms quadratic in ϕ_0 and $A_{0\mu}$ is less simple than for the pure scalar case of the previous section. As in Section 2, the heavy modes give rise to a temperature-dependent mass term, $\frac{\Delta m^2}{2} | \phi |^2$, for the scalar field. The contributions to $\frac{\Delta m^2}{2}$ are shown diagrammatically in Figure 2. The effect of this term is to replace m_0^2 in the Euclidean time-independent effective action obtained from (31) by $m_0^2(1 - \frac{T^2}{T_c^2})$, just as in (28).

However there is a novelty here in that the heavy scalar modes also induce a vector mass $\frac{\Delta\mu^2}{2}A_\mu A^\mu$ in the tadpole appr ximation, (Figure 3). This induced mass is of order e^2T^2. Unfortunately, the tadpole diagram does not describe the total effects of the temperature-dependent self mass to our order of approximation. Non-local contributions like the photon self-energy have to be included as they also give terms $O(e^2T^2)$ for large T. All the relevant one-loop diagrams which contribute to $O(e^2T^2)$ are shown in Figure 4. Further problems arise because the heat bath gives a preferential inertial frame which leads to temporal and spatial components of A_μ being decoupled, giving rise to two independent self-mass terms $\Pi_L(k)$, $\Pi_T(k)$ for momentum k [13]. The same preferential reference frame makes the Π's non-analytic in k_0. The result is that depending on how one takes the zero momentum limit in the inverse fourier transforms of the mass, different masses are obtained [13]. Only the tadpole term, with no momentum dependence is immune from this uncertainty. Yet another complication arises because (unlike the mean field calculations) the background fields are not constant. This means that the self energy diagrams of Figure 4 have to be evaluated with non-zero external momenta. The effect of these difficulties is to make it hard to explicitly evaluate the gauge mass, even to $O(e^2T^2)$.

There is no easy resolution to these problems in the context of the approximation we are making here. The simplest approach is to restrict ourselves to the regime, $\lambda \gg e^2$, in which the gauge field contributions cannot be large [3]. Terms of order e^2T^2 are then constrained by $e^2\eta^2(e^2m_0^2/\lambda)$ and the vector mass is approximately unchanged. At the same time the vector loop gives a small contribution to the effective scalar mass. The effect is to replace $\bar{I}$ of (36) by

$$\bar{I} = -\int d^3x\mathcal{L}(\phi_0(x), A_{0\mu}(x)) \tag{37}$$

where $\mathcal{L}$ is derived from (31) by:
i) going Euclidean
ii) removing Euclidean time dependence
iii) deleting the massive modes
iv) replacing m_0^2 by $m_0^2(1-\frac{T^2}{T_c^2})$
v) implementing gauge fixing.

[3]The qualitative details of our later discussions are not changed by introducing a gauge mass anyway

Thus we finally obtain:

$$Z = \int \mathcal{D}\phi_0 \mathcal{D}A_{0\mu} exp(-\beta \overline{I}[\phi_0, A_{0\mu}]) \tag{38}$$

where

$$\overline{I}[\phi_0, A_{0\mu}] = \int d^3x [\frac{1}{4}F_{0ij}F_0^{ij} - \frac{1}{2}(\partial_i\phi_0)(\partial^i\phi_0^*)$$
$$-\frac{ie}{2}A_{0i}[\phi_0\partial^i\phi_0^* - \phi_0^*\partial^i\phi_0] - \frac{1}{2}m_0^2(1-\frac{T^2}{T_c^2}) \mid \phi_0 \mid^2$$
$$-\frac{1}{2}e^2 \mid \phi_0 \mid^2 A_{0i}A_0^i + \frac{\lambda}{4!} \mid \phi_0 \mid^4 + \frac{1}{\xi}(\partial_i A_0^i)^2] \tag{39}$$

in the covariant gauge for example and the term $\frac{1}{\xi}(\partial_i A_0^i)$ describes our gauge fixing.

