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Functional Representation for the Isometries of de Sitter Space*

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We examine the Schrödinger picture for a spinless field theory in two-dimensional de Sitter space and construct an ultraviolet finite functional representation for the de Sitter Lie algebra. The one-parameter family of de Sitter vacua is found to be only phase-invariant, except for one value of the parameter where the state is truly invariant.

I. INTRODUCTION

Quantum field theory is a quantum mechanical system with an infinite number of degrees of freedom. Bosonic models can therefore be formulated, analogously to ordinary quantum mechanics, in a Schrödinger picture where the fixed-time ($t=0$) field operator $\Phi(x)$ and its conjugate momentum $\Pi(x)$ are realized through multiplication by $\varphi(x)$ and functional differentiation with respect to $\varphi(x)$, respectively, both operations acting on "wave functionals" of φ, $\Psi(\varphi) = \langle \varphi | \Psi \rangle$.

While this approach allows using insight gained from quantum mechanics in quantum field theory, it has the shortcoming that the renormalization procedure required for dynamical calculations is difficult to implement, even though it has been recently established [1]. Kinematical calculations, on the other hand, are more tractable, and many structural features of various quantum field theories have been exposed in this way, e.g., vacuum angle in gauge theories,[1] topological obstructions to Gauss's law,[1] confinement [3], etc. Also, the Schrödinger formulation lends itself to variational approximations,[2] both static and time-dependent [5], with which one can study symmetry breaking,[2] soliton effects,[2] phase transitions,[2] inflationary cosmic evolution [6], etc., typically with a Gaussian trial wave functional.

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[1] For a review, see Jackiw in Ref. [2].

[2] For a review, see Jackiw in Ref. [4].

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Recently, another application of this formalism was given: a functional representation for the infinite parameter conformal group in two dimensions was constructed. As a by-product, well-defined conformal generators were obtained, and this in turn allowed computing their commutator, determining unambiguously the center [Schwinger term] in the algebra, without normal ordering with respect to any preselected Fock vacuum [7].

We view this as an improvement over conventional procedures for several reasons. First, the determination of algebraic relations between symmetry generators should be carried out intrinsically, not making reference to extraneous constructs like a Fock vacuum; indeed, if operators are defined by normal ordering, the center in the conformal algebra depends on the Fock state which determines the normal ordering conventions; see below [8]. Second, for theories in an external, time-dependent background field it is not possible to define the Fock vacuum unambiguously.

For example, in the presence of a background classical gravitational field the Hamiltonian for matter fields is time-dependent and the concept of an energy ground state is not applicable. If the spacetime were asymptotically flat, one might prescribe conventional boundary conditions by requiring that the asymptotic isometries of Poincaré invariance be implemented in the vacuum state; i.e., asymptotically it should be annihilated by the generators of the Poincaré group. However, for a wide class of interesting situations the asymptotic metric is nontrivial and the choice of boundary conditions becomes arbitrary.

This happens in de Sitter space, which may be relevant to the extreme conditions of the very early universe for which the initial conditions are certainly unknown [9]. Nevertheless, non-trivial isometries are present and vacuum states may be required to be invariant under the corresponding transformations. However, to implement such a constraint, it is necessary to well-define the generators of the isometries independently of any vacuum states. This is what we achieve here, by formulating the problem in the Schrödinger picture. This also lets us clarify the nature of de Sitter invariant states.

There appear in the literature assertions that de Sitter invariance leaves a one-parameter freedom [10]. These statements are variously made about states, or about expectation values in these states—for example, propagators.[3] In fact, one can describe the situation more precisely. We find that with respect to our intrinsically defined generators, only one value of the parameter gives an invariant state [i.e., it is annihilated by the generators]. For other values of the parameter, states are only invariant up to an infinite phase [i.e., they are eigenvectors of the generators with infinite eigenvalues]. Consequently, expectation values are invariant for all values of the parameter, but there is only one invariant state-the "Euclidean" vacuum [9]. Other states acquire an [infinite] 1-cocycle when they are transformed;[4] they may be called "phase-invariant."

[3] A monograph on the subject is by Birrell and Davies [11]. Two research articles relevant to our investigation are by Ratra [12] and Allen [13].

[4] The role of cocycles in quantum theory is described by Jackiw in Ref. [2].

