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Bosonic Superconducting Cosmic Strings

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Abstract

We study the classical solutions of bosonic superconducting strings for quartic and Coleman–Weinberg effective potentials. We map the parameter space of solutions, and discuss and quantify back reaction and critical currents. The quench transition is generally first order. We consider static loop configurations in which electromagnetic stress balances the loop string tension. Such static loops are shown to exist only in a very small region of the parameter space. We give accurate results for the energy per length of a non-superconducting gauged string for arbitrary ratio of Higgs to vector mass.

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I. Introduction

Recently there has been considerable interest in superconducting cosmic strings. First proposed by Witten[1], these are cosmic strings endowed with dynamical properties which allow an effective Higgs mechanism for electromagnetism to occur on the string. There are several novel ways in which such objects might be detected[2], or have substantial effects on the formation of structure in the early Universe[3]. Superconducting strings are of two varieties: fermionic or bosonic. In the fermionic case, superconductivity arises because of the occurrence of charged Jackiw–Rossi zero modes which effectively behave as Nambu–Goldstone bosons in $1+1$ dimensions and give a longitudinal component to the photon field on the string. In some sense, this is a "natural phenomenon" in that it relies only upon certain systematic conditions being met, e.g., the presence of charged fermions with particular couplings to the vortex Higgs. The rest is guaranteed by topology, anomalies, index theorems and the like.

Bosonic superconductivity requires that some charged field develop a VEV (vacuum expectation value) in a region transversely localized on the string. This is a dynamical effect and must be engineered (in the scalar potential) to occur. It then becomes of interest to inquire how natural the phenomenon is, i.e., does one have to fine tune the parameters to have such a condensate form, or is the parameter space where it occurs large? Furthermore, interesting dynamical questions arise, e.g., what determines the saturation current and does the system undergo a first or second order phase transition when the critical current is exceeded and the superconductivity quenches? Are there solutions with and without significant back reaction of the charged condensate upon the vortex itself? Can a superconducting cosmic string loop with a sufficiently large current and attendant electromagnetic field energy become stabilized against its string tension, i.e., is there a stable "floating solution"[4,5]? The latter question is a very delicate one because it involves the various parameters of the theory in a nontrivial way, and it is the question which led us to initiate the present study.

The aim of this work is to give a comprehensive analysis of the microphysical phenomenon of bosonic superconducting strings. We do this by the use of accurate variational solutions for the various scalar and vector fields, an approach which significantly reduces the number of degrees of freedom and makes the analysis tractable. In the cases where we have direct comparison with analytic or very accurate numerical results there is excellent agreement between those results and ours. In short, we trust the results of our variational calculation.

The paper is organized as follows: in Sec. II, by using a variational analysis, we address the question of which regions of parameter space (for the scalar potential) permit bosonic superconducting cosmic strings; in Sec. III we study the dynamics of bosonic superconductivity; in Secs. IV and V we address the questions of critical currents and the possibility of floating or static solutions; in Sec. VI we translate our analysis from the natural space of dimensionless parameters we introduce to the parameters of the scalar potential; in Sec. VII we summarize our work and make some concluding remarks. In Table I we summarize the dimensionless parameters we introduce to simplify our analysis, and in Table II we summarize the constraints which must be satisfied for a bosonic superconducting vortex solution to exist.

II. Variational Analysis of Bosonic Superconducting Cosmic Strings

Vortices arise when the first homotopy class $\Pi_1(G/H)$ associated with a symmetry breaking $G \to H$ is nontrivial. Typically $\Pi_1(G/H)$ is the set of integers corresponding to the winding numbers of scalar field configurations. The simplest realization of this is the breaking of $U(1)'$ by a complex scalar field. If $U(1)'$ is gauged we have a Nielsen-Olesen flux tube[6]; if not we have a global or "axion" string[7]. For even the simplest quartic potential admitting symmetry breaking and flux tubes the classical profile of the solution is not completely known, nor would its knowledge be expected to be of great utility. There have been previous studies which obtain exact[8] or very accurate[9,10] results, and we will compare our results to these to test our variational ansatz approach. However, in the application to bosonic superconducting strings[1] no such results exist, and we must test our variational ansatz by checking the stability of our results when additional terms are added to the ansatz.

We begin with a variational study of the usual non-superconducting flux tubes (of both varieties), and then examine the effects of bosonic superconductivity. The

Lagrangian density that describes the interactions of a $U(1)'$ charged scalar field Φ in a general potential $V_\Phi(\Phi)$ takes the form:

$$L_\Phi = -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + (D_\mu\Phi)^*(D^\mu\Phi) - V_\Phi(\Phi) \tag{2.1}$$

where $D_\mu = \partial_\mu - iqA'_\mu$, $F'_{\mu\nu} = \partial_\nu A'_\mu - \partial_\mu A'_\nu$, A'_μ is the $U(1)'$ gauge field, and q is the charge of the Φ field. With cylindrical symmetry the Hamiltonian (per unit length in the z direction) is:

$$\tilde{H}_\Phi = \int r\, dr d\theta \left[\left|\frac{\partial\Phi}{\partial r}\right|^2 + \left|\frac{1}{r}\frac{\partial\Phi}{\partial\theta} - iqA'_\theta\Phi\right|^2 + V_\Phi(\Phi) + \frac{B'^2}{2}\right] \tag{2.2}$$

(henceforth we use $\tilde{H}$ to designate a Hamiltonian per unit length). Here B' is the magnetic field associated with the vortex. To study the global case we set $q = 0$ (whence $B' = 0$). Following the standard conventions, we write Φ (and other complex scalar fields) as:

$$\Phi = (\phi_1 + i\phi_2)/\sqrt{2} \tag{2.3}$$

where ϕ_1 and ϕ_2 are real fields. Note that this convention (the factor of $1/\sqrt{2}$) is used so that the quantum theory has the usual equal–time commutation relations. Although our analysis will be classical, it is important to track the $1/\sqrt{2}$ factor in order to compare with other results.

The general vortex solution has the form:

$$\Phi = \frac{\bar{v}}{\sqrt{2}}P(r)e^{i\eta} \tag{2.4}$$

where $P(r)$ is real, and $\bar{v}$ minimizes the potential $V(v) = V_\Phi(ve^{i\eta}/\sqrt{2})$ (we distinguish between the potential $V_\Phi(\Phi)$ for the complex field and the potential $V(v)$ for the real component to preserve consistency with the standard normalization conventions). Requiring Φ to be single-valued upon traversing a circular path restricts the possible changes in phase, i.e., $\Delta\eta = 2\pi N$, where N is an integer. We may then take the phase of Φ to uniformly wind over the path: $\eta = N\theta$. At the boundaries we have $P(r \to \infty) \to 1$ and $P(r \to 0) \to O(r^{|N|})$. Also, in the gauged case we have $A'_\theta(r \to \infty) \to N/qr$ and $A'_\theta(r \to 0) \to O(r)$.

Now, $P(r)$ can in principle be determined from the equations of motion associated with Eqn(2.1)[10,11], but it is simpler to adopt a variational approach. Using a combination of powers of exponentials of the form $e^{-\mu r}$ one can always engineer a function with the above short and long distance limits required for $P(r)$. For example, for $N = 1$ we choose:

$$P(r) = (1 - e^{-\mu r}) \tag{2.5}$$

We shall consider the case $N = 1$ throughout the rest of this paper. This is the simplest choice for N, and in the gauged case there is a large portion of parameter space (corresponding to vortices that exhibit Type II superconductivity in the Ginzburg-Landau theory[12]) in which a vortex with $|N| \geq 2$ is unstable and decays into $|N| = 1$ vortices[10]. Later we argue that global strings with $|N| \geq 2$ are unstable, which further motivates restricting our analysis to $N = 1$ vortices.

A. The Global Case

Adopting expression(2.5) as a variational ansatz we find the expectation of the Hamiltonian in the ungauged $U(1)'$ case, expanding in powers of $e^{-\mu r}$, to be:

$$\begin{aligned} < \tilde{H}_\Phi > \; &= \; \frac{1}{4}\pi v^2 + \pi v^2 I_\theta(\mu, \lambda) + V(v)(\pi R_\infty^2) \\ &+ \; 2\pi \int_0^\infty r\, dr V'(v)(-v e^{-\mu r}) + 2\pi \int_0^\infty r\, dr \frac{1}{2} V''(v)(v^2 e^{-2\mu r}) + ... \end{aligned} \tag{2.6}$$

or, upon performing the integrals in the potential terms:

$$\left\langle \tilde{H}_\Phi \right\rangle = \frac{1}{4}\pi v^2 + \pi v^2 I_\theta(\mu, \lambda) + V(v)(\pi R_\infty^2) + 2\pi \sum_{n=1} \frac{(-1)^n v^n V^{[n]}(v)}{n^2 n! \mu^2} \tag{2.7}$$

Here πR_∞^2 is the infinite area normal to the z-axis and this term which acts as a cosmological constant is the dominant contribution to the energy. Hence the variational calculation for v requires that $v = \bar{v}$, where $V'(\bar{v}) = 0$, and the vanishing of the effective cosmological term requires as usual that $V(\bar{v}) = 0$. The term $I_\theta(\mu, \lambda)$ is log–divergent in the global case with λ representing a large-scale cut-off. Upon varying with respect to μ the λ dependence disappears and one has:

$$\frac{\partial I_\theta(\mu, \lambda)}{\partial \mu} = \left. \frac{1}{\mu} \right|_{\lambda \to \infty} . \tag{2.8}$$

So upon variation of Eqn(2.7) we obtain the extremal solution for μ:

$$\mu^2 = 4\sum_{n=2} \frac{(-1)^n \bar{v}^{n-2} V^{[n]}(\bar{v})}{n^2 n!} \tag{2.9}$$

where $\bar{v}$ solves $V'(\bar{v}) = 0$ (note that the $n = 1$ term in the series is then zero).

The dominant contribution to the energy per length is typically the angular contribution, I_θ, which is easy to calculate for arbitrary N. We take

$$\Phi = \frac{\bar{v}}{\sqrt{2}}(1 - e^{-\mu r})^N e^{iN\theta} \tag{2.10}$$

as our ansatz. For $\mu\lambda >> 1$ we have:

$$I_\theta = N^2 \int_0^{\mu\lambda} \frac{(1 - e^{-y})^{2N}}{y} dy \approx N^2 \ln(\mu\lambda) \tag{2.11}$$

We then see, in our variational approximation, that the ratio of the energy per length of a vortex with vorticity $|N|$ to that of $|N|$ vortices with unit vorticity (vorticity is conserved) is $\simeq N$, and the decay into $|N|$ vortices each of unit vorticity is likely for vortices with $N \geq 2$.

Therefore, upon substituting Eqn(2.9) into Eqn(2.7) the mass per unit length of the $N = 1$ vortex takes the form:

$$\left\langle \tilde{H}_\Phi \right\rangle = \left(\frac{3}{4} + I_\theta(\mu, \lambda)\right) \pi\bar{v}^2 = \left(\frac{3}{4} + \ln(\mu\lambda)\right) \pi\bar{v}^2 \tag{2.12}$$

(note the normalization conventions chosen here are the standard ones; if one normalizes $\langle\Phi\rangle = v'$ then one obtains for the logarithmic contribution to the energy per unit length: $2\ln(\mu\lambda)\pi v'^2$).

Typically μ is of order the mass of the Higgs at the minimum of the potential. For example, choosing the usual form for the scalar potential,

$$V_\Phi(\Phi) = -m_\Phi^2|\Phi|^2 + \frac{\lambda_\Phi|\Phi|^4}{3!} + \frac{3m_\Phi^4}{2\lambda_\Phi} \tag{2.13}$$

whence:

$$V(v) = -\frac{m_\Phi^2 v^2}{2} + \frac{\lambda_\Phi v^4}{4!} + \frac{3m_\Phi^4}{2\lambda_\Phi} \tag{2.14}$$

yields $\bar{v}^2 = 3!m_\Phi^2/\lambda_\Phi$ and we find:

$$\mu^2 = \frac{V''(\bar{v})}{2} - \frac{2\bar{v}V'''(\bar{v})}{27} + \frac{\bar{v}^2 V''''(\bar{v})}{96} \approx 0.62\, m_\Phi^2 \tag{2.15}$$

B. The Gauged Case

In the case where $U(1)'$ is gauged we obtain the expectation value of the Hamiltonian:

$$\begin{aligned} \left\langle \tilde{H}_\Phi \right\rangle &= \frac{1}{4}\pi v^2 + \pi v^2 I_\theta(\mu, v) + \langle B'^2/2\rangle + V(v)(\pi R_\infty^2) \\ &+ 2\pi \int_0^\infty r\, dr V'(v)(-ve^{-\mu r}) + 2\pi \int_0^\infty r\, dr \frac{1}{2} V''(v)(v^2 e^{-2\mu r}) + ... \end{aligned} \tag{2.16}$$

where now the difference with the global case is the nontrivial dependence in $I_\theta(\mu, v) + \langle B'^2/2\rangle$ upon v/μ, and it is not possible to write a systematic solution for μ. Here $< B'^2/2 >$ represents the magnetic field contribution.

We now extend our variational analysis by making an ansatz for the gauge field:

$$A'_\theta = \frac{(1 - e^{-hr})^2}{qr} \tag{2.17}$$

where h is another variational parameter. This ansatz for A'_θ corresponds to a magnetic flux tube of width $\sim h^{-1}$ and total flux $2\pi/q$. It is now convenient to define the following dimensionless parameters:

$$a = \frac{m_\Phi^2}{\mu^2}; \qquad b = \frac{(q\bar{v})^2}{m_\Phi^2} = \frac{6q^2}{\lambda_\Phi} = \frac{2m_V^2}{m_H^2}; \qquad s = \frac{\mu}{h} \tag{2.18}$$

where m_H is the physical mass of the Higgs particle ($m_H = \sqrt{2}m_\Phi$), and $m_V = q\bar{v}$ is the vector boson mass. For $b > 2$ ($b < 2$) the vortices correspond to Type I (Type II) superconductivity in the Ginzburg–Landau theory. Physically, the width of the vortex is $\sim \mu^{-1} \sim \sqrt{a} m_\Phi^{-1}$, and s is $\sim$ (width of the magnetic flux tube)/(width of the vortex).

Using our previous ansatz for Φ and our ansatz for the gauge field we readily find the angular integral contribution to the energy per length:

$$I_\theta = G(s) = \ln(3^4(s+4)^2(2s+3)^4(s+2)^8/2^{11}(s+2)(s+1)^4(s+3)^8) \tag{2.19}$$

Of course, there is no large distance logarithmic divergence here since the gauge field cancels the contribution of the Higgs at large distances. The contribution to the energy from the scalar potential is found to be $(89/288)\pi\bar{v}^2 a$. and the energy in the magnetic field is given by:

$$\left\langle \frac{B'^2}{2} \right\rangle = \frac{4\pi\bar{v}^2 \ln(\frac{9}{8})}{abs^2} \tag{2.20}$$

Collecting terms, the energy per length of the gauge vortex is:

$$\left\langle \tilde{H}_\Phi \right\rangle = \pi\bar{v}^2[\frac{1}{4} + G(s) + \frac{4\ln(\frac{9}{8})}{abs^2} + \frac{89}{288}a] \tag{2.21}$$

where $\bar{v}$ is determined by $V'(\bar{v}) = 0$, and as before $V(\bar{v}) = 0$. From Eqn(2.21) and Eqn(2.18) we obtain the variational equations:

$$\frac{\partial G(s)}{\partial s} - \frac{89a}{144s} = 0 \tag{2.22}$$

$$\frac{\partial G(s)}{\partial s} - (8/abs^3)\ln(\tfrac{9}{8}) = 0 \tag{2.23}$$

by varying with respect to the parameters μ and h, respectively. Subtracting these equations gives the simple relation:

$$a^2 s^2 b = \frac{1152}{89}\ln(\tfrac{9}{8}) \tag{2.24}$$

These equations can be solved by selecting a value for s, solving for a in Eqn(2.22), and using the above relation to obtain b. The energy per length of the vortex is easily computed in this manner, and is plotted as a function of b in Fig. 1. In addition, we plot the parameters a and s, as a function of b, in Figs. 2 and 3.

For $b << 1$ (equivalent to $q \to 0$), the energy per length is dominated by the angular kinetic energy. We also see from Fig. 2 that $a = m_\Phi^2/\mu^2$ approaches the value we obtained in the global case ($a \simeq 1.6$), which is reassuring. In general, a is of the order of unity. Near $b \sim 1$, where all the terms in the Hamiltonian are important, we find agreement of our results with those of others [8,9] to better than 2%. Over the entire natural range of b ($0.01 \lesssim b \lesssim 100$; to be discussed later) we find agreement of our results with the semi-quantitative results of Bogomol'nyi and Vainshtein[10], giving us confidence that our energy per length, and hence our variational analysis, is accurate. As a further check, we also added another term to the scalar and vector field ansatze; the energy decreased by $\approx 1\%$ for $b \sim 1$, and smaller changes were observed for all other values of b.

C. The Superconducting Condensate

The discussion of this subsection is independent of the choice of global or gauged vortex. We now consider the bosonic superconducting cosmic strings which arise in a $U(1) \otimes U(1)'$ gauge theory with the general scalar potential[1] for which the Lagrangian density takes the form:

$$L = L_{\Phi} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \sigma)^* (D^\mu \sigma) - U_\sigma(\sigma) - f|\sigma|^2 |\Phi|^2 \tag{2.25}$$

where $D_\mu = \partial_\mu - ieA_\mu$, $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$, and A_μ is the $U(1)$ gauge field. The field σ carries $U(1)$ (ordinary electromagnetic) charge e and no $U(1)'$ charge, the field Φ carries no $U(1)$ charge and $U(1)'$ charge q (we have not written the photon vector potential explicitly). We now obtain the Hamiltonian per unit length:

$$\tilde{H} = \tilde{H}_\Phi + \tilde{H}_\sigma + f \int_0^{2\pi} \int_0^\infty r dr d\theta |\Phi|^2 |\sigma|^2 \tag{2.26}$$

where:

$$\tilde{H}_\sigma = \int_0^{2\pi} \int_0^\infty r \, dr d\theta \left[\left| \frac{\partial \sigma}{\partial r} \right|^2 + \frac{1}{r^2} \left| \frac{\partial \sigma}{\partial \theta} \right|^2 + U_\sigma(\sigma) \right] \tag{2.27}$$

(in this expression we have neglected the photon field).

We presently assume that $U_\sigma(\sigma)$ is an unstable, quartic scalar potential,

$$U_\sigma(\sigma) = -m_\sigma^2 |\sigma|^2 + \frac{\lambda_\sigma}{3!} |\sigma|^4 \tag{2.28}$$

and the overall stability of the theory against the breaking of electromagnetism (far from the flux tube) is controlled by this term and the $f|\sigma|^2|\Phi|^2$ term in Eqn(2.25). The condition that $U(1)$ remains unbroken outside the flux tube is that $f\bar{v}^2\sigma^2/2 + U_\sigma(\sigma)$ has no global minimum for nonzero σ. Nonetheless, the basis of bosonic superconductivity is that in the core region in which $\langle\Phi\rangle \to 0$ the f term no longer stabilizes the σ field, and it may be energetically favorable for a condensate to form. Of course, since $\langle\sigma\rangle \to 0$ as $r \to \infty$, it costs kinetic energy to allow the σ field to develop a nonzero condensate at $r \to 0$, and *a priori* it is not clear whether the gain in potential energy wins out over the cost in kinetic energy.

