Fermi National Accelerator Laboratory

Fermilab-Pub-87/120-T
August, 1987

The Sphaleron Strikes Back

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Abstract The issue of sphaleron induced baryon decay, and various paradoxes related to the instanton method of computation are addressed. By various examples we argue that there is no contradiction between the instanton estimates and sphaleron estimates, and argue that for the electroweak theory these estimates correspond to different approximations for distinct phenomena. We also investigate numerically the nature of the classical decay of a sphaleron in the 1+1 Abelian Higgs model.

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

1 Introduction

Recent investigations into the standard model have suggested the possibility of significant violations of baryon number at temperatures near 1 TeV and above.[1-3] Baryon number violation is caused by the *winding* of the weak gauge fields, the two being related by the anomaly:[4]

$$B(t_2) - B(t_1) = \frac{n_f g^2}{32\pi^2} \int_{t_1}^{t_2} dt\ d^3x\ \mathrm{tr} F\tilde{F} = \frac{n_f g^2}{8\pi^2} \int_{t_1}^{t_2} dt\ d^3x\ \mathrm{tr} \mathbf{E} \cdot \mathbf{B}. \qquad (1)$$

Each time the fields wind, baryon number is changed by n_f units where n_f is the number of families of quarks and leptons. The difference of baryon number and lepton number, however, is exactly conserved.

Creating these gauge fields costs energy, and the classical potential-energy barrier for winding the gauge fields[1] is of order $E_0 \sim M_w/\alpha_w$. At zero temperature, such processes can occur only by quantum tunneling and so are exponentially suppressed by the Euclidean action $S/\hbar \sim \Delta E\ \Delta t/\hbar \sim 1/\alpha_w$. Such tunneling may be analyzed using instantons and, in the classic analysis of t' Hooft,[5] baryon number violation at zero temperature was estimated to be so small that it is unlikely to have ever occurred in the lifetime of the observed universe. At finite temperature, however, one can pass over the barrier *classically* with a Boltzmann factor $\exp(-\beta E_0)$. In a classically-allowed transition, there is no exponential suppression due to quantum tunneling. It is to this possibility that previous work has been addressed. The barrier has been identified with a classically unstable, static solution to the equations of motion known as the sphaleron.[1] Estimates of the rate at which this barrier is crossed suggest that baryon-number violation is significant enough to easily dissipate any baryon excess in a universe with $B - L = 0$.

Though the idea sounds simple once proposed, the formalism is more confusing. When working in Euclidean space, the only configurations which at first sight seem to have something to do with winding are the instantons.[6] Shuryak,[7] and Gross, Pisarski, and Yaffe,[8] have extended the analysis of instantons to finite temperature, and the Euclidean action of *any* configuration which winds once in Euclidean time is still bounded below by $2\pi/\alpha_w$. Thus, t' Hooft's conclusion appears unchanged: the rate of baryon number violation is essentially zero. Ellis, Flores, Rudaz, and Seckel[9] have recently put forward this argument in the case of baryon-number violation in electroweak theory.

Our main goal will be to demonstrate that the leap to this conclusion is misplaced. In a previous work,[3] we examined the formalism of how a dynamic process, the winding of the gauge fields, could be related to the expansion of the path-integral about a *static* configuration — the sphaleron. In the current paper, we shall examine the relation of these processes to instanton physics. We conclude that instanton estimates are not relevant to this phenomenon.

Gross, Pisarski, and Yaffe were concerned with the calculation of equilibrium quantities, such as the dependence of the free energy on θ_{QCD}. Equilibrium quantities requiring winding are indeed exponentially suppressed. Time-dependent correlations, however, are more subtle. We shall investigate in detail a simple toy model where, though winding is suppressed in Euclidean time, it is unsuppressed when analytically continued to real time. We shall see that real-time winding is unsuppressed because it comes from the sector with *zero* Euclidean winding number.

This demonstration will explain the appearance of real-time winding in certain operators, such as time-dependent correlations of baryon number, which can be non-zero without Euclidean winding. But there is an alternative way to investigate baryon-number violation which *requires* Euclidean winding.[10] A process which winds the gauge fields once will emit a member of each weak fermion doublet. In a theory with one generation of quarks and leptons, for example, three quarks and a lepton will be emitted. If there is significant baryon-number violation, then there should be significant S-matrix elements involving the appearance of three quarks and a lepton. But one can show that such an amplitude $\langle qqql \cdots \rangle$ can only receive a contribution from configurations with non-trivial Euclidean winding, and so, it seems, should be exponentially suppressed.

As we shall see, this argument is flawed because the process mediated by the sphaleron is a classical one and so involves a large number of quanta. To cross the barrier, the fields must at some time configure themselves into a physical sphaleron with energy $\sim M_w/\alpha_w$. The size of a sphaleron is $r \sim 1/M_w$, and so the typical Fourier component is $k \sim M_w$. When the sphaleron decays, the momenta of particles in the final state will be typically M_w. The sphaleron will therefore decay into $\sim 1/\alpha_w$ W's and Z's, producing the quarks and lepton as a side-effect due to the anomaly.

The amplitudes of interest are then of the form $\langle qqql\ A^{1/\alpha}\rangle$ rather than simply $\langle qqql\rangle$. We shall argue that instanton estimates to amplitudes break down in the classical, many-quanta limit where the sphaleron estimates are made. By attempting to extrapolate instanton estimates to this limit, we shall see qualitatively how they can yield unsuppressed amplitudes. Conversely, by attempting to extrapolate the sphaleron estimates to the few-quanta limit, we shall see qualitatively how they yield the instanton suppression. We conclude that the inclusive rate for baryon-number violation is unsuppressed.

In section 2, we examine the toy system of a quantum-mechanical pendulum at high temperature to establish our first claim that Euclidean winding is not directly related to real-time winding. In an exactly soluble version of this model, we shall see how winding that is suppressed in Euclidean time becomes unsuppressed when analytically continued to real time. We also discuss the physics of unsuppressed winding in a model with more than just a single degree of freedom: the sine-Gordon model on a finite ring in 1+1 dimensions.

In section 3, we turn to the argument that S-matrix elements that violate baryon number should be suppressed. A simple problem in integration highlights the breakdown of instanton estimates in the many-quanta limit. Using coherent states, we then study the few-quanta limit of sphaleron estimates.

In section 4, we address two other issues concerning the sphaleron estimate: electric damping and thermal collisions.

In section 5, we investigate the classical evolution of the decay of the sphaleron. This yields a specific example of a solution to the Minkowski equations of motion which winds. Our arena for this investigation will be the Abelian Higgs model in 1+1 dimensions, which is numerically more tractable than the Weinberg-Salam model in 3+1 dimensions. Our interest is to check if there is anything singular or sick about this evolution. We find that the transition does wind the fields once but has some interesting structure peculiar to 1+1 dimensions.

In section 6 we offer our conclusions.

We end this introduction with a brief review of the sphaleron solution and the corresponding estimates of baryon-number violation. The particular sphaleron solution found by Klinkhamer and Manton[1] is identified as the barrier between neighboring vacua for the following reasons: (1) it is a static, unstable solution to the equations of motion, (2) there exists a path through configuration space from one vacua to the next for which the sphaleron is the point of maximum potential energy, and (3) moving from one vacuum to the sphaleron changes baryon number by exactly one half the amount of moving from one vacuum to the next. The last is determined by computing the Chern-Simons charge of the sphaleron:

$$Q = n_f \frac{g^2}{32\pi^2} \int d^3x K^o \tag{2}$$

where the Chern-Simons current is

$$K^\mu = \epsilon^{\mu\nu\rho\sigma}(F^a_{\nu\rho}W^a_\sigma - \frac{2}{3} g \epsilon_{abc} W^a_\nu W^b_\rho W^c_\sigma). \tag{3}$$

This charge is related to baryon-number violation by $\Delta B = \Delta Q$.

