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DUALITY AND MULTI-GLUON SCATTERING

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Abstract

For the *six* gluon scattering process we give explicit and simple expressions for the amplitude and its square. To achieve this we use an analogy with string theories to identify a unique procedure for writing the multi-gluon scattering amplitudes in terms of a sum of gauge invariant dual sub-amplitudes multiplied by an appropriate color (Chan-Paton) factor. The sub-amplitudes defined in this way are invariant under cyclic permutations, satisfy powerful identities which relate different non-cyclic permutations and factorize in the soft gluon limit, the two gluon collinear limit and on multi-gluon poles. Also, to leading order in the number of colors these sub-amplitudes sum *Incoherently* in the square of the full matrix element. The results contained here are important for Monte-Carlo studies of multi-jet processes at hadron colliders as well as for understanding the general structure of QCD.

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1 Introduction

The calculation of multi-gluon scattering processes in QCD is extremely complicated owing to the cancellations that occur because of the gauge invariance of the theory. In this paper we present simple and explicit analytical results for the *six* gluon scattering amplitude in the helicity representation and its square summed over the colors and helicities of the gluons. This is achieved by using an analogue with string theories to identify gauge invariant, dual sub-amplitudes for multi-gluon processes. The sub-amplitudes are obtained by rewriting the color factors of the Feynman diagrams in terms of traces of color matrices in the fundamental representation of the gauge group. To evaluate the sub-amplitudes the polarization vectors for the gluons are written in terms of Weyl spinors and the calculus of spinor products is employed. The dual sub-amplitudes so defined and calculated have many remarkable properties that are generally expected only of the *full* amplitude. The most important property being the factorization of the sub-amplitudes in the soft gluon limit, in the two gluon collinear limit and on the three gluon poles. The simple form of the sub-amplitudes and their many surprising and beautiful properties suggests that there is a hidden simplicity in QCD which is yet to be discovered. Also, the results obtained in this paper are the first time the explicit matrix element squared has been derived for any six parton scattering process in QCD.

These sub-amplitudes and their squares are also useful for Monte Carlo studies of multi-jet physics. The present (Cern $Sp\bar{p}S$ and Fermilab Tevatron) and future hadron colliders (SSC or LHC) have or will have many multi-jet events. These events hold great promise for quantitative tests of Quantum Chromodynamics (QCD) as well as being significant backgrounds to many other processes of interest in the standard model and to the discovery of new physics [1]. Up to now only the two and three jet final

$p_1, p_2 \ldots p_n$ and helicities $\epsilon_1, \epsilon_2 \ldots \epsilon_n$, can be written as

$$\mathcal{M}_n = \sum_{perm'} tr\,(\lambda^{a_1}\lambda^{a_2}\ldots\lambda^{a_n})\; m(p_1,\epsilon_1; p_2,\epsilon_2; \cdots; p_n,\epsilon_n), \tag{2.1}$$

where the sum, *perm'*, is over all $(n-1)!$ *non-cyclic* permutations of $1, 2, \ldots, n$ and the λ's are the matrices of the symmetry group in the fundamental representation. The proof of this statement is very simple using the identities $[\lambda^a, \lambda^b] = i f_{abc}\lambda^c$ and $tr(\lambda^a\lambda^b) = \frac{1}{2}\delta^{ab}$. In any tree level Feynman diagram, replace the color structure function at some vertex using $f_{abc} = -2i\; tr(\lambda^a\lambda^b\lambda^c - \lambda^c\lambda^b\lambda^a)$. Now each leg attached to this vertex has a λ matrix associated with it. At the other end of each of these legs there is either another vertex or this is an external leg. If there is another vertex, use the λ associated with this internal leg to write the structure function of this vertex $f_{cde}\;\lambda^e$ as $-i\;[\lambda^d, \lambda^e]$. Continue this processes until all vertices have been treated in this manner. Then this Feynman diagram has been placed in the form of eqn(2.1). Repeating this procedure for all Feynman diagrams for a given process completes the proof.