Presently we assume that the terms involving the σ field are sufficiently weak that they do not back react upon the Φ field, and the σ condensate can be studied in the fixed background of the Φ vortex solution just discussed. We refer to this as the *concrete vortex* approximation. Later we study the validity of this approximation.

With standard normalization we introduce the real part of σ as $Re\ \sigma = u/\sqrt{2}$ and $U(u) = U_\sigma(u/\sqrt{2})$. The standard potential becomes:

$$U(u) = -\frac{m_\sigma^2}{2}u^2 + \frac{\lambda_\sigma u^4}{4!} \tag{2.29}$$

We examine the properties of the equation of motion in the absence of superconducting currents:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + m_\sigma^2 u - \frac{\lambda_\sigma u^3}{3!} - f|\Phi|^2 u = 0 \tag{2.30}$$

At this point it is very useful to introduce dimensionless parameters which rescale the various dimensionful parameters relative to the size of the vortex:

$$\alpha = m_\sigma^2/\mu^2 = am_\sigma^2/m_\Phi^2;\ \beta = f\bar{v}^2/2\mu^2 = 3af/\lambda_\Phi;\ \tilde{\sigma}^2 = \frac{u^2\lambda_\sigma}{\mu^2};\ y = \mu r \tag{2.31}$$

where the additional relations follow from substituting in $\mu^2 = m_\Phi^2/a$. The equation of motion now becomes:

$$\frac{d^2\tilde{\sigma}}{dy^2} + \frac{1}{y}\frac{d\tilde{\sigma}}{dy} - \tilde{\sigma}[\beta P(y)^2 - \alpha] - \frac{\tilde{\sigma}^3}{3!} = 0 \tag{2.32}$$

As $y \to 0$ we find $\tilde{\sigma} \to \tilde{\sigma}_0 + O(y^2)$. For $y \gtrsim 1$ and $\tilde{\sigma}^2 << 6(\beta - \alpha)$ we have $\tilde{\sigma}^2 \propto K_0(y\sqrt{\beta-\alpha})$, and

$$\tilde{\sigma}^2 \propto \frac{e^{-2y\sqrt{\beta-\alpha}}}{y\sqrt{\beta-\alpha}} \tag{2.33}$$

in the limit $y\sqrt{\beta-\alpha} >> 1$. Here K_0 is a modified Bessel function of zeroth order. We also note that for $\alpha = \beta$ and $y \gtrsim 1$, $\tilde{\sigma}^2 \to 6/y^2$.

To investigate the dynamics of the superconducting condensate we make a variational ansatz of the form:

$$\sigma(t,z,r,\theta) = \frac{\sigma_0}{\sqrt{2}} e^{-\kappa r}(1 + \kappa r + \kappa' r^2 + \kappa'' r^3) e^{i\phi(z,t)} \tag{2.34}$$

with four variational parameters σ_0, κ, κ', and κ'' (we do not endow σ with vorticity). We choose four variational parameters so that convergence can be checked with the second and third order ansatz. This ansatz has the correct short distance limit and κ^{-1} represents the size of the σ condensate. The fact that the charged field σ acquires a vacuum expectation value in the core of the string signals that the string is superconducting and $\phi(z,t)$ is a massless mode which supplies the longitudinal degree of freedom for the photon on the string.

To describe the essential physics we find it useful to consider a simple, truncated ansatz:

$$\sigma(t,z,r,\theta) = \frac{\sigma_0}{\sqrt{2}} e^{-\kappa r} e^{i\phi(z,t)} \tag{2.35}$$

with which we can discuss many results analytically. We presently use this ansatz and also make use of the previously obtained profile for the vortex field, Φ, to obtain the expectation value for the Hamiltonian:

$$\left\langle \tilde{H} \right\rangle = \left\langle \tilde{H}_\Phi \right\rangle + \frac{1}{4}\pi\sigma_0^2 + \frac{\pi \tilde{U}(\sigma_0)}{\kappa^2} + \frac{\pi f v^2 \sigma_0^2}{8\kappa^2} F(\mu/\kappa) \tag{2.36}$$

Here $\tilde{U}$ is derived from U upon performing the cylindrical integration normal to the string axis. If, for example, $U(u)$ has the polynomial expansion:

$$U(u) = \sum_{n\ even} U_n u^n \tag{2.37}$$

then we have:

$$\tilde{U}(\sigma_0) = 2 \sum_{n\ even} \frac{U_n}{n^2} \sigma_0^n. \tag{2.38}$$

The function $F(x)$ represents the overlap of the Φ profile turning on to its asymptotic value $\bar{v}$ over a distance scale μ^{-1} from the flux tube and the σ profile turning off over a distance scale κ^{-1}. It takes the form:

$$F(x) = \left[1 - \frac{2}{(1+x/2)^2} + \frac{1}{(1+x)^2}\right] = \left[\frac{x^4 + 6x^3 + 6x^2}{x^4 + 6x^3 + 13x^2 + 12x + 4}\right]. \quad (2.39)$$

where $x \equiv \mu/\kappa \sim$ (width of the σ condensate)/(width of the vortex). We see that $F(x)$ is positive over its range and we further note the limits:

$$\begin{aligned} F(x) &\approx 3x^2/2 \quad (0 \le x \le .5); \\ F(x) &\approx x/3 \quad (.5 \le x \le 2); \\ F(x) &\to 1 \quad (x \to \infty) \end{aligned} \quad (2.40)$$

The full potential for both fields is:

$$\begin{aligned} V_{\Phi,\sigma}(\Phi,\sigma) &= -m_\Phi^2|\Phi|^2 + \frac{\lambda_\Phi|\Phi|^4}{3!} - m_\sigma^2|\sigma|^2 + \frac{\lambda_\sigma}{3!}|\sigma|^4 + f|\Phi|^2|\sigma|^2 + \frac{3m_\Phi^4}{2\lambda_\Phi} \\ V(v,u) &= -m_\Phi^2 v^2/2 + \frac{\lambda_\Phi v^4}{4!} - m_\sigma^2 u^2/2 + \frac{\lambda_\sigma u^4}{4!} + fv^2u^2/4 + \frac{3m_\Phi^4}{2\lambda_\Phi} \end{aligned} \quad (2.41)$$

The overall theory must be such that far from flux tubes the $U(1)'$ symmetry is spontaneously broken while $U(1)$ remains unbroken. This thus leads to constraints on the parameters in Eqn(2.41). The most important constraint is that the ground-state has a true global minimum corresponding to $\langle\sigma\rangle = 0$ and $\langle\Phi\rangle \neq 0$. The condition for this is:

$$\frac{m_\Phi^4}{\lambda_\Phi} > \frac{m_\sigma^4}{\lambda_\sigma} \qquad \textbf{(constraint 1.)} \qquad (2.42)$$

and:

$$f\bar{v}^2/2 = \frac{3fm_\Phi^2}{\lambda_\Phi} > m_\sigma^2 \qquad \text{or} \qquad \beta > \alpha \qquad \textbf{(constraint 2.)} \qquad (2.43)$$

Note that a local minimum is still allowed at $\langle\sigma\rangle = 3!m_\sigma^2/\lambda_\sigma$ and $\langle\Phi\rangle = 0$. However, **constraint (1)** guarantees this to be a false vacuum. This potential also has an extremum corresponding to $\langle\sigma\rangle \neq 0$ and $\langle\Phi\rangle \neq 0$. However, this extremum can be the global minimum only if $\langle\sigma\rangle$ is imaginary.

We then find for $\tilde{U}(\sigma_0)$:

$$\tilde{U}(\sigma_0) = -\frac{m_\sigma^2 \sigma_0^2}{4} + \frac{\lambda_\sigma \sigma_0^4}{192} \tag{2.44}$$

Our problem is thus the minimization of the energy per length:

$$\begin{aligned} E(\sigma_0, \kappa) &= \frac{1}{4}\pi\sigma_0^2 + \frac{\pi\tilde{U}(\sigma_0)}{\kappa^2} + \frac{\pi f v^2 \sigma_0^2}{8\kappa^2} F(\mu/\kappa) \\ &= -\frac{\pi}{4}\left[\alpha x^2 - \beta x^2 F(x) - 1\right]\sigma_0^2 + \frac{\pi\lambda_\sigma x^2 \sigma_0^4}{192\mu^2} \end{aligned} \tag{2.45}$$

Note that a nontrivial minimum will occur if the overall coefficient of σ_0^2 is negative, or:

$$\alpha x^2 - \beta x^2 F(x) - 1 > 0 \qquad \textbf{(constraint 3.)} \tag{2.46}$$

We can thus determine a lower limit to α. Using **constraint (2)** we have the condition:

$$\alpha x^2(1 - F(x)) - 1 > 0 \tag{2.47}$$

It is readily verified that the $max(x^2(1 - F(x)) = \lim_{x\to\infty} x^2(1 - F(x)) = 7$, and thus the lower limit to α is:

$$\alpha \geq \frac{1}{7} \tag{2.48}$$

[It should be noted that in his discussion, Witten considers the limit $\lambda_\sigma \approx 0$ and $\alpha \approx \beta$, and argues that the solution exists because the 2–dimensional Schroedinger equation with negative definite potential admits a normalizeable boundstate (in the $\alpha = \beta$ limit this corresponds to a negative coefficient to u^2 in the above potential). Strictly speaking, however, the absence of the λ_σ term causes the overall theory to be unstable (**constraint (1)** cannot be satisfied). The region external to the vortex is a false vacuum and σ_0 grows without bound, eventually expelling the Φ field to infinity; the vortex ceases to exist. Therefore, the λ_σ term must always be present at some level and the normalization of σ_0 is always determined as above. Moreover, such a term is induced by interactions and one cannot have the strict $\alpha = \beta$ case without Coleman-Weinberg symmetry breaking by the σ field in far vacuum. Indeed, we see below that our variational calculation picks out the family of solutions with $\alpha \approx \beta$, except when $\alpha < 1/7$. Why is our result in apparent conflict with the theorem that such solutions should always exist for any α including $\alpha \to 0$? The answer is that our variational ansatz cannot probe the extreme weak potential case. We will see, however, that we obtain sufficient information about the parameter space that we can infer its structure as $\alpha \to 0$ by Witten's analytic result.]

We thus find the extremal solution for σ_0^2:

$$\sigma_0^2 = \frac{24\mu^2}{\lambda_\sigma}\left(\alpha - \beta F(x) - 1/x^2\right) \tag{2.49}$$

which is a valid solution provided it is positive. This is just a restatement of **constraint (3)**. Substituting the solution for σ_0^2 into the Eqn(2.45) gives the energy per length as a function of κ:

$$E(x) = -\frac{3\pi\mu^2}{\lambda_\sigma x^2}\left(\alpha x^2 - \beta x^2 F(x) - 1\right)^2 \tag{2.50}$$

which leads to the extremal equation for κ:

$$\alpha x^2 - \beta\left[x^2 F(x) + x^3 F'(x)\right] + 1 = 0 \tag{2.51}$$

We have scanned over the parameter space defined by α, β and x for solutions consistent with **constraint (2)**, and **constraint (3)**. The allowed solution space and the width of the σ condensate are shown in Figs. 4 and 5. Our procedure consisted of choosing a value of β, solving for α using Eqn(2.51) and then scanning over values of x, checking for consistency with the constraints. We can also obtain the equation for the outer boundary of solutions, parameterized by x, by setting $\sigma_0^2 = 0$:

$$\beta(x) = 2/x^3 F'(x) \tag{2.52}$$

$$\alpha(x) = \beta(x)F(x) + 1/x^2 \tag{2.53}$$

The parameter space of solutions from the full ansatz Eqn(2.34) was also determined. This was done by looking for a global minimum in the energy with $\sigma_0^2 > 0$ and $\beta > \alpha$. The outer boundary obtained using the full ansatz is also shown in Fig. 4, and it is not significantly different from the boundary determined by our truncated ansatz. We also note that while the energy given by the full ansatz Eqn(2.34) has converged to within a few percent, our simple ansatz can give an energy per length which differs from that of the full ansatz by $\sim 50\%$ for unfortunate choices of α and β. However, both ansatze yield energies that agree to within a few percent near $\alpha = \beta$, which happens to be one of the most important regions for the rest of this study. At this point we abandon the more complicated ansatz.

The line of solutions corresponding to $\alpha = \beta$ is already known from Witten's argument and it extends in reality down to $\alpha = \beta = 0$. We see that the parameter space of solutions is restricted for small α and β and grows to the indicated wedge for larger values. Our definitions of α and β have been very convenient because the allowed parameter space, in the concrete vortex approximation, is independent of whether or not the string is gauged. However, it should be mentioned that if m_σ^2 and $f\bar{v}^2/2$ were normalized by m_Φ^2, instead of by our variational parameter μ^2, global strings would have a fixed-wedge of allowable parameter space while the gauge strings would have a wedge of a size determined by the ratio of scalar and vector boson masses. The parameter space with this normalization is obtained by the mapping $(\alpha,\, \beta) \to (\alpha/a,\, \beta/a)$, where a can be obtained from Fig. 2 in the gauged case, and $a \approx 1.6$ in the global case. We then see, with this normalization, that the gauged and global strings have the same available parameter space for $b \lesssim 0.1$, and for $b \gtrsim 0.1$ the gauged parameter space can be significantly larger than the global parameter space. In Sec. VI we will discuss how one translates from α, β back to physical parameters in the scalar potential.

D. Back Reaction onto the Vortex

In the previous analysis we have viewed the Φ background solution as fixed, i.e., the width of the vortex, μ^{-1}, is held fixed as a parameter determined from the potential for Φ alone. In this section we relax our concrete vortex approximation and vary the full Hamiltonian with respect to μ to study the validity of this approximation. We also map out the regions of solution space where there are significant deviations from our concrete vortex approximation; fortunately, they occupy only a small fraction of the entire parameter space of solutions. Since the full Hamiltonian depends upon whether or not the string is gauged, we consider each case separately.

1. The Global Case

The full Hamiltonian for the global string is:

$$\left\langle \tilde{H} \right\rangle = \frac{\pi \bar{v}^2}{4} + \pi \bar{v}^2 I_\theta(\mu) + 2\pi \sum_{n=1} \frac{(-1)^n \bar{v}^n V^{[n]}(\bar{v})}{n^2 n! \mu^2} - \frac{3\pi\mu^2}{\lambda_\sigma x^2} \left(\alpha x^2 - \beta x^2 F(x) - 1\right)^2 \tag{2.54}$$

It is now useful to introduce a new variational parameter which is the ratio of the value obtained in the concrete vortex approximation ($\equiv \mu_0$) to the true value of μ. We thus introduce:

$$y = \mu_0/\mu; \qquad \text{where} \qquad \mu_0^2 = 4 \sum_{n=2} \frac{(-1)^n \bar{v}^{n-2} V^{[n]}(\bar{v})}{n^2 n!} \tag{2.55}$$

Also, we define α', β', and x' to be ratios with μ_0, i.e., $x' = \mu_0/\kappa = yx$, and correspondingly, $\alpha' = \alpha/y^2$, $\beta' = \beta/y^2$. Removing overall constant factors and additive constants we have the variational Hamiltonian in y and x':

$$\left\langle \tilde{H} \right\rangle = -\ln y + y^2/2 - \frac{\gamma}{x'^2}\left(\alpha' x'^2 - \beta' x'^2 F(x'/y) - 1\right)^2 \tag{2.56}$$

We have also introduced the parameter $\gamma = 3\mu_0^2/\bar{v}^2\lambda_\sigma$; note the ratio $\mu_0/\bar{v}$ has already been determined implicitly in the above discussion for the general quartic polynomial potential: $\gamma \approx 0.31\lambda_\Phi/\lambda_\sigma$. In the concrete vortex approximation, $y = 1$, the primed parameters coincide with the unprimed ones; this redefinition has the advantage of minimizing the y dependence of the additional term.

The joint extremal equations in x' and y are:

$$\alpha' x'^2 - \beta' \left[x'^2 F(x'/y) + x'^3 F'(x'/y)/y\right] + 1 = 0 \tag{2.57}$$

$$1 - y^2 + 2\beta'\gamma W x' F'(x'/y)/y = 0 \tag{2.58}$$

where $W = \left(\alpha' x'^2 - \beta' x'^2 F(x'/y) - 1\right)$. We also recover our previous constraints recast in the present variables: $W > 0$, and $\beta' - \alpha' > 0$. Note that since $W > 0$ we must have $y > 1$, i.e., the vortex is always larger than the size given by our concrete approximation.

Upon specification of γ the allowed values of α' are restricted by **constraint(1)**:

$$\alpha' < \frac{0.9}{\sqrt{\gamma}} \tag{2.59}$$

We present in Figs. 6, 7, 8, and 9 both α' and β' as a function of y for two choices of γ : 10^{-3} and 10^{-8} . These results were obtained by solving for α' and β' in the extremal equations and scanning over x' and y (always checking that the constraints are satisfied). We see, as a general rule, that back reaction can only be significant if the above inequality approaches equality. In other words, a necessary condition for back reaction to be important is:

$$\frac{m_\Phi^4}{\lambda_\Phi} \approx \frac{m_\sigma^4}{\lambda_\sigma} \tag{2.60}$$

This should come as no surprise since in this limit the vacuum energy associated with a σ condensate is about the same as that associated with a Φ condensate. We further see, upon comparison of the graphs with the same γ, that large values of y (i.e., significant back reaction) only occur when condition(2.60) is satisfied and:

$$\alpha' \approx \beta' \qquad (\text{or} \qquad m_\sigma^2 \approx \frac{3fm_\Phi^2}{\lambda_\Phi}) \tag{2.61}$$

For the choice $\gamma = 10^{-4}$, 10^{-6}, and 10^{-8} we also show the α', β' parameter space where $y \geq 1.2$ in Fig. 4. We see that our concrete vortex approximation is valid over most of the parameter space, and solutions with even mild back reaction are very rare, and are restricted to be near the line $\alpha = \beta$.

2. The Gauged Case

The full Hamiltonian per length for the gauged string is:

$$\left\langle \tilde{H} \right\rangle = \frac{\pi\bar{v}^2}{4} + \pi\bar{v}^2 G(s) + \frac{4\pi\bar{v}^2 \ln(\frac{9}{8})}{abs^2} + \frac{89}{288}\pi\bar{v}^2 a - \frac{3\pi\mu^2}{\lambda_\sigma x^2}\left(\alpha x^2 - \beta x^2 F(x) - 1\right)^2 \tag{2.62}$$

We redefine variables as in the global case, again use $y = \mu_0/\mu$, and further define $s' = sy$. We replace a with $a'y^2$, where a' is the value of a in the concrete vortex approximation. Dropping constant factors and additive constants we now have the variational Hamiltonian in y, x', and s':

$$\left\langle \tilde{H} \right\rangle = G(s'/y) + \frac{4\ln(\frac{9}{8})}{a'bs'^2} + \frac{89}{288}a'y^2 - \frac{\gamma}{x'^2}\left(\alpha' x'^2 - \beta' x'^2 F(x'/y) - 1\right)^2 \quad (2.63)$$

We have defined γ as before, and in this case $\gamma = \lambda_\Phi / 2\lambda_\sigma a'$. The constraint on α' which follows from **constraint 1** now takes the form:

$$\alpha' < \sqrt{\frac{a'}{2\gamma}} \quad (2.64)$$

The method of solution in this case was to numerically search for a global minimum in the energy. Results very similar to the global case were obtained. Again, the main result is that if we are to have a superconducting solution and $y >> 1$, we require:

$$\frac{m_\Phi^4}{\lambda_\Phi} \approx \frac{m_\sigma^4}{\lambda_\sigma}; \qquad \text{and} \qquad \alpha' \approx \beta' \quad (2.65)$$

To summarize the back reaction issue we can say that for all but a small portion of the parameter space of solutions back reaction of the σ condensate upon the vortex itself is not important. Only for $\alpha \approx \beta$ is back reaction potentially significant.