The sphaleron is studied in pure $SU(2)$ theory with the Weinberg angle treated as a perturbation. The size of the sphaleron is $\sim 1/M_w$ and its energy is

$$E_{sp} = 2M_W \frac{A(\lambda/g^2)}{\alpha_W} \tag{4}$$

where $A \sim 2$. This puts E between 8 and 14 TeV at zero temperature depending on the Higgs self-coupling λ. Using a Boltzmann factor, the rate per unit volume for crossing the barrier might therefore be of the order

$$R \sim T^4 \exp\left(-E_{sp}(T)/T\right) \tag{5}$$

where we replace M_w by the effective temperature-dependent mass $M_w(T)$ of finite-temperature field theory. As one approaches the critical temperature T_c of Weinberg-Salam, $M_w(T) \to 0$, the size of the sphaleron becomes infinite, the classical energy barrier becomes zero, and the estimate (5) becomes order 1.

A direct computation[3] of this rate is possible by weak coupling methods in the temperature range $M_w(T) \ll T \ll M_w(T)/\alpha_w$ for the case $\lambda \sim g$. In this range, the simple formula (5) is reduced by several orders of magnitude, but the rate is nonetheless sufficiently large to be relevant for cosmology. At higher temperatures, it is difficult to conclude anything by direct computation since infrared divergences of the finite temperature theory render weak coupling methods almost useless. In fact, in the symmetric phase above the critical temperature, there is no sphaleron solution. Nevertheless, we put forward arguments in Ref. 3 that, at very high temperatures, the rate may go as

$$R \sim \alpha_w^p T^4 \tag{6}$$

where p is a number, $p \sim 3 - 4$. This is in contradistinction to instanton estimates, which at first sight would seem most valid and familiar in the symmetric phase.

2 Real-Time vs. Euclidean Winding

2.1 The Pendulum

We now turn to our first investigation of the relation between winding in Euclidean time and winding in real-time. Our model for this study will be the quantum mechanics of a rigid pendulum, where the pendulum may swing all the way around in a circle. This model has many of the paradoxical features of the electroweak theory. At zero temperature, in the presence of the gravitational potential shown in Fig. 1, the pendulum can only wind about by quantum tunnelling; the rate may be calculated by instanton analysis. At finite temperature, the pendulum is given an ensemble of energies weighted by $\exp(-\beta E)$. At high temperatures, large compared to the potential difference between the top and bottom of the orbit, we expect that the pendulum will wind around with probability close to one. On the other hand, in Euclidian space, a solution which winds must have a singular high-temperature limit since it is required to do so in Euclidean time $\beta \to 0$. We shall show that the Euclidian action for such a winding solution grows as $2\pi^2 T/\hbar$ in the high-temperature limit, and therefore the instanton is exponentially suppressed. Our purpose is to show how the formalism nonetheless reproduces one's physical intuition of a rapidly-rotating pendulum.

A pendulum is described by a spatial coordinate x constrained to a circle, so that points are identified modulo 2π. We shall allow the range of x to be $-\infty < x < \infty$ so that we can follow the pendulum as it wraps around several times, but we must remember that the physical coordinate is identified modulo 2π. We shall keep track of $\hbar$ in this section to help identify the classical limit.

Note that the sphaleron for the pendulum is given by

$$x_{sp}(t) = \pi. \tag{7}$$

This static, unstable solution is the top of the swing of the pendulum. The Boltzmann factor for crossing the barrier is just $\exp(-\beta E_{sp})$.

Let us now suppose that there are no external forces such as gravity acting on the pendulum (or that the temperature is high enough that these can be ignored) so that the action is simple and quadratic:

$$S_E = \int_0^{\hbar\beta} d\tau \; \frac{1}{2}\dot{x}^2 \tag{8}$$

where we have chosen units in which the moment of inertia is 1. So, taking the winding number n of a path to be

$$n = \frac{1}{2\pi}\int_0^{\hbar\beta} d\tau \; \dot{x}, \tag{9}$$

we can write the partition function as

$$Z = \sum_n \int [\mathcal{D}x]_n \exp(-S_E/\hbar) \tag{10}$$

where in each term we integrate only over paths with winding number n. The instanton solutions and actions are

$$x_n(\tau) = 2\pi n\tau/\hbar\beta \qquad S_n = 2\pi^2 n^2/\hbar\beta. \tag{11}$$

Let us now modify the system by adding a θ term which couples to the winding number:

$$S_E = \int_0^{\hbar\beta} d\tau[\frac{1}{2}\dot{x}^2 + i\hbar\frac{\theta}{2\pi}\dot{x}]. \tag{12}$$

The advantage of this simple system is that the action is quadratic, so we can obtain exact results for quantities of interest.

Now let us consider a measure of the real-time rate at which the pendulum wraps around the circle:

$$A(t) \equiv \langle \mathcal{T} \left[\left(\frac{x(t) - x(0)}{2\pi}\right)^2\right] \rangle \tag{13}$$

where the $\mathcal{T}$ denotes time-ordering. When the temperature is large, we expect this last quantity to be given by its classical value. Classically, the ordering prescription is irrelevant and the equipartition theorem gives $\frac{1}{2}\bar{v^2} = \frac{1}{2}T$ so that

$$A(t) = \frac{t^2 T}{4\pi} + o(\hbar). \tag{14}$$

The dependence of the free energy on θ, however, is measured by

$$\frac{\partial^2 F}{\partial \theta^2} = \langle n^2 \rangle = A(-i\hbar\beta). \tag{15}$$

This quantity is exponentially small as can be seen explicitly by replacing, in each winding-number sector, $x(\tau)$ by $x_n(\tau) + \delta x(\tau)$ where δx is periodic. The partition function then factorizes exactly into

$$Z = \left(\sum_n \exp(-2\pi^2 n^2/\hbar^2\beta) \exp(in\theta)\right) \int [\mathcal{D}\delta x] \exp\left(-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \; \frac{1}{2}(\delta\dot{x})^2\right). \tag{16}$$

The expectation of n^2 is then just

$$\frac{\sum n^2 \exp(-2\pi^2 n^2/\hbar^2\beta) \exp(in\theta)}{\sum \exp(-2\pi^2 n^2/\hbar^2\beta) \exp(in\theta)} = 2\exp(-2\pi^2/\hbar^2\beta)\cos\theta + o\left(e^{-4\pi^2/\hbar^2\beta}\right), \tag{17}$$

and so $A(-i\hbar\beta)$ is exponentially suppressed.

Let us now, by calculating $A(t)$ for arbitrary t, see how the unsuppressed classical result is consistent with the suppressed value of $A(-i\hbar\beta)$. We have seen that the contributions from $n \neq 0$ will be exponentially suppressed, so let us concentrate on the $n = 0$ sector:

$$A(t) = \langle \mathcal{T} \left[\left(\frac{\delta x(t) - \delta x(0)}{2\pi}\right)^2\right] \rangle \; + \; o\left(e^{-2\pi^2/\hbar^2\beta}\right). \tag{18}$$

We need the propagator $\langle \mathcal{T}\delta x \delta x \rangle$ in a free theory at finite temperature. This is well known to be

$$\frac{i}{k^2 - (m/\hbar)^2 + i\epsilon} + \frac{1}{\exp(\hbar\beta|k_0|) - 1}\delta\left(k^2 - (m/\hbar)^2\right) \tag{19}$$

where we have temporarily included a mass as an infrared cut-off. In configuration space, this gives

$$\langle \delta x(t_1)\delta x(t_2)\rangle = \frac{\hbar}{2m}\left[\frac{\exp(im|t_1 - t_2|/\hbar)}{\exp(\beta m) - 1} - \frac{\exp(-im|t_1 - t_2|/\hbar)}{\exp(-\beta m) - 1}\right]. \tag{20}$$

So

$$A(t) = \frac{\hbar}{4\pi^2 m}\left[\frac{\exp(-im|t|/\hbar)-1}{\exp(-\beta m)-1} - \frac{\exp(im|t|/\hbar)-1}{\exp(\beta m)-1}\right] + o\left(e^{-2\pi^2/\hbar^2\beta}\right). \quad (21)$$

We can now safely take the cut-off m to zero:

$$A(t) = \frac{1}{4\pi^2}\left[\frac{t^2}{\beta} + i\hbar t\right] + o\left(e^{-2\pi^2/\hbar^2\beta}\right) \quad (22)$$

The classical and high-temperature limits are the same and reproduce the previous result $A(t) \approx t^2 T/4\pi$. Moreover, $A(-i\hbar\beta) = o\left(e^{-2\pi^2/\hbar^2\beta}\right)$ as it should.