The sub-amplitudes $m(1, 2, \ldots, n) \equiv m(p_1,\epsilon_1; p_2,\epsilon_2; \ldots p_n,\epsilon_n)$ of eqn(2.1) satisfy a number of important properties and relationships.

(1) $m(1, 2, \ldots, n)$ is gauge invariant.

(2) $m(1, 2, \ldots, n)$ is invariant under cyclic permutations of $1, 2, \ldots, n$

(3) $m(n, n-1, \ldots, 1) = (-1)^n\, m(1, 2, \ldots, n)$

(4) The Ward Identity:

$$m(1,2,3,\ldots,n) + m(2,1,3,\ldots,n) + m(2,3,1,\ldots,n) + \cdots + m(2,3,\ldots,1,n) = 0 \tag{2.2}$$

of these dual sub-amplitudes will assume a particularly simple form.

The gauge invariance and properties under cyclic and reverse permutations allows the calculation of far fewer than the $(n-1)!$ sub-amplitudes that appear in the dual expansion. In fact the number of sub-amplitudes that are needed is just the number of different orderings of positive and negative helicities around a circle. Of course some of the sub-amplitudes vanish because of the partial helicity conservation of tree level Yang-Mills and others are simply related to one another through the properties (2) through (4).

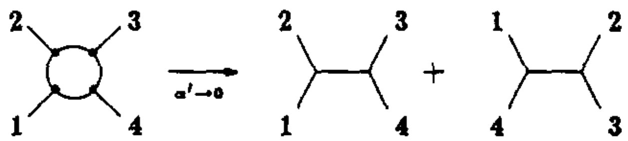

Figure 1: The zero-slope limit of the four gluon string diagram in terms of Feynman diagrams (tri-gluon couplings only).

3 Evaluation of the Sub-Amplitudes

We use the helicity basis for the polarization vectors which was introduced by Xu, Zhang and Chang[8] which is an important improvement over the CALKUL technique [9]. This is achieved by introducing massless spinors, $|p\pm\rangle$, which have momentum p and helicity ± 1. The adjoint of this spinor is $\langle p\mp|$. The *spinor products* are the scalar quantities obtained by multiplying $\langle p-|$ with $|q+\rangle$ or $\langle p+|$ with $|q-\rangle$.

(1) $k \cdot \epsilon_\pm(k,q) = 0,$

$\epsilon_\pm(k,q) \cdot \epsilon^*_\mp(k,q) = 0$ and $\epsilon_\pm(k,q) \cdot \epsilon^*_\pm(k,q) = -1.$

(2) $\epsilon^\mu(k,q') = \epsilon^\mu(k,q) + \beta(k,q',q)\ k^\mu.$

(3) $q \cdot \epsilon_\pm(k,q) = 0.$

(4) $\epsilon_\pm(k_1,q) \cdot \epsilon_\pm(k_2,q) = 0.$

(5) $\epsilon_\pm(k_1,k_2) \cdot \epsilon_\mp(k_2,q) = 0.$

The properties in (1) are the standard properties of polarization vectors. Whereas (2) together with the gauge invariance of the sub-amplitudes, i.e. $m(1,2,\cdots,n)|_{\epsilon^i=p^i} = 0$, implies that β is irrelevant and hence we can choose different reference momenta for each of the gluons and different reference momenta for a given gluon in different sub-amplitudes. Property (3) eliminates many terms if the reference momenta are chosen to be other light-like momentum vectors in the calculation. Whereas, (4) and (5) suggest that for a given sub-amplitude calculation all gluons with the same helicity should have the same reference momentum and that this reference momentum should be the momentum of a gluon with opposite helicity. Of course for a given sub-amplitude it is an art to choosing the reference momenta of the gluons so as to minimize the complexity of the resulting expression, but in general minimizing the number of nonzero $\epsilon_i \cdot \epsilon_j$'s is the most useful choice.