E. Coleman–Weinberg Effective Potentials

It is interesting to consider the possibility of vortices and associated superconductivity, in the case of Coleman–Weinberg[13] symmetry breaking. Here the fields Φ and σ have zero renormalized mass but the radiative corrections due to the interactions with their gauge bosons produce an unstable effective potential at the one–loop level. We may consider the effective potential to be[13]:

$$\begin{aligned} V(\Phi,\sigma) &= \frac{\lambda_\Phi}{3!}|\Phi|^4 + \frac{3q^4}{16\pi^2}|\Phi|^4\left(\ln(2|\Phi|^2/v^2) - \frac{25}{6}\right) \\ &+ \frac{\lambda_\sigma}{3!}|\sigma|^4 + \frac{3e^4}{16\pi^2}|\sigma|^4\left(\ln(2|\sigma|^2/v'^2) - \frac{25}{6}\right) + f|\Phi|^2|\sigma|^2 \end{aligned} \quad (2.66)$$

The condition that $\Phi = ve^{i\theta}/\sqrt{2}$ minimize the Φ part of the potential implies that:

$$0 = \frac{\lambda_\Phi}{6} - \frac{11q^4}{16\pi^2} \quad (2.67)$$

at which point the vacuum energy density is:

$$E_{vac} = -\frac{3q^4}{128\pi^2}v^4 \tag{2.68}$$

In general we wish for the σ field to minimize the potential at some other mass scale, $v'/\sqrt{2}$. Consequently, we have the relationship:

$$0 = \frac{\lambda_\sigma}{6} - \frac{11e^4}{16\pi^2} \tag{2.69}$$

At the σ minimum we have:

$$E_{vac} = -\frac{3e^4}{128\pi^2}v'^4 \tag{2.70}$$

and the stability of the theory at the $\sigma = 0,\ \Phi \neq 0$ minimum requires (**constraint (1')**):

$$qv > e'v' \tag{2.71}$$

which is the analog of **constraint (1)**. Besides $f > 0$, no further constraints are obtained. Unlike the case with ordinary scalar potentials, we find that the second non-trivial constraint arises from requiring that the extremum $\sigma = v_1/\sqrt{2}$, $\Phi = v_2 e^{i\theta}/\sqrt{2}$ (v_1, $v_2 \neq 0$) not be the true minimum. Upon extremizing the potential, v_1 and v_2 can be solved from:

$$8A(v_1/v)^2 \ln(v_1/v) + C(v_2/v')^2 = 0 \tag{2.72}$$

$$8(v_2/v')^2 \ln(v_2/v') + C(v_1/v)^2 = 0 \tag{2.73}$$

where $A = (qv/ev')^4 > 1$, $C = \beta/\chi > 0$, and we have introduced the parameter χ:

$$\chi = \frac{3e^4 v'^2}{64\pi^2\mu^2} \tag{2.74}$$

which is the analogue of α (β is defined as before, i.e., $\beta = fv^2/2\mu^2$). Because of the positivity of A and C, it is clear that $v_1 < v$ and $v_2 < v'$. To find the parameter space (A, C) that represents the global minimum, we scan through the space (v_1/v, v_2/v'), solve for A and C, and then check that the energy is lower than that given by Eqn(2.68). This parameter space, which is *not* acceptable, is shown in Fig. 10. We see from Fig. 10 that a necessary condition (though not sufficient if $A \approx 1$) for stability of the theory at the $\sigma = 0$, $\Phi \neq 0$ minimum is:

$$\beta \gtrsim 1.2\chi \tag{2.75}$$

which is the analog for Coleman–Weinberg breaking of **constraint (2)**.

Now we use the variational ansatz of the preceding analysis. We again recover Eqn(2.9) and find presently for μ:

$$\mu^2 = \frac{q^4v^2}{\pi^2}\left(\frac{3}{16} - \frac{5}{36} + \frac{11}{256} - O(10^{-3})\right) \approx .092\frac{q^4v^2}{\pi^2} \tag{2.76}$$

in the global case. The series does not terminate because of the expansion of the logarithm; we keep here terms to $O(e^{-4\mu r})$.

Similarly, for the σ field we find the energy functional per unit length:

$$E(\sigma_0,\kappa) = \frac{\pi}{4}(1+\beta x^2F(x))\sigma_0^2 - \frac{9e^4x^2\sigma_0^4}{1024\mu^2\pi} + \frac{3e^4x^2\sigma_0^4}{512\mu^2\pi}\ln(\sigma_0^2/v'^2) \tag{2.77}$$

We see here another manifestation of the modified **constraint (2)**; if $\beta \to 0$ this energy is unbounded below for $\kappa \to 0$; the lower limit on f prevents this catastrophe.

We rescale the above Hamiltonian by letting $\sigma_0^2 = v'^2\varsigma_0^2/\chi$. The energy then takes the form:

$$E = \pi v'^2(2\varsigma_0^2[1+\beta x^2F(x)] + \varsigma_0^4x^2[\ln(\frac{\varsigma_0^2}{\chi}) - \frac{3}{2}])/8\chi \tag{2.78}$$

Upon extremization of the energy with respect to ς_0 and x we obtain:

$$\beta[2F(x) + xF'(x)] + \varsigma_0^2[\ln(\varsigma_0^2/\chi) - 3/2] = 0 \tag{2.79}$$

$$1 + \beta x^2F(x) + \varsigma_0^2x^2[\ln(\varsigma_0^2/\chi) - 1] = 0 \tag{2.80}$$

The solution to the upper boundary in this case does not correspond to $\varsigma_0^2 = 0$, and the boundary is more difficult to obtain. Given values of χ and β, ς_0^2 typically has two extremal solutions. However, the solution of interest can be obtained by the following requirement: $E < 0$. The extremal equations can be solved for β and χ in terms of x and ς_0. It is then possible to scan over the variational parameter space (x, ς_0) and search for solutions satisfying the modified **constraint (2)**. Results from such a scan are shown in Fig. 11. We see that the parameter space (χ, β) is very similar to the parameter space (α, β) obtained with Higgs potentials.

III. Dynamics of Bosonic Superconductivity

To recapitulate what we have done to this point, using our variational ansatze we have mapped out the regions of parameter space which allow bosonic superconductivity, for both ordinary and Coleman-Weinberg scalar potentials (see Figs. 4 and 12). Further, we have explored the regions of parameter space for the scalar potential where the σ condensate significantly modifies the vortex solution itself, which occurs for $\alpha \approx \beta$ and $m_\Phi^4/\lambda_\Phi \approx m_\sigma^4/\lambda_\sigma$. Throughout these analyses the superconducting current was taken to be zero. Presently we obtain the expression for the energy associated with superconducting currents, i.e., the kinetic energy of the charge carriers and the energy in the magnetic field. Throughout we will use our truncated ansatz. First we directly solve Maxwell's equations without resort to a Green's function expression (which Witten[1] does); however, the usual UV singularities still occur and must be dealt with in a self-consistent manner.

Consider phase fluctuations about the σ condensate obtained above:

$$\sigma(r,\theta,z,t) \to \frac{\sigma_0(r)}{\sqrt{2}}\exp(i\phi(z,t)) \tag{3.1}$$

and we obtain the effective action for $\phi(z,t)$ in the case of an infinite straight z-axis string:

$$\begin{aligned} I \;&=\; \frac{1}{2}\int 2\pi r\; drdzdt\; \sigma_0(r)^2\left\{(\partial_t\phi - eA_0)^2 - (\partial_z\phi - eA_z)^2\right\} \\ &-\frac{1}{4}\int d^4x\; F_{\mu\nu}F^{\mu\nu} \;\; + O(1/\kappa^2) \end{aligned} \tag{3.2}$$

We thus obtain the equation of motion for the ϕ field and the vector potential:

$$\partial_0^2\phi - \partial_z^2\phi \;=\; e(\partial_0 A_0 - \partial_z A_z) \tag{3.3}$$

$$\partial_\mu F^{\mu\nu} \;=\; eK(\partial^z\phi - eA^z)\delta^2(x_\perp)\delta^{z\nu} \tag{3.4}$$

where:

$$K = \int 2\pi r dr\; \sigma_0(r)^2 \qquad (= \pi\sigma_0^2/2\kappa^2) \tag{3.5}$$

the latter result holding with our simple ansatz. The interesting solution for our purposes corresponds to a conducting wire of length L with ϕ topological winding number N; that is:

$$\phi = \frac{2\pi N}{L} z; \qquad A^0 = A^x = A^y = 0 \tag{3.6}$$

and we obtain the remaining component of the vector potential:

$$A_z = -(constant) \ln\left(\sqrt{x^2 + y^2}/L\right) \tag{3.7}$$

which describes a circumferential B field:

$$B_x = -(constant)\frac{y}{x^2 + y^2} \qquad B_y = (constant)\frac{x}{x^2 + y^2} \tag{3.8}$$

and the constant is determined by self–consistency with the current by way of Stokes's theorem. The current is:

$$j_z = eK\left(\partial_z \phi - eA_z(0)\right)\delta^2(x_\perp) \tag{3.9}$$

and we must interpret $A_z(0)$. Witten similarly encounters this subtlety in his Green's function solution and we simply define:

$$A_z(0) = (constant) \ln(\kappa L) \tag{3.10}$$

(the lower cutoff is the size of our "wire", $\sim \kappa^{-1}$). Then using Stokes's theorem we obtain the result:

$$(constant) = \frac{eKN}{L}\left(1 + \frac{e^2 K}{2\pi}\ln(\kappa L)\right)^{-1} \tag{3.11}$$

We make the following definition:

$$\omega = \frac{1}{e^2 K}\left[1 + \frac{e^2 K}{2\pi}\ln(\kappa L)\right] \tag{3.12}$$

which is the "inductance per unit length" and the result for the B field becomes:

$$B_x = -\frac{N}{eL\omega}\frac{y}{x^2 + y^2}; \qquad B_y = \frac{N}{eL\omega}\frac{x}{x^2 + y^2}. \tag{3.13}$$

Now, we may compute the resulting energy from the Hamiltonian:

$$H = \frac{1}{2}\int 2\pi r\ drdz\sigma_0(r)^2(\partial_z\phi - eA_z)^2 + \int d^3x\ \frac{B^2}{2} \tag{3.14}$$

The $\partial_z\phi - eA_z$ contribution is just $2\pi^2N^2/Le^4\omega^2K$ and the $B^2/2$ contribution is logarithmically divergent in the transverse dimensions which we cut-off at L, which yields $\pi N^2\ln(\kappa L)/Le^2\omega^2$. These terms combine to give the effective Hamiltonian per length of string:

$$\begin{aligned}\left\langle \tilde{H}\right\rangle &= \frac{2\pi^2N^2}{L^2K\omega^2e^4}\left(1+\frac{e^2K}{2\pi}\ln(\kappa L)\right)\\ &= \frac{2\pi^2N^2}{L^2e^2\omega} = \frac{1}{2}\omega I^2\end{aligned} \tag{3.15}$$

where the current $I = \int dA\ j_z = 2\pi N/eL\omega$. We use electromagnetic units here, and elsewhere, that correspond to $e^2/4\pi = \alpha_{EM}$, where $\alpha_{EM} \simeq 1/137$ is the electromagnetic coupling constant. For reference, e GeV $= 2.43 \times 10^5$ Amperes.

The ratio of the energy in the magnetic field to the KE of the charge carriers is $e^2K\ln(\kappa \mathrm{L})/2\pi$, and with our simple ansatz $K = \pi\sigma_0^2/2\kappa^2$. Upon using our previous extremal solution for σ_0^2 and taking $\lambda_\sigma \lesssim 1$ (perturbativity) we find that for a large portion of parameter space $K >> 1$, as shown in Fig. 12. For the loops of interest L is a macroscopic (or even cosmological) length, implying that $L\kappa \gg 1$, and so the field energy is the dominant contribution when $K \gg 1$. The energy associated with the supercurrent, in this case, is no different from that of an ordinary wire with current I. The only region of our parameter space (α, β) that the kinetic energy of the charge carriers can have a significant contribution is near the upper boundary of solutions, where the σ condensate is starting to become energetically unfavorable and $K \to 0$.

Superconducting currents can be induced in a cosmic string if it moves through a magnetic field in the Universe. Whether or not there were primordial magnetic fields in the early Universe is still an open and very important question which has been considered elsewhere[14]. We mention, however, that superconducting strings inherently have a non-zero winding number in the σ field (and hence current) since the phase of the σ field must have been uncorrelated on the scale of the horizon $\xi \sim t_{SSB}$ at the time of the symmetry breaking ($t = t_{SSB}$) that gave rise to cosmic strings. This results in a winding number in a length of string L of of at least

$\sim (L/\xi)^{1/2}$, which leads to a minimum current that must be present (and which one might refer to as 'the Kibble current').

IV. Critical Currents

We now examine the breakdown of bosonic superconductivity. Collecting terms in the Hamiltonian associated with the σ condensate gives:

$$\left\langle \tilde{H}_\sigma \right\rangle = -\frac{\pi}{4}(\alpha x^2 - \beta x^2 F(x) - 1)\sigma_0^2 + \frac{\pi \lambda_\sigma x^2 \sigma_0^4}{192\mu^2} + \frac{2\pi^2 N^2 K}{L^2(1 + \frac{e^2 K}{2\pi}\ln(\kappa L))} \tag{4.1}$$

using our simple ansatz for σ and using the ordinary scalar potential. (Since this is a Hamiltonian per length, the last term is of the order of $1/L^2$.) The current cannot be arbitrarily large because there will be a (critical) current beyond which it will be energetically favorable for the σ field to become zero everywhere. Since $\left\langle \tilde{H}_\sigma \right\rangle$ vanishes for $\sigma = 0$, the transition should occur when $\left\langle \tilde{H}_\sigma \right\rangle$ becomes nonnegative. Although N is topological in nature, it can unwind in processes where σ goes through zero (where the phase ϕ is not well defined), which on energetic grounds should occur when the above Hamiltonian approaches zero.

Notice, however, that we should display the full σ_0 behavior in this expression by restoring the expression for K obtained previously:

$$\left\langle \tilde{H}_\sigma \right\rangle = -\frac{\pi}{4}(\alpha x^2 - \beta x^2 F(x) - 1)\sigma_0^2 + \frac{\pi \lambda_\sigma x^2 \sigma_0^4}{192\mu^2} + \frac{\pi^3 N^2 \sigma_0^2}{\kappa^2 L^2[1 + e^2\sigma_0^2 \ln(\kappa L)/4\kappa^2]} \tag{4.2}$$

If we could neglect the logarithm in the denominator of the last term, we could conclude that the phase transition when the critical current is reached is second order, i.e., the terms involving σ_0^2 would cancel. However, the logarithm is large and one can see by plotting this expression (for some fixed x) as a function of σ_0 that there is a second local minimum for a wide range of currents. This minimum can correspond to a positive energy and thus be a false vacuum state. Therefore, the string should undergo a *first order phase transition*, presumably through the nucleation of bubbles which are regions of true vacuum and are not superconducting. Of course, false vacua can be metastable and the full analysis requires considering variations in both x and σ_0. This study is currently in progress[15]. We will content ourselves to defining absolute stability when $\langle H_\sigma \rangle < 0$. This gives a definition of the critical current, i.e., when $\langle H_\sigma \rangle = 0$ we have a critical situation, though this may not correspond to a physically reasonable critical current and may be an underestimate.

This critical current can easily be estimated by using the unperturbed extremal solution for σ_0^2 from Eqn(2.49) and equating the r.h.s. of Eqn(4.2) to zero. The result for the critical current found in this way is:

$$I_{crit} = \mu\sqrt{\frac{6\pi}{\lambda_\sigma \omega}}(\alpha x - \beta F(x)x - x^{-1}) \tag{4.3}$$

In the limit $K >> 1$ the critical current takes the form:

$$I_{crit} = \pi\mu\sqrt{\frac{12}{\lambda_\sigma \ln(\kappa L)}}(\alpha x - \beta F(x)x - x^{-1}) \tag{4.4}$$

This limit is convenient to display since the λ_σ and logarithmic dependence is now multiplicative. The factor involving x, α, and β can be computed, and in general is of the order of α. The allowed parameter space for the critical current in this limit, as a function of α, is shown in Fig. 13.

V. Static Loops

We will now explore the possibility that a loop of string can be stabilized against its string tension by electromagnetic stresses and achieve a static, or floating state[4,5]. That this could occur is easy to see. Neglecting numerical factors, the energy of a loop is $\sim$ (string tension $\sim \bar{v}^2$) $\times L + LI^2/2$, the first term representing the energy due to the string tension and the second due to the electromagnetic field. As a loop oscillates it radiates both electromagnetic and gravitational radiation, and in the process must shrink in size. Conservation of the winding number N means that the supercurrent $I \propto N/L$ must increase, and so the magnetic field energy varies as $1/L$, whereas the potential energy of the string varies as L. Assuming that the string remains superconducting (so that N is constant) the loop will reach a state of minimum energy for $L = L_{static} \sim N/\bar{v}$, where it can no longer decrease its energy by shrinking, and the string tension is balanced by electromagnetic stresses.

To consider loops of string in our present framework which is strictly only applicable to infinitely long strings, we require that the scale of the fields be much smaller than the curvature of the loop: $\mu L >> 1$, $\kappa L >> 1$, $hL >> 1$. The ideal approach to search for static loops would be to extremize the full Hamiltonian with respect to σ, κ, μ, and L, and search for solutions consistent with our constraints. However, we will restrict ourselves to the case where the loop is stabilized by currents that are smaller than the critical current, and so we will not address the question of metastable, supercritical static loops. By so restricting ourselves, we are able to use our previous results, i.e., our variational parameters do not change in the presence of the current. This amounts to requiring that the energy in the current not 'back react' upon the rest of the Hamiltonian. In what follows we consider the case of the usual scalar potential, although some of the results are essentially independent of the form of the potential.

We write the energy of a superconducting loop, in units of $\pi\bar{v}^2$, as follows:

$$E(L) = [A_\Phi + w\ln(\mu L)]L - B_\sigma L + \frac{C_\sigma}{L[1 + \frac{e^2K}{2\pi}\ln(\kappa L)]} \tag{5.1}$$

where B_σ includes all the sigma dependent terms except for those associated with the charge carriers and magnetic field, given by C_σ, and A_Φ contains the remaining terms depending only upon Φ, which are independent of the loop length L. In the global case $w = 1$, and in the gauge case $w = 0$. The quantites A_Φ, B_σ, and C_σ are determined from Eqns(2.21) or (2.12) and (4.1). In general the energy per length of a superconducting string (with subcritical current) must be less than that of an ordinary cosmic string since the σ contribution to Eqn(5.1) is necessarily negative.

Note that the coefficients A_Φ, B_σ, and C_σ are all positive. The length of loop that minimizes the above energy, and represents the static state, is:

$$L_{static} = \left(\frac{C_\sigma\delta}{[A_\Phi + w + w\ln(\mu L) - B_\sigma][1 + \frac{e^2K}{2\pi}\ln(\kappa L_{static})]}\right)^{1/2} \tag{5.2}$$

where

$$\delta = 1 + \frac{\frac{e^2K}{2\pi}}{1 + \frac{e^2K}{2\pi}\ln(\kappa L_{static})} \tag{5.3}$$

Requiring L_{static} to be real gives us the condition that $A_\Phi + w[1 + \ln(\mu L)] > B_\sigma$.