In Section II we explain our method for implementing in quantum field theory an invariance group of transformations, without making a commitment to a specific Fock basis. Also, we review our earlier work on conformal transformations [7], as an example of our procedure and because some of our results are used in the present investigation. Section III is devoted to a two-dimensional field theory in de Sitter space, and well-defined generators of de Sitter transformations are constructed. In Section IV, we examine the transformation properties of the vacua in de Sitter space and exhibit the behavior described above. Finally, additional remarks about various properties of field theory in de Sitter space comprise the concluding Section V. There we study conformal symmetries of a massless field in de Sitter space. Only the Euclidean vacuum is invariant, the others are not even phase-invariant. Also, we show how the family of de Sitter vacua passes in the limit of flat space to the corresponding states in Minkowski space.

II. Intrinsic Renormalization of Symmetry Generators

When a dynamical system, governed by a local Lagrangian, is invariant against a continuous group of transformations, Noether's theorem allows for the construction of conserved quantities, Q, which generate these transformations. In a canonical description, the generators are polynomials of the canonical coordinates Φ and momenta Π. In a field theory, such polynomials $Q(\Phi, \Pi)$ are not well-defined owing to ultraviolet singularities when field operators at the same point are multiplied. As a first step toward arriving at a well-defined expression, the generator is regulated, $Q \to Q^R$, for example, by splitting points in products. Before the regulators are removed, the generators must be renormalized. For linear field theories, which may still be non-trivial owing to interaction with an external background, a subtraction suffices; however, the form of the subtraction must be specified. This is conventionally achieved by studying matrix elements of the regulated generator in a Fock vacuum. We propose that alternatively, field states $|\varphi\rangle$, $\Phi|\varphi\rangle = \varphi|\varphi\rangle$, be used for this purpose.

The matrix element of $Q^R(\Phi, \Pi)$ between field states is given by

$$\langle \varphi_1 | Q^R(\Phi, \Pi) | \varphi_2 \rangle = Q^R\left(\varphi_1, \frac{1}{i}\frac{\delta}{\delta\varphi_1}\right) \delta(\varphi_1 - \varphi_2), \tag{2.1}$$

where the functional delta-function is represented by a functional integral.

$$\delta(\varphi_1 - \varphi_2) = \int \mathscr{D}\alpha \exp i \int dx\, \alpha(x)((\varphi_1(x) - \varphi_2(x)). \tag{2.2}$$

Because (2.1) is a functional distribution, it is not apparent that operator products at coincident points are ill-defined. Hence, (2.1) is not useful for extracting singularities in Q. However, we may also consider the functional representation of the finite transformation.

$$U^R(\varphi_1, \varphi_2) \equiv \langle \varphi_1 | e^{-i\tau Q^R} | \varphi_2 \rangle. \tag{2.3}$$

This is a functional of φ_1 and φ_2, not a distribution, and its behavior as the regulators are removed can be determined. U satisfies a functional Schrödinger-like equation,

$$i\partial_\tau U^R(\varphi_1, \varphi_2) = Q^R\left(\varphi_1, \frac{1}{i}\frac{\delta}{\varphi_1}\right) U^R(\varphi_1, \varphi_2) \tag{2.4a}$$

with boundary condition

$$U^R(\varphi_1, \varphi_2)\,|_{\tau=0} = \delta(\varphi_1 - \varphi_2). \tag{2.4b}$$

For simple systems, when Q^R is quadratic in the field operators, (2.4) may be solved explicitly, and the limit when the regulators are removed is easily determined. Typically, one finds that infinities appear, but they are confined to a phase τq^R; i.e., $e^{i\tau q^R} U^R(\varphi_1, \varphi_2) = \langle \varphi_1 | \, e^{-i\tau(Q^R - q^R)} | \, \varphi_2 \rangle$ possesses a well-defined limit, and the regulators may be removed. This allows defining the renormalized generator as

$$:Q: \equiv \lim_R \, (Q^R - q^R), \tag{2.5}$$

while the renormalized representation functional is

$$:U:(\varphi_1, \varphi_2) \equiv \langle \varphi_1 | e^{\;i\tau:Q:} | \, \varphi_2 \rangle = \lim_R e^{i\tau q^R} U^R(\varphi_1, \varphi_2). \tag{2.6}$$

Note that no reference to a Fock vacuum is made and the colons do not signify normal ordering.

