Fermi National Accelerator Laboratory

FERMILAB-Conf-88/088-A

May, 1988

STABILITY OF SELF–GRAVITATING BOSONS

Philippe Jetzer[*][†]

NASA/Fermilab Astrophysics Center

Fermi National Accelerator Laboratory

P.O. Box 500, Batavia, Illinois, 60510, USA

Abstract

I study the problem of the dynamical stability of the equilibrium solutions for the bosonic stellar configurations in the framework of general relativity. Following the method developed by Chandrasekhar[1], I find a variational principle for determining the eigenfrequencies of the oscillations. Using the variational principle, one can find an upper bound for the central densities where dynamical instability occurs. For the non–interacting massive complex scalar fields the equilibrium configurations are dynamically unstable for central densities bigger than $\rho = 1.04 \times 10^{98} m^2\ g/cm^3$ (m is the boson mass in grams) whereas for the quartic self–interacting case the bound is given by $\rho = 0.53 \times 10^{98} m^2\ g/cm^3$ (for a value of the quartic coupling constant: $\tilde{\lambda} = 3.8 \times 10^{12} m^2$).

---

[*] Supported by the Swiss National Science Foundation. Address after October 1, 1988: CERN, Theory Division, CH–1211, Genève 23, Switzerland

[†] Submitted to the Fifth Marcel Grossmann Meeting, Perth, Australia, August, 1988.

I. INTRODUCTION

In an early work, Ruffini and Bonazzola[2] found spherically symmetric gravitational equilibrium configurations in asymptotically flat space–times for non–interacting massive complex fields by solving the coupled system of Einstein–Klein–Gordon equations. They used the simplifying assumptions that the radial scalar field solution is nodeless and that the configuration is at zero temperature. For the boson stars, like for their fermionic counterparts, there is also a critical mass and a critical particle number above which mass gravitational collapse occurs. These quantities are respectively given by $M_{crit} = 0.633 M^2_{Planck}/m$ and $N_{crit} = 0.653 M^2_{Planck}/m^2$.

The extension to the self–interacting case, for a potential $V(|\phi|) = \frac{1}{4}\tilde{\lambda}|\phi|^4$ ($\tilde{\lambda} > 0$), has been considered in the work of Colpi, Shapiro and Wasserman[3]. The consequence of switching on a quartic self–interaction between the scalar particles is to increase the above limits on the critical mass and particle number. As it was already shown in Ref. [2], it is not consistent with the Klein–Gordon equation to describe the spherically symmetric boson condensate as a perfect fluid with a given equation of state. As a consequence, we cannot apply to the boson stars the theorem on stability of a fluid star which is based upon the assumption of a perfect fluid behaviour. Nevertheless, if we plot the mass for the equilibrium configurations against central density for these stars, we find, except for the scales, a behaviour remarkably similar to those of neutron–stars: the mass quickly raises to a maximum (for $\rho = \rho_{crit}$), drops a little, oscillates and approaches an asymptotic value at large central densities, the same happening for the particle number. These similarities lead naturally to the conjecture[4] that the stability behaviour is similar to the fermionic case, namely that the boundary between stable and unstable configurations being given by $\rho = \rho_{crit}$.

Here we study the problem of the dynamical stability of the bosonic equilibrium configurations in the framework of general relativity[5]. We discuss both the case of a non–interacting massive complex scalar field, as well as the case with a quartic self–interacting potential. However, the analytic part of our results can immediately be extended to any interacting potential for complex scalar fields. We analyze the time evolution of infinitesimal radial oscillations, which conserve the total number of particles, starting from the system described by the scalar wave equation coupled to the Einstein's field equations. We follow closely the method developed by Chandrasekhar[1]; the main modification being that we cannot use the energy–momentum tensor of a perfect fluid, but instead we use the energy–momentum tensor of a quartic self–interacting massive complex field (the non–interacting case being given by setting $\tilde{\lambda} = 0$). We find an eigenvalue equation which determines the normal modes of the radial oscillations, and as in the perfect fluid case we find a variational principle for determining the eigenfrequencies of the oscillations. This allows then to find numerically, using suitable trial functions, upper bounds to the central density, of the equilibrium configurations, from which on dynamical instability will occur.