4 The Statistical Properties of Strings around the Phase Transition

We now wish to evaluate the partition function (38) in further detail. To do this we will apply the saddle point method. The dominant contributions to the integral will come from the field configurations that satisfy the stationary equations:

$$\frac{\delta I}{\delta \phi}|_{\phi=\phi_{saddle}, A^\mu = A^\mu_{saddle}} = 0$$
$$\frac{\delta I}{\delta A_\mu}|_{\phi=\phi_{saddle}, A^\mu = A^\mu_{saddle}} = 0$$

that is, from the field configurations that satisfy the equations of motion:

$$\partial^i F_{ji} = \frac{1}{2}ie(\phi^*\partial_j\phi - \phi\partial_j\phi^*) - e^2A_j \mid \phi \mid^2 \tag{40}$$

$$(\partial_i + ieA_i)^2\phi = -m_0^2(1-\frac{T^2}{T_c^2})\phi + \frac{\lambda}{3!} \mid \phi \mid^2 \phi \tag{41}$$

The contribution of any solution of these equations to the partition function can be found by substitution into (38). The solution ϕ=const, A=0, being the solution of minimum energy, gives the maximum contribution. Away from the critical temperature the partition function will be well approximated by this term alone. However as we approach T_c, taking into account only the contribution of the absolute maximum is no longer a good approximation. We now have to find all the maxima of the functional and sum their contributions. This becomes necessary because, although the secondary maxima will be individually weighed heavily against the constant field contribution, their sum, because of the large number of different non-constant configurations, will be larger.

In principle in evaluating the partition function (38) we should consider the contributions of all the different types of solutions to equations (40,41). We will not do this, but instead will only consider the string like solutions. We do this because, here we only wish to consider the effect of the topologically stable defects on the phase transition [4]. Equations (40,41) contain string-like solutions at temperatures $T < T_c$. The simplest string solution is an infinitely straight static one running along the z-axis, say. This can be expressed as [14]:

$$\phi = \mid \phi(r) \mid e^{i\theta} \tag{42}$$

$$\underline{A} = \frac{\underline{r} \wedge \underline{k}}{r} \mid \underline{A}(r) \mid \tag{43}$$

where $\underline{k}$ is a unit vector in the z-direction. Imposing the gauge conditions $(A_{0,0}(x) = 0,\ \underline{\nabla}.\underline{A_0}(x) = 0)$, and substituting (42,43) into (40,41) we obtain:

$$-\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr} \mid \phi_0 \mid) + [(\frac{1}{r} - e \mid A_0 \mid)^2 - m_0^2(1 - \frac{T^2}{T_c^2}) + \frac{\lambda}{3!} \mid \phi_0 \mid^2] \mid \phi_0 \mid = 0 \tag{44}$$

$$-\frac{d}{dr}(\frac{1}{r}\frac{d}{dr}(r \mid A_0 \mid)) - e\frac{1}{r} \mid \phi_0 \mid^2 + \mid A_0 \mid e^2 \mid \phi_0 \mid^2 = 0 \tag{45}$$

Before continuing with the evaluation of the partition function we will first discuss the form of the thermal string solutions and how they vary with temperature. The solutions to (44,45) are shown schematically in Figure 5. At large distances from the string

$$Lim_{r\to\infty} \mid \phi(r) \mid \longrightarrow \eta$$

[4]The distribution of non-topologically stable field configurations we believe is not cosmologically interesting. They would rapidly disappear as the universe expanded and cooled. Only the trapped singularities would remain for a cosmologically interesting time.