A by-product of this procedure is that one can determine modifications—anomalies or cocycles—in the realization of the transformation group's Lie algebra in our quantum field theory. The formal charges Q generally satisfy commutation relations which follow the abstract Lie algebra of the group.

$$[Q_1, Q_2] = iQ_{(1,2)}. \tag{2.7}$$

A composition law for the regulated charges is defined analogously to (2.7).

$$[Q_1^R, Q_2^R] = iQ_{(1,2)}^R. \tag{2.8}$$

[This leads to an infinite Lie algebra. The regulated charges no longer generate symmetry transformations, but they do generate canonical transformations.] Evidently the renormalized charges satisfy

$$[:Q_1:, :Q_2:] = i:Q_{(1,2)}: + i \lim_R q_{(1,2)}^R. \tag{2.9}$$

When the limit of $q_{(1,2)}^R$ is non-vanishing, a quantum mechanical anomaly—a center in the algebra—emerges. Note that before the limit is taken, the center is "trivial": it can be removed by redefining the charges, or equivalently by redefining the phases of the representation functionals; indeed, that is how q^R arose in the first place and thus may be similarly eliminated. However, after the regulators are removed, the extension can be "non-trivial"; it cannot be removed.

Let us review how this works for the infinite two-dimensional conformal group which transforms $x \pm t$ into arbitrary functions of $x \pm t$. We concentrate on the subgroup that acts only on $x + t$, the infinitesimal transformation being $\delta_f(x+t) = f(x+t)$. The corresponding field transformation law is

$$\delta_f \chi = (f\chi)' \tag{2.10}$$

$$\chi \equiv \frac{1}{\sqrt{2}} (\Pi + \Phi'). \tag{2.11}$$

The field χ satisfies the canonical commutation relation

$$[\chi(x), \chi(y)] = i\delta'(x-y) \equiv k(x, y). \tag{2.12}$$

[The dash denotes differentiation with respect to argument.] The generator is formally given by

$$Q_f = \tfrac{1}{2} \int dx\, f(x)\, \chi^2(x). \tag{2.13}$$

The formal commutator algebra of the generators, which is established with the help of (2.12), follows the Lie algebra of the abstract group.

$$[Q_f, Q_g] = iQ_{(f, g)} \tag{2.14}$$

$$(f, g) \equiv fg' - gf'. \tag{2.15}$$

However, closure of the conformal generator algebra in a two-dimensional local quantum field theory violates positivity and Lorentz invariance, which puts into evidence the need to regulate the formal expression (2.13). We define the regulated generators by

$$Q_f^R = \tfrac{1}{2} \int dx\, dy \chi(x)\, F(x, y)\, \chi(y) \equiv Q_F, \tag{2.16}$$

where the symmetric bilocal function $F(x, y)$ tends to the local limit $f(x)\, \delta(x-y)$ as the regulators are removed.

The functional representation for Q_F is

$$\langle \varphi_1 | Q_F | \varphi_2 \rangle = \frac{1}{4} \int dx\, dy \left[\frac{1}{i} \frac{\delta}{\delta\varphi_1(x)} + \varphi_1'(x) \right] \times F(x, y) \left[\frac{1}{i} \frac{\delta}{\delta\varphi_1(y)} + \varphi_1'(y) \right] \delta(\varphi_1 - \varphi_2) \tag{2.17}$$

and the representation functional may be found.

$$U(\varphi_1, \varphi_2; F) \equiv \langle \varphi_1 | e^{-iQ_F} | \varphi_2 \rangle = N_F \exp - \int dx\, dy\, \varphi_1(x)\, k(x, y)\, \varphi_2(y)$$

$$\times \exp \frac{i}{2} \int dx\, dy (\varphi_1(x) - \varphi_2(x))\, K_F(x, y)(\varphi_1(y) - \varphi_2(y)). \tag{2.18}$$

Here

$$N_F = \det{}^{1/2} F^{-1/2} \left(\frac{k_F}{2\pi i \sin k_F/2} \right) F^{-1/2} \tag{2.19a}$$

$$k_F \equiv F^{1/2} k F^{1/2} \tag{2.19b}$$

$$K_F = F^{-1/2} \left(k_F \operatorname{ctn} \frac{k_F}{2} \right) F^{-1/2}. \tag{2.19c}$$

[A matrix notation is being used for the kernels k and F.]