4 Four and Five Gluon Scattering

In the rest of this paper we will use the shorthand notation for the spinor products, $\langle ij \rangle = \langle p_i - | p_j + \rangle$ and $[ij] = \langle p_i + | p_j - \rangle$; then using the techniques of the last section it is easy to derive the following results. For the *four* gluon process, expand the color

In squaring the four gluon amplitude and summing over colors the $O(N^{-2})$ terms in eqn(2.3) can be shown to vanish by using only the general properties, especially the Ward Identity, of the sub-amplitudes. Therefore,

$$\sum_{colors} |\mathcal{M}_4|^2 = \frac{N^2(N^2-1)}{16} \sum_{perm'} |m(1,2,3,4)|^2, \tag{4.2}$$

and the square of each sub-amplitude is very simple because the spinor product is the square root of twice the dot product. The final result is the standard four gluon matrix element squared.

$$\sum_{hel.} \sum_{colors} |\mathcal{M}_4|^2 = N^2(N^2-1)\, g^4 \left(\sum_{i>j} S_{ij}^4 \right) \sum_{perm'} \frac{1}{S_{12}S_{23}S_{34}S_{41}}. \tag{4.3}$$

Here we have not averaged over incoming helicities or colors.

For five gluon scattering only those Feynman diagrams, or part there of, with color structure the same as the diagrams of Fig. 2 contribute to the $m(1,2,3,4,5)$ sub-amplitude. This is easily seen by rewriting the color factors for the Feynman diagrams as

$$f^{abX} f^{XcY} f^{Yde} = 2i\, tr([\lambda^a, \lambda^b][\lambda^c, [\lambda^d, \lambda^e]]).$$

Again, it is a straight forward, simple calculation [4] to show that the only nonzero sub-amplitudes have either two or three negative helicity gluons and that the three positive - two negative helicity sub-amplitude is given by

$$m_{3+2-}(1,2,3,4,5) = 4\sqrt{2} i g^3 \frac{\langle IJ \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}. \tag{4.4}$$

Where I and J are again the momenta of the negative helicity gluons and the denominator ordering is determined by the order of the momenta in the sub-amplitude. The two positive - three negative helicity amplitude is obtained from this last equation by or

$$f^{abX}f^{cdY}f^{egZ}f^{XYZ} \;=\; 2\,tr([\lambda^a,\lambda^b][\lambda^c,\lambda^d][\lambda^e,\lambda^g]) - 2\,tr([\lambda^e,\lambda^g][\lambda^c,\lambda^d][\lambda^a,\lambda^b]).$$

Then, by using the appropriate reference momenta for the polarization vectors it is easy to see that the only non-zero sub-amplitudes are those with four positive - two negative, two positive - four negative and three positive - three negative helicities. After a lengthy calculation we have obtained the following expressions for the six gluon sub-amplitudes.

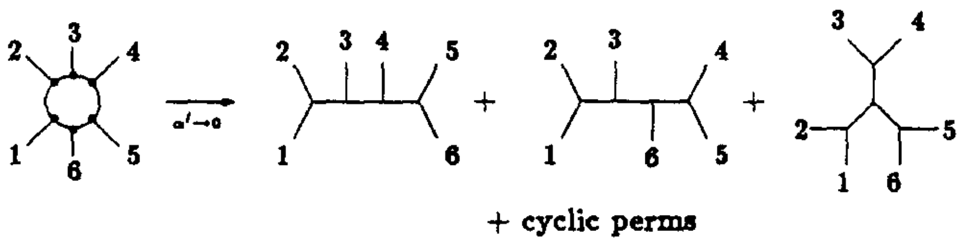

Figure 3: The zero-slope limit of the six gluon string diagram in terms of Feynman diagrams (tri-gluon couplings only).

The sub-amplitudes for the four positive - two negative helicity processes are a straight forward generalization of the four and five-gluon sub-amplitudes;

$$m_{4+2-}(1,2,3,4,5,6) \;=\; 8ig^4\frac{\langle IJ\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}. \tag{5.1}$$

Again, I and J represent the momenta of the negative helicity gluons. Different permutations can be obtained as before by keeping fixed the numerator and permuting the momenta in the denominator. The two positive - four negative helicity sub-amplitude is obtained from eqn(5.1) by complex conjugation.