where $\eta^2 = \frac{m_s^2}{\lambda}$ and $m_s^2(T) = m_0^2(1 - \frac{T^2}{T_c^2})$. At the core $\mid \phi \mid$ vanishes. The thickness of the core is determined by m_s^{-1}, the Compton wavelength of the Higgs particle. The magnetic field is restricted to the core, the skin depth being determined by m_v^{-1}, the inverse of the vector mass m_v,

$$m_v = e\eta = \frac{e}{\sqrt{\lambda}} m_s.$$

The energy per unit length of the vortex has two separate components, that due to the scalar field,

$$\sigma_s = O(\eta^2(T))$$

and that due to the vector field,

$$\sigma_v = O(\frac{e^2\eta^4}{m_v^2}) = O(\eta^2(T)) \tag{46}$$

with a comparable coefficient. Note that if we had found an additional $\underline{A_0}$ mass term of the form, $\triangle\mu^2 = \alpha e^2 T^2$, it's effect would be to replace σ_v of (46) by

$$\sigma_v = O(\frac{\eta^4(T)}{\eta^2(T) + \alpha T^2}) \tag{47}$$

The effect of increasing the temperature is firstly to cause σ_s and σ_v to decrease to zero (at $T = T_c$), thus reducing the energy per unit length of the string. (This would be true even if we had found the $\triangle\mu^2$ correction.) Secondly it makes the strings become wider as T_c is approached.

We now return to evaluating the contribution to the partition function (38) due to strings. We can write it as

$$Z \approx \sum exp(-\beta I_{st}[\phi_s, A_s]) \tag{48}$$

where we have restricted the sum to field configurations satisfying (40,41) which are nodal lines. Equation (44,45) gives the field configurations for an infinite string; string solutions which are not straight, but curved so smoothly that any segment of length of order the width will also be straight, are, to a very good approximation also solutions of the equations (40,41), whose energy per unit length is approximately the same as the infinitely straight string. Next we note that since, in this paper, we are interested in the effect of the topologically stable strings on the phase transition, the strings must be either in loops or infinite in length (in the absence of monopoles). To proceed further in evaluating the partition function it will be necessary to neglect the interaction energy of the strings when they are more than a distance l (l=width of string) apart, and to include it as an infinite repulsive/attractive force when strings are within a distance l [5]. We do this by restricting our strings to be non-self-intersecting. Placing the strings in the volume V on a cubic lattice (for convenience only) of spacing l, we can write:

$$Z = \sum_n W(n) exp(-\beta \sigma l n) \tag{49}$$

where W(n) denotes the number of different configurations of a string of length nl with the above properties and $\sigma = \sigma_v + \sigma_s$ the total energy per unit length.

Let us first consider the contribution of loops. At high string segment density, non-self-intersecting random walks will be approximated by Brownian walks [18]. The case of non-self intersecting walks at very low densities has been studied by polymer physicists (see [19] for example). If $P(t)$ denotes the fraction of walks of length nl, which start and end at the same point, it follows from these studies that:

$$P(n) = Cn^{-q} \qquad \begin{aligned} q &= \frac{3}{2} \textit{ at high density} \\ &= \frac{7}{4} \textit{ at low density} \end{aligned}$$

where C is a normalization factor. This results in the number of distinct configurations of a single loop of size nl being:

$$w_1(n) = \frac{1}{2} C n^{-q-1} a^n \qquad a = 5 \textit{ on a cubic lattice} \tag{50}$$

The extra divisor of $2n$ in (50) arises from the fact that an n-step loop is both non-orientatable and has n possible starting points. The contribution

[5]This is effectively what occurs in the numerical simulations. When two strings come within a distance l of each other, they intercommute. We are really replacing an exponential force of repulsion by a step function type force which acts over a distance l

of single loops to the partition function Z is thus, from (49) and (50):

$$Z_1 = \frac{1}{2}\frac{CV}{l^3}\sum_{n=1}^{\infty} n^{-q-1} exp(-\beta l n \sigma_{eff}) \tag{51}$$

where

$$\begin{aligned} \sigma_{eff} &= \sigma - \frac{\ln a}{\beta l} \\ &= \sigma - \frac{T}{l}\ln a \\ &= \sigma(1 - \frac{T}{T_{st}}) \end{aligned} \tag{52}$$

and

$$T_{st} = \frac{\sigma l}{ln a} \tag{53}$$

The n steps need not constitute a single loop but two or more loops. Because of the lack of interaction energy the partition function for a 'gas' of loops is:

$$\begin{aligned} Z_{loop} &= 1 + Z_1 + \frac{1}{2!}Z_1^2 + \frac{1}{3!}Z_1^3 + ... \\ &= exp(Z_1) \\ &= exp[\frac{1}{2}\frac{CV}{l^3}\sum_{n=1}^{\infty} n^{-q-1} e^{-\beta n l \sigma_{eff}}] \end{aligned} \tag{54}$$