In the local limit, K_F attains a well-defined expression.

$$K_F(x, y) \to K_f(x, y) = \frac{1}{f(x)} \left\{ \int \frac{d\lambda}{2\pi} \left(\lambda \operatorname{ctn} \frac{1}{2} \lambda \right) \exp \left(-i\lambda \int_y^x \frac{dz}{f(z)} \right) \right\} \frac{1}{f(y)}. \tag{2.20}$$

The normalization constant N_F, however, diverges. The divergence resides in an unimportant constant factor Z [which may be removed by redefining the functional measure of integration] and in a phase.

$$q^F = \tfrac{1}{4} \operatorname{tr} F\omega \tag{2.21}$$

$$\omega(x, y) \equiv \int \frac{dp}{2\pi} e^{ip(x-y)} |p| = -P \frac{1}{\pi(x-y)^2} \tag{2.22}$$

[P means principal value.] Thus,

$$:U:(\varphi_1, \varphi_2; F) \equiv Z^{-1} e^{iq^F} U(\varphi_1, \varphi_2; F)$$

$$= Z^{-1/2} \langle \varphi_1 | e^{-i(Q_F - q^F)} | \varphi_2 \rangle Z^{-1/2} \tag{2.23}$$

possess a well-defined local limit, and we are led to define the following renormalized generator,

$$:Q_f: \equiv \lim_{F \to f} (Q_F - \tfrac{1}{4} \operatorname{tr} F\omega) \tag{2.24}$$

and an extended Lie algebra replaces (2.14).

$$[:Q_f:, :Q_g:] = i:Q_{(f,g)}: - \frac{i}{48\pi} \int dx (fg''' - gf'''). \tag{2.25}$$

[Of course, the subtraction is ambiguous up to terms that are finite in the local limit; these are obviously "trivial" and may be adjusted at will.]

The above approach is to be contrasted with the conventional one, wherein the subtraction is given by the expectation of Q_F in a Fock vacuum. In the functional Schrödinger picture, a Fock vacuum is a Gaussian functional, $\Psi_\Omega(\varphi)$, with covariance Ω. [For simplicity, here we take Ω to be real.]

$$\Psi_\Omega(\varphi) \equiv \langle \varphi | \Omega \rangle = \left(\det{}^{1/4} \frac{1}{\pi} \Omega \right) \exp - \tfrac{1}{2} \int dx\, dy\, \varphi(x)\, \Omega(x, y)\, \varphi(y). \quad (2.26)$$

This is annihilated by $A(x)$.

$$\begin{aligned} A(x) &= \frac{1}{\sqrt{2}} \int dy [\Omega^{1/2}(x, y)\, \Phi(y) + i\Omega^{-1/2}(x, y)\, \Pi(y)] \\ &= \frac{1}{\sqrt{2}} \int dy \left[\Omega^{1/2}(x, y)\, \varphi(y) + \Omega^{-1/2}(x, y) \frac{\delta}{\delta\varphi(y)} \right]. \end{aligned} \quad (2.27)$$

[The usual annihilation operator is the Fourier transform of (2.27).] The problem is that without specifying a dynamical Hamiltonian, which determines a unique ground state, the covariance is undetermined: for any Ω, $A^\dagger$ and A are creation and annihilation operators.

$$[A(x), A^\dagger(y)] = \delta(x - y). \quad (2.28)$$

The conventional subtraction depends on Ω,

$$\begin{aligned} q_F^\Omega &\equiv \langle \Omega | Q_F | \Omega \rangle = \tfrac{1}{4} \operatorname{tr} F(\Omega - (\Omega - k)\, V(\Omega + k)) \\ V(x, y) &\equiv \langle \Omega | \varphi(x)\, \varphi(y) |\, \Omega \rangle = \tfrac{1}{2} \Omega^{-1}(x, y) \\ q_F^\Omega &= \tfrac{1}{8} \operatorname{tr} F(\Omega + k\Omega^{-1} k), \end{aligned} \quad (2.29)$$

as do the conventionally renormalized generators.

$$:Q_f^\Omega: \equiv \lim_{F \to f} (Q_F - q_F^\Omega). \quad (2.30)$$

The Ω dependence survives in the center of the algebra (2.25) [8]. For example, for translation invariant states,

$$\Omega(x, y) = \int \frac{dp}{2\pi} e^{-ip(x-y)} \Omega(p), \quad (2.31)$$

the last term in (2.25) is replaced by

$$\frac{1}{4} \int dx\, dy\, f(x)\, g(y) \int \frac{dk}{2\pi} e^{ik(x-y)} \int \frac{dq}{2\pi} \left[\left(q + \frac{k}{2}\right) C\left(q - \frac{k}{2}\right) - \left(q - \frac{k}{2}\right) C\left(q + \frac{k}{2}\right) \right]$$

$$C(p) \equiv \frac{1}{2} \left[\Omega(p) + \frac{p^2}{\Omega(p)} \right]. \quad (2.32)$$

[Note that linear growth with p is the least divergent behavior possible for $C(p)$.]