The moral is that large real-time winding comes from the sector with Euclidean winding-number zero. The fluctuations in Euclidean time must be small since otherwise we would find suppression due to a large Euclidean action; in the case at hand, $|A(-i\tau)| \leq \hbar^2\beta/4\pi^2$ for $0 \leq \tau \leq \hbar\beta$. When analytically continued to real-time, however, the fluctuations become large.

2.2 Interpreting the Instanton

The instanton is certainly related to the communication of the region $x \sim 0$ to the region $x \sim 2\pi$. We have seen that the instanton is suppressed but the communication is not. Why should this be so?

We can address this question by considering the Euclidian path integral which takes us from $x = x_o$ to $x = x_o + 2\pi$, integrating over x_o. This quantity is precisely the contribution to the partition function from the winding-number 1 sector of the theory.

$$Z_1 = \int dx_o < x_o + 2\pi \mid \exp(-\beta H) \mid x_o >= \int_{x(\beta)=x(0)+2\pi} \exp(-S_E) \quad (23)$$

We know Z_1 to be small $\sim \exp(-2\pi^2/\beta)$ at high temperature. (We have returned to the convention $\hbar = 1$ for the remainder of this paper.)

If we analyze Z_1 in terms of energy eigenstates, this result is perhaps a bit surprising. Expressed in terms of energy eigenstates, Z_1 becomes

$$Z_1 = \sum_E \int dx_o \; \Psi_E^*(x_o + 2\pi)\Psi_E(x_o) \; \exp(-\beta E). \quad (24)$$

We expect that in general there will be a large overlap between high energy states which have coordinates differing by 2π. The reason the sum in Eq. 24

nonetheless gives a small result is because the phases interfere destructively. For instance, in the absence of external forces,

$$\Psi_E(x) = \frac{1}{\sqrt{L}} \exp(ikx) \tag{25}$$

where k is the spatial momentum, $E = k^2/2$, and we have temporarily restricted x to $-L/2 < x < L/2$, $L \to \infty$. Notice that the overlap probability

$$P = \left| \int dx \ \Psi^*_E(x + 2\pi)\Psi_E(x) \right|^2 \tag{26}$$

is one. But now consider Eq. 24. The sum over E becomes an integral over k which may be evaluated explicitly:

$$Z_1 = L \int dk \ \exp(i2\pi k) \exp(-\beta k^2/2). \tag{27}$$

If we normalize by the zero-winding contribution, we get

$$\frac{Z_1}{Z_0} = \exp(-2\pi^2/\beta). \tag{28}$$

The action integral has been suppressed by cancelation of the varying phases of the overlap amplitude. The width of the interval in k over which this cancelation occurs is $k \sim \sqrt{T}$.

2.3 The Sine-Gordon Model

In the analysis of the previous sections, we have concerned ourselves with a theory having only one degree of freedom. In the case of a quantum field theory we expect that the other degrees of freedom will play a role. In this section, we examine such a system by considering the sine-Gordon model in 1+1 dimensions on a finite-length ring. We shall see that this system also has instantons which are suppressed in the high-temperature limit, but real-time winding is controlled by a Boltzmann factor which is unsuppressed. Though we cannot solve this system exactly as we did for the pendulum, we shall give physical arguments for the lack of exponential suppression.

The sine-Gordon model has the action

$$S = \frac{1}{\lambda} \int d^2x \left(\frac{1}{2}(\partial\phi(x))^2 + m^2(1 - \cos(\phi(x)) \right) \tag{29}$$

In this equation the coupling constant is λ, which we shall take as small. The size of the system will be taken as L in the spatial direction and β in the Euclidian time direction, corresponding to a finite temperature.

The sine-Gordon theory is labelled by topologically distinct vacua for which

$$\langle\phi(x)\rangle = 2\pi N \tag{30}$$

where N is an integer. In infinite volume, these different vacua cannot be connected with instanton solutions.

We can estimate the instanton contribution to the action in finite volume. First consider the limit where the temperature is small, $T << m$. In this case we expect that the time extent of the instanton is $\delta t \sim 1/m$. The spatial extent is $\delta x \sim L$ since ϕ must change by $2\pi\delta n$ over all of space. So

$$S_{inst} \sim \frac{mL}{\lambda}, \qquad T << m. \tag{31}$$

In the high temperature limit $T >> m$, the instanton must change in Euclidean time $\delta t \sim \beta$, so that here we have

$$S_{inst} \sim \frac{L}{\beta\lambda}, \qquad T >> m. \tag{32}$$

In any range of temperature, the instanton contribution to the topologically non-trivial sector of the path integral is suppressed by $\exp(-S_{inst})$ which is exponentially small in the limit of small λ but finite L.

Once again, though instantons are exponentially suppressed, we expect the transition rate to be unsuppressed at high temperature. The theory has kink/anti-kink solitons where the kink and anti-kink interpolate between the different ground-states. The energy of a kink is of order $E \sim m/\lambda$ so that the kink/anti-kink contribution to the partition function goes as

$$P \sim \exp(-2\beta m/\lambda). \tag{33}$$

At high temperature, we expect many kink/anti-kink pairs. These can change the average value of ϕ by moving around the ring as in Figs. 2 and 3, and the cost of producing them is just the Boltzmann factor of Eq. 33.

On average, of course, as many pairs will move around the ring one way as the other, and the average winding will be zero. This is because we have included no preference for the direction of winding. To do so, we need a chemical potential term

$$\mu N \sim \frac{\mu}{2\pi L} \int dx\ \phi(x). \tag{34}$$

In electroweak theory, μ would be the chemical potential of the baryons and leptons, and this term reflects the fact that every winding creates baryons and leptons and so costs energy μ. In the case at hand, the chemical potential creates a force which pushes the kinks in one direction around the ring and the anti-kinks in the other.

3 Euclidean Winding Revisited

3.1 The Instanton and Many Quanta

Rather than examining correlations of winding number, as in the previous section, we now examine S-matrix elements which explicitly show the destruction or creation of baryons. Such matrix elements can only get contributions from non-trivial Euclidean winding, and so the Euclidean action is always bounded below by $2\pi^2/\alpha$. It therefore seems that all such S-matrix elements must be suppressed, regardless of the temperature.

As mentioned in the introduction, the flaw in this argument is that the decay of the sphaleron involves a large number ($\sim 1/\alpha$) of quanta; the relevant matrix elements are schematically of the form $\langle qqql\ A^{1/\alpha}\rangle$. Formally, the estimate of an amplitude as $\exp(-S_{inst})$ requires the assumption that the current term $J \cdot A$ used for calculating Green's functions yield only a small perturbation on the action of the instanton. That assumption breaks down if J is "big." As we shall see, this occurs for amplitudes involving a large number of quanta.