From these weights we can calculate the average length of string in loops.

$$\begin{aligned} \overline{N}_{loops} &= [\sum_{n=0}^{\infty} \frac{ln(Z_1)^n}{n!}]/Z_{loop} \\ &= Z_1 l \end{aligned} \tag{55}$$

Similarly the mean number of loops of size nl is:

$$R(nl) = \frac{1}{2}\frac{CV}{l^3} n^{-q-1} exp(-\beta n l \sigma_{eff}) \tag{56}$$

Note that this is in agreement with the numerical simulations of [7]. It also agrees with the statistical properties of strings derived using a rather different approach by Mitchell and Turok [1] if we neglect string interactions completely. At high temperatures for which $\sigma_{eff} \approx 0$, R appears to be a scale invariant distribution while at lower temperatures loops of large length are exponentially suppressed.

Now let us consider the contribution of 'infinite' strings to the partition function [6]. Since for large n:

$$a^n - n^{-q-1}a^n \approx a^n \tag{57}$$

we can write it as:

$$\begin{aligned} Z_\infty &= \lim_{n\to\infty} \sum_k [\frac{CV}{2l^3}]^k e^{-\beta k n l \sigma_{eff}} \frac{1}{k!} \\ &= \lim_{n\to\infty} exp[\frac{CV}{2l^3} e^{-\beta n l \sigma_{eff}}] \end{aligned} \tag{58}$$

We can immediately see that when $\sigma_{eff} \gg 0$, these strings make a negligible contribution. As σ_{eff} tends to zero however, the contribution of the infinite strings will be more and more important. We can evaluate the ratio of string length in loops to the total string length produced at their formation by noting that at this temperature $\sigma_{eff} = 0$. We then obtain (using equations (54),(55) and similar equations for the infinite strings)

$$\frac{\Lambda_{loop}}{\Lambda_{total}} \sim 10\% \tag{59}$$

This result is however, sensitive to the lattice type the strings are laid on. In general though, we still expect most of the string length to be in infinite strings at this temperature.

We note that Z_∞ and Z_{loop} both diverge at temperatures greater than T_{st} (the temperature at which $\sigma_{eff} = 0$). Above this temperature there are large fluctuations in the ϕ field and it is no longer appropriate to describe the field in terms of string-like configurations. Thus we can think of T_{st} as the temperature at which our strings are formed. This temperature is (neglecting the $\ln a$):

$$T_{st} \simeq \sigma(T)l(T) \simeq \gamma\eta^2 m^{-1} \tag{60}$$

[6]By infinite we mean strings which are as large as the box in which our calculations are being performed

for some $\gamma \sim O(1)$, and $m = min(m_s, m_v)$ (in our case $\lambda >> e^2$, $m = m_s$). Since the right hand side of (60) vanishes at $T = T_c$, it follows that:

$$T_{st} < T_c \tag{61}$$

as it must be. The difference between T_{st} and T_c is small. Explicitly,

$$1 - \frac{T_{st}^2}{T_c^2} = O(\lambda), \qquad m = m_s \tag{62}$$

or

$$1 - \frac{T_{st}^2}{T_c^2} = O(e^2), \qquad m = m_v \tag{63}$$

In each case we have ignored terms $O(\frac{e^2}{\lambda})$ in the coefficients on the right hand side, but even if $e^2 = \lambda$ say, those coefficients only change by a factor of order unity. As we have so many uncertain coefficients it is not useful to be more specific.