The three positive - three negative helicity sub-amplitudes are not as simple, but like the two positive - two negative helicity sub-amplitudes they can be written down and hence the Altarelli-Parisi [13] behaviour of the squared amplitude when two gluons become collinear, is

$$m_{3+3-}(1,2,3,4,5,6) = \frac{8ig^4}{t_{123}t_{234}t_{345}}$$
$$\left[\frac{a_1}{\langle 12\rangle\langle 23\rangle[45][56]} + \frac{a_2}{\langle 23\rangle\langle 34\rangle[56][61]} + \frac{a_3}{\langle 34\rangle\langle 45\rangle[61][12]}\right.$$
$$\left. + \frac{a_4}{\langle 45\rangle\langle 56\rangle[12][23]} + \frac{a_5}{\langle 56\rangle\langle 61\rangle[23][34]} + \frac{a_6}{\langle 61\rangle\langle 12\rangle[34][45]}\right]. \tag{5.4}$$

where the coefficients a_1 through a_6 are given in Table II. In this representation the two particle propagators always appear as a spinor product, i.e. as a square root of the propagator, therefore the square of this sub-amplitude only diverges like a single power of the propagator when two gluons become collinear. This is the Altarelli-Parisi behaviour for the sub-amplitudes. Further properties of these amplitudes will be discussed in the next section.

The six gluon sub-amplitudes satisfy the three distinct Ward Identities obtained from the following equation

$$m(1,2,3,4,5,6) + m(2,1,3,4,5,6) + m(2,3,1,4,5,6)$$
$$+ m(2,3,4,1,5,6) + m(2,3,4,5,1,6) = 0 \tag{5.5}$$

using the helicity ordering of the first term as either $m(1+,2+,3+,4+,5-,6-)$, $m(1+,2+,3+,4-,5-,6-)$ or $m(1+,2-,3+,4-,5+,6-)$. These three Identities are extremely powerful and relate sub-amplitudes with different orderings of the helicities.

the numerator as

$$\langle (n-1)\ 2\rangle\langle n\ 1\rangle \;=\; \langle (n-1)\ 1\rangle\langle n\ 2\rangle \;+\; \langle (n-1)\ n\rangle\langle 1\ 2\rangle.$$

The two terms thus generated are exactly the extra terms needed for the n-gluon Ward Identity for this helicity structure. This provides further evidence that this is indeed the sub-amplitude for the $(n-2)$ positive - two negative helicity gluon scattering process.

6 Factorization Properties of the Sub-Amplitudes

The most important and remarkable properties of the Yang-Mills dual sub-amplitudes are their factorization properties, whose origin can be traced back to the string picture. In this section we show that the sub-amplitudes discussed in this paper factorize in

(1) the soft gluon limit,

(2) when two gluons become collinear and

(3) when three gluons add to form an on mass-shell gluon
i.e. on the three gluon pole.

For arbitrary n-gluon scattering these factorization properties of the sub-amplitudes will extend up to factorization on the $[n/2]$-gluon poles.

First, we consider the soft gluon limit. Consider the sub-amplitudes when gluon 1 has an energy which is small compared to all the other energies in the process. Then the five and six gluon sub-amplitudes calculated here, satisfy

$$m(1^+,2\ldots,n) \;\stackrel{1^+\ soft}{\longrightarrow}\; \left\{\frac{g\sqrt{2}\ \langle n\ 2\rangle}{\langle n\ 1\rangle\langle 1\ 2\rangle}\right\}\; m(2,3\ldots,n) \tag{6.1}$$

$$m(1^-,2\ldots,n) \;\stackrel{1^-\ soft}{\longrightarrow}\; \left\{\frac{g\sqrt{2}\ [n\ 2]}{[n\ 1][1\ 2]}\right\}\; m(2,3\ldots,n). \tag{6.2}$$

