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The B Parameter Beyond the Leading Order of $1/N$ Expansion

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ABSTRACT

We calculate the next to the leading $1/N$ corrections to the B parameter which measures the size of the $\Delta S = 2$ matrix element $\langle \bar{K}^o \mid (\bar{s}d)_{V-A}(\bar{s}d)_{V-A} \mid K^o \rangle$. These corrections turn out to be small which assures the validity of the $1/N$ expansion. We find $B = 0.70 \pm 0.07$ to be compared with the leading order value $B = 0.75$. We clarify the differences between these results and the results of the usual chiral perturbative calculations which give $\mid B \mid \approx 1/3$ at the tree level but appear to signal the breakdown of chiral perturbation theory at the one-loop level.

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**Chargé de Recherches au FNRS.

One of the most important parameters in the K-meson system is the so-called B parameter which in the standard model measures the size of the $K^o - \bar{K}^o$ mixing. As such, it plays a crucial role in the evaluation of the $K_L - K_S$ mass difference and in predicting the size of the CP violation present in the standard model.

The scale independent parameter B is defined by

$$B = B(\mu^2)[\alpha(\mu^2)]^{-a} \qquad\qquad a = \frac{9(N-1)}{N(11N-6)} \tag{1}$$

where α is the QCD running coupling constant, N is the number of colors, μ is a normalization scale and the function $B(\mu^2)$ is defined as follows

$$\langle \bar{K}^o \mid Q(\mu) \mid K^o \rangle \equiv \langle \bar{K}^o \mid (\bar{s}d)_{V-A}(\bar{s}d)_{V-A} \mid K^o \rangle \equiv B(\mu^2)\frac{16}{3}F_K^2 m_K^2. \tag{2}$$

Here $V-A$ refers to $\gamma_\mu(1-\gamma_5)$, F_K is the kaon decay constant ($F_K = 120MeV$) and m_K is the kaon mass. The second factor on the r.h.s. of Eq.(1) represents the μ dependence of the Wilson coefficient function associated with the $\Delta S = 2$ operator $Q(\mu)$ in Eq. (2). For $N = 3$ one obtains the result of ref. [1].

Due to the non-perturbative nature of the problem, it was a common practice until recently to evaluate $B(\mu^2)$ in simple models which led to a variety of values for B ranging from 1/3 to 2 [2]. Among these the vacuum insertion estimate of ref. [3] ($B(\mu^2) = 1$) and the $PCAC - SU(3)$ estimate of refs. [4,5] ($\mid B \mid \simeq 1/3$) are the ones best known.

During the last two years efforts have been made to calculate the parameter B in QCD using the $1/N$ expansion [6], the QCD sum rules [7–10] and the lattice approach [11,12]. In the leading order of the $1/N$ expansion one finds $B = 3/4$ [6,13] and a similar value in the quenched version of the lattice QCD ($B = 0.7 \pm 0.3$) has recently been obtained [12]. There is still no consensus on the value of B resulting from the QCD sum rules. Whereas in ref. [7] the value $B = 0.33$ has been obtained using the two-point functions, in refs. [8–10] higher values of B($B = 1.2 \pm 0.1, B = 0.58 \pm 0.16$ and $B = 0.84 \pm 0.08$ respectively) have been found on the basis of the three-point functions.

In this letter we will extend the calculation of ref. [6] beyond the leading order of $1/N$ expansion using the approach developed in refs. [14,15]. In this approach the matrix element $\langle \bar{K}^o \mid Q(\mu) \mid K^o \rangle$ can be calculated using a truncated chiral

Lagrangian describing the low energy strong interactions of pseudoscalar mesons [14–16].

$$L_{tr} = \frac{f_\pi^2}{4}\Big[tr(D_\mu U D_\mu U^+) + r\ tr(m(U+U^+)) - \frac{r}{\Lambda_\chi^2} tr(m(D^2U + D^2U^+))\Big] \quad (3)$$

where

$$U = exp(i\Pi/f_\pi) \quad , \quad \Pi = \sum_{a=1}^{8} \lambda^a \pi^a \quad (4)$$

is the unitary chiral matrix describing the octet of pseudoscalar mesons[1)]. $D_\mu U$ is the usual weak covariant derivative and m is the 3×3 real and diagonal quark mass matrix. From the structure of this Lagrangian we can read off the consistent meson representation of the quark current $(\bar{s}d)_{V-A}$:

$$(\bar{s}d)_{V-A} = i\frac{f_\pi^2}{2}\Big\{(\partial_\mu U)U^+ - U(\partial_\mu U^+) - \frac{r}{\Lambda_\chi^2}[m(\partial_\mu U^+) - (\partial_\mu U)m]\Big\}_{ds}. \quad (5)$$