II. EQUILIBRIUM CONFIGURATIONS

The many particle system is described by a second–quantized complex scalar field coupled to gravity. The action for this system is given by

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} + \frac{g^{\mu\nu}}{2} \phi^*_{;\mu} \phi_{;\nu} - \frac{m^2}{2} |\phi|^2 - \frac{\tilde{\lambda}}{4} |\phi|^4 \right] \tag{1}$$

This action is invariant under a global $U(1)$ phase transformation, $\phi \rightarrow e^{i\theta}\phi$, which implies the conservation of its generator Q, the number of particles minus the number of antiparticles. The corresponding energy–momentum tensor is

$$T^{\mu}_{\nu} = \frac{1}{2} g^{\mu\sigma} \left(\phi^*_{;\sigma} \phi_{;\nu} + \phi_{;\sigma} \phi^*_{;\nu} \right) - \frac{1}{2} \delta^{\mu}_{\nu} \left[g^{\lambda\sigma} \phi^*_{;\lambda} \phi_{;\sigma} + m^2 |\phi|^2 + \frac{\tilde{\lambda}}{2} |\phi|^4 \right] \tag{2}$$

For our considerations it is more convenient to write the complex scalar field as $\phi = (\phi_1 + i\phi_2) e^{-i\omega t}$, where ϕ_1 and ϕ_2 are two real fields, which are functions of r and t alone since we consider only spherically symmetric equilibrium configurations. In fact, we expect them to correspond to solutions with minimal energy. We express the metric in Schwarzschild coordinates

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{3}$$

where ν and λ are functions of r and t only ($g^{00} = -e^{-\nu}$ and $g^{rr} = e^{-\lambda}$). For $r \to \infty$, we require $e^{\nu(r,t)} \to 1$ as well as $e^{\lambda(r,t)} \to 1$.

At the equilibrium configuration, the functions ν, λ are time independent. In order to recover the solutions found in Refs. [2] and [3], we set $\phi_2 = 0$ at the equilibrium as well as $\phi_1(r,t) = \phi_0(r)$. In this way, we have only particles and the charge is then the total number of particles N_0. We will denote all the equilibrium quantities by a subscript o.

We are left with two Einstein equations

$$\left(re^{-\lambda_0}\right)' = 1 - 4\pi G r^2\left[\left(m^2 + e^{-\nu_0}w^2 + \frac{\tilde{\lambda}}{2}\phi_0^2\right)\phi_0^2 + e^{-\lambda_0}\phi_0'^2\right] \tag{4}$$

$$\frac{e^{-\lambda_0}}{r}\nu_0' = \frac{1}{r^2}\left(1 - e^{-\lambda_0}\right) + 4\pi G\left[\left(e^{-\nu_0}w^2 - m^2 - \frac{\tilde{\lambda}}{2}\phi_0^2\right)\phi_0^2 + e^{-\lambda_0}\phi_0'^2\right] \tag{5}$$

and the scalar wave equation

$$\phi_0'' + \left(\frac{2}{r} + \frac{\nu_0' - \lambda_0'}{2}\right)\phi_0' - e^{\lambda_0}\left(m^2 - e^{-\nu_0}w^2 + \tilde{\lambda}\phi_0^2\right)\phi_0 = 0 \tag{6}$$

The total particle number is given by

$$N_0 = 4\pi w\int_0^{\infty} dr\; r^2\phi_0^2 e^{1/2(\lambda_0 - \nu_0)} \tag{7}$$

For the numerical integration of the set of equations (4), (5) and (6), it is convenient to introduce the new variables $x = mr, \sigma = (4\pi G)^{1/2}\phi_0$, with the factor w^2/m^2 being absorbed into the definition of the metric function e^{ν_0}. Note that the coupling constant $\tilde{\lambda}$ is also replaced by $\Lambda = \frac{\tilde{\lambda}M^2_{Planck}}{4\pi m^2}$.

The appropriate initial conditions and boundary conditions are: $\sigma(0) = const$, $\sigma'(0) = 0, e^{\lambda_0(0)} = 1, e^{\nu_0(\infty)} = \frac{w^2}{m^2}$ (since we absorbed w^2/m^2 in e^{ν_0}). One finds then the solution of the system of equations, which constitute an eigenvalue problem for $e^{\nu_0(0)}$ and $e^{\nu_0(\infty)}$ with the constraint that $\sigma(x)$ has no nodes.

III. EIGENVALUE EQUATION AND VARIATIONAL PRINCIPLE

We consider now the situation where the equilibrium configuration is perturbed in a way such that the spherical symmetry is still preserved. These perturbations will give rise to motions in the radial direction.