At this temperature most of the string length is in infinite strings and equation (49) implies that the loops have a scale invariant distribution. This is in complete agreement with the numerical simulations of Vilenkin and Vachaspati [17]. We can also calculate the width of the strings at formation (this will be of the same order as their mean separation). By direct substitution:

$$\begin{aligned}
m_s(T_{st}) &= O(\sqrt{\lambda} m_s(T=0)), \quad m = m_s, \\
m_s(T_{st}) &= O(e m_s(T=0)), \qquad m = m_v, \\
&\textit{and} \\
m_v(T_{st}) &= O(\sqrt{\lambda} m_v(T=0)), \quad m = m_s, \\
m_v(T_{st}) &= O(e m_v(T=0)), \qquad m = m_v,
\end{aligned}$$

That is, the network of strings at the phase transition has the separation of the centres of the flux tubes scaled up by a factor $O(\frac{1}{\sqrt{\lambda}})$, (recall $e^2 < \lambda$), compared to the closest packing of cold strings. The factor $O(\frac{1}{\sqrt{\lambda}})$ can be obtained by other considerations [3], and this reinforces our belief in the chain of approximations given above.

At temperatures less than T_{st} it is thermodynamically less favourable to have infinite strings and more favourable to have small loops.

From a cosmological point of view a more interesting question to address would be what happens as the universe cools through T_{st}. Our calculations were for flat space but seem to suggest that, when the expanding universe was very hot, we would be unable to recognize any string configurations. As we go through T_{st} strings would be formed. Initially most of the string length would be in infinite strings, but as the universe cooled those strings would grind themselves up as fast as possible into the smallest loops they could make. Eventually the strings would no longer be in thermal equilibrium as they would not be able to chop themselves up fast enough, but still as much length as possible would go into the smallest possible loops. This picture, if true, makes the string domination scenario of Kibble [15] and Bennett [16] seem unlikely. This picture also seems to be confirmed by the simulations of Albrecht and Turok [8] and those of Bennett and Bouchett [9].

5 Discussion

Investigating the effects of finite temperature corrections to field theories is an extremely useful method for determining the properties and distribution of topological defects in the early universe.

In this paper we have studied the effect of temperature on Nielsen-Olesen type vortices. We found that the strings developed an effective temperature dependent tension, $\sigma_{eff}(\beta)$, and a temperature dependent width. We then went on to evaluate the contribution of these string solutions to the partition function of a scalar gauge theory. We discovered that the partition function diverged at a temperature we called T_{st}. At this temperature the heat capacity also diverges and there are therefore large fluctuations in the energy. As has been emphasized recently by Mitchell and Turok [1] our description then breaks down. We conjecture however that the critical exponent for the heat capacity is small [7]. It follows that fluctuations are small until we are at temperatures very close to T_{st}. It is for this reason that we believe that our results for the distribution of strings are accurate.

[7] This is to be contrasted with their results which do not take into account the variation in the width of the string with temperature.

Above T_{st} we believe it is no longer appropriate to describe the field in terms of string like configurations and therefore interpret T_{st} as the string formation temperature (note also $T_{st} = O(T_{Ginsburg})$). From the partition function for the vortex solutions we evaluated the equilibrium distribution of strings. We discovered that as we approach T_{st}, most of the string length goes into infinite strings (occupying about 90 % of the total string length), and that we had a scale invariant distribution of loops, both with approximately Brownian trajectories [david]. This is under the assumption that interactions between different string segments are neglected. The result agrees with Mitchell and Turok [1], who used a phase space calculation to evaluate the string network configuration, and with computer simulations [7,17]. Under the same assumption we obtain the result that at low temperatures most of the string length is in loops and that there are exponentially few large loops, again in agreement with the simulations and [1]. One improvement that our technique allows over the results of [1] is that we can incorporate some of the string interactions . We do this by making our strings non-self-intersecting. In fact, at high densities this improvement is irrelevant because non-self-intersecting walks are well approximated by Brownian walks [18]. The reason is that the excluded volume to the walk becomes approximately homogeneous at high densities, so to a good approximation, there is an equal probability of strings going in any direction, or being Brownian. However, at low densities the two types of walks are different. Non-self-intersecting walks have been investigated in polymer physics (see Wiegel [19] and references therein) and at low densities they find that q in equation(54) is $\frac{7}{4}$, for example. It would be interesting to know if the computer simulations of [7] see this difference.