The leading in $1/N$ contributions to any quantity of interest are obtained simply from the tree diagrams whereas the next to leading order corrections are found by calculating the one loop contributions. More generally the $1/N$ expansion corresponds to the loop expansion characterized by the inverse powers of $f_\pi^2 (f_\pi^2 \sim N)$ with the strong interaction vertices given fully by the truncated Lagrangian of Eq. (3). Since we have truncated on the pseudoscalar mesons, the effective low energy meson theory describing the strong interactions appears non-renormalizable and a physical ultraviolet cutoff, denoted by M, must be introduced in order to make the meson loop diagrams finite. In the evaluation of the matrix element $\langle \bar{K}^o \mid Q(\mu) \mid K\rangle$ we will identify the scale μ with the cutoff scale M . The physical picture behind our approach has been discussed in detail in ref. [14, 15] and the present letter constitutes the application to the case at hand. As discussed in ref. [15] there are some similarities between our calculations and the usual chiral perturbative calculations found in the literature [17]. On the other hand there exist also important differences and we will return to them at the end of this letter.

1) We neglect the $\eta_0 - \eta_8$ mixing. In this limit, all the η_0 contributions at the one loop level turn out to cancel.

The parameter r can be eliminated in favor of the meson masses

$$m_\pi^2 = \frac{r}{2}(m_u + m_d) \qquad , \qquad m_K^2 \approx \frac{r}{2} m_s$$

$$m_8^2 = \frac{4}{3} m_K^2 - \frac{1}{3} m_\pi^2. \tag{6}$$

The parameters f_π and Λ_χ can be generally expressed in terms of the physical pion (F_π) and kaon (F_K) decay constants [14,15]. Using L_{tr} of Eq. (3) one finds in the $SU(2)$ limit :

$$F_K = f_\pi \big[1 + \frac{m_K^2}{\Lambda_\chi^2} - \frac{3}{8}\frac{1}{f_\pi^2}(2\ I_2(m_K^2) + I_2(m_\pi^2) + I_2(m_8^2))\big] \tag{7}$$

$$\frac{F_K}{F_\pi} = 1 + \frac{m_K^2 - m_\pi^2}{\Lambda_\chi^2} - \frac{1}{8}\frac{1}{f_\pi^2}[2I_2(m_K^2) - 5I_2(m_\pi^2) + 3I_2(m_8^2)] \tag{8}$$

where

$$I_2(m_i^2) = \frac{i}{(2\pi)^4}\int \frac{d^4q}{(q^2 - m_i^2)} = \frac{1}{(4\pi)^2}[M^2 - m_i^2 ln(1 + \frac{M^2}{m_i^2})] \tag{9}$$

with M being the physical cut-off of the truncated meson theory. For later purposes we will also need Z_K, the wave function renormalization of the kaon field ($K_R^o = Z_K^{1/2} K^o$) :

$$Z_K = 1 - \frac{1}{4}\frac{1}{f_\pi^2}[2\ I_2(m_K^2) + I_2(m_\pi^2) + I_2(m_8^2)]. \tag{10}$$

We are now ready to evaluate the parameter B using the $1/N$ expansion approach.

A. Leading Order

In the leading order the only contribution comes from the tree diagram in Fig. 1 where the solid square is a two-meson weak vertex representing the $\Delta S = 2$ operator Q with the quark current $(\bar{s}d)_{V-A}$ given by the leading term in Eq. (5):

$$[(\bar{s}d)_{V-A}]_{Leading} = -f_\pi[\partial_\mu\pi + \frac{1}{2}\frac{r}{\Lambda_\chi^2}(m\partial_\mu\pi + \partial_\mu\pi m)]_{ds}. \tag{11}$$

A straightforward calculation now gives

$$\langle \bar{K}^o \mid Q \mid K^o \rangle \mid_{Leading} = 4 f_\pi^2\big[1 + 2\frac{m_K^2}{\Lambda_\chi^2}\big]\ m_K^2 = 4F_K^2 m_K^2 \tag{12}$$

where in obtaining the last equality the leading order expression for F_K (the first two terms in Eq. (7)) has been used. Comparing this result with Eq. (2) we indeed find

$$[B]_{leading} = [B(\mu^2)]_{leading} = \frac{3}{4}. \tag{13}$$

which agrees with ref. [6] where the leading contribution has been obtained by using a modified vacuum insertion method in which the Fierz contributions have been set to zero. In obtaining the above result we have used the fact that in the strict large N limit $a = 0$.