(6.3)

$$m(1^+,2^-,3,\ldots) \overset{1^+ \parallel 2^-}{\longrightarrow} \left\{\frac{ig\sqrt{2}\, z^2\langle 12\rangle}{\sqrt{z(1-z)}}\right\} \frac{-i}{S_{12}}\, m(P^+,3,\ldots) \tag{6.6}$$

$$+ \left\{\frac{ig\sqrt{2}\, (1-z)^2\, [12]}{\sqrt{z(1-z)}}\right\} \frac{-i}{S_{12}}\, m(P^-,3,\ldots)$$

$$m(1^-,2^-,3,\ldots) \overset{1^- \parallel 2^-}{\longrightarrow} \left\{\frac{ig\sqrt{2}\, \langle 12\rangle}{\sqrt{z(1-z)}}\right\} \frac{-i}{S_{12}}\, m(P^-,3,\ldots). \tag{6.7}$$

Note that either $\langle 12\rangle$ or $[12]$ appears in the numerator of each term. Also, it is useful to interpret the factor in braces as the "three gluon sub-amplitude" in the limit when two gluons become collinear. This three gluon sub-amplitude has the square root suppression of the pole as well as having the square root of the appropriate Altarelli-Parisi gluon-fusion function. From this result and the incoherence of the sub-amplitudes in the square of the matrix element the standard results of Altarelli and Parisi are obtained in a simple manner.

The sub-amplitudes also factorize in the three particle channel; here let $P = 1+2+3$, then as $P^2 \to 0$ it is easy to see that

$$m(1,2,3,4,5,6) \;\to\; \frac{1}{2}\; m(1,2,3,-P)\; \frac{-i}{P^2}\; m(P,4,5,6) \tag{6.8}$$

for the helicity structure three positive and three negative. Since helicity is conserved in the four gluon process, the helicity of the intermediate gluon is determined for this helicity structure and the four positive - two negative helicity sub-amplitude has no three particle poles.

Of course the full matrix element must also factorize. This is trivial in Feynman diagram language but here it is not so obvious because of the way we have added diagrams together. The color factors almost factorizes for an $SU(N)$ gauge group,

$$tr\,(\lambda^1\lambda^2\ldots\lambda^n) \;=\; 2\sum_x tr\,(\lambda^1\ldots\lambda^m\lambda^x)\,tr\,(\lambda^x\lambda^{m+1}\ldots\lambda^n) \tag{6.9}$$

Thus, the complete matrix element squared, summed over helicities and colors, is given by

$$\begin{aligned}\sum_{hel.}\sum_{colors} |\mathcal{M}_6|^2 = \quad & \frac{N^4(N^2-1)}{32}\Big[\sum_{perm'}\sum_{i>j} H_0^{ij}(1,2,3,4,5,6) \\ & + \sum_{all\ perm}\{\frac{1}{6}H_1(1,2,3,4,5,6) + H_2(1,2,3,4,5,6) \\ & + \frac{1}{2}H_3(1,2,3,4,5,6)\ \}\Big] \qquad (7.3)\end{aligned}$$

where the subscripts on the functions, H, determine the helicity structure of the squared sub-amplitudes. H_0^{ij} is the four positive - two negative (gluons I and J) helicity structure, H_1 is the alternating helicity structure $(1^+2^-3^+4^-5^+6^-)$, H_2 is the mixed helicity structure $(1^+2^+3^-4^+5^-6^-)$ and H_3 is the adjacent structure $(1^+2^+3^+4^-5^-6^-)$. These H functions can be calculated either numerically from the sub-amplitudes, eqns(5.1, 5.2), or for the leading color terms from the analytic form of the square of the sub-amplitudes given below.

To calculate the squares of the sub-amplitudes many properties of the spinor products developed by Xu et al[8] were used. In fact very compact expression in terms of the Lorentz invariants, S_{ij}, were obtained for two out of the four sub-amplitudes squared. The other two sub-amplitude structures are not as compact but consist of less than two hundred terms when expressed purely in terms of the elementary kinematical invariants S_{ij} and t_{ijk}. Here, we give the two simpler squares (the others are in the Appendix).