If we allow for the possibility of having open strings as well as closed, then we can estimate their distribution. We then obtain that the number of string configurations is maximized by having an exponentially suppressed distribution of open strings, (i.e the longest ones are suppressed). In a GUT model where the monopole mass is of the same order as the string tension, this would be a good estimate for the distribution of monopoles connected by strings in the early universe. The result agrees well with the numerical simulations of such a phase transition [20]. It is also worth noting that the temperature of the open strings is higher than that of the closed string.

The values of m_s and m_v which are obtained at T_{st} are very important in determining the relevant length scales at the phase transition. We have demonstrated that both masses become scaled up by a factor $O(\sqrt{\lambda})$ over the zero temperature result. This means that the network of strings at the phase transition has the separation of the centres of the flux tubes scaled up by a factor of $O(\frac{1}{\sqrt{\lambda}})$. This is just the Ginzburg length scale, that was used by Kibble [3], in determining the initial length scale for the distribution of the Higgs fields at the phase transition. It is very encouraging that we also obtain this result and believe it adds support to Kibble's mechanism describing the phase transition.

We would like to make one comment about the imaginary time formalism. At high temperatures we were lead to investigate τ - independent solutions for the fields. This was a because, at high temperatures the width of the string $\xi \gg \beta$, so to a good approximation the variation of the fields over the interval β can be taken to be zero. At lower temperatures negelecting the temperature τ dependence of the solutions is nolonger justifiable. It is unclear what the interpretation of the τ dependent solutions would be in terms of the real time t, because we can not perform a Wick rotation to reobtain the real time in the theory.

The results we have obtained have applications outside of cosmic strings and the early universe. Patel has used some of the formalism of section 4 in investigating confinement in QCD [21]. Similar methods have also been used to investigate the lambda transition in liquid He^4 [22] and [19] (for further references see [19]).

Finally we feel it is worth pointing out that this analysis is for local strings rather than global strings. It is the former type which we expect to be important in galaxy forming scenarios, so it is important that features like scaling at high temperatures and exponential suppression of large loops at low temperatures were obtained. The calculations presented in this paper were in flat space. The next step is to allow for a curved spacetime in the action, and to see how this affects the distribution of the strings. This work is currently in progress.

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Appendix A

Here we present an alternative derivation of the partition functions of Sections 2 and 3. We start by deriving that for the scalar field theory of Section 1.

As in Section 1 we start from the path integral representation of the partition function:

$$Z = \int \mathcal{D}\phi exp[I_\beta(\phi)]$$

This time we split I_β into two parts, I_{ren} the finite temperature renormalized action and I_{count} the part containing the counterterms. To $O(\lambda)$ for example:

$$I_{ren} = \int_0^\beta d\tau \int d^3x[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2(T)\phi^2 - \frac{\lambda}{4!}\phi^4] \qquad (64)$$

$$I_{count} = \int_0^\beta d\tau \int d^3x \frac{\lambda}{2}A\phi^2$$

We now evaluate the partition function by expanding I about a field configuration ϕ_0 that satisfies the equation of motion derived from I_{ren}. We obtain:

$$\begin{aligned} I = \;& I[\phi_0] + \tfrac{1}{2} < \tfrac{\delta^2 I_{ren}}{\delta\phi_1\delta\phi_2}\eta_1\eta_2 >_{1,2} + ... \\ & + < \tfrac{A\lambda}{2}\phi_{01}^2 + A\lambda\eta_1\phi_{01} + \tfrac{A\lambda}{2}\eta_1^2 >_1 \end{aligned}$$