Consider a simple example of saddle-point approximations from calculus (that is, 0+0 dimensional space-time):

$$I_n \equiv \int_{-\infty}^{+\infty} dx\ e^{-S(x)} x^{2n} \qquad \text{where} \qquad S(x) = g^{-2}\left[1 + (x - g^{-1})^2\right]. \tag{35}$$

The minimum of S is $S_{min} = 1/g^2$ and occurs at $x_o = 1/g$. This is intended to be analogous to the instanton case.

The integrand of I_0 is bounded above by $\exp(-S_{min})$, which disappears in the small g limit, and I_0 is correctly estimated as having this exponential dependence. But let us now consider I_n for large n. Specifically, take $n = 1/g^2$. We now need

$$I_{1/g^2} = \int_{-\infty}^{+\infty} dx\; e^{-[S(x)-g^{-2}\ln x]}. \tag{36}$$

The second term in the exponent is order $1/g^2$ and cannot be treated perturbatively. Plugging in $x = x_o$, we know the integrand can be at least as big as

$$e^{-1/g^2}\left(\frac{1}{g}\right)^{2/g^2} \sim \left(\frac{1}{g^2}\right)!, \tag{37}$$

and so the exponential dependence of I_{1/g^2} is $\gtrsim (1/g^2)!$. So I_{1/g^2} does not vanish in the small g limit — quite the opposite.

One can continue to play with the dependence on g if one interprets I_n as analogous to the amplitude for scattering of n quanta into n quanta. To get a rate, we should square I_n and divide by $n!$ for the final particles. For $n = 1/g^2$, this gives $\gtrsim (1/g^2)!$.

One may follow through the same sort of argument for the g dependence of instanton amplitudes in electroweak theory. The usefulness of the instanton estimate is even more obscured by the complications of the momentum dependence of the amplitude, the necessity of analytically continuing the result to real-time, and the phase-space and initial-particle distribution integrals to be done. All of these issues are potentially fraught with subtleties.

The moral is that instanton methods are not effective for amplitudes involving many quanta, and we expect, based on the classical picture of the sphaleron, that these are precisely the amplitudes of interest.

3.2 The Sphaleron and Few Quanta

Having seen that instanton estimates can become unsuppressed in the limit of many quanta, we shall now complete our study of the relationship between instantons and sphalerons by showing that sphaleron estimates become suppressed in the limit of few quanta. The basic argument is that the number of quanta in a classical coherent state is Poisson distributed. The strength of events involving different numbers of particles should be roughly

$$P_n = \frac{\langle n\rangle^n}{n!}e^{-\langle n\rangle}. \tag{38}$$

For n small, this is suppressed by $\exp(-\langle n\rangle)$. Since $\langle n\rangle \sim 1/\alpha_w$, we find a suppression similar to the instanton one.

To understand the branching ratio P_n more formally, we shall briefly consider the description of the sphaleron as a coherent state.[11] The sphaleron is, to good approximation, a classical object. As such, it decays classically and its decay rate may be computed by following its classical evolution. The value of this rate depends on whether or not there is significant damping at the temperature of interest. Without damping, we expect that the rate is order $1/M_w$. With damping, this rate may be reduced by 2 to 3 orders of magnitude. Once the sphaleron has decayed, we have classical waves radiating away from the original position of the sphaleron. What is the probability that the total number of quanta in these waves will be measured as small (say 3 rather than $1/\alpha$)?

To address this question, let us write the classical final state in the language of quantum field theory using the coherent state representation:[12]

$$|\, out\rangle = e^{-\langle n\rangle/2}\exp\left(\int \frac{d^3k}{(2\pi)^3\sqrt{2E}}\ \Phi_{cl}(k)\ a^\dagger(k)\right)\,|\, O\rangle. \tag{39}$$

In this formula, E is the energy of the quanta created by $a^\dagger(k)$. $\Phi(k)$ generically represents the Fourier transform of the final-state classical field; we have ignored all indices and problems associated with gauge freedom (see Ref. 11 for details). The factor $\exp(-\langle n\rangle/2)$ normalizes the state, where the average number of quanta $\langle n\rangle$ is given by

$$\langle n\rangle = \int \frac{d^3k}{(2\pi)^3 2E}|\Phi(k)|^2. \tag{40}$$

The classical coherent state which we have written down is not an energy and momentum eigenstate. In the classical limit when $\langle n\rangle \to \infty$, the state has well defined energy and momentum. We work in a frame where

$$\int \frac{d^3k}{(2\pi)^3 2E}E|\Phi(k)|^2 = E_{sphal} \tag{41}$$

and where the average value of the spatial momentum vanishes.

The overlap of $| \, out \rangle$ with an n-particle state may now be computed as

$$
\begin{aligned}
|\langle out \mid k_1, k_2, \cdots k_n \rangle|^2 &= |\langle out \mid a^\dagger(k_1) \cdots a^\dagger(k_n) \mid O \rangle|^2 \\
&= e^{-\langle n \rangle} \frac{|\,\Phi(k_1)\,|^2}{(2\pi)^3 2E_1} \cdots \frac{|\Phi(k_n)|^2}{(2\pi)^3 2E_n}.
\end{aligned} \tag{42}
$$

Notice that since $|\Phi|^2 \sim 1/\alpha_W$, this branching fraction is not analytic in the weak coupling limit. Upon integrating over final three momenta, and remembering a $1/n!$ for identical particles in the final state, we see that the integrated branching probabilities are Poisson distributed as in Eq. (38).

The Poisson distribution for the final state is typical of a classical distribution of particles. The surprising issue here is how large $\langle n \rangle$ is, and the fact that this leads to a tremendous suppression of sphaleron decays into a small number of particles.

4 Other Issues

In this section, we cover a few other areas of possible confusion in the physics and formalism of the sphaleron approximation. We shall discuss the role of electric screening and thermal collisions on the transition.

4.1 Electric Screening

The first issue is electric screening, which has been suggested by Ellis, *et. al.* as a possible source of tremendous suppression for sphaleron processes. Electric screening plays an important role in instanton physics because, in order for the Euclidean winding $\int_0^\beta d\tau \; \mathbf{E} \cdot \mathbf{B}$ to be order 1 for very small β, one needs large electric and magnetic fields. These large electric fields are screened, and so the contribution of such a configuration is suppressed. This effect gives a contribution to the effective free energy of $\sim (RT)^2$ for instantons of size R.[8] If one were to plug in the sphaleron size $R \sim 1/M_w$, one would find a suppression of the form $\exp(-T^2/M_w^2)$.

However, as we have previously discussed, the winding does not happen in Euclidean time but in real time, and so the winding is not constrained to happen within time β. Winding can occur with small electric fields present for a long time rather than large electric fields present for a short time. How do damping effects in the plasma determine this? The sphaleron is a purely magnetic configuration and so is not itself electrically screened. The process which changes baryon number, however, is a dynamic one that evolves through the sphaleron. In the creation and decay of the magnetic fields of the sphaleron, electric fields must be produced, and these electric fields will be screened.

Indeed, we found in Ref. 3 that, for the temperatures we studied, such damping effects slow the sphaleron decay rate by a few orders of magnitude. (Specifically, we found that Landau damping was the most important effect. Landau damping has a similar origin to electric screening and is proportional to the size of the electric fields.) Since the decay of the magnetic configuration is damped, the electric fields produced are smaller and the time for the decay is longer. These effects cancel in the winding number $\int dt\ \mathbf{E}\cdot\mathbf{B}$. The suppression of the rate due to the longer decay time is *algebraic*, not exponential.