A stable static configuration may be obtained if the critical current is not exceeded in the static configuration: this means that the sum of the last two terms in

Eqn(5.1) must be negative. We define the critical loop size, $L_{critical}$, to be the loop size for which the loop current is equal to the critical current:

$$L_{critical} = \left(\frac{C_\sigma}{B_\sigma[1 + \frac{e^2K}{2\pi} \ln(\kappa L_{critical})]} \right)^{1/2} \tag{5.4}$$

(In calculating $L_{critical}$ we have ignored the back reaction of the current upon the vortex and the condensate; including the back reaction may modify this result slightly.) Since we are seeking subcritical floating solutions, we must have $L_{static} > L_{critical}$ (to be perfectly safe we should probably require $L_{static} \gtrsim 3L_{critical}$ so that the back reaction of the current upon the vortex is less than $\sim$ 10%, and can be neglected). Using $L_{static} > L_{critical}$ as the criteron, we see that static loops are possible only if:

$$0 < A_\Phi + w[1 + \ln(\mu L)] - B_\sigma < \delta B_\sigma \frac{[1 + \frac{e^2K}{2\pi}\ln(\kappa L_{critical})]}{[1 + \frac{e^2K}{2\pi}\ln(\kappa L_{static})]} \tag{5.5}$$

If we require $\kappa L/\pi > 10$ so that our variational approach is not invalidated by curvature effects, we must have $1 \leq \delta < 1.3$. It is then apparent that the energy in the vortex is very close to that in the condensate (true for global or gauged loops), which is not surprising since a static string implies an equal balance of the energy between the σ and Φ fields. The energy per length of a static loop is

$$\frac{E(L)}{L} = [A_\Phi + w\ln(\mu L) - B_\sigma][1 + \frac{1}{\delta}] + \frac{w}{\delta} \tag{5.6}$$

For a given set of parameters that specify the potential A_Φ and B_σ are fixed. However, C_σ depends upon the winding number. In the canonical scenario of cosmological loop production, loops of size L_0 are continuously formed by breaking off from infinite strands of string when the age of the Universe is about $t_0 \sim L_0$[16,17]. This results in a loop having a winding number N of at least $\sim (t_0/\xi)^{1/2}$. Taking $N \propto (t_0/\xi)^{1/2}$ leads to a spectrum of static length sizes: $L_{static} \propto t_0^{1/2}$.

We now check to see if the concrete vortex approximation is appropriate for our study of static loops (and find out that it is not). The concrete vortex approximation is valid when:

$$\frac{\pi}{4}\beta x^2 F(x)\sigma_0^2 << A_\Phi + w\ln(\mu L) \tag{5.7}$$

Because we are considering static loops, $A_\Phi + w\ln(\mu L) \approx B_\sigma$, and inequality(5.7) becomes:

$$\beta x^2 F(x) << \frac{1}{2}[\alpha x^2 - \beta x^2 F(x) - 1] \tag{5.8}$$

A scan of the parameter space reveals that this condition is never satisfied, and there is always significant back reaction onto the vortex. This immediately locates the only possible region of parameter space where static loops might exist:

$$\frac{m_\Phi^4}{\lambda_\Phi} \approx \frac{m_\sigma^4}{\lambda_\sigma} \; (\Rightarrow \quad \alpha \approx \sqrt{a/2\gamma}) \qquad \text{and} \qquad \alpha \approx \beta \tag{5.9}$$

Since there is back reaction we must now consider the global and gauged case separately.

A. The Global Case

For a global loop, $w = 1$, and for an ordinary scalar potential:

$$A_\Phi = \frac{1}{4} - \ln(y) + \frac{y^2}{2} \tag{5.10}$$

and upon using our simple ansatz for σ:

$$B_\sigma = \frac{\gamma}{x'^2}\left(\alpha' x'^2 - \beta' x'^2 F(x'/y) - 1\right)^2 \tag{5.11}$$

From Eqn(5.5) it follows that the criterion for stable static global loops is:

$$A_\Phi + 1 + \ln(\mu_0 L) - B_\sigma < 1.3 B_\sigma \tag{5.12}$$

Since the size of the loop enters into the criterion, for a given potential there is always a maximum length L_{max} beyond which stable, static global loops do not exist:

$$\mu_0 L_{max} = \exp(2.3 B_\sigma - A_\Phi - 1) \tag{5.13}$$

For lengths bigger than $\sim L_{max}$ the critical current is reached before the loop shrinks to its static length. Recall that there is a minimum length we can consider without having to worry about loop curvature effects:

$$\mu_0 L_{min} = 10\pi y \tag{5.14}$$

This leads to yet another constraint: $L_{max} > L_{min}$. We note that small values of the ratio L_{max}/L_{min} imply that static global loops were only be produced during a very short cosmological time, early in the history of the Universe.

As mentioned before, the variational equations can be solved for the parameters α' and β' in terms of x and y. For the choice $\gamma = 10^{-4}$ we show the α', β' parameter space for $L_{max}/L_{min} \geq 1$, 1000, and 10^{10} in Figs. 14, 15, and 16. We see that loops much larger than the minimum length occupy an increasingly smaller portion of parameter space.

B. The Gauged Case

In the gauged case we do not encounter the peculiarity encountered with static global loops, i.e., the energy per length of the vortex is independent of the size of the loop. Static loops are more natural in this sense, as the existence of floating solutions only depends upon the parameters of the potential, and not also upon L_{static} as it does in the global case. However, here too floating solutions only occupy a very small portion of parameter space. From Eqn(5.5) it follows that our criterion for static gauge loops is:

$$\frac{A_\Phi}{B_\sigma} < 2.3 \tag{5.15}$$

The full Hamiltonian, given by Eqn(2.62), was numerically minimized for $\gamma = 10^{-4}$ for three different values of b: 0.01, 1, and 100. Superconducting solutions satisfying the above constraint are shown in Figs. 17, 18, and 19, where we plot α' vs. β' for each value of b. In each case, the allowed region of parameter space for floating loops is very small. To gain a proper perspective of just how small this parameter space is, we note that for the aforementioned cases the area of our static solutions to that of our superconducting solutions is $\sim 1:1000$.

To summarize floating loops, we find that there is a small region of the parameter space of solutions where stable, static solutions exist, as specified by:

$$f \simeq \sqrt{\lambda_\Phi \lambda_\sigma}/3 \qquad m_\sigma \simeq m_\Phi (\lambda_\sigma/\lambda_\Phi)^{1/4} \tag{5.16}$$

VI. Model Building

In this section we describe a procedure for choosing the 5 parameters of the potential:

$$V(v,u) = -m_\Phi^2 v^2/2 + \frac{\lambda_\Phi v^4}{4!} - m_\sigma^2 u^2/2 + \frac{\lambda_\sigma u^4}{4!} + f v^2 u^2/4 + \frac{3m_\Phi^4}{2\lambda_\Phi} \tag{6.1}$$

such that superconducting vortex solutions exist. However, we first discuss the natural values for the parameters f, λ_Φ, and λ_σ. The coupling f can be arbitrarily small since it is multiplicatively renormalized (note that this requires no mixing between the $U(1)$ and $U(1)'$ gauge bosons and is special to our model; it may not be a general feature of this mechanism in other settings), but it cannot be larger than ~ 1, the point perturbativity is lost. It is clear that λ_σ and λ_Φ cannot be arbitrarily small since the exchange of σ and Φ loops require $\sim f^2$ counterterms. If we try to define the renormalized theories with λ_σ and λ_Φ smaller than $\sim f^2$ we will have the values of physical quantities like σ determined by effective potentials of the Coleman–Weinberg type rather than by the tree approximation potentials, and we will effectively recover the same constraints. Moreover, for small f the gauge loop corrections require that λ_σ exceed $\sim e^4$ and λ_Φ exceed $\sim q^4$. For global loops $q = 0$, and for gauged loops $q^2 = bm_\Phi^2/\bar{v}^2$ (which requires $\lambda_\Phi \lesssim b^{-2}$). We then see, in the gauged case, that the natural range for b is: $10^{-2} \lesssim b \lesssim 10^2$ (where we also assume that q is of the general order of e). For $b \gtrsim 10^2$ the appropriate effective potential is that of the Coleman–Weinberg type, and for $b \lesssim 10^{-2}$ a non-linear σ model approximation becomes appropriate.

We presently describe a method, though not unique, that allows one to construct a potential which permits bosonic superconductivity. This can easily be done with the aid of several of our graphs for the case that there is negligible back reaction—which is a good approximation unless $\alpha \approx \beta$, and in any case is always a good starting point.

One can first pick the quartic couplings and f (consistent with perturbativity and the Coleman–Weinberg limit), and b if the string is gauged. This determines β:

$$\beta = 3fa/\lambda_\Phi \tag{6.2}$$

where a can be determined from Fig. 2. In the global case, $a \approx 1.6$, while in the gauge case, a is typically of the order of unity. **Constraint (1)** and Fig. 4 restrict the possible choices of α:

$$max[1/7,\ \alpha(\beta)] \leq \alpha \leq min[\beta, a\sqrt{\lambda_\sigma/\lambda_\Phi}] \tag{6.3}$$

where $\alpha(\beta)$ is the upper boundary in Fig. 4, which for $\beta \geq 50$ is given by: $\alpha(\beta) \simeq 1.43\beta^{0.52}$. For solutions to exist at all, λ_σ, λ_Φ, f, and b must be selected accordingly, i.e., $a\sqrt{\lambda_\sigma/\lambda_\Phi} \geq 1/7$. Finally, any values of m_Φ and m_σ may be picked consistent with:

$$m_\sigma^2/m_\Phi^2 = \alpha/a \tag{6.4}$$

If for the parameters chosen, $\alpha \simeq \beta$ and $\alpha \simeq a\sqrt{\lambda_\sigma/\lambda_\Phi}$, back reaction is likely to be important, and one may wish to take it into account (see Sec. IID).

As an aside, we mention how one can arrange the potential to have static loops. If the parameters λ_σ, λ_Φ, and m_Φ are selected, the other parameters are essentially determined:

$$m_\sigma \simeq m_\Phi \left(\frac{\lambda_\sigma}{\lambda_\Phi}\right)^{1/4} \qquad \text{and} \qquad f \simeq \frac{1}{3}\sqrt{\lambda_\sigma\lambda_\Phi} \tag{6.5}$$

The Coleman–Weinberg limit is automatically satisfied in this case.

VII. Concluding Remarks

Let us summarize our work. By using simple variational ansatze we have studied a number of important properties of cosmic strings. First, we have computed (to an accuracy of better than 2%) the energy per length of ordinary gauge cosmic strings as a function of $b = 6q^2/\lambda_\Phi = 2m_V^2/m_H^2$; our results are displayed in Figs. 1 and 2. It is very apparent that the energy per length is insensitive to b. Since the critical temperature for the phase transition which produces cosmic strings $T_c \sim \bar{v}(1+b)^{-1/2} \propto (1+b)^{-1/2}(\text{energy/length})^{1/2}$, this implies that for cosmic string of fixed string tension, one can by appropriate tuning of λ_Φ (i.e., $b \gg 1$) make T_c much smaller than its natural value $\sim (\text{energy/length})^{1/2}$. This fact may be of some importance if one is interested in producing cosmic string in inflationary Universe models[18], where the temperatures reached after inflation are typically much, much smaller than $\bar{v} \sim 10^{15} - 10^{16}$ GeV, the scale associated with the energy per length required for 'cosmologically interesting' strings.

Second, we have mapped out the scalar potential parameter space for bosonic superconductivity. The parameter space of solutions is shown in Figs. 4 and 11, and the constraints are summarized in Table II. From our analysis it is quite apparent that bosonic superconductivity does not require a fine-tuning of the parameters in the scalar potential, and in fact may be quite a generic phenomenon.

Our study of the dynamics of bosonic superconductivity included a quantitative discussion of the critical current, which we define to be the current such that the energy associated with the σ condensate becomes non-negative (at which point it becomes energetically favorable to the system to disperse the σ condensate). Our analysis indicates that in general the quench transition is likely to be first order; however, the question of metastability of supercritical currents must still be addressed.

With the exception of a small region of the solution parameter space, the 'back reaction' of the σ condensate upon the vortex itself is small (see Fig. 4), and the σ condensate can be treated as existing on a 'concrete flux tube'. For $\alpha \approx \beta$ and $m_\sigma^4/\lambda_\sigma \approx m_\Phi^4/\lambda_\Phi$ (so that $f \sim (\lambda_\sigma\lambda_\Phi)^{1/2}/3$) the back reaction can be significant. In this regime, gauge or global strings may be able to achieve a static (or floating) configuration for subcritical currents; however, since the back reaction is significant the analysis is difficult, and it is probably still premature to say with confidence that such states are possible. We can say with confidence that the parameter space which allows floating configurations is very tiny (see Figs. 14-19).

There are still a number of important issues to be addressed. Precisely how does a superconducting string quench when the critical current is exceeded, and does the quench lead to detectable effects (e.g., UHE cosmic rays[2])? While it has been shown that fermionic loops cannot achieve a floating state by the support of the kinetic energy of the charge carriers alone, no thorough analysis similar to ours has yet been performed which includes the electromagnetic stresses also. Our results for the bosonic case would suggest that the possibility is unlikely.

Finally, there is no particular obstacle to extending variational analyses of this type to a large number of related cosmic string issues. For example, of considerable importance is a microphysical understanding of cusps[19]. Cusps arise as singularities in the world sheet description of cosmic strings, but are clearly nonsingular configurations of the Φ and σ fields. It seems interesting to us to develop a similar analysis of the σ field in the presence of, say, a *concrete cusp* in the Φ field to answer the question of what, if any, are the changes in the critical current and the energetics of the σ condensate at the cusp. For example, we wish to know if superconductivity is destroyed by a cusp of given extrinsic curvature for a given value of the local current. This study is currently in progress.

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References

1. E. Witten, *Nucl. Phys.* **B249**, 557 (1985).
2. C.T. Hill, D.N. Schramm, and T.P. Walker, *Phys. Rev. D*, in press (1987).
3. J. Ostriker, C. Thompson, and E. Witten, *Phys. Lett.* **180B**, 231 (1986).
4. E. Copeland, M. Hindmarsh, and N. Turok, *Phys. Rev. Lett.* **58**, 1910 (1987).
5. H. Hodges and M.S. Turner, unpublished report, Dec. 1986.
6. H. Nielsen and P. Olesen, *Nucl. Phys.* **B61**, 45 (1973).
7. A. Vilenkin and A.E. Everett, *Phys. Rev. Lett.* **48**, 1867 (1982).
8. H. de Vega and F. Schaposnik, *Phys. Rev.* **D14**, 1100 (1976).
9. L. Jacobs and C. Rebbi, *Phys. Rev.* **B19**, 4486 (1978).
10. E. Bogomol'nyi and A. Vainshtein, *Sov. J. Nucl. Phys.* **23**, 588 (1976).
11. D. Garfinkle, *Phys. Rev.* **D32**, 1323 (1985).
12. V.L. Ginzburg and L.D. Landau, *Zh. Eksp. Teor. Fiz.* **20**, 1064 (1950).
13. S. Coleman and E. Weinberg, *Phys. Rev.* **D7**, 1888 (1973).
14. M.S. Turner and L. Widrow, in preparation.
15. F. Accetta, Ph.D. thesis, in preparation.
16. A. Albrecht and N. Turok, *Phys. Rev. Lett.* **54**, 1868 (1985).
17. A. Vilenkin, *Phys. Rep.* **121**, 263 (1985).
18. For a recent comprehensive review of inflation, see, e.g., M.S. Turner, in *Architecture of Fundamental Interactions at Short Distances*, eds. P. Ramond and R. Stora (Amsterdam, North-Holland, 1987).
19. D.N. Spergel, J. Goodman, and T. Piran, *Nucl. Phys. B*, in press (1987).

Figure Captions

1. In (a) we show the energy per length of the gauged vortex over the natural range of $b = 6q^2/\lambda_\Phi = 2m_V^2/m_H^2$. Our numerical results are well fit (to better than 5 %) by energy per length $= 1.19\pi\bar{v}^2 b^{-0.195}$; here $\bar{v} = \sqrt{6m_\Phi^2/\lambda_\Phi}$ is the VEV of the real part of the Φ field. For completeness, in (b) we show the energy per length over an extended range in b.

2. The parameter $a = m_\Phi^2/\mu^2$ vs. b for the gauged vortex. The width of the vortex $\mu^{-1} \sim \sqrt{a} m_\Phi^{-1}$.

3. The variational parameter s ($\approx$ the ratio of the size of the magnetic flux tube to that of the vortex) is shown vs. b.

4. The allowed α, β parameter space for superconducting solutions with the ordinary unstable scalar potential. The region between the two solid lines is the allowed region mapped out by our simple ansatz; the dashed line is the upper boundary obtained from our full four-parameter ansatz. For $\alpha \geq 10$, the upper boundary from the full ansatz is $\beta \simeq 0.5\alpha^{1.93}$; for $\alpha \leq 10$ the upper boundary is $\beta \simeq 1.32\alpha^{1.35}$. The solid triangles indicate regions of parameter space where the back reaction of the σ condensate on the vortex is significant (so defined by $y \geq 1.2$) for $\gamma = 10^{-4}$, 10^{-6}, and 10^{-8}.

5. The variational parameter x ($\approx$ the ratio of the size of the condensate to that of the vortex) is shown vs. α.

6. The allowed α' parameter space as a function of y for a global string with $\gamma = 10^{-3}$. Significant deviations from the concrete vortex model occur for large y.

7. The allowed β' parameter space as a function of y for a global string with $\gamma = 10^{-3}$.

8. The allowed α' parameter space as a function of y for a global string with $\gamma = 10^{-8}$.

9. The allowed β' parameter space as a function of y for a global string with $\gamma = 10^{-8}$.

10. The region of parameter space ($A = (qv/ev')^4$, $C = \beta/\chi$) that corresponds to a true vacuum with $< \sigma >, < \Phi > \neq 0$. This region, which corresponds to electromagnetism being broken far from the vortex, is strictly disallowed.

11. The χ, β parameter space of solutions for superconducting vortices for Coleman-Weinberg potentials.

12. The α, β parameter space of solutions consistent with $K\lambda_\sigma > 100$. In this region the magnetic field energy dominates that of the KE of the charge carriers. Solid lines indicate the entire parameter space of solutions.

13. The critical currents of bosonic superconducting strings are shown as a function of α, for all possible choices of β consistent with $K\lambda_\sigma > 100$.

14. The allowed α', β' parameter space for a floating global string with $\gamma = 10^{-4}$ and $L_{max}/L_{min} \geq 1$.

15. The allowed α', β' parameter space for a floating global string with $\gamma = 10^{-4}$ and $L_{max}/L_{min} \geq 1000$.

16. The allowed α', β' parameter space for a floating global string with $\gamma = 10^{-4}$ and $L_{max}/L_{min} \geq 10^{10}$.