With our approach, a unique [up to finite terms] covariance is selected in (2.22): $\Omega=\omega$, $C(p)=|p|$. This corresponds to the ground state of the massless theory. Evidently, a generator renormalized with an arbitrary Ω cannot be exponentiated in the full functional space, though it does exist in the Fock space built on the Fock vacuum $|\Omega\rangle$.

It should be clear that different Fock vacua, which are related by a Bogoliubov transformation, are in general inequivalent, in the sense that

$$\langle\Omega_1|\Omega_2\rangle=\det{}^{1/4}\Omega_1\ \det{}^{1/4}\Omega_2\ \det{}^{-1/2}\left(\frac{\Omega_1+\Omega_2}{2}\right)$$

vanishes.

Below we shall make use of the results in Eqs. (2.10)–(2.25); hence, let us elaborate on them. Observe that alternatively to regulating as in (2.16), the following procedure may be adopted, which is in fact equivalent. Instead of the definition (2.11) for χ, use a regulated formula,

$$\chi_+^{\Delta}(x)\equiv\frac{1}{\sqrt{2}}\left[\Pi(x)-i\int dy\, k_{\Delta}(x,y)\,\Phi(y)\right] \tag{2.33}$$

where the regularization consists of replacing k by k_{Δ},

$$k_{\Delta}\equiv\Delta k\Delta, \tag{2.34}$$

with Δ being some symmetric well-behaved kernel, thus leaving k_{Δ} antisymmetric. As the regulators are removed Δ approaches the δ function. The regulated generator is defined by

$$Q_f^{\Delta}=\tfrac{1}{2}\int \chi_+^{\Delta}\, f\chi_+^{\Delta}. \tag{2.35}$$

[In a self-evident notation, we view f as the kernel $f(x)\,\delta(x-y)$.] That this is equivalent to (2.16) is seen by making a canonical transformation.

$$\tilde{\Pi}(x)=\int dy\,\Delta^{-1}(x,y)\,\Pi(y),\qquad \tilde{\Phi}(x)=\int dy\,\Delta(x,y)\,\Phi(y),\qquad \tilde{\chi}=\frac{1}{\sqrt{2}}(\tilde{\Pi}+\tilde{\Phi}'). \tag{2.36}$$

In terms of the new variables, Q_f^{Δ} takes the form

$$Q_f^{\Delta}=\tfrac{1}{2}\int \tilde{\chi}f_{\Delta}\tilde{\chi} \tag{2.37}$$

$$f_{\Delta}\equiv\Delta f\Delta. \tag{2.38}$$

This is the same as (2.16) with f_{Δ} replacing F. It immediately follows that

$$\langle\varphi_1|e^{-iQ_f^{\Delta}}|\varphi_2\rangle=(\det\Delta)\,U(\tilde{\varphi}_1,\tilde{\varphi}_2;f_{\Delta}) \tag{2.39}$$

so that the subtraction in (2.24), with $F = f_\Delta$, renormalizes the regulated generator (2.35).

The above development concerns transformations that the act on $x + t$, but leave $x - t$ unchanged. We may, of course, treat the latter in an identical fashion simply by replacing $\chi_+ \equiv (1/\sqrt{2})(\Pi + \Phi')$ with $\chi_- \equiv (1/\sqrt{2})(\Pi - \Phi')$ or after regularization using instead of χ_+^Δ, the regulated version of χ_-.

$$\chi_-^\Delta \equiv \frac{1}{\sqrt{2}} (\Pi + ik_\Delta \Phi). \tag{2.40}$$

The representation functional is as in (2.18), with $-k$ replacing k; therefore, the renormalizing subtraction is again as in (2.21) and (2.24).

Since χ_+^Δ commutes with χ_-^Δ, so do the generators effecting the $x + t$ transformations with those that generate changes in $x - t$. Therefore, when both $x + t$ and $x - t$ are transformed, the genrator is the sum of the two separate generators and the renormalizing subtraction is the sum of the separate subtractions. The transformation functional is obtained by composing, through functional integration over φ, the two separate functionals U, each one taking the form (2.18) and (2.19), but with opposite signs for k and with different transformation functions F.