The equations governing the small perturbations are obtained by expanding all functions to first order and then by linearizing the equations. We obtain the following linearized set of equations for the perturbed quantities

$$\left(re^{-\lambda_0}\delta\lambda\right)' = -8\pi G r^2 \Big[\frac{1}{2}\delta\nu e^{-\nu_0} w^2 \phi_0^2 + \frac{1}{2}\delta\lambda e^{-\lambda_0}(\phi_0')^2 - e^{-\nu_0}\left(w^2\phi_0\delta\phi_1 + w\phi_0\delta\dot{\phi}_2\right) - m^2\phi_0\delta\phi_1 - \tilde{\lambda}\phi_0^3\delta\phi_1 - e^{-\lambda_0}\phi_0'\delta\phi_1'\Big] \tag{8}$$

$$\frac{e^{-\lambda_0}}{r}\left(\delta\nu' - \nu_0'\delta\lambda\right) = \frac{e^{-\lambda_0}}{r^2}\delta\lambda + 8\pi G\Big[-\frac{1}{2}\delta\nu e^{-\nu_0} w^2\phi_0^2 - \frac{1}{2}\delta\lambda e^{-\lambda_0}(\phi_0')^2 + e^{-\nu_0}(w^2\phi_0\delta\phi_1 + w\phi_0\delta\dot{\phi}_2) - m^2\phi_0\delta\phi_1 - \tilde{\lambda}\phi_0^3\delta\phi_1 + e^{-\lambda_0}\phi_0'\delta\phi_1'\Big] \tag{9}$$

and

$$\delta\dot{\lambda} = 8\pi G r\left(\phi_0'\delta\dot{\phi}_1 + w\phi_0\delta\phi_2' - w\phi_0'\delta\phi_2\right) \tag{10}$$

The linearized equations for $\delta\phi_1$ and $\delta\phi_2$ are

$$\delta\phi_1'' + \left(\frac{2}{r} + \frac{\nu_0' - \lambda_0'}{2}\right)\delta\phi_1' + \frac{1}{2}\phi_0'(\delta\nu' - \delta\lambda') + \delta\lambda e^{\lambda_0}\left[w^2 e^{-\nu_0}\phi_0 - m^2\phi_0 - \tilde{\lambda}\phi_0^3\right] + e^{\lambda_0}\left(w^2e^{-\nu_0} - m^2 - 3\tilde{\lambda}\phi_0^2\right)\delta\phi_1$$

$$- \quad e^{\lambda_0-\nu_0}\phi_0 w^2\delta\nu - e^{\lambda_0-\nu_0}\delta\ddot{\phi}_1 + 2e^{\lambda_0-\nu_0}w\delta\dot{\phi}_2 = 0 \tag{11}$$

and

$$\begin{aligned}\delta\phi_2'' \quad &+ \quad \left(\frac{2}{r}+\frac{\nu_0'-\lambda_0'}{2}\right)\delta\phi_2' + e^{\lambda_0}\left(w^2e^{-\nu_0} - m^2 - \tilde{\lambda}\phi_0^2\right)\delta\phi_2 \\ &- \quad e^{\lambda_0-\nu_0}\delta\ddot{\phi}_2 + \frac{1}{2}e^{\lambda_0-\nu_0}w\left[\phi_0\left(\delta\dot{\nu}-\delta\dot{\lambda}\right) - 4\delta\dot{\phi}_1\right] = 0 \ .\end{aligned} \tag{12}$$

It might seem that we have a set of five equations (8), (9), (10), (11) and (12) for four unknown functions $(\delta\nu, \delta\lambda, \delta\phi_1, \delta\phi_2)$. It turns out, in fact, that these equations are not all independent, for instance, one can show that eq. (12) is a consequence of the other equations[5]. The basic system of four equations can be reduced to a system of two coupled equations involving only the real components of the perturbation of the scalar field.