17. The allowed α', β' parameter space for a floating gauged string with $\gamma = 10^{-4}$ and $b = 0.01$.

18. The allowed α', β' parameter space for a floating gauged string with $\gamma = 10^{-4}$ and $b = 1$.

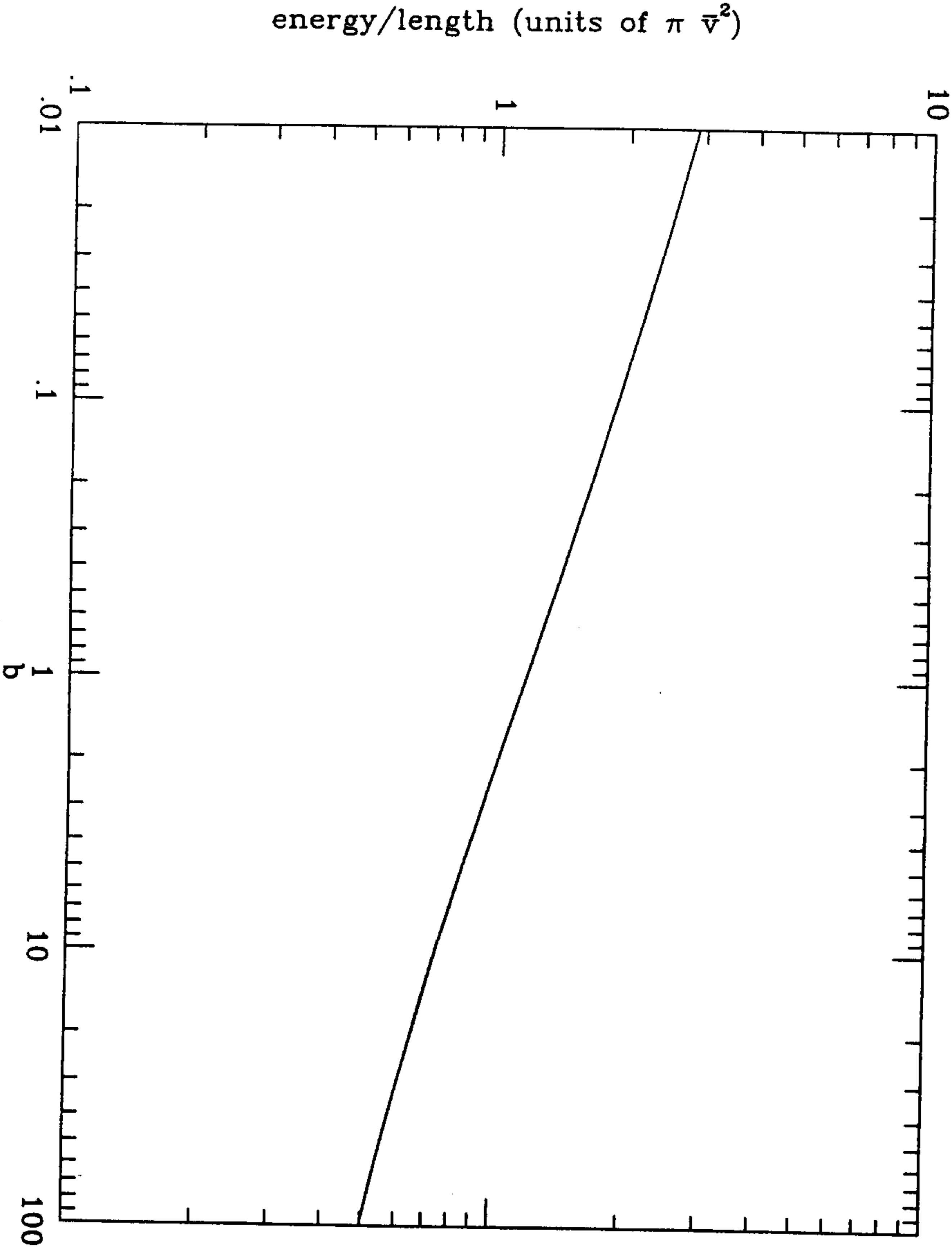
19. The allowed α', β' parameter space for a floating gauged string with $\gamma = 10^{-4}$ and $b = 100$.

Table I. Summary of Parameters for the Scalar Potential: $V(v,u) = -m_\Phi^2 v^2/2 + \frac{\lambda_\Phi v^4}{4!} + \frac{3m_\Phi^4}{2\lambda_\Phi} - m_\sigma^2 u^2/2 + \frac{\lambda_\sigma u^4}{4!} + fu^2v^2/4$. Primed quantities are related to their unprimed counterparts by: $a' = a/y^2$, $s' = sy$, $\alpha' = \alpha/y^2$, $x' = yx$, and $\beta' = \beta/y^2$ (see Sec. IID).

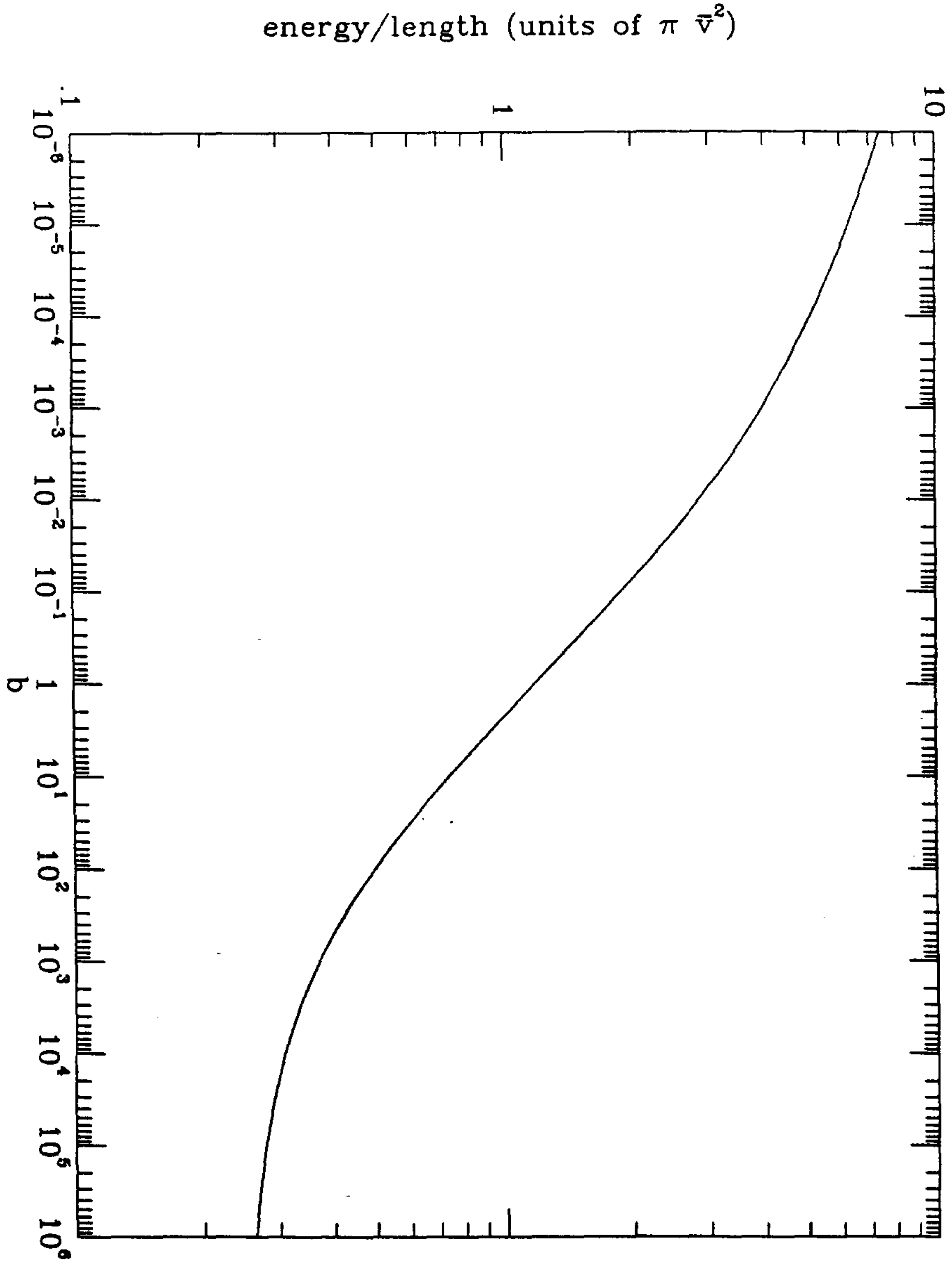
Parameter:	Definition:	Comments:
$\bar{v}$	$\bar{v}^2 = 6m_\Phi^2/\lambda_\Phi$	VEV of the real part of Φ
a	m_Φ^2/μ^2	size of vortex $\approx \sqrt{a}/m_\Phi$ for a global string $a \simeq 1.6$; for a gauged string $a \sim$ order unity
b	$q^2\bar{v}^2/m_\Phi^2 = 6q^2/\lambda_\Phi$	2 times the ratio of the squares of vector to scalar masses
s	μ/h	ratio of the size of the magnetic flux tube to that of the vortex
α	$m_\sigma^2/\mu^2 \approx am_\sigma^2/m_\Phi^2$	convenient normalization for m_σ^2 by vortex scale μ^2
β	$f\bar{v}^2/2\mu^2 \approx 3af/\lambda_\Phi$	convenient normalization for interaction by vortex scale μ^2
x	μ/κ	ratio of the size of σ condensate to that of vortex
y	μ_0/μ	true size of vortex to that obtained in concrete vortex approximation
γ	$3\mu_0^2/\bar{v}^2\lambda_\sigma = (\lambda_\Phi/\lambda_\sigma)/2a$	in terms of γ constraint 1 is: $\alpha < (a/2\gamma)^{1/2}$
χ	$3e^4v'^2/64\pi^2\mu^2$	analogue of α in the Coleman–Weinberg case

Table II. Summary of Constraints on a Bosonic Superconducting Cosmic String with Unstable Quartic Scalar Potential

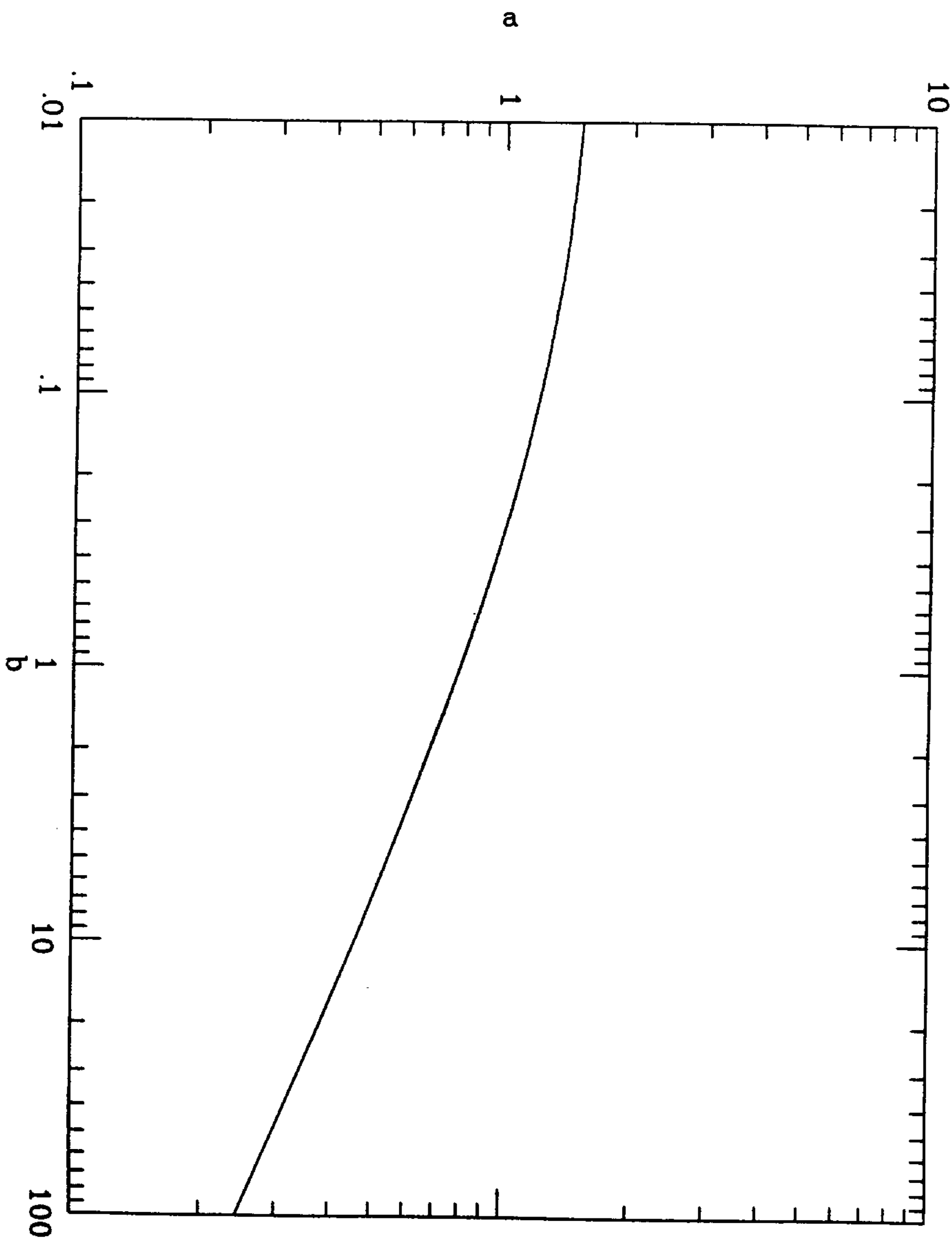
	Constraint:	Comments:
(1)	$m_\Phi^4/\lambda_\Phi > m_\sigma^4/\lambda_\sigma$	Stability of vacuum against breaking of electromagnetism
(2)	$\beta > \alpha$	Stability of vacuum against breaking of electromagnetism
(3)	$\sigma_0^2 > 0$	Existence of true condensate for $\alpha \geq 10$ requires $\alpha \leq \beta \leq 0.5\alpha^{1.93}$
(4)	$m_\sigma \simeq (\lambda_\sigma/\lambda_\Phi)^{1/4} m_\Phi$ $f \simeq \sqrt{\lambda_\Phi \lambda_\sigma}/3$	Existence of floating solutions



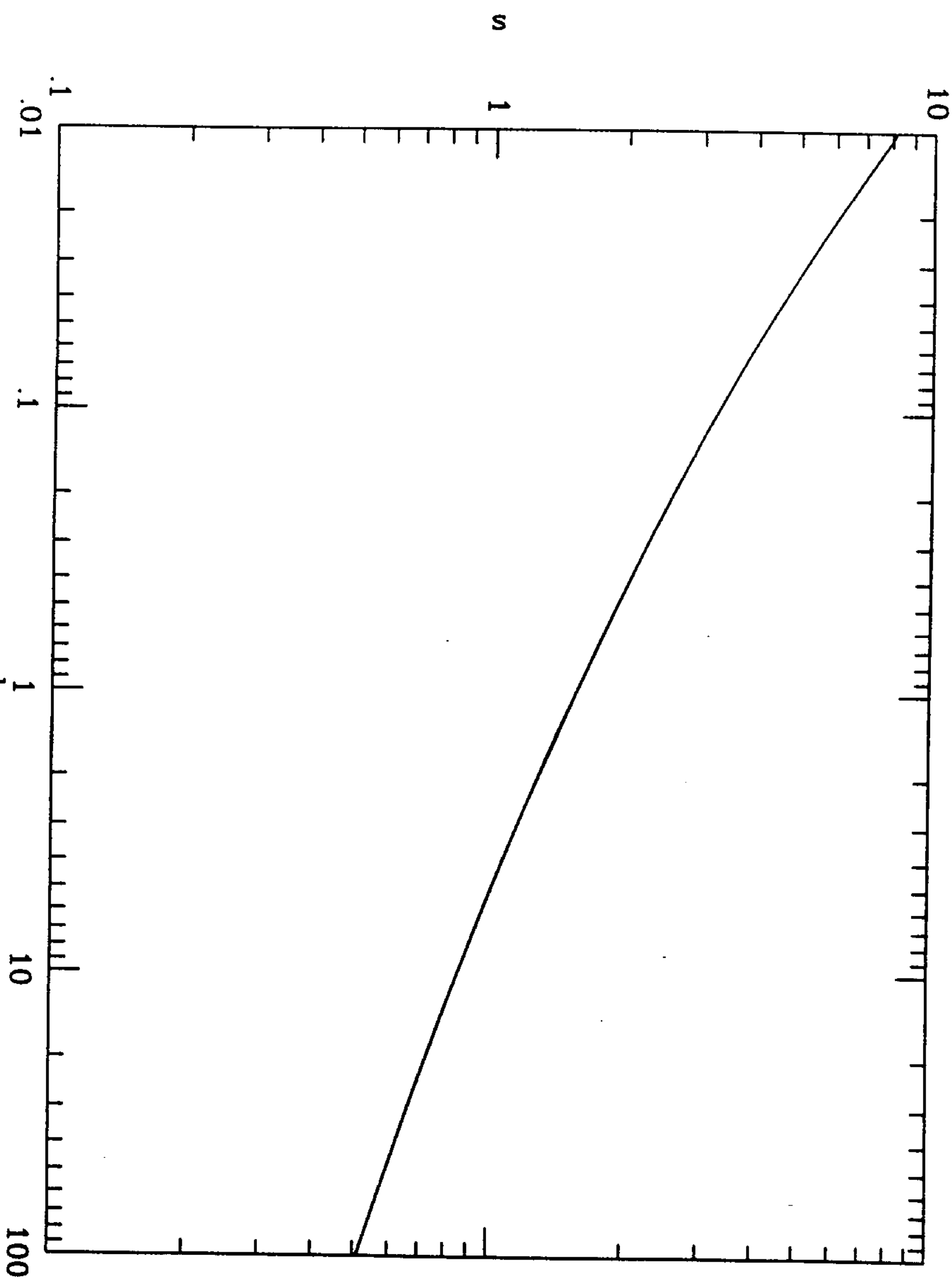
- FIG 1a -


energy/length (units of π v̄²)
.1
1
10
b
10⁻⁶
10⁻⁵
10⁻⁴
10⁻³
10⁻²
10⁻¹
1
10¹
10²
10³
10⁴
10⁵
10⁶