Our result differs substantially from the one obtained in ref. [4] ($\mid B \mid \approx 1/3$) where the matrix element $\langle \bar{K}^o \mid Q \mid K^o \rangle$ has been related to the experimentally measured $\Delta I = 3/2$ amplitude $A(K^+ \to \pi^+\pi^0)$[2] by assuming $SU(3)$ symmetry and using a soft pion limit. As shown in ref. [5] these approximations correspond to the leading order of chiral expansion. In our approach this limit ($\Lambda_\chi \to \infty, F_K = F_\pi = f_\pi$.) would also imply a rather small value for B

$$[B(\Lambda_\chi \to \infty)]_{leading} = \frac{3}{4}\left[\frac{F_\pi}{F_K}\right]^2 \approx 0.46. \tag{14}$$

The remaining difference results from the fact that in our approach $[A(K^+ \to \pi^+\pi^0)]_{leading} = 1.4\ [A(K^+ \to \pi^+\pi^0)]_{exp}$. As we have shown in ref. [15] the inclusion of $1/N$ corrections makes the amplitude $A(K^+ \to \pi^+\pi^0)$ compatible with the data. On the other hand as we will now show the result in eq. (13) is rather stable with respect to these corrections.

B. Next-to-Leading Order

There are two kinds of one loop diagrams shown in Fig. 2 which contribute in the next-to-leading order to the matrix element $\langle \bar{K}^o \mid Q \mid K^o \rangle$. The diagrams in Fig. 2a contain only a four meson weak vertex which can be inferred from Eqs. (2) and (5):

$$[weak\ vertex]_4 = \frac{2}{3}(\partial_\mu\pi)_{ds}[\pi(\partial_\mu\pi)\pi - \frac{1}{2}(\partial_\mu\pi)\pi^2 - \frac{1}{2}\pi^2(\partial_\mu\pi)]_{ds}$$

[2] Note that in the short distance analysis the $\Delta S = 2$ and $\Delta I = 3/2$ operators have the same anomalous dimensions and consequently the same μ-dependence.This feature allows an estimate of the μ-independent parameter B.

$$-\frac{1}{4}[(\partial_\mu\pi)\pi - \pi(\partial_\mu\pi)]_{ds}[(\partial_\mu\pi)\pi - \pi(\partial_\mu\pi)]_{ds}. \tag{15}$$

The diagrams in Fig. 2b contain in addition to the two-meson vertex of Fig. 1 also a four-meson vertex from L_{tr} of Eq. (3):

$$[L_{tr}]_4 = \frac{1}{24}\frac{1}{f_\pi^2}[tr(\pi\ \partial_\mu\pi\ \pi\ \partial_\mu\pi - \pi^2\ \partial_\mu\pi\ \partial_\mu\pi) + \frac{1}{2}\ r\ tr(m\ \pi^4)]. \tag{16}$$

The diagrams in which both internal lines originate from the same quark current are *factorizable*. The remaining diagrams are *non-factorizable*. Clearly all the diagrams in Fig. 2b are *non-factorizable* whereas the inspection of the vertex in Eq. (15) shows that Fig. 2a contains both the factorizable and non-factorizable diagrams.

The factorizable diagrams of Fig. 2a together with the leading diagram of Fig. 1 (also factorizable) give

$$\langle \bar{K}^o \mid Q \mid K^o\rangle_F =$$

$$= 2[f_\pi^2(1 + 2\frac{m_K^2}{\Lambda_\chi^2}) - 2\ I_2(m_K^2) - I_2(m_8^2) - I_2(m_\pi^2)]\langle \bar{K}^o \mid \partial_\mu K^o \partial_\mu K^o \mid K^o\rangle$$

$$= 2F_K^2\langle \bar{K}^o \mid (\partial_\mu K^o)_R(\partial_\mu K^o)_R \mid K^o\rangle = 4F_K^2 m_K^2 \tag{17}$$

where we have used Eqs. (7) and (10). We have thus explicitly shown that all the *factorizable* next-to-leading contributions to $\langle \bar{K}^o \mid Q \mid K^o\rangle$ can be absorbed into the leading term by using the physical kaon decay constant and the physical kaon fields. Note also that the next-to-leading factorizable contributions do not contain the usual contribution from the Fierz rearrangement of the $\Delta S = 2$ quark operator as it is a part of the non-factorizable contributions given below.