At very high temperatures, $T \gtrsim M_w(T)/\alpha_w$, the sphaleron is *magnetically* screened by the plasma. At these temperatures, however, the approximations which dictate that the system must create an approximate sphaleron in order to pass over the barrier break down. The system may be able to pass through small configurations which are unscreened (see Ref. 3 for details). In any case, the process proceeds quickly enough for $T < M_w(T)/\alpha_w$ to be cosmologically significant.

4.2 Thermal Collisions

We shall now examine the issue of thermal collisions. Suppose that, before a sphaleron decays, a collision with the thermal bath knocks it back over the barrier so that there is no net transition. (See Fig. 4.) It seems that counting the number of sphalerons might then overcount the number of net transitions.

To explore this possibility, we delineate four important length-scales of the problem: M_w^{-1}, $(gT)^{-1}$, $(\alpha_w T)^{-1}$, and $[\alpha_w^2 T \ln(T/M_w)]^{-1}$. M_w^{-1} is the size of the sphaleron; $(gT)^{-1}$ is the electric screening length and the scale for Landau damping; $(\alpha_w T)^{-1}$ is the magnetic screening length and $\alpha_w T \ll M_w$ is required for the validity of perturbation theory about the sphaleron[3]; and $[\alpha_w^2 T \ln(T?M_w)]^{-1}$ is the mean free path of Ws and Zs in the thermal bath. In Ref. 3, we found that we could analyze the sphaleron for $M_w \ll T \ll M_w/\alpha_w$. In this range, the mean free path is the longest scale of the four discussed. At the lower end of the range, the decay time of the sphaleron is order M_w^{-1} and is shorter than the mean free path. At the higher end, damping is important and increases the decay time by a factor of order $(gT/M_w)^2$. In the limit $T \ll M_w/\alpha_w$, the decay time is still small compared to the mean free path. So, for the temperatures studied, thermal collisions during the decay of the sphaleron should not yield a significant change in the estimate of the transition rate. (We should emphasize that, in the range $M_w \ll T \ll M_w/\alpha_w$, the estimated rate for baryon-number violation is as large as 10^{10} times the expansion rate of the universe.[3] One would need a very significant effect to shut this process down.)

5 Classical Evolution of the Sphaleron

In this section, we turn away from instanton-related issues and investigate the classical evolution of the decaying sphaleron. We are interested to see if there is anything singular or otherwise bizarre about this evolution. The classical decay of the sphaleron is difficult to study in the Weinberg-Salam model beyond the analysis of small fluctuations. This is because it is difficult to solve the classical equations of motion for the sphaleron as it decays. At some time it may be necessary to embark on a computation of this problem in classical time evolution, but at present such an effort seems unwarranted. We have therefore chosen to study a problem which is more numerically tractable: the Abelian Higgs model in 1+1 dimensions.

The Abelian Higgs model in 1+1 dimensions is a theory with instantons.[13] It has been used as a model for sphaleron processes, and the one-loop computation of sphaleron-induced decay has been performed analytically.[14] We are interested in tracing the decay of the sphaleron after it has left the neighborhood of the sphaleron configuration. We shall do so classically. Classical physics can be misleading in 1+1 dimensions where quantum fluctuations dominate in the infrared. Our hope is that the classical decay in 1+1 will be analogous to the classical decay in 3+1, where the classical approximation is a good one.

We take as the action

$$S = \int d^2x \left(-\frac{1}{4}F_{\mu\nu}^2 + (D_\mu\Phi)^2 - V(\Phi) + \overline{\Psi}\gamma_\mu(i\partial^\mu + g\gamma_5 A^\mu)\Psi\right) \tag{43}$$

where

$$V(\Phi) = \lambda(\mid\Phi\mid^2 - \frac{c^2}{2})^2. \tag{44}$$

For computational reasons, we shall assume the system is finite and periodic in the spatial direction with length L. Note that the gauge field has been coupled axially to the fermions so that the fermion number current is anomalous:

$$\partial_\mu F^\mu = \partial_\mu(\overline{\Psi}\gamma^\mu\Psi) = -\frac{g}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu}. \tag{45}$$

where $\epsilon_{\mu\nu}$ is the two-dimensional antisymmetric symbol. The topological charge corresponding to the anomaly is

$$Q_{top} = -\frac{g}{4\pi}\int dx A_1(x) \tag{46}$$

where A_1 is the spatial vector potential. The combination of baryon number minus topological charge is conserved so that

$$\frac{d}{dt}(Q_F - Q_{top}) = 0. \tag{47}$$

A sphaleron solution for this theory has been constructed in Ref 14. It is

$$\Phi = ie^{-i\pi x/L}\frac{c}{\sqrt{2}}\tanh\frac{M_H x}{2} \tag{48}$$

and

$$A_1 = \frac{\pi}{gL}. \tag{49}$$

We here work in the gauge $A_0 = 0$, and the parameter M_H is given as

$$M_H^2 = 2\lambda c^2. \tag{50}$$

The topological charge of this sphaleron solution is 1/2. It is also possible to show that the sphaleron corresponds to an energy saddle point along a non-contractible loop which connects topologically distinct vacua of the theory.

We shall investigate the decay in the absence of fermions. It is then straightforward to solve the equations of motion numerically from initial conditions. We assume that the sphaleron describes the fields at $t = 0$ and then assume some small initial value for the time derivatives of the fields. We work in the sector of the theory where there is no external electric field. To check our results, we verify that the electric charge and energy are conserved to the accuracy permitted by the numerical evaluation.

We consider one such numerical simulation here. We work in units where $g = 1$ and have chosen the parameters $\lambda = 1$ and $c = 2$. The spatial extent of our box is $L = 24$. Fig. 5 shows our choice of the intial time derivative of Φ. Fig. 6 shows the energy density at t=0, 7, and 35. As one can see, the sphaleron indeed decays and spreads out through the box. Notice, however, the small bump that remains in the t=35 curve; this bump is persistent. It turns out to be the quasi-stable *breather* of real ϕ^4 theory.[15] It is interesting that the breather, originally discovered for a real scalar field, apparently remains quasi-stable when embedded in the Abelian Higgs model, a theory with a complex scalar filed.

The topological charge Q should change by one unit in the transition. Fig. 7 shows the topological charge evolved both forward and backward in time. The average value has indeed changed by one unit, but there are undamped oscillations of order one!

These oscillations are an interesting feature special to 1+1 dimensions. The reason for their appearance is that plane waves can carry topological density. Let us ignore, for the moment, the nonlinear debris such as the breather and consider a sphaleron that has decayed into asymptotic plane waves. In the classical 1+1 Abelian Higgs model, the gauge symmetry is broken and the A field has a massive, longitudinal mode. (This is not true of the quantum system.) Let us write the A field of the decayed sphaleron as a superposition of plane waves:

$$A_\mu(x) = \int \frac{dk}{2\pi} f_\mu(k) \cos\left(k \cdot x + \phi(k)\right). \tag{51}$$

Writing $f_\mu(k) = f(k)\lambda_\mu(k)$, where λ_μ is the longitudinal polarization vector, the oscillation of the topological charge Q is

$$\dot{Q} \sim g \int dx\, \tilde{F} \sim g M_w f(0) \cos(M_w t + \phi_o). \tag{52}$$

The size of the oscillations is then

$$\Delta Q \sim g f(0). \tag{53}$$

This is the source of the oscillations in Fig. 7. Note that the period of oscillations in this case is $2\pi/M_w = \pi$ as predicted. We believe that the variations in the amplitude are due to the interaction with the non-linear breather.