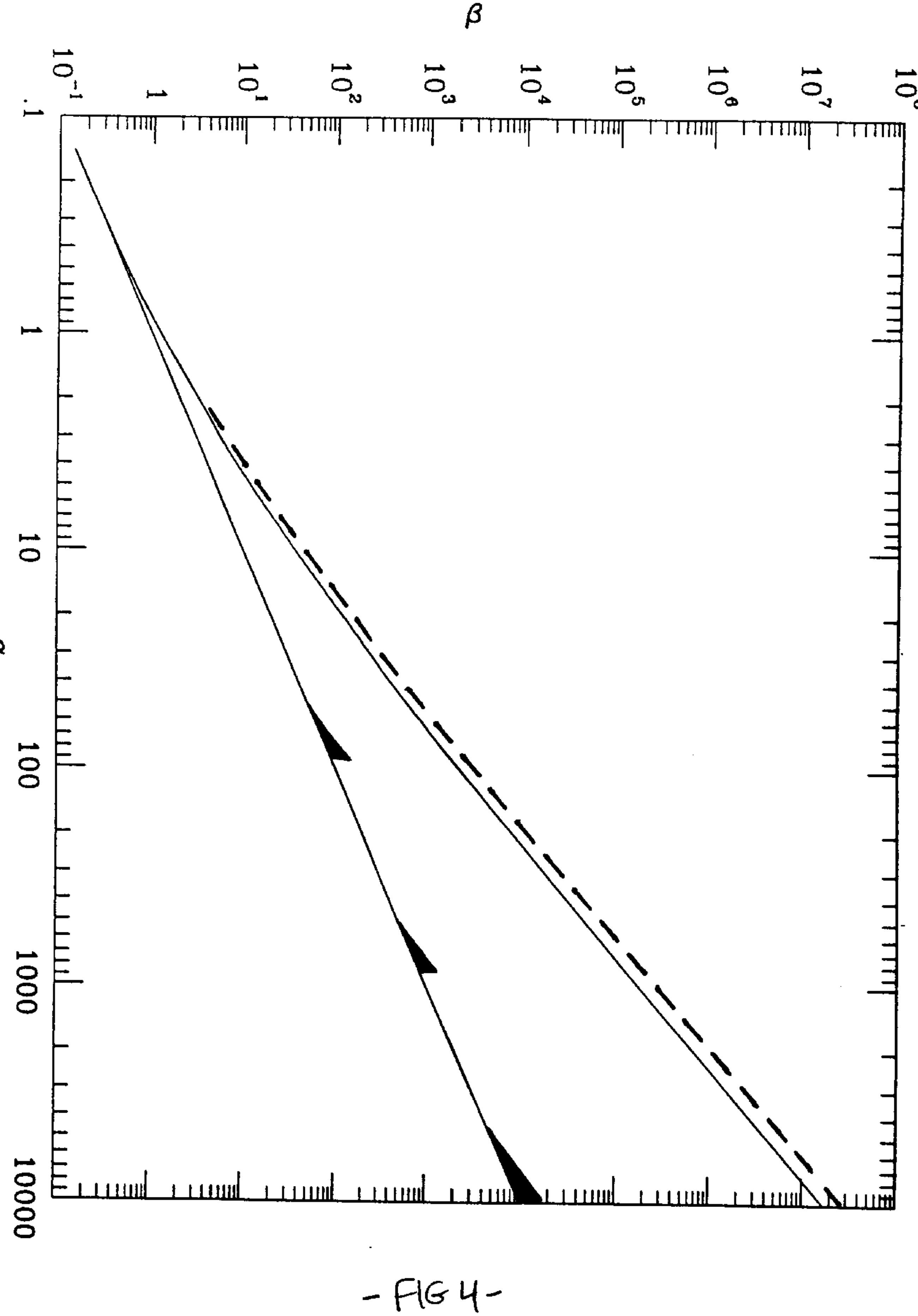
-FIG 1b-



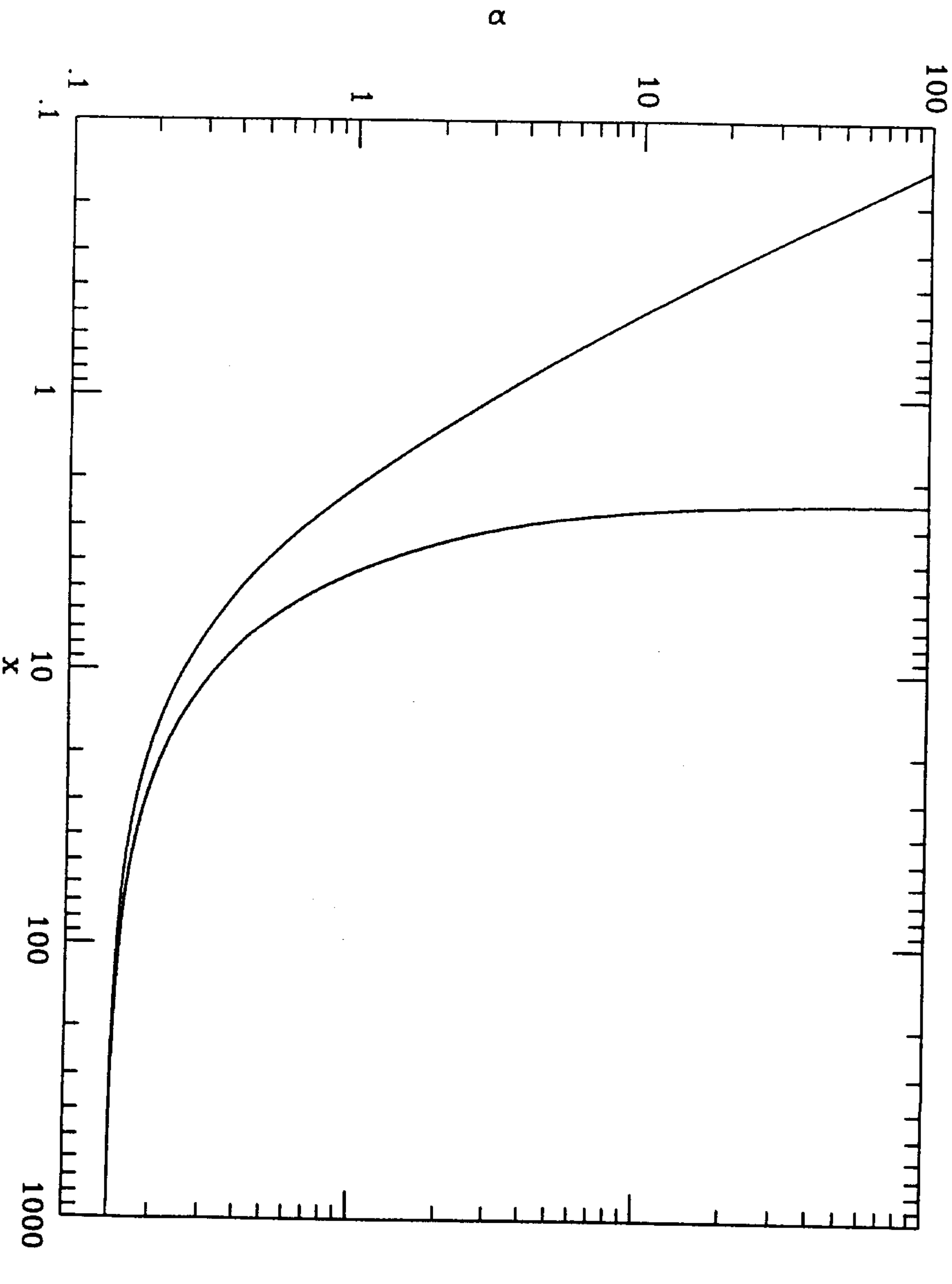
-FIG 2-



- FIG 3 -



- FIG 4 -



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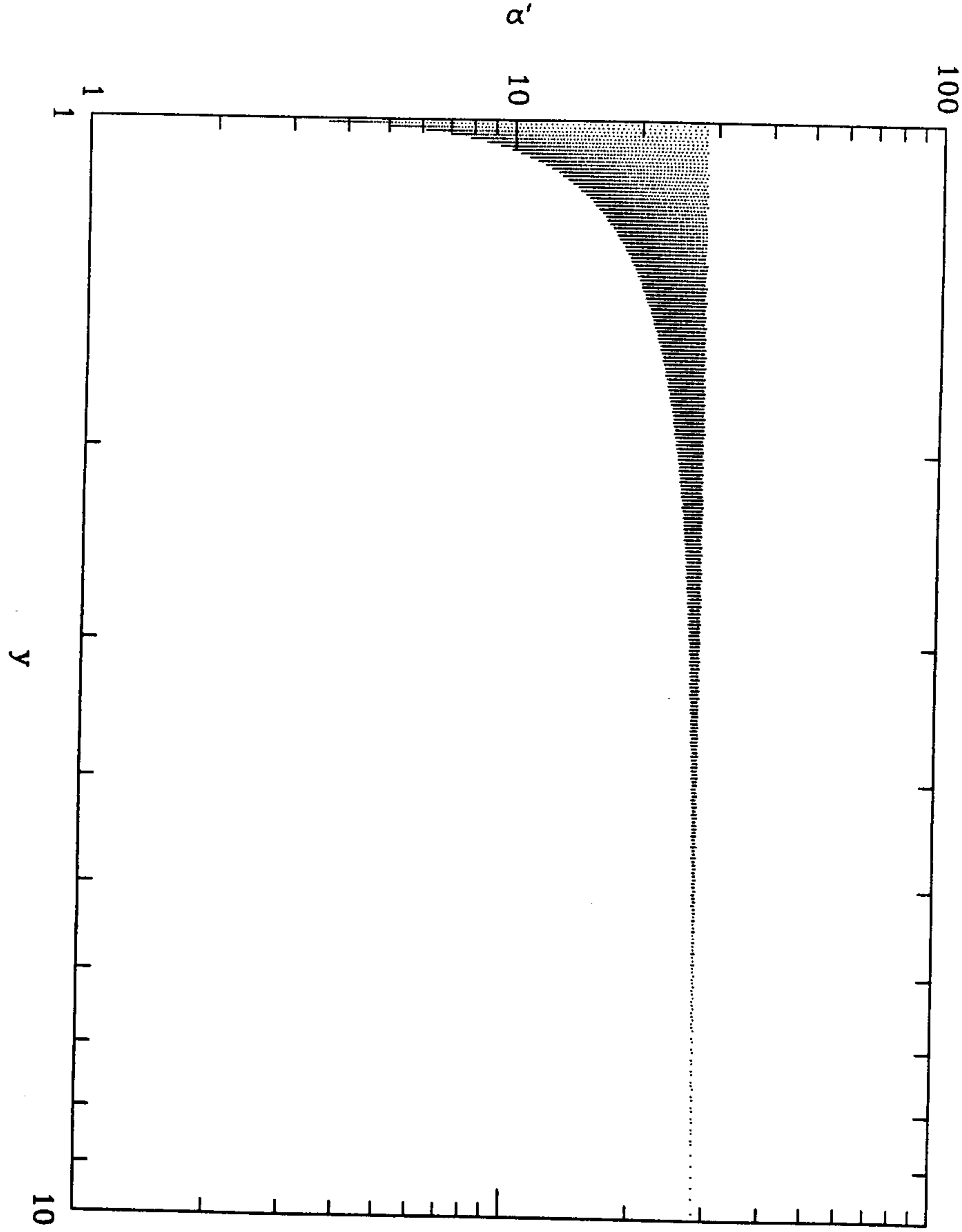

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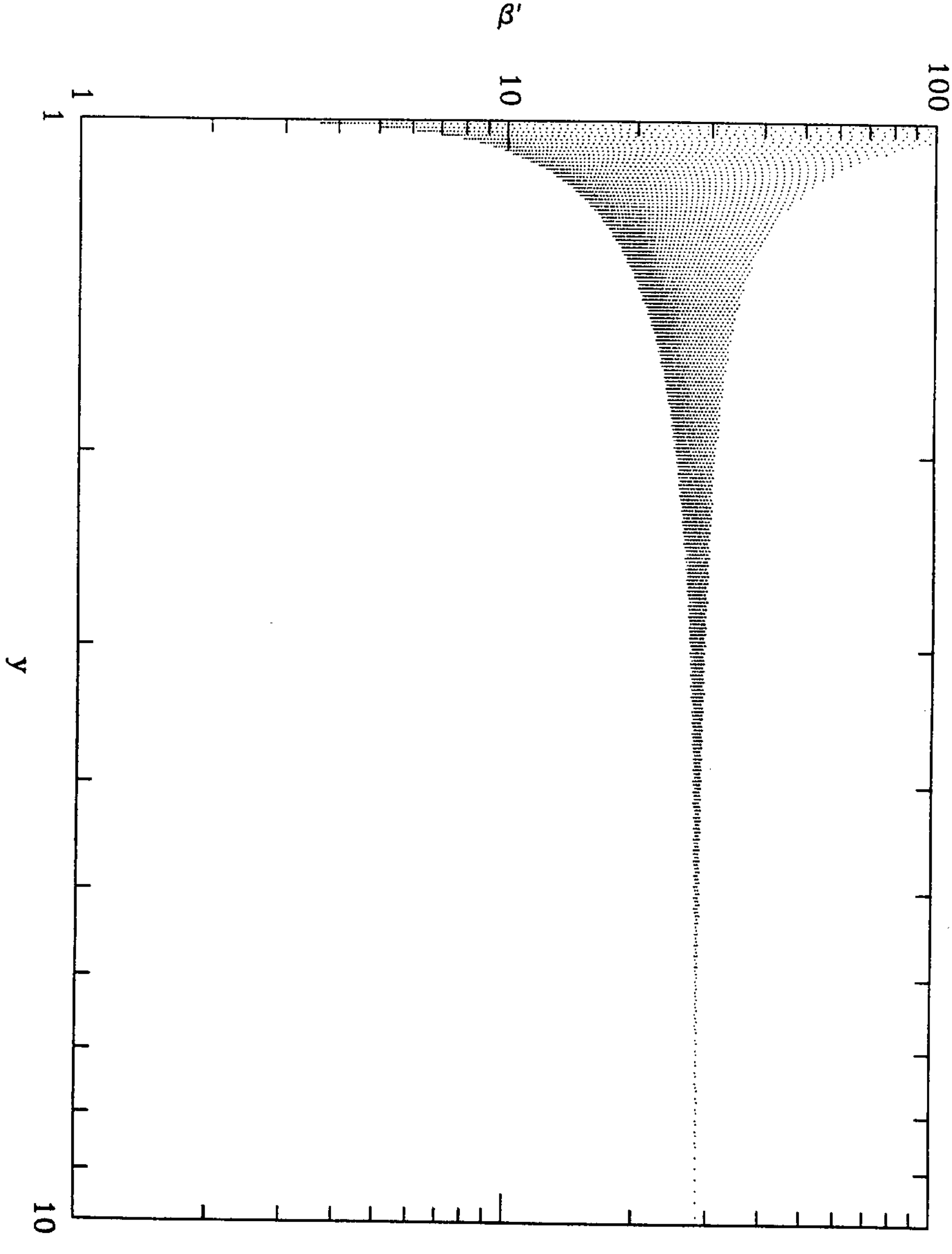

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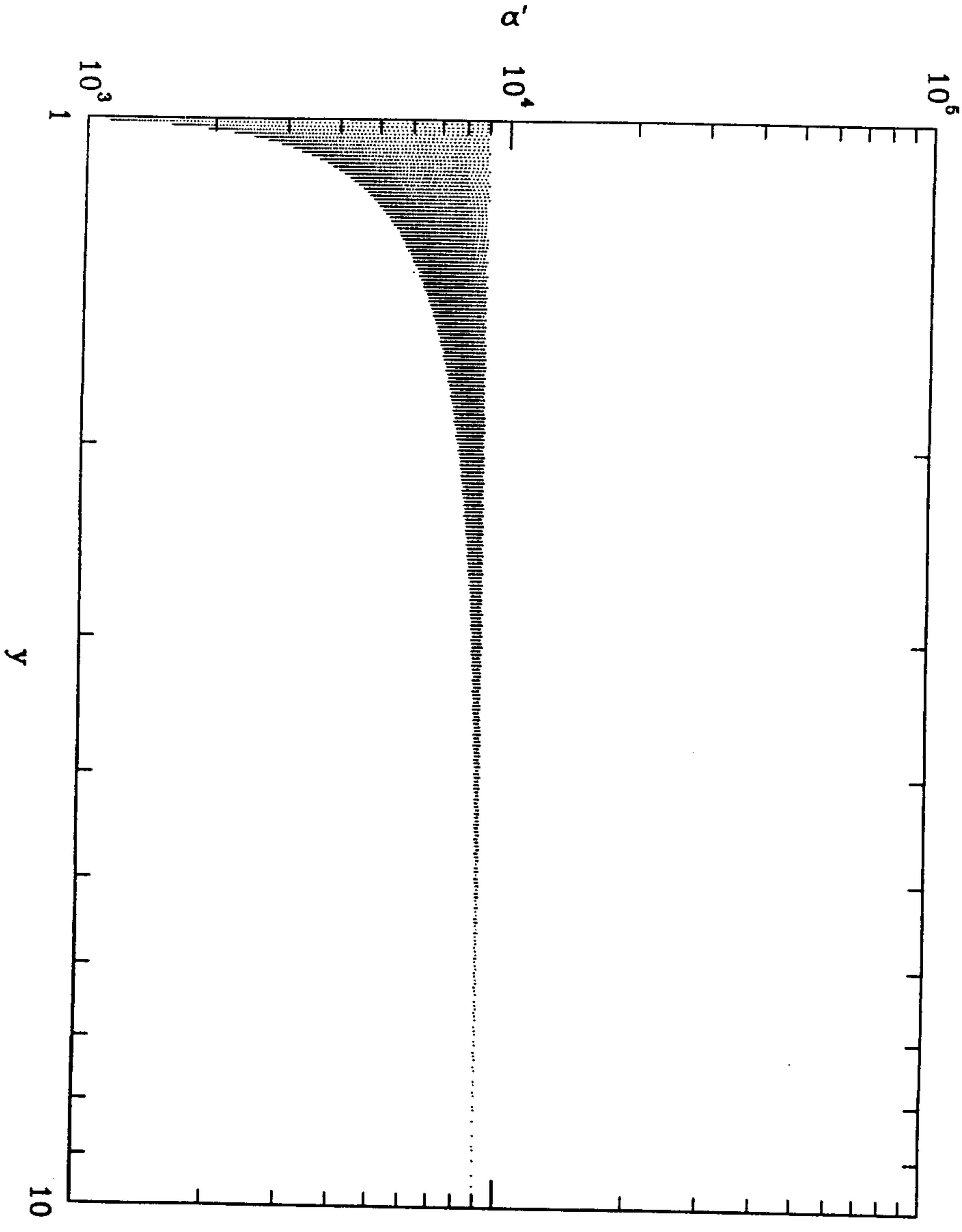
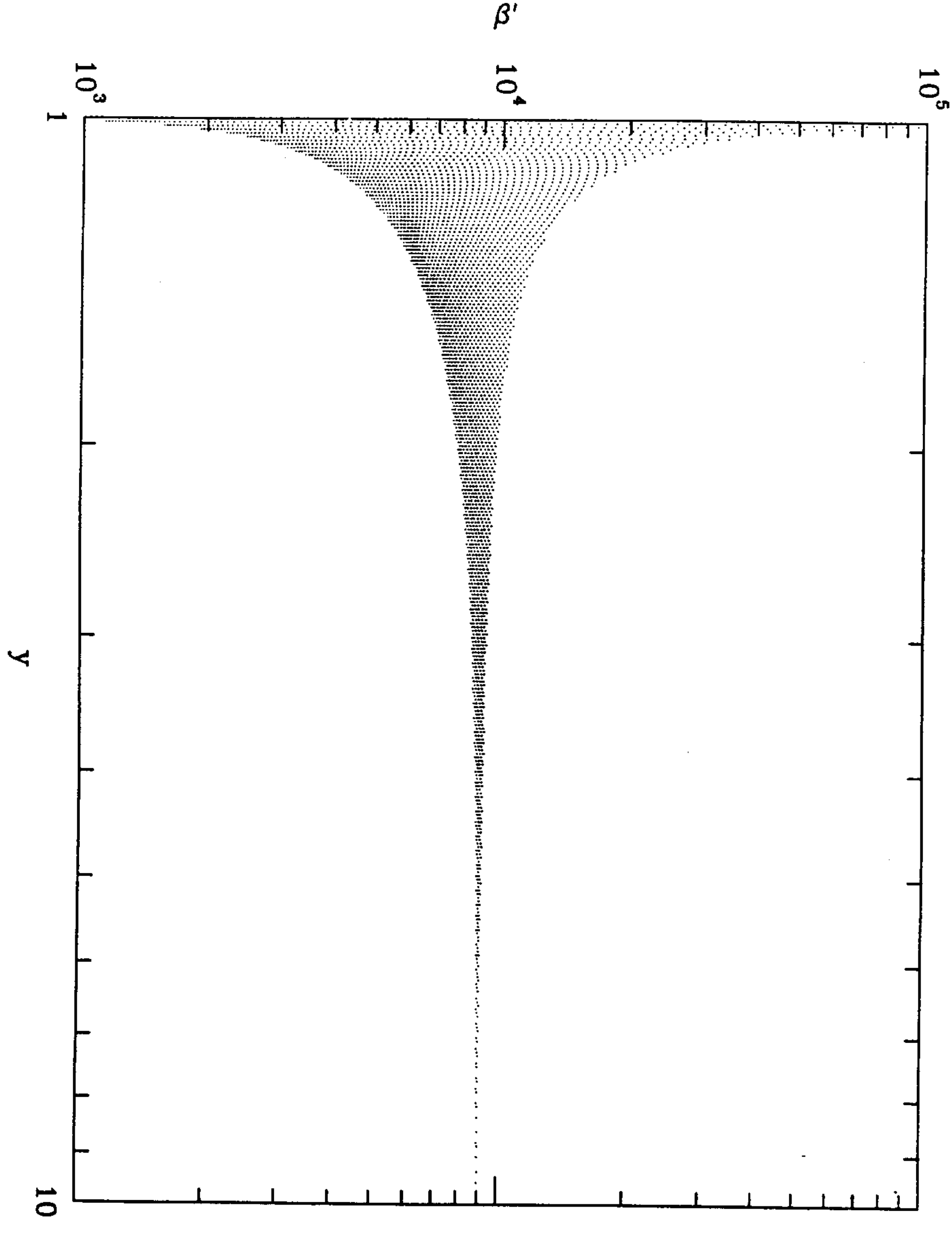
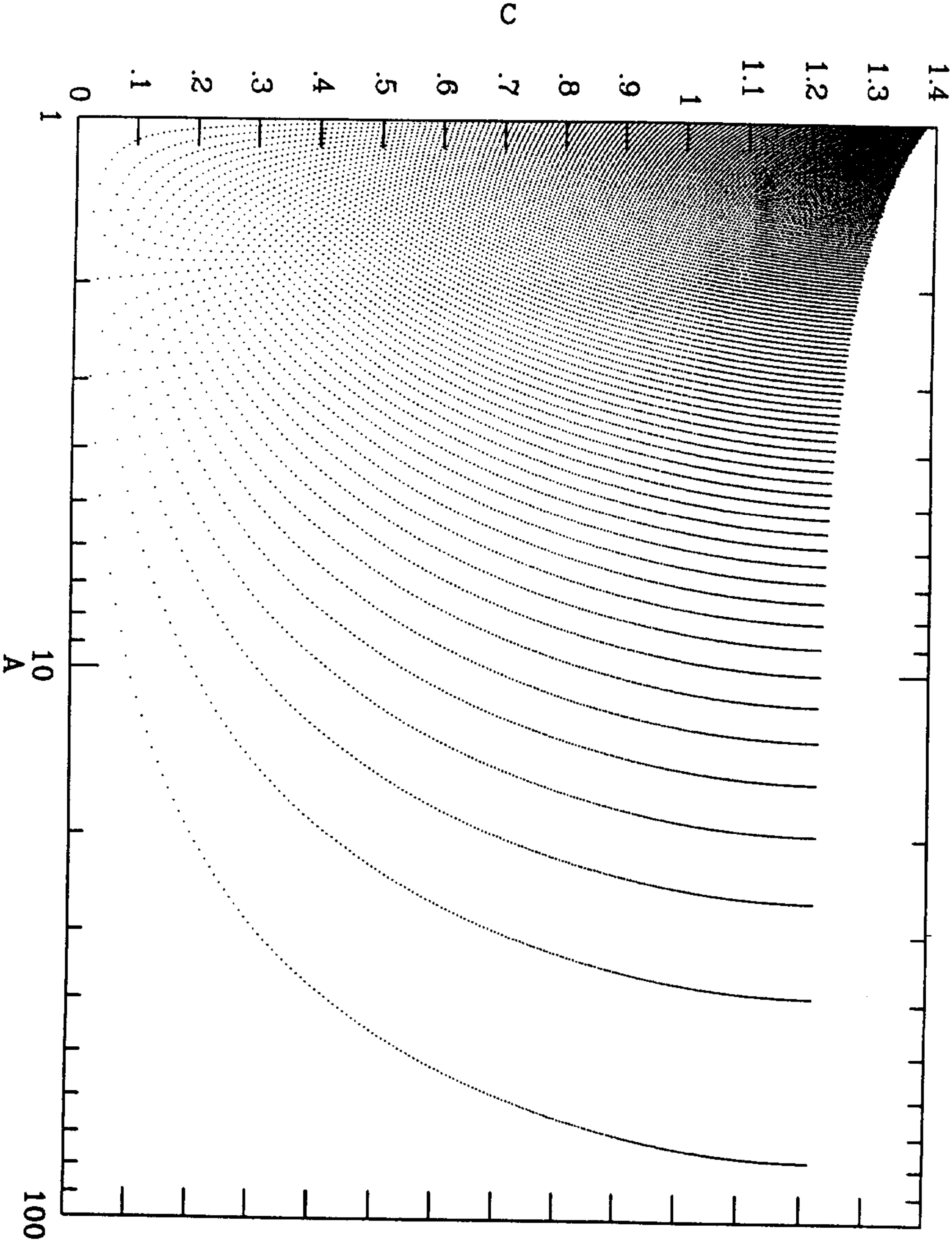