We notice that eq. (10) integrates directly in time if we write $\delta\phi_2$ as

$$\delta\phi_2(r,t) \equiv \phi_0(r)\dot{g}(r,t) \tag{13}$$

we get

$$\delta\lambda = 8\pi G r\left(\phi_0'\delta\phi_1 + w\phi_0^2 g'\right) \quad . \tag{14}$$

By suitably combining the above equations, we get the two final equations, where we write $f_2(r,t) \equiv g'(r,t)$ and $f_1(r,t) \equiv \delta\phi_1(r,t)$.

$$\begin{aligned} f_2'' \quad &+ \quad f_2'\left(\frac{3}{2}(\nu_0'-\lambda_0') + \frac{2}{r} + \frac{2\phi_0'}{\phi_0}\right) - e^{\lambda_0-\nu_0}\ddot{f}_2 \\ &+ \quad f_2\Bigg[\frac{2(\nu_0'-\lambda_0')}{r} - \frac{2}{r^2} + \frac{(\nu_0''-\lambda_0'')}{2} + 2\frac{\phi_0''}{\phi_0} - 2\left(\frac{\phi_0'}{\phi_0}\right)^2 \\ &+ \quad \frac{(\nu_0'-\lambda_0')}{2}\left((\nu_0'-\lambda_0') + 4\frac{\phi_0'}{\phi_0}\right) + 4\pi G r w^2\phi_0^2 e^{\lambda_0-\nu_0}\left(\nu_0'-\lambda_0'+\frac{2}{r}\right)\Bigg] \\ &- \quad f_1'\frac{2}{\phi_0}e^{\lambda_0-\nu_0}w + f_1\Bigg[\frac{2}{\phi_0^2}\phi_0' w e^{\lambda_0-\nu_0} - 8\pi G r e^{\lambda_0}\left(m^2\phi_0 + \tilde{\lambda}\phi_0^3\right)we^{\lambda_0-\nu_0} \\ &+ \quad 4\pi G r\phi_0'\left(\nu_0'-\lambda_0'+\frac{2}{r}\right)we^{\lambda_0-\nu_0}\Bigg] = 0 \ . \end{aligned} \tag{15}$$

and

$$
\begin{aligned}
f_1'' &+ f_1'\left(\frac{2}{r}+\frac{1}{2}(\nu_0'-\lambda_0')\right)+f_1\Big[e^{\lambda_0}\left(-3w^2e^{-\nu_0}-m^2-3\tilde{\lambda}\phi_0^2\right)\\
&- 16\pi G r e^{\lambda_0}\phi_0'\left(m^2\phi_0+\tilde{\lambda}\phi_0^3\right)+4\pi G r\left(\phi_0'\right)^2\left(\nu_0'-\lambda_0'+\frac{2}{r}\right)\Big]\\
&- \ddot{f}_1 e^{\lambda_0-\nu_0}+f_2' 2w\phi_0+f_2\Big[4\pi G r\phi_0'\left(\nu_0'-\lambda_0'+\frac{2}{r}\right)w\phi_0^2\\
&+ 8\pi G r e^{\lambda_0}\left(-m^2\phi_0-\tilde{\lambda}\phi_0^3\right)w\phi_0^2+4w\phi_0'\\
&+ 4\frac{\phi_0 w}{r}+(\nu_0'-\lambda_0')w\phi_0\Big]=0 \ . \qquad (16)
\end{aligned}
$$

We expect the radial perturbations not to change the total particle number N_0. We, therefore, set $\delta N_0 = 0$. It turns out that this leads to boundary conditions for f_2[5]. For $r \to \infty$ we require $r^2\phi_0^2 f_2 \to 0$, and for $r \to 0$ we have $r^2 f_2 \to 0$.

To further deal with the partial differential equations (15) and (16) we suppose that $f_1(r,t)$ and $f_2(r,t)$ have a time dependence of the form $e^{i\sigma t}$. Inserting this into the equations leads to an eigenvalue problem with eigenvalue σ^2 and corresponding eigenfunctions $f_1(r)$ and $f_2(r)$. The appropriate boundary conditions are: $r \to \infty$ $f_1 \to 0, r^2\phi_0^2 f_2 \to 0; r \to 0$ $f_1 = const, r^2 f_2 \to 0$. The operator associated with the eigenvalue problem is real and symmetric, thus the eigenvalues σ^2 are real. Dynamical instability will occur whenever $\sigma^2 \le 0$. One can find the following variational principle ($G_1 = r^2 e^{1/2(\nu_0-\lambda_0)}$, $G = r^2\phi_0^2 e^{3/2(\nu_0-\lambda_0)}$)