Let us consider the same analysis in 3+1 dimensions. We find

$$\begin{aligned} \dot{Q} &\sim g^2 \int d^3x \ \operatorname{tr} F\tilde{F} \\ &\sim g^2 \int d^3k \ \epsilon^{\mu\nu\rho\sigma} k_\mu f^a_\nu(\vec{k}) \tilde{k}_\rho f^a_\sigma(-\vec{k}) \cos\left(2\omega_k t + \phi(\vec{k}) + \phi(-\vec{k})\right) \end{aligned} \tag{54}$$

where $\tilde{k} = (\omega_k, -\vec{k})$ is the parity reflection of k. At large times t, the cosine will be highly oscillatory, and so $\dot{Q} \to 0$ at large times as long as f is smooth. Thus, unlike in 1+1 dimensions, any oscillations in 3+1 dimensions will damp away.

6 Conclusions

In this paper, we have argued that there is no contradiction between instanton and sphaleron estimates of baryon-number changing processes in the electroweak theory. These estimates are in fact complementary, being valid in different temperature regions. For the particular quantities of interest here, instantons provide useful estimates only at low temperatures. We have also considered a variety of model problems and have shown that the sphaleron analysis yields qualitatively and semi-quantitatively correct results.

To summarize, the sphaleron appears to provide a viable baryon-number changing mechanism, and in a range of temperatures, $M_w(T) \ll T \ll M_w(T)/\alpha_w$, the rate of baryon-number change may be reliably computed in a weak coupling limit.

Acknowledgements

We gratefully acknowledge useful conversations on these issues with W. Bardeen, J. Ellis, R. Flores, K. Freese, E. Kolb, E. Mottola, R. Pisarski, D. Seckel, M. Shaposhnikov, H. Thacker, and F. Wilczek. P. Arnold wishes to thank T. Banks, S. Dimopoulos, B. Ratra, and G. Starkman for discussions concerning the pendulum model and real-time vs. Euclidean winding. We also thank the hospitality of the Aspen Center for Physics where this work was completed.

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Figure Captions

Fig. 1: The gravitational potential felt by a simple circular pendulum.

Fig. 2: A single kink/anti-kink pair circles the ring, changing ϕ_{avg}.

Fig. 3: Multiple kink/anti-kink pairs move partway around the ring, changing ϕ_{avg}.

Fig. 4: The system passes over the barrier, collides with the thermal bath, and is knocked back, producing no net transition.

Fig. 5: The initial values of Re$\dot{\Phi}$ (solid) and Im$\dot{\Phi}$ (dashed) used in our simulation.

Fig. 6: The energy densities at t=0, 7, and 35.

Fig. 7: The topological charge as a function of time.