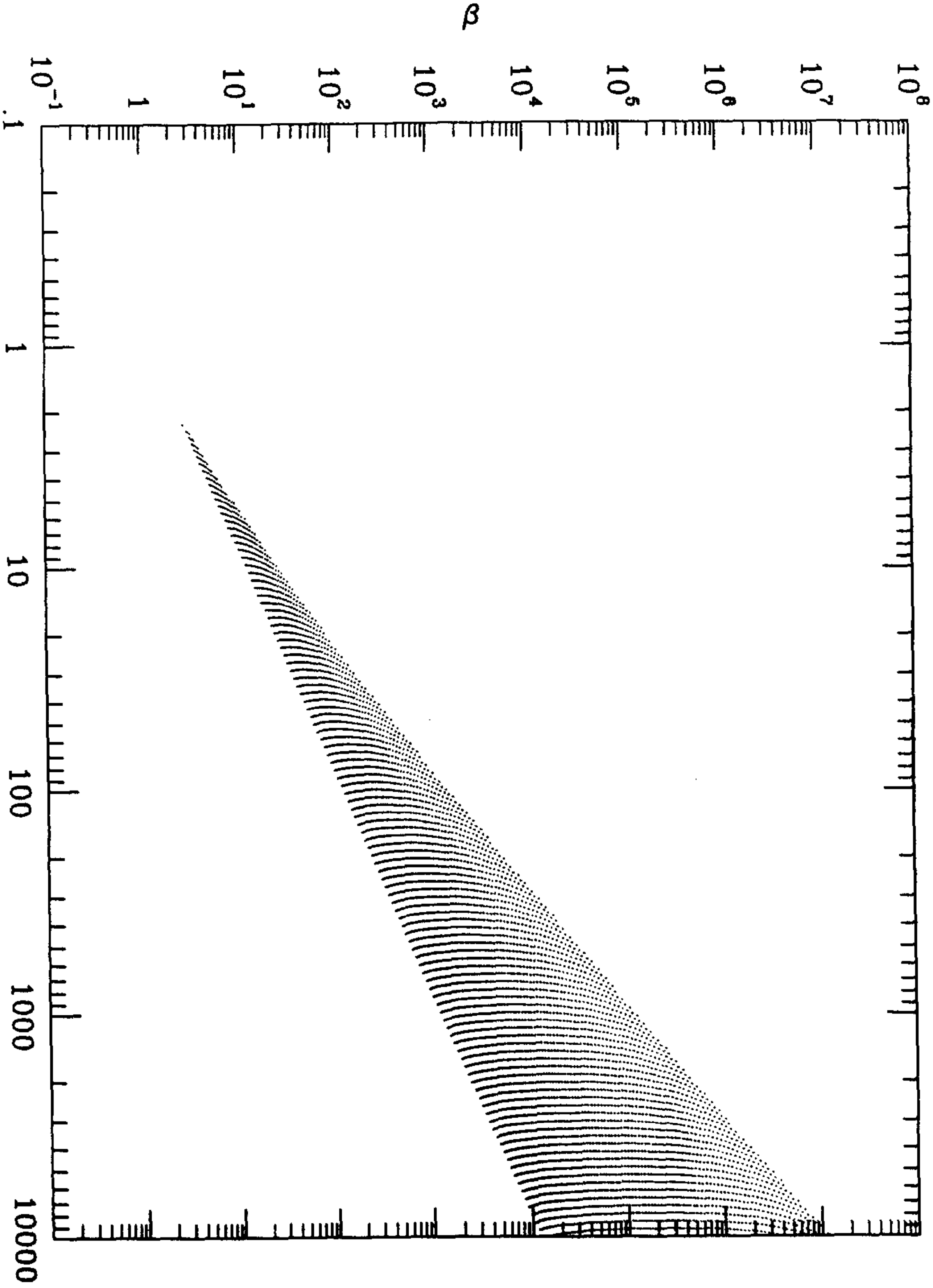
- FIG 8 -



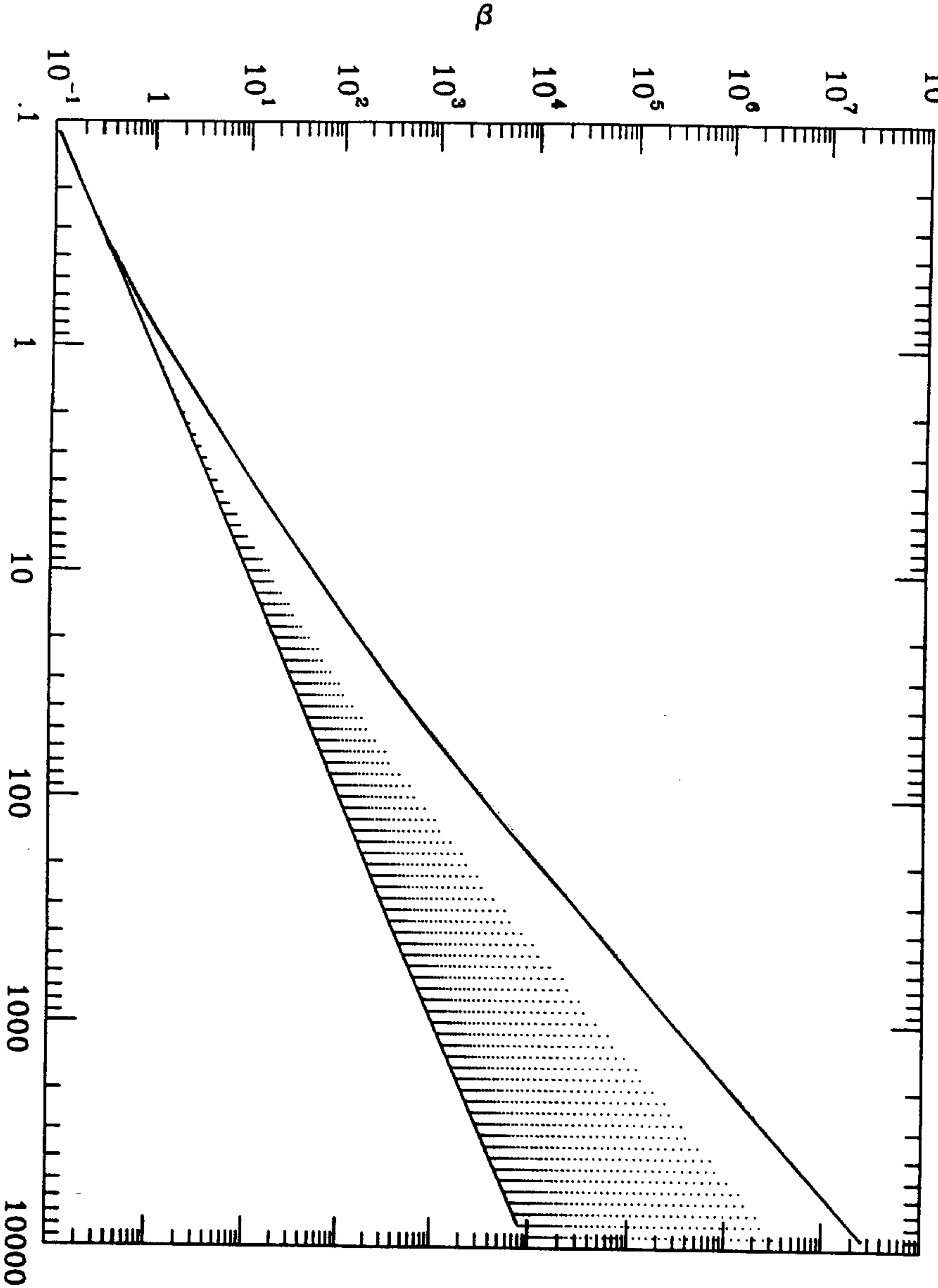
- FIG 9 -



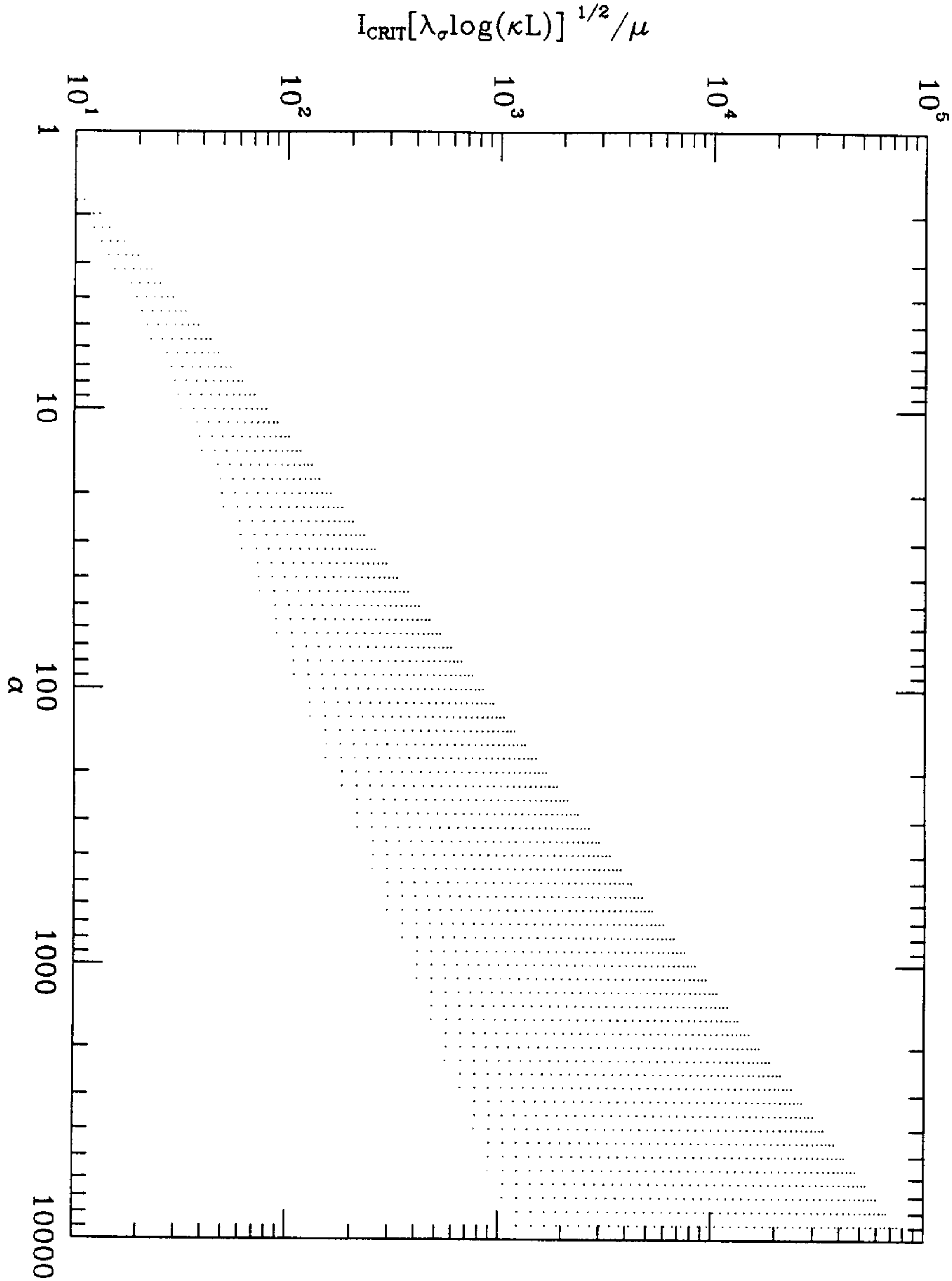
-FIG 10-



-FIG 11-

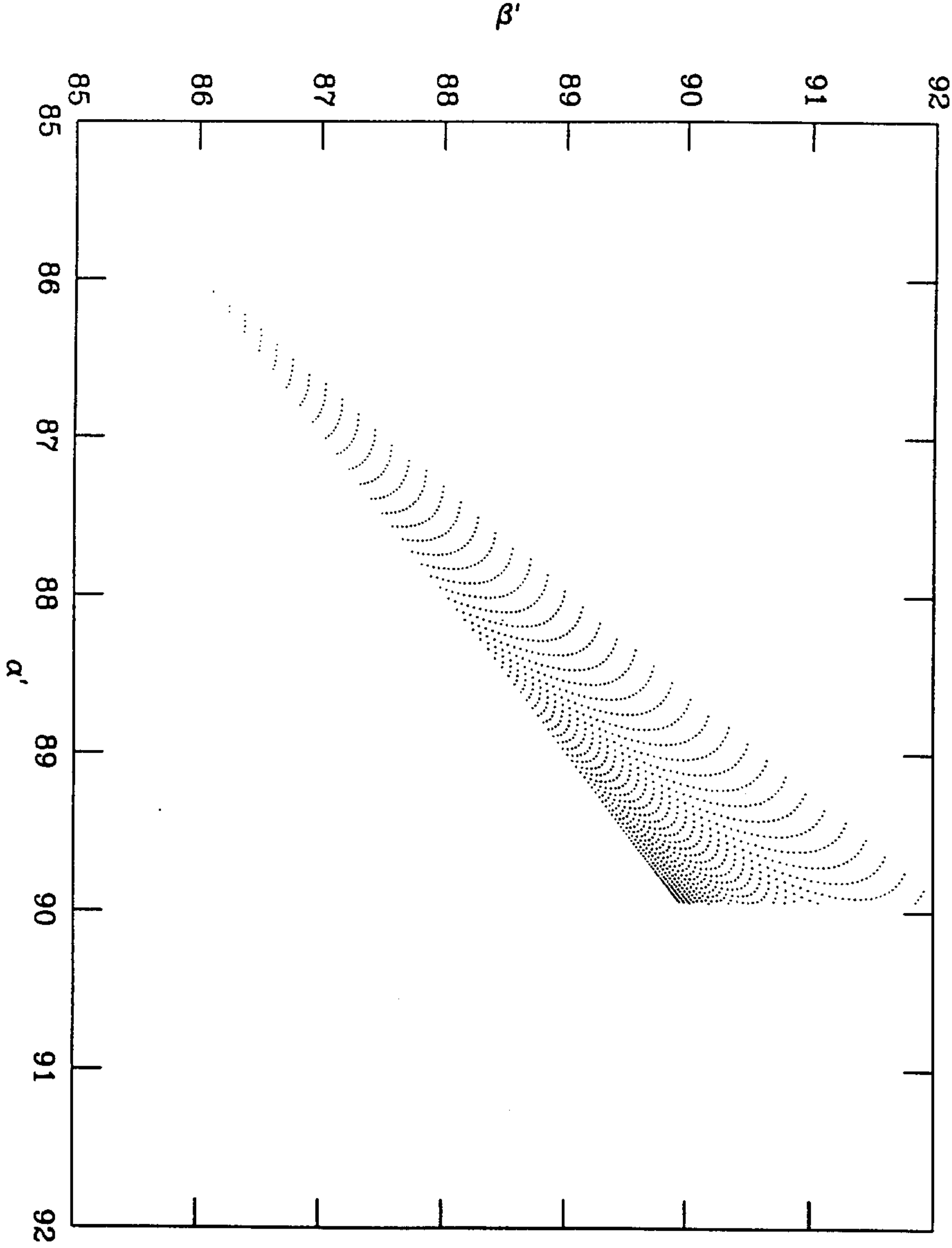


-FIG 12-

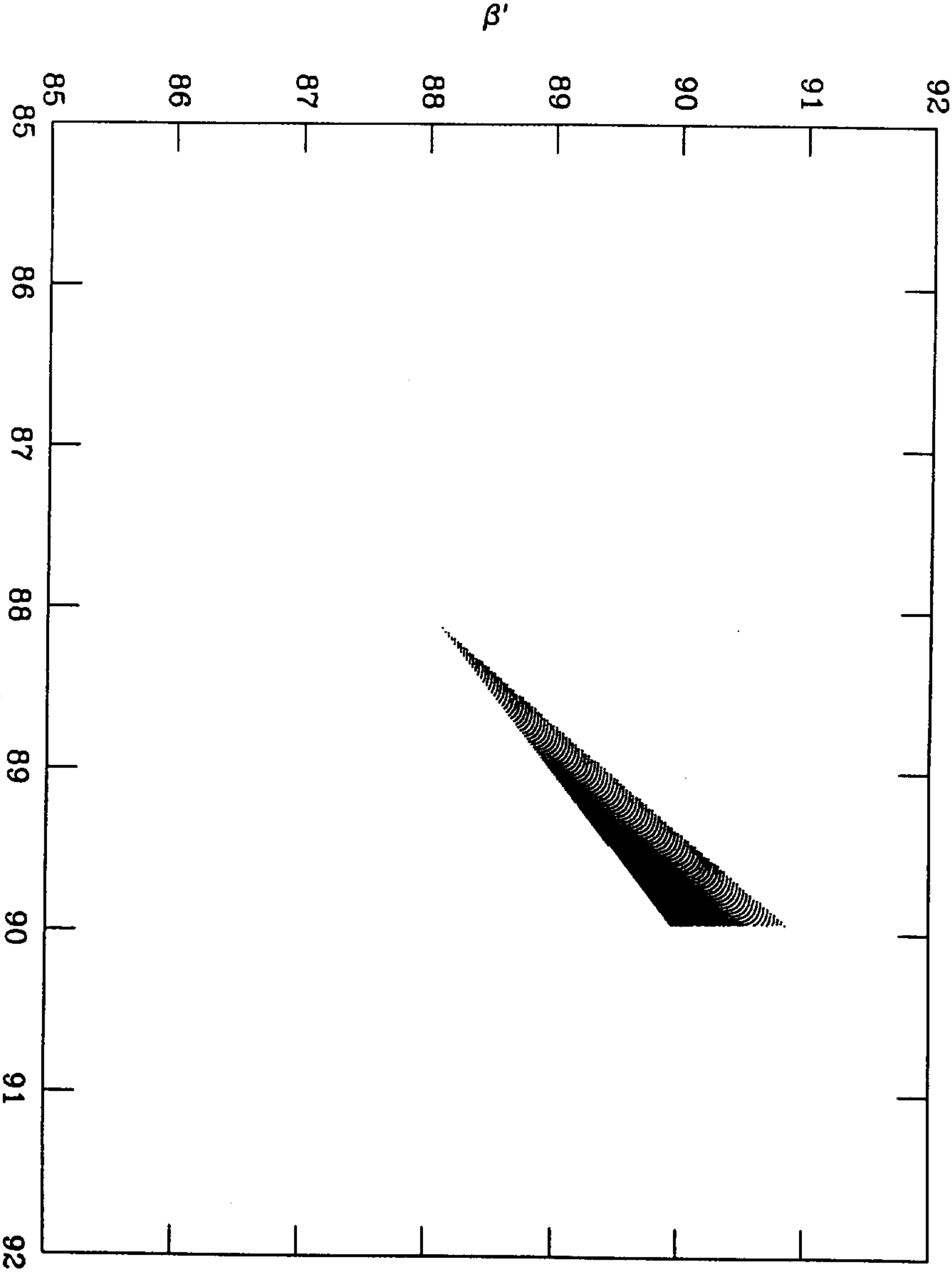


$I_{CRIT}[\lambda_\sigma \log(\kappa L)]^{1/2}/\mu$