where

$$\eta = \phi - \phi_0$$
$$\phi_1 = \phi(x_1)$$

and $< .. >_{1,..N}$ means integrate over $d^3x_1 d\tau_1 .... d^3x_N d\tau_N$. This expansion is then substituted into the partition function. The resulting functional is evaluated by introducing a current j coupled to the field η. To $O(\lambda)$ one obtains:

$$Z \propto exp(I[\phi_0])exp(< \frac{\lambda}{4}\phi_{01}^2 D_{11} + \frac{A\lambda}{2}\phi_{01}^2 >_1)$$

where

$$D_{12} = -\int \frac{d^3p}{(2\pi)^3}\frac{1}{\beta}\sum_n \frac{e^{i(x_1-x_2)p}}{(\frac{2\pi n}{\beta})^2 + p^2 + m^2}$$

is the finite temperature propagator. Now m in this equation was chosen to be the finite temperature mass and so:

$$\frac{A\lambda}{2} = -\frac{\lambda}{4}D_{11}$$

Thus we obtain the contribution of one saddle point to the partition function as:

$$Z \propto exp(I_{ren}[\phi_0])$$

This is the contribution of one saddle point. If we make a 'dilute gas' approximation we obtain:

$$Z = A\sum_{\phi_0} exp(I_{ren}[\phi_0])$$

where A is a normalisation constant and the sum is over all field configurations satisfying the equation of motion derived from I_{ren}. At high temperatures it is a good approximation to neglect the τ variation of our solutions. This is because the solutions have to be periodic in β and at high temperatures β will be much smaller than the spatial width of the solutions[8]. Thus we have reobtained the result of section 2.

It is straight forward to apply this same scheme to evaluating the partition for scalar QED and reobtain the results of section 3.

[8]We thank N.Turok for pointing this out to us

Figure Captions

(1) Diagrammatic representation of the lowest order mass correction. (Here dashed lines denote the heavy modes and the solid lines the light modes)

(2) Feynman graphs for the tadpole corrections to the scalar mass. (Here again dashed lines denote the heavy modes, solid lines the light mode and wavy lines refer to heavy gauge field modes)

(3) Feynman graph for the tadpole correction to the gauge field mass.

(4) The remaining Feynman graphs of $O(e^2)$.

(5) An example of the field configurations for a vortex solution at temperature T.