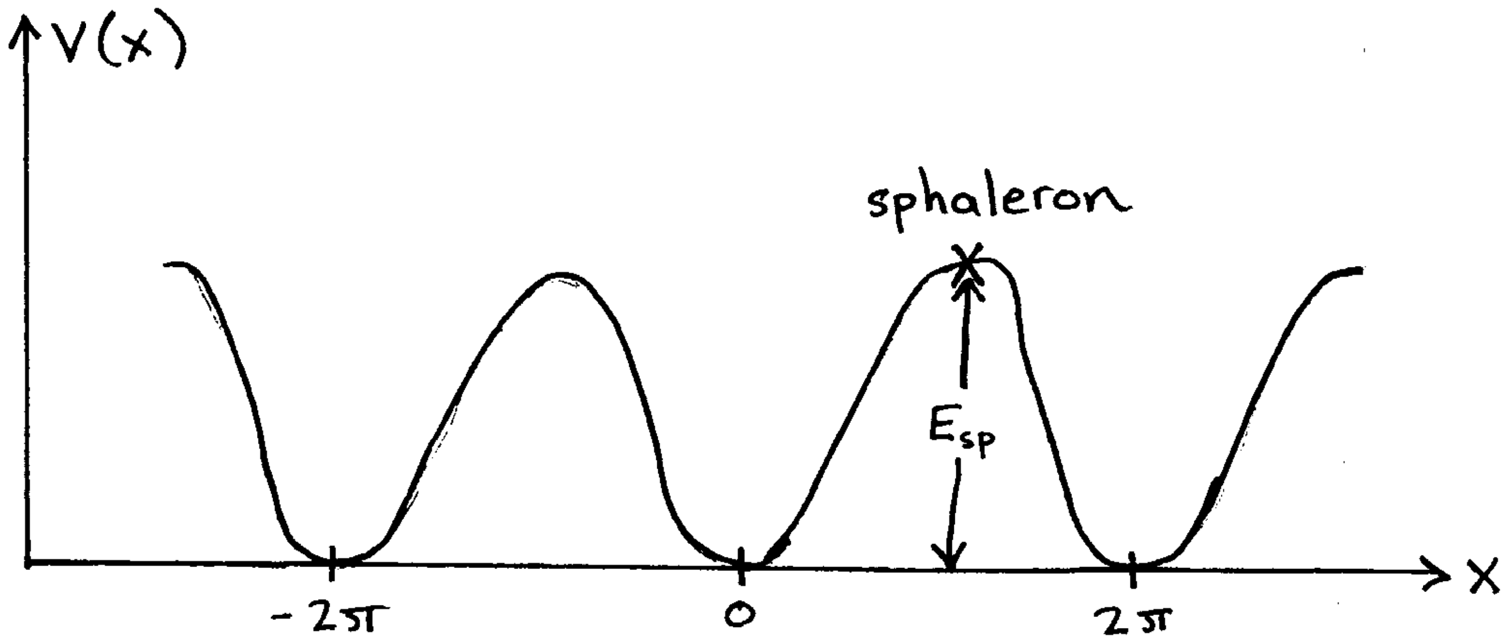

Fig. 1

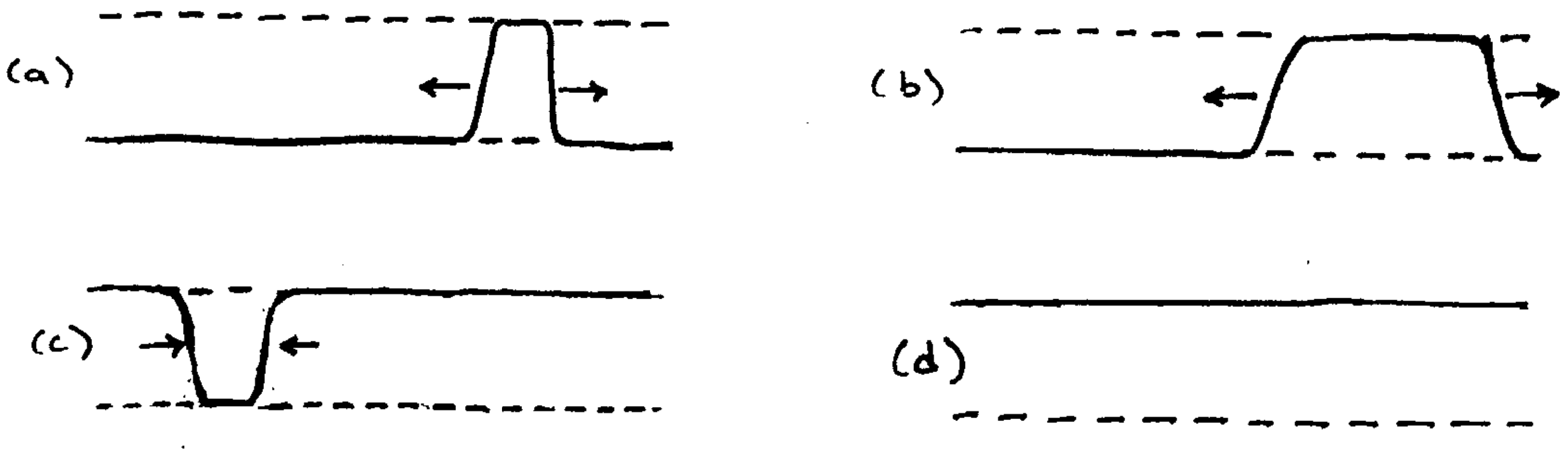

Fig. 2

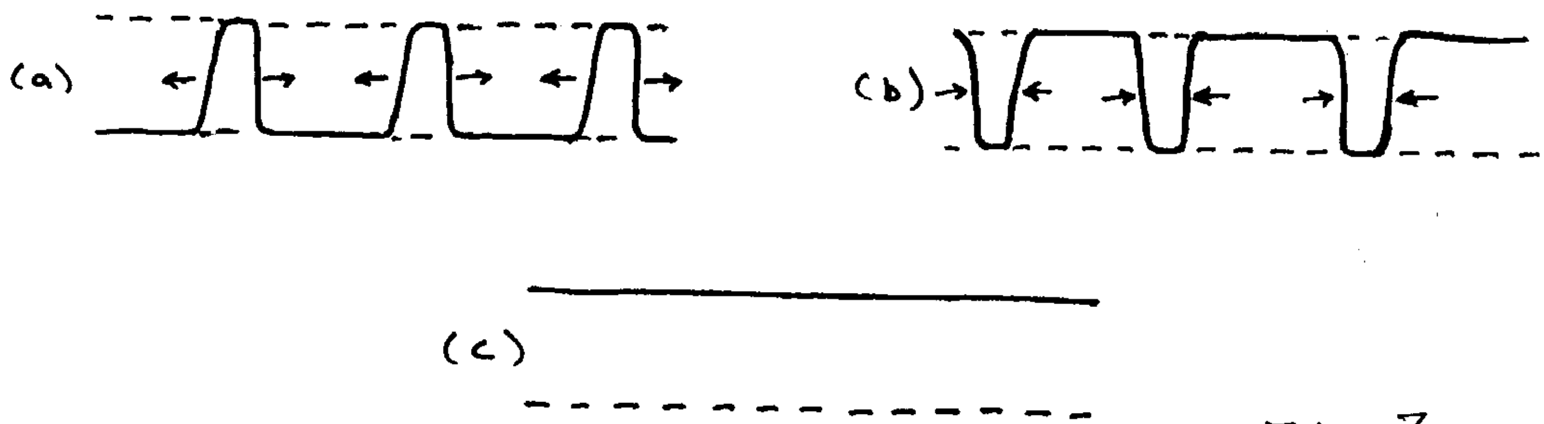

Fig. 3

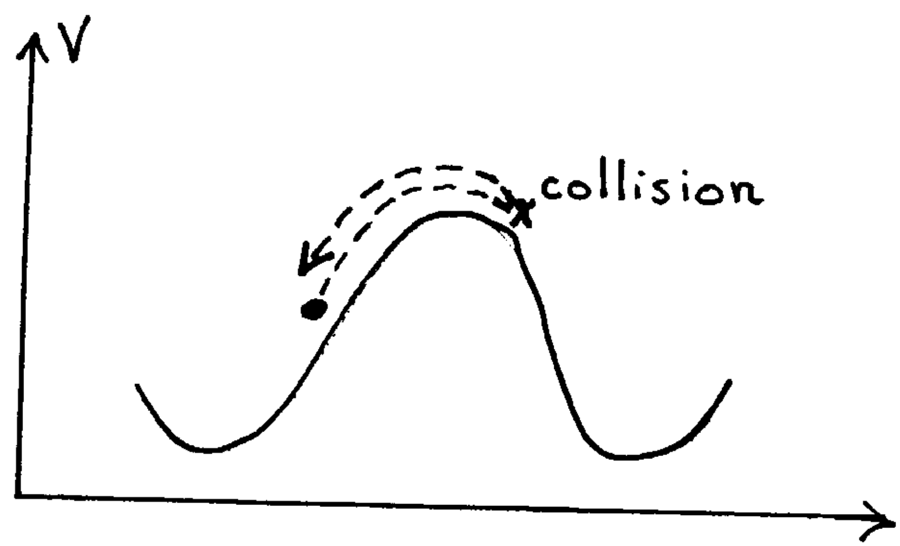