The non-factorizable diagrams of Fig. 2a give

$$\langle \bar{K}^o \mid Q \mid K^o\rangle_{NF}(2a) = \frac{16}{3}I_4(m_K^2) - 3I_4(m_8^2) - I_4(m_\pi^2) - m_K^2[I_2(m_\pi^2) + 3I_2(m_8^2)] \tag{18}$$

and from Fig. 2b we find

$$\langle \bar{K}^o \mid Q \mid K^o\rangle_{NF}(2b) = -\frac{4}{3}[I_4(m_K^2) + 3m_K^2 I_2(m_K^2) + 3m_K^4 I_3(m_K^2)]. \tag{19}$$

Here the functions $I_3(m_i^2)$ and $I_4(m_i^2)$ are defined as follows

$$I_3(m_i^2) = \frac{i}{(2\pi)^4}\int \frac{d^4q}{(q^2-m_i^2)^2} = \frac{1}{(4\pi)^2}\Big[\frac{M^2}{M^2+m_i^2} - ln(1+\frac{M^2}{m_i^2})\Big]. \qquad (20)$$

and

$$I_4(m_i^2) = \frac{i}{(2\pi)^4}\int \frac{q^2\ d^4q}{(q^2-m_i^2)} = m_i^2 I_2(m_i^2) - \frac{1}{2}\frac{M^4}{(4\pi)^2}. \qquad (21)$$

The quartic cut-off dependences present in Eqs. (18) and (19) cancel each other in the sum as required by chiral invariance.[3] Adding the next-to-leading contributions given in Eqs. (18) and (19) to the factorizable contribution of Eq. (17) we finally obtain ($M^2 = \mu^2$)

$$B(\mu^2) = \frac{3}{4}[1 - \frac{1}{4F_K^2}[3(1+\frac{m_8^2}{m_K^2})I_2(m_8^2) + (1+\frac{m_\pi^2}{m_K^2})I_2(m_\pi^2) + 4m_K^2 I_3(m_K^2)]]. \quad (22)$$

This is the main result of our paper which together with Eq. (1) gives the B parameter as obtained by calculating the first two terms in the $1/N$ expansion.

In Table 1 we give $B(\mu^2)$ and B obtained using Eqs. (22) and (1) respectively for different values of $\mu = M$ and Λ_{QCD}. The μ dependent factor multiplying $B(\mu^2)$ in Eq. (1) results from the usual QCD renormalization group analysis done within the quark picture.[4] The very weak μ dependence of B shows that the quark and the meson pictures of strong interactions match well as required for the consistency of our calculation. We find

$$B = \begin{cases} 0.73 \pm 0.03 & \Lambda_{QCD} = 0.2\ GeV \\ 0.66 \pm 0.02 & \Lambda_{QCD} = 0.3\ GeV \end{cases} \qquad (23)$$

which implies that the next-to-leading order corrections are small. We also note that the main uncertainty comes from the value of Λ_{QCD}.

The small next-to-leading corrections to $\langle \bar{K}^o \mid Q \mid K^o \rangle$ found by us should be contrasted with the large one-loop corrections found in ref. [5]. In order to

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[3] The additional term usually added to the strong interaction Lagrangian given in Eq. (3) to assure the invariance of the measure (see ref. [18]) does not contribute to B at the one loop level.

[4] In our numerical calculations we have used the large N limit for a of Eq. (1) (i.e. $a = 3/11$) and α_{QCD} corresponding to three effective flavours.

understand this difference it is useful to take $M^2 \gg m_i^2$ and set $m_\pi^2 = 0$ in Eqs. (17)-(19). This gives

$$\langle \bar{K}^o \mid Q(M) \mid K^o \rangle = 4 \; F_K^2 m_K^2 [1 - \frac{1}{(4\pi F_K)^2}(2M^2 - \frac{10}{3} m_K^2 ln \frac{M^2}{m_K^2})] \, . \qquad (24)$$

Now in ref. [5] the usual chiral perturbation theory supplemented by the dimensional regularization has been used. In such an approach the quadratic dependence on the physical cut-off is lost so that the functions $I_i(m_i^2)$ take the form

$$m_i^{-2} I_4(m_i^2) = m_i^2 I_3(m_i^2) = I_2(m_i^2) = \frac{1}{(4\pi)^2} m_i^2 ln \frac{m_i^2}{\tilde{\mu}^2} \qquad (25)$$

where $\tilde{\mu}$ is the subtraction point. Using Eq. (25) in Eqs. (17)-(19), setting $m_\pi^2 = 0$ and dropping the $1/\Lambda_\chi^2$ term, as it was done in ref. [5] we find (DR = dimensional regularization)

$$[\langle \bar{K}^o \mid Q \mid K^o \rangle]_{DR} \; = 4 \; F_K^2 m_K^2 [1 - \frac{1}{(4\pi F_K)^2} \frac{10}{3} m_K^2 ln \frac{m_K^2}{\tilde{\mu}^2}] \qquad (26)$$

$$= 4 \; f_\pi^2 m_K^2 [1 - \frac{1}{(4\pi f_\pi)^2} \frac{35}{6} m_K^2 ln \frac{m_K^2}{\tilde{\mu}^2}] \qquad (27)$$

where the last equation follows from the use of Eq. (7).