First, the four positive - two negative helicity sub-amplitude squared is

$$|m_{4+2-}(1,2,3,4,5,6)|^2 = 64\ g^8 \frac{S_{IJ}^4}{S_{12}S_{23}S_{34}S_{45}S_{56}S_{61}} \qquad (7.4)$$

where I and J are the negative helicity gluons. Of course the two positive - four negative helicity sub-amplitudes are given by the same expression with I and J now being the

The smallness of the non-leading color terms and the fact that the leading color terms are just the squares of the simple sub-amplitudes implies that the square of this matrix element is easy to obtain.

The double poles of S_{34} and S_{61} in $|m(1^+,2^+,3^+,4^-,5^-,6^-)|^2$, eqn(7.5), are only apparent. This can be seen by using the identity

$$tr(\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}) = t_{123}t_{234}t_{345} - t_{123}S_{34}S_{61} - t_{234}S_{45}S_{12} - t_{345}S_{56}S_{23},$$

here $\hat{\imath} = p_i \cdot \gamma$, and realizing that for adjacent momenta this trace goes to zero as the square root of the Lorentz invariant, S_{ij} , as this invariant goes to zero. The Altarelli - Parisi relationship can be obtained from this squared *sub-amplitude* by using

$$\int \frac{d\phi}{2\pi} \; tr^2(\hat{1}\hat{2}\hat{3}\hat{4}\hat{5}\hat{6}) \;\rightarrow\; 2\; S_{12}S_{23}S_{34}S_{45}S_{56}S_{61}$$

as any two momenta that are adjacent in the trace become parallel and the integral is the standard azimuthal averaging for these two momenta.

8 Conclusion

Here we have presented an extremely powerful technique for evaluating multi-gluon scattering processes by using an analogue with string theories to identify gauge invariant sub-amplitudes. Not only are these sub-amplitudes straight forward to calculate but they are simple and satisfy many important properties. The most remarkable properties are their factorization in the soft gluon limit, the two gluon collinear limit and on multi-particle poles. This suggests that there is a hidden simplicity in QCD yet to be discovered. We have demonstrated the power of these techniques and simplicity of the results by presenting the amplitude and its square for the four, five and six gluon

for more than five external particles so the sub-amplitudes were evaluated using Feynman perturbation theory.

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$$\pi_r : (123456) \rightarrow (654321).$$

π_r is a symmetry of all of the three matrix elements squared, while $\pi_\pm$ are symmetries of $|m(+-+-+-)|^2$ only. Whenever either of the π's appears as an entry in Table A, this entry has to be filled in such a way as to enforce these symmetries. For example, the term proportional to T_3^{-2} in $|m(+-+-+-)|^2$ is given by $S_{35}^2 S_{26}^2 \{1Z4Z\}^2$, and in $|m(+++---)|^2$ is given by $S_{12}^2 S_{45}^2 \{3X6X\}^2$.

Finally, the symbol $\chi.c.$ after a product of traces is the chiral conjugate and its meaning is clear from the following example:

$$\{356Y4Y\}\{124Y3Y\} + \chi.c. = \{356Y4Y\}\{124Y3Y\} + \{356Y4Y\gamma_5\}\{124Y3Y\gamma_5\}.$$

To express the amplitude squared in terms of the elementary kinematical invariants S_{ij} and t_{ijk}, it is necessary to expand the traces appearing in the Table. Below we give the set of identities that we have used to carry out this expansion which generates eqn(7.5) for $|m(1+,2+,3+,4-,5-,6-)|^2$ and fewer than two hundred terms for the other sub-amplitudes squared.