Fig. 4

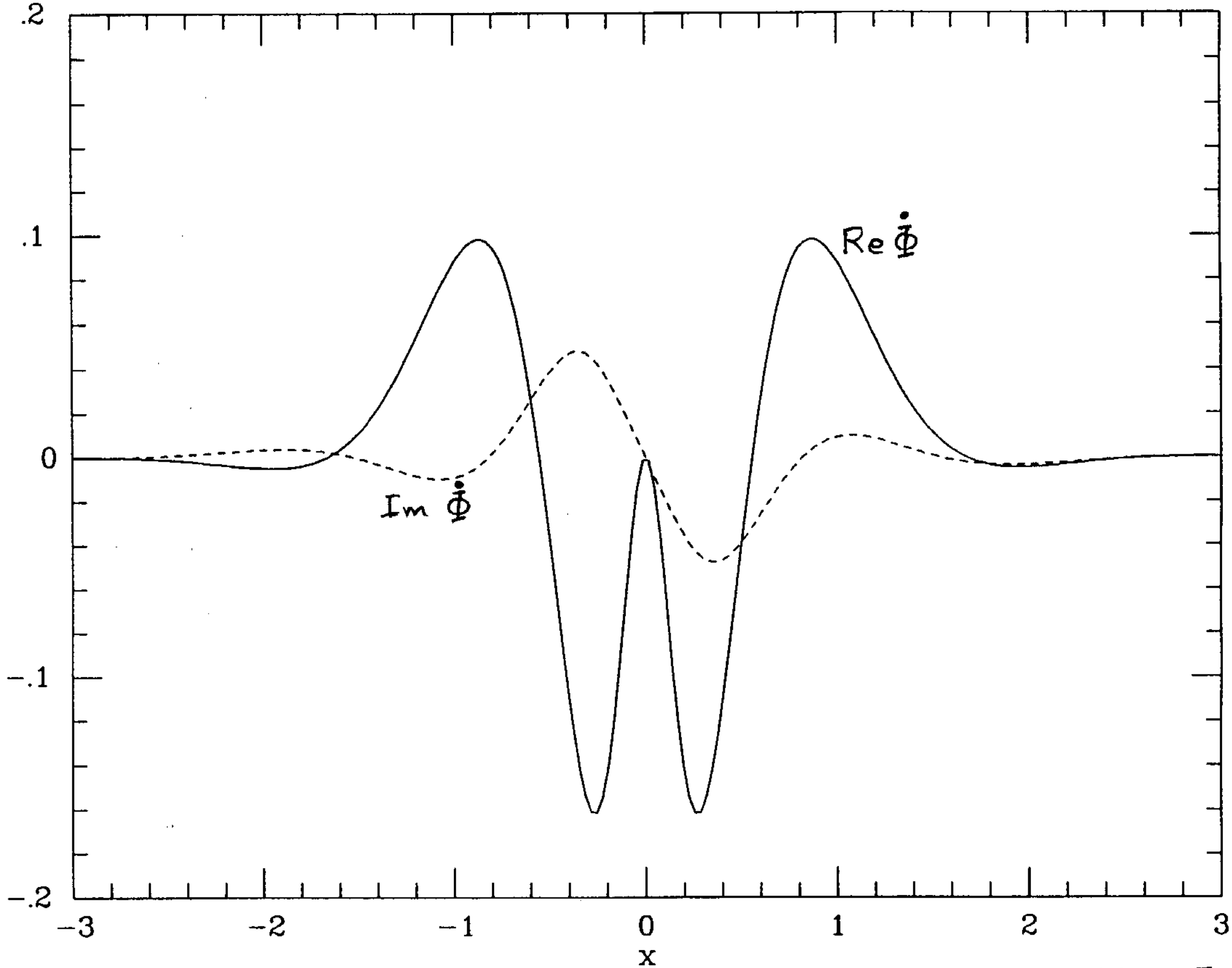

Fig. 5

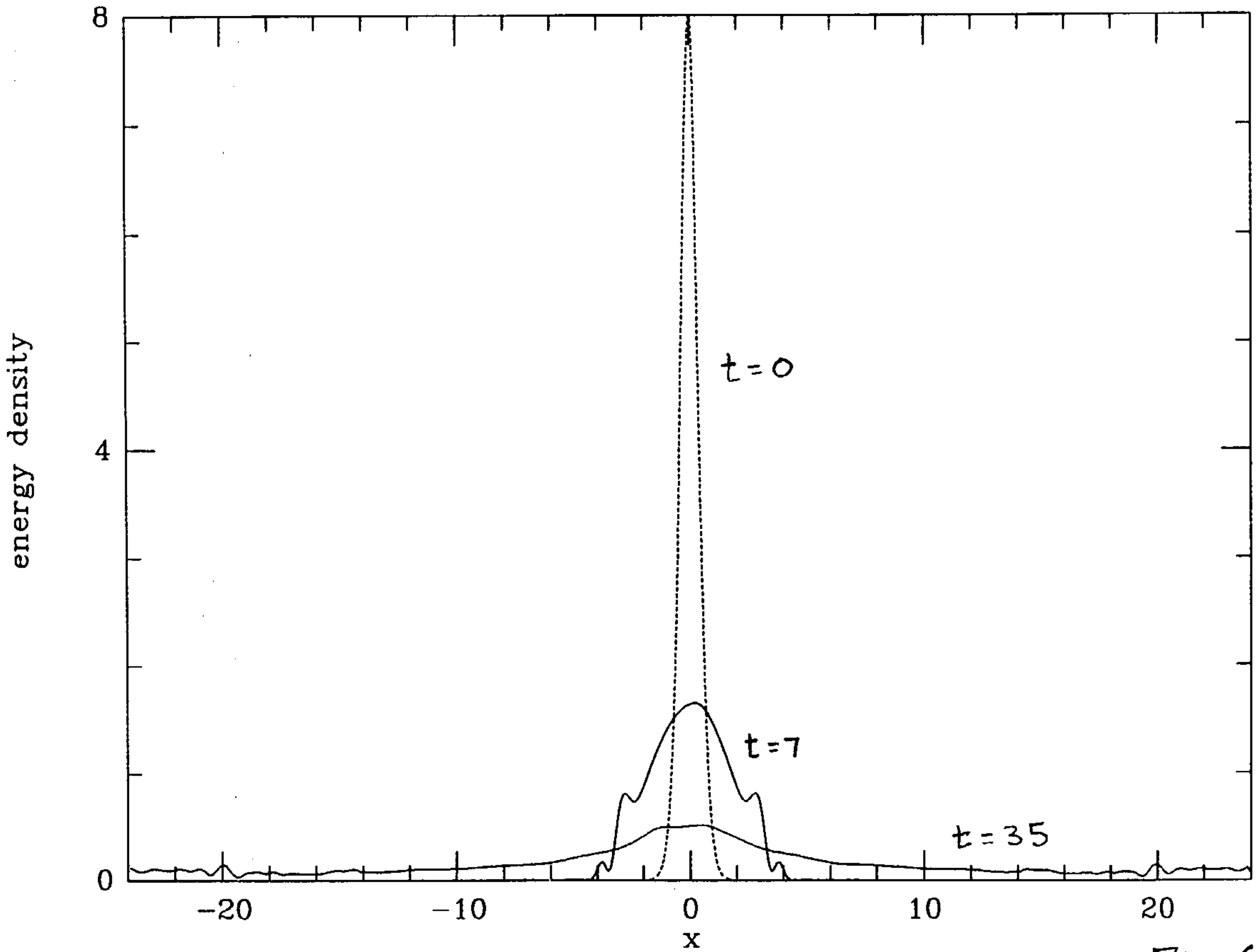

Fig. 6

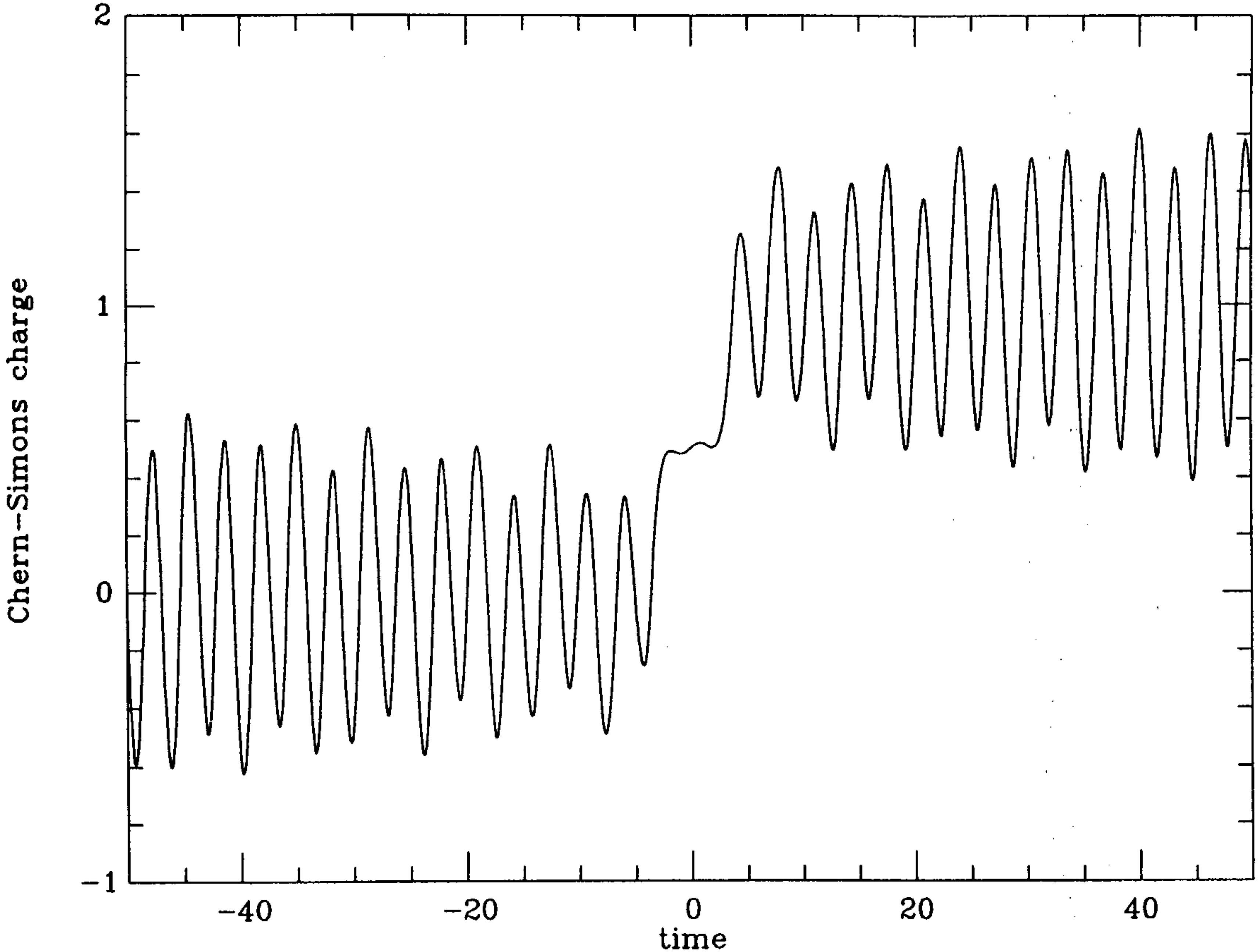

Fig. 7