The correction term in the parenthesis of Eq. (27) agrees precisely with the corresponding term in Eq. (25) of ref. [5][5)]. For $\tilde{\mu} = 0(1GeV)$ the correction term in Eq. (27) is 0(150%) indicating a breakdown of chiral perturbation theory. On the other hand as seen in Eq. (26) a substantially smaller correction 0(50%) is obtained if f_π is expressed in terms of the physical kaon decay constant. In any case the corrections in Eqs. (26) and (27) are substantially larger than the ones given in Eq. (24) where the quadratic dependence on the physical cut-off has been kept. Indeed the quadratic and logarithmic dependences on M cancel each other to large extent in Eq. (24) so that not only is the correction small but in addition the M dependence of $\langle \bar{K}^o \mid Q(M) \mid K^o \rangle$ is weak in contrast to the strong $\tilde{\mu}$ dependence seen in Eqs. (26) and (27). This weak M dependence of $\langle \bar{K}^o \mid Q(M) \mid K^o \rangle$ is consistent with the weak $\mu = M$ dependence of the short

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5) In ref. [5] $f = \sqrt{2} f_\pi$

distance factor in Eq. (1) (a small anomalous dimension of the operator Q) so that a nearly M independent result for B can be obtained.

It should be emphasized that the M^2 dependence in Eq. (24) is exactly the same as in the hadronic matrix element entering the $\Delta I = 3/2$ amplitude $A(K^+ \to \pi^+\pi^0)$. However the logarithmic contributions to the latter amplitude are smaller than the ones in Eq. (24). Consequently the loop corrections further suppress the $\Delta I = 3/2$ amplitude as desired without affecting considerably the leading order result for B.

From the point of view of our approach the dimensional regularization makes extra infrared subtractions of quadratically divergent terms. These subtractions are not permitted in the full integration of the loop contributions to the truncated theory. As we have just seen the quadratic dependence on the physical cut-off stabilizes the $1/N$ expansion and was also an essential ingredient in the matching of the meson and quark pictures. It should also be emphasized that this quadratic dependence is fully consistent with chiral symmetry [19].

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$\mu[GeV]$	$B(\mu^2)$	$B(\Lambda_{QCD} = 0.2\ GeV)$	$B(\Lambda_{QCD} = 0.3\ GeV)$
0.6	0.68	0.77	0.68
0.7	0.64	0.75	0.67
0.8	0.58	0.70	0.64

Table 1 : The values of $B(\mu^2)$ and B for various values of $\mu = M$ and Λ_{QCD}

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Figure captions

Fig. 1 The tree diagram contributing to the matrix element $\langle \bar{K}^o \mid Q \mid K^o \rangle$. The solid square represents the operator Q with the current $(\bar{s}d)_{V-A}$ given in Eq. (5).

Fig. 2 The one-loop diagrams contributing to the matrix element $\langle \bar{K}^o \mid Q \mid K^o \rangle$. The solid square in Fig. 2a represents the vertex of Eq. (15). The solid circle in Fig. 2b is the vertex of Eq. (16).

Fig. 1

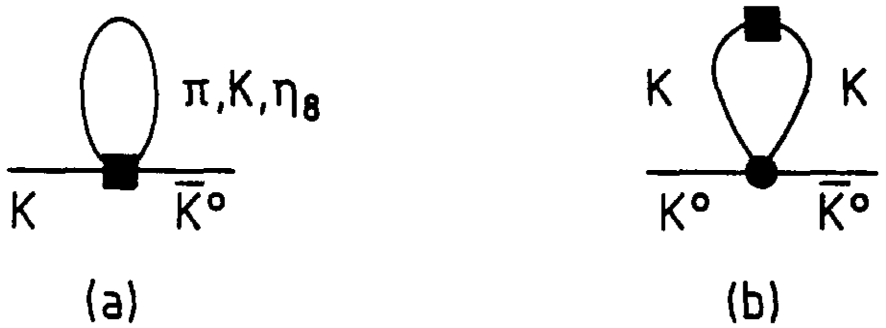

Fig. 2