10^1
10^2
10^3
10^4
10^5
1
10
100
1000
10000
α

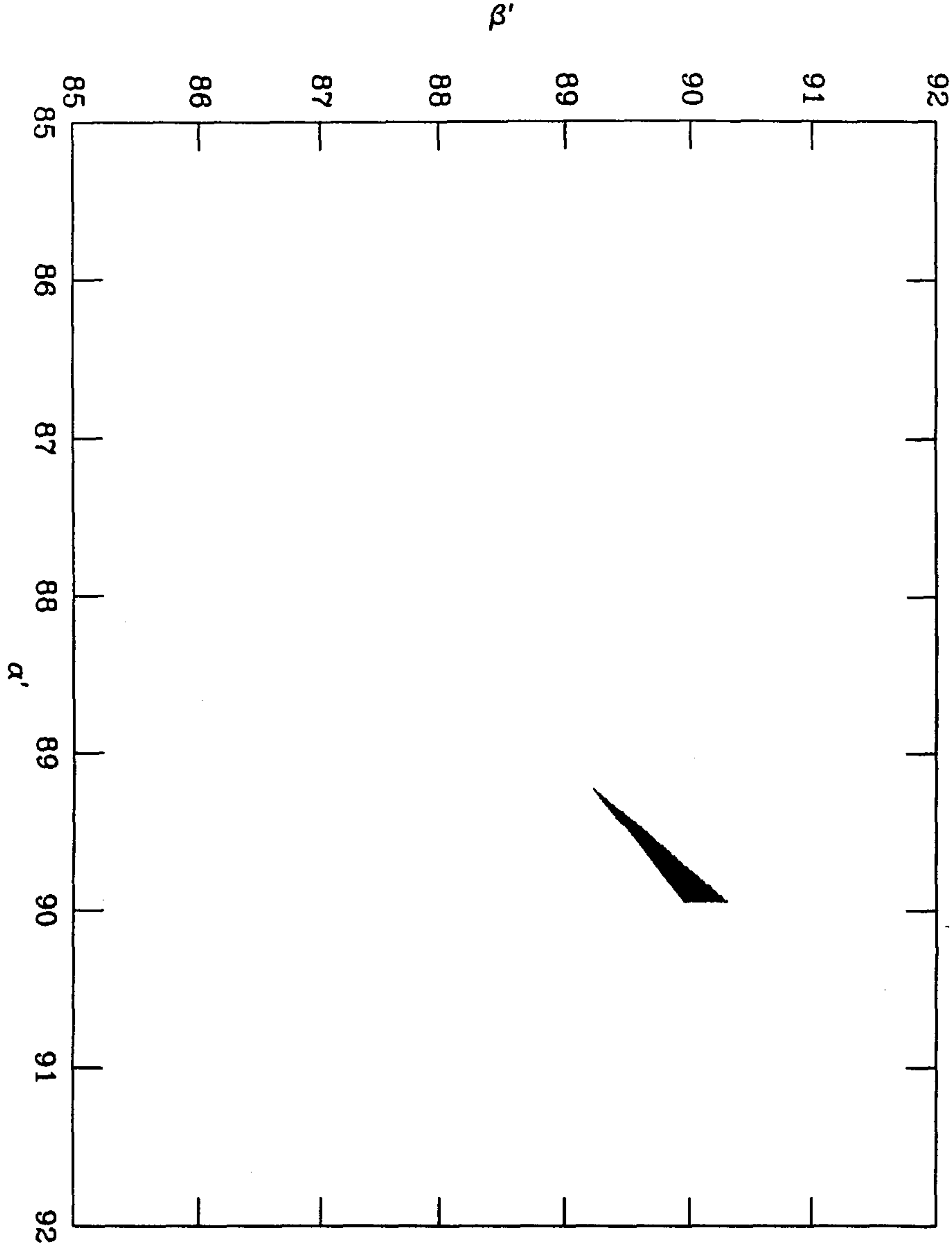
- FIG 13 -



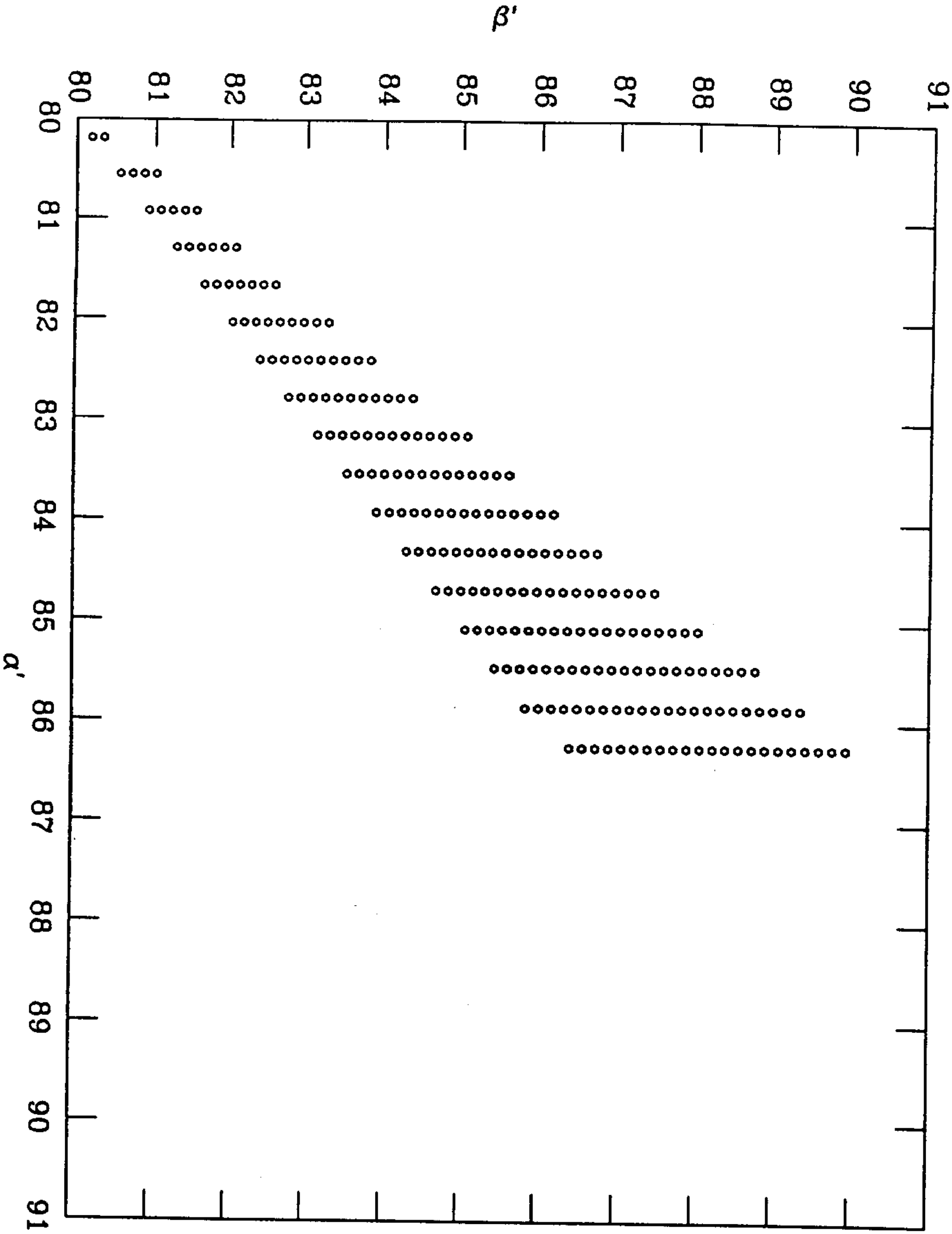
-FIG 14-



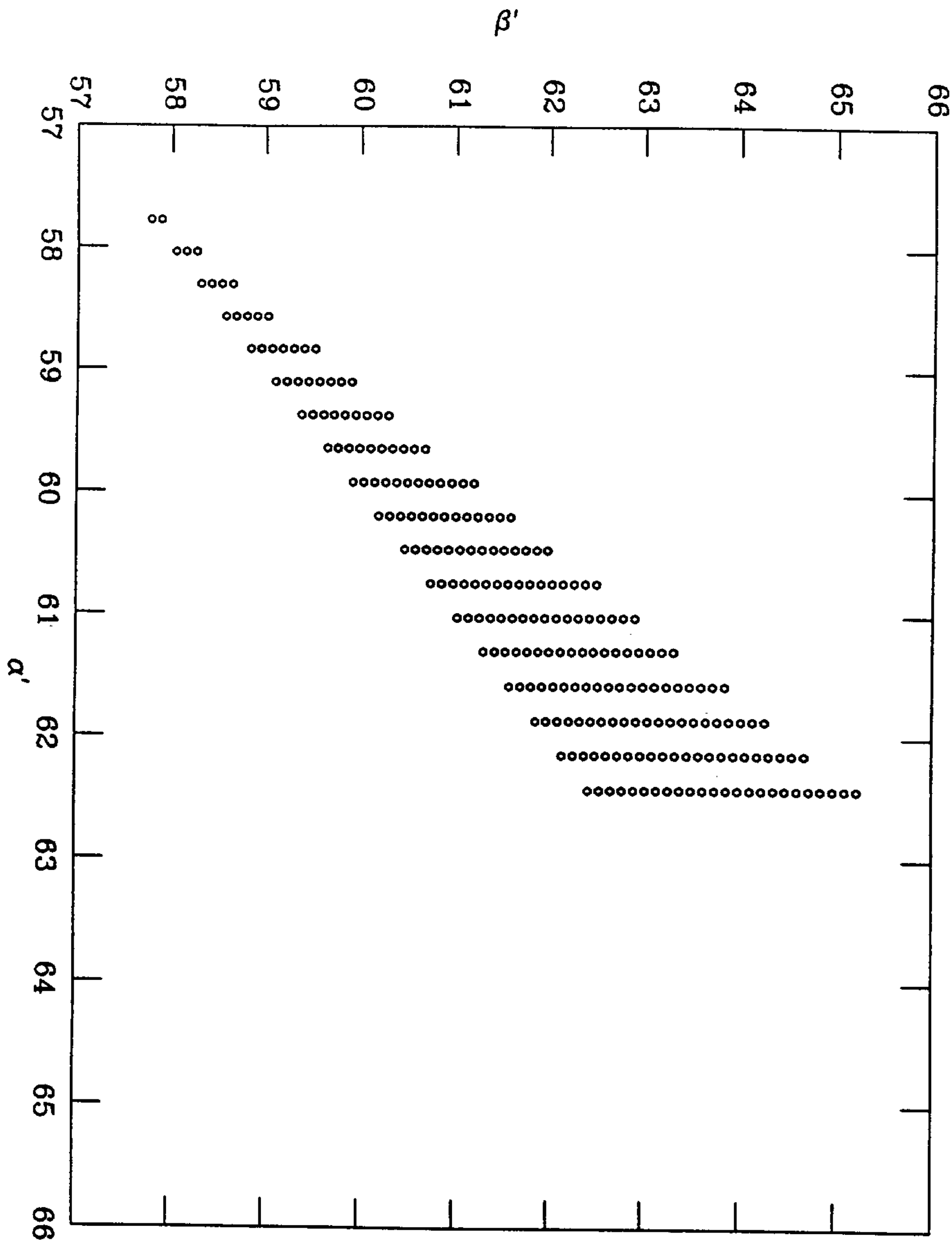
-FIG 15-



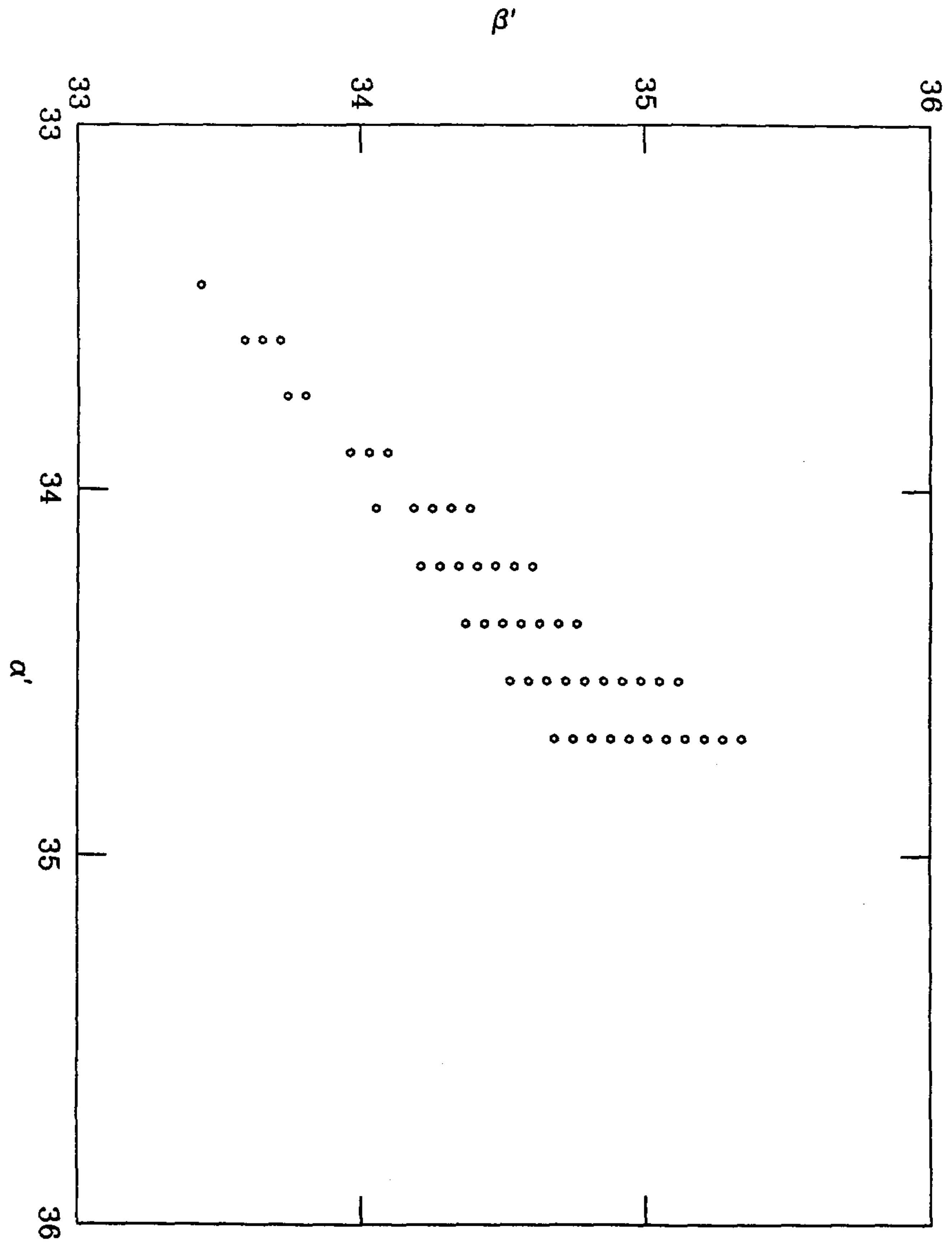
-FIG 16-



-FIG 17-



-FIG 18-



-FIG 19-