$$
\begin{aligned}
\sigma^2\int_0^\infty \frac{1}{2}e^{(\lambda_0-\nu_0)}\left(G_1 f_1^2+G_2 f_2^2\right)dr &= \int_0^\infty\Big[\frac{1}{2}G_1 f_1'^2+\frac{1}{2}G_1 f_1^2 C_1(r)+\frac{1}{2}G_2 f_2'^2+\frac{1}{2}G_2 f_2^2 C_2(r)\\
&+ f_2 f_1'\left(2\phi_0 w G_1\right)+f_2 f_1 G_1 C_3(r)\Big]dr \qquad (17)
\end{aligned}
$$

with

$$
\begin{aligned}
C_1(r) &= e^{\lambda_0}\Big[\left(3w^2e^{-\nu_0}+m^2+3\tilde{\lambda}\phi_0^2\right)+16\pi G e^{\lambda_0} r\phi_0'\left(m^2\phi_0+\tilde{\lambda}\phi_0^3\right)\\
&- 4\pi G r\left(\phi_0'\right)^2\left(\nu_0'-\lambda_0'+\frac{2}{r}\right)\Big] \qquad (18)
\end{aligned}
$$

$$C_2(r) = \left[\frac{2(\lambda_0' - \nu_0')}{r} + \frac{2}{r^2} - \frac{(\nu_0'' - \lambda_0'')}{2} - 2\frac{\phi_0''}{\phi_0} + 2\left(\frac{\phi_0'}{\phi_0}\right)^2 + \frac{(\lambda_0' - \nu_0')}{2}\left[(\nu_0' - \lambda_0') + 4\frac{\phi_0'}{\phi_0}\right] - 4\pi G r w^2 \phi_0^2 e^{\lambda_0 - \nu_0}\left(\nu_0' - \lambda_0' + \frac{2}{r}\right)\right] \qquad (19)$$

$$C_3(r) = -2\phi_0' w + 8\pi G r e^{\lambda_0}\left(m^2\phi_0 + \bar{\lambda}\phi_0^3\right) w\phi_0^2 - 4\pi G r \phi_0'\left(\nu_0' - \lambda_0' + \frac{2}{r}\right) w\phi_0^2 \qquad (20)$$

A sufficient condition for dynamical instability to occur is that the right hand side of eq. (17) vanishes (or becomes negative) for some chosen pair of trial functions f_1, f_2, which satisfy the appropriate boundary conditions.

We tried for both cases we considered, namely $\Lambda = 0$ and $\Lambda = 30$, with different pairs of trial functions. In the case $\Lambda = 0$, an upper bound for the occurrence of dynamical instability is given by $\sigma(0) = 0.535$ corresponding to a central density $\rho = 1.04 \times 10^{98} m^2\ g/cm^3$ (m is the boson mass in grams). (For comparison: The central density of the critical mass is $\sigma(0) = 0.271$ and the point from which on the binding energy becomes positive corresponds to $\sigma(0) = 0.540$.) In the interacting case with $\Lambda = 30 (\Lambda = \bar{\lambda} M_{Planck}^2/4\pi m^2, \bar{\lambda} = 3.8 \times 10^{12} m^2)$ the upper bound is given by $\sigma(0) = 0.285$, corresponding to a central density $\rho = 0.53 \times 10^{98} m^2\ g/cm^3$. ($\sigma(0) = 0.158$ for the critical mass, and $\sigma(0) = 0.312$ for the point from which on the binding energy is positive.)

In both cases, the bounds are above the critical density corresponding to the maximum mass, but below the point from which on the binding energy becomes positive. A weak point of the variational method is that we cannot easily estimate how far our bounds are from the actual values. Our results can thus not be conclusive about the conjecture whether the border between stability and instability is given by the central density corresponding to the maximum mass, although they are not very far from it.

I am grateful to Professor S.Chandrasekhar for many helpful discussions and for his continued support. I would like to thank Professor R. Ruffini for his interest in the work and for the invitation to participate at the fifth Marcel Grossmann Meeting. This work was supported in part by DOE and by NASA at Fermilab.

References

[1] S. Chandrasekhar, Phys. Rev. Lett **12**, (1964) 114, 437; Ap. J **140** (1964) 417.

[2] R. Ruffini and S. Bonazzola, Phys. Rev. **187** (1969) 1767;

[3] M. Colpi, S.L. Shapiro and I. Wasserman, Phys. Rev. Lett. **57**, (1986) 2485.

[4] J.D. Breit, S. Gupta and A. Zaks, Phys. Lett. **140B**, (1984) 329;

[5] P. Jetzer, "Dynamical Instability of Bosonic Stellar Configurations," Fermilab preprint 66-A, May 1988.