As an immediate consequence of (9.1) and (9.2)

$$\mathrm{tr}^2(i_1 i_2 \cdots i_{2n}) - \mathrm{tr}^2(i_1 i_2 \cdots i_{2n}\gamma_5) = 4\, S_{i_1 i_2} S_{i_2 i_3} \cdots S_{i_{2n} i_1}. \tag{9.3}$$

A straight forward generalization of this identity is

$$\begin{aligned} \mathrm{tr}(i_1 i_2 \cdots i_{2n}\gamma_5)\,\mathrm{tr}(j_1 j_2 \cdots j_{2m}\gamma_5) &= \mathrm{tr}(i_1 i_2 \cdots i_{2n})\,\mathrm{tr}(j_1 j_2 \cdots j_{2m}) \\ &- 2\Big[[i_1\, i_2]\langle i_2\, i_3\rangle \cdots \langle i_{2n}\, i_1\rangle \langle j_1\, j_2\rangle [j_2\, j_3] \cdots [j_{2m}\, j_1] + c.c.\Big]. \end{aligned} \tag{9.4}$$

These two identities reduce all of the traces containing a γ_5 and thus one can show that

$$\{462135\gamma_5\}\{642315\gamma_5\} = \{462135\}\{642315\}$$

Table A: The Square of the three positive - three negative helicity Sub-Amplitudes.

$Pole$	$\|m(1^+2^-3^+4^-5^+6^-)\|^2/16$	$\|m(1^+2^+3^-4^+5^-6^-)\|^2/16$	$\|m(1^+2^+3^+4^-5^-6^-)\|^2/16$
$\frac{1}{T_1^2}$	$S_{13}^2S_{46}^2\{2Z5Z\}^2$	$S_{12}^2S_{56}^2\{3Y4Y\}^2$	0
$\frac{1}{T_2^2}$	π_+	$S_{24}^2S_{56}^2\{1Y3Y\}^2$	$S_{23}^2S_{56}^2\{1X4X\}^2$
$\frac{1}{T_3^2}$	π_-	π_r	π_r
$\frac{1}{T_2T_3}$	$2\{351Z426Z\}^2+\chi.c.$	$2\{356Y421Y\}^2+\chi.c.$	$2\{321X456X\}^2+\chi.c.$
$\frac{1}{T_1T_3}$	π_+	$2S_{12}^2\{356Y4Y\}^2+\chi.c.$	0
$\frac{1}{T_1T_2}$	π_-	π_r	0
$\frac{1}{T_1S}$	$2t_{234}S_{13}S_{46}\{2Z5Z\}\{135Z264Z\}+\pi_r$	$2S_{12}^2S_{56}t_{234}\{3Y4Y\}\{356Y4Y\}+\pi_r$	0
$\frac{1}{T_2S}$	π_+	$2S_{24}S_{56}^2t_{345}\{1Y3Y\}\{124Y3Y\}+$ $2S_{24}S_{56}t_{123}\{1Y3Y\}\{356Y421Y\}$	$2t_{123}S_{23}S_{56}\{1X4X\}\{321X456X\}$
$\frac{1}{T_3S}$	π_-	π_r	π_r
$\frac{1}{S^2}$	$2S_{34}S_{61}(\{264Z135Z\}\{246Z315Z\}+\chi.c.)+$ $t_{123}^2S_{15}S_{35}S_{24}S_{26}\{1Z4Z\}\{3Z6Z\}+$ $2t_{234}t_{345}S_{13}S_{46}\{2Z5Z\}\{426Z351Z\}+$ $\pi_++\pi_-$	$2S_{12}S_{34}S_{56}S_{61}(\{356Y4Y\}\{124Y3Y\}+\chi.c.)+$ $t_{123}^2S_{12}S_{24}S_{35}S_{56}\{1Y3Y\}\{4Y6Y\}+$ $2t_{234}t_{345}S_{12}S_{56}\{3Y4Y\}\{356Y421Y\}+$ $[2S_{12}S_{23}S_{56}(\{356Y4Y\}\{356Y421Y\}+\chi.c.)+\pi_r]+$ $[t_{234}S_{12}S_{56}S_{35}\{4Y6Y\}\ (S_{12}t_{234}\{3Y4Y\}+2t_{123}\{124Y3Y\})+\pi_r]$	$t_{123}^2S_{12}S_{23}S_{45}S_{56}\{1X4X\}\{3X6X\}$