$$\zeta = O(m_s^{-1}) \qquad \lambda = O(m_v^{-1})$$

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Figure 1

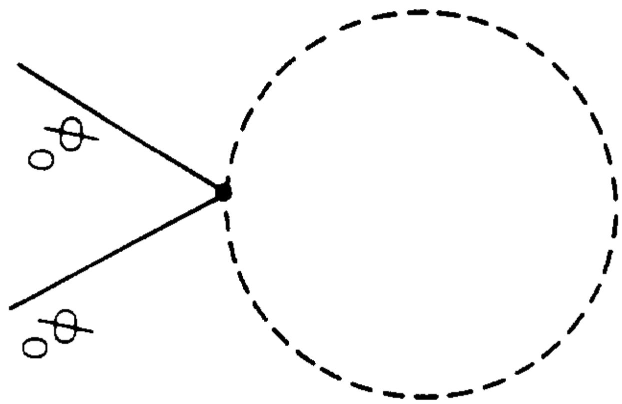

Figure 2

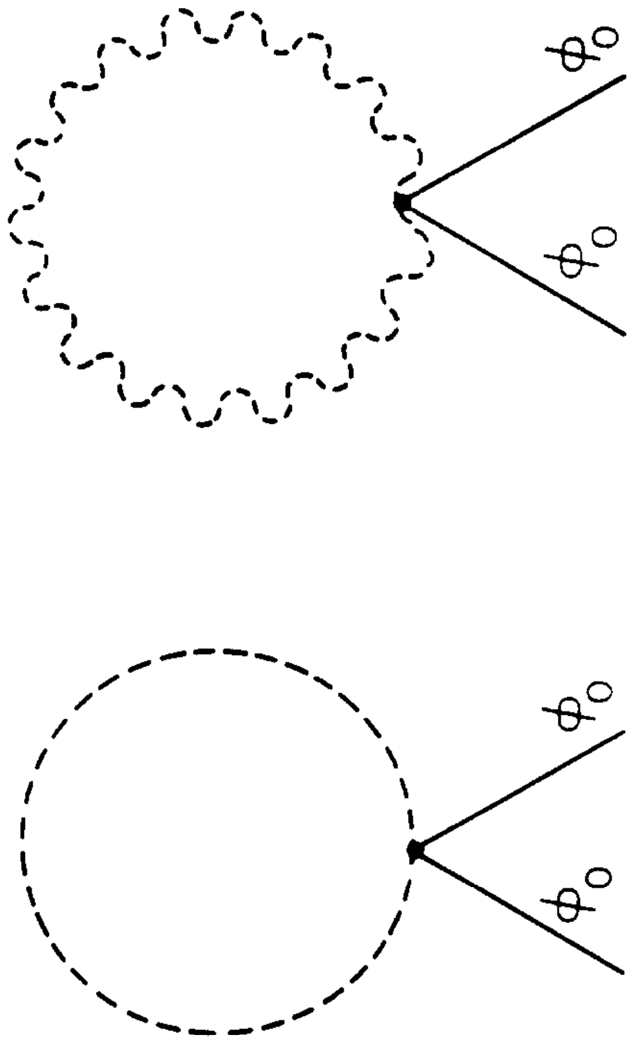

Figure 3

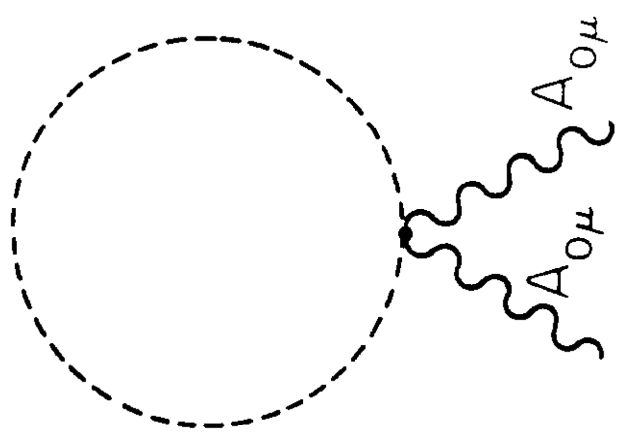

Figure 4

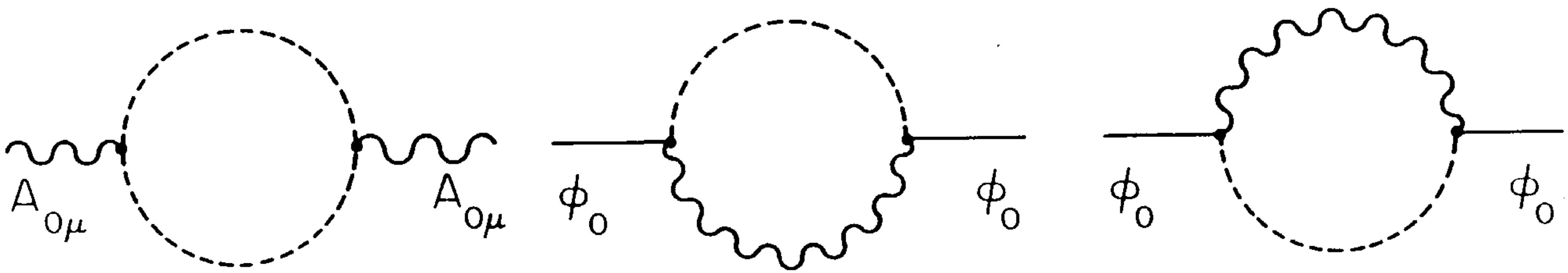

Figure 5