Note finally that our entire regularization and renormalization procedure addresses ultraviolet infinities. Owing to the infinite range of x, there may be additional infrared divergences for particular forms of f. We ignore these.

III. Isometry Generators in de Sitter Space

Consider 1 + 1 de Sitter space described by the metric,

$$ds^2 = dt^2 - e^{2ht} dx^2 \tag{3.1}$$

which is a particular open slicing of the general space, defined by the surface of a hyperboloid of revolution in 1 + 2 dimensions. The spacetime possesses constant curvature $2h^2$. It is convenient to pass to conformal coordinates, by defining a "conformal time."

$$\tilde{t} = e^{-ht}/h \tag{3.2}$$

$$ds^2 = \frac{1}{\tilde{t}^2 h^2} (d\tilde{t}^2 - dx^2). \tag{3.3}$$

Except at the end of Section V, we shall work with (3.3), and hence we suppress the tilde.

The Killing equation

$$f_{\alpha;\beta} + f_{\beta;\alpha} = 0 \tag{3.4}$$

for the infinitesimal isometries f^μ admits three linearly independent solutions. The simplest corresponds to spatial translations.

$$f_1^\mu = (0, 1). \tag{3.5}$$

The other two, a dilatation and a restricted spatial conformal transformation,

$$f_2^\mu = (t, x) \tag{3.6}$$

$$f_3^\mu = (tx, \tfrac{1}{2}(t^2 + x^2)) \tag{3.7}$$

together with (3.5) close on the $SO(2, 1)$ de Sitter Lie algebra.

$$\begin{gathered} [f_1, f_2]^\mu = f_1^\mu \qquad [f_1, f_3]^\mu = f_2^\mu \qquad [f_2, f_3]^\mu = f_3^\mu \\ [f_i, f_j]^\mu \equiv f_i^\alpha \partial_\alpha f_j^\mu - f_j^\alpha \partial_\alpha f_i^\mu. \end{gathered} \tag{3.8}$$

A scalar field, with mass m, without self-interaction but in de Sitter space, is governed by the Lagrange density

$$\mathscr{L} = \sqrt{-g}\, \tfrac{1}{2}[g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - m^2 \varphi^2] \tag{3.9}$$

and the covariantly conserved energy-momentum tensor is

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - \tfrac{1}{2} g_{\mu\nu}(g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - m^2 \varphi^2). \tag{3.10}$$

Time-independent generators of symmetry transformations are constructed from the Killing vectors.

$$Q_f = \int dx\, \sqrt{-g}\, T^0{}_\nu f^\nu. \tag{3.11}$$

In this way, we are lead to three generators, which formally are given by

$$Q_f = \int dx \{ \tfrac{1}{2}(\Pi^2 + \Phi'^2 + a^2 \Phi^2) f^0 + \Pi \Phi' f^1 \}, \tag{3.12}$$

where the canonical momentum Π is defined by $\Pi \equiv \delta\mathscr{L}/\delta\dot{\Phi} = \dot{\Phi}$ and $a \equiv m/(ht)$. Observe that Q_1 is just the total momentum.

To regulate (3.12), we rewrite it as

$$Q_f = \tfrac{1}{2} \int dx \{ (f^0 + f^1) \chi_+^2 + (f^0 - f^1) \chi_-^2 + a^2 \Phi^2 f^0 \}. \tag{3.13}$$

Hence, we define the regulated formula by [see (2.35)]

$$\begin{aligned} Q_f^\Delta &= \tfrac{1}{2} \int \{ \chi_+^\Delta (f^0 + f^1) \chi_+^\Delta + \chi_-^\Delta (f^0 - f^1) \chi_-^\Delta + a^2 \Phi f^0 \Phi \} \\ &= \tfrac{1}{2} \int (\Pi f^0 \Pi + \Phi k_\Delta f^0 k_\Delta \Phi + a^2 \Phi f^0 \Phi - i \Pi f^1 k_\Delta \Phi + i \Phi k_\Delta f^1 \Pi). \end{aligned} \tag{3.14}$$

Since χ_+^{Δ} and χ_-^{Δ} commute with each other, $\langle \varphi_1 | e^{-iQ_f^{\Delta}} | \varphi_2 \rangle$ may be readily evaluated in the massless case $m = 0$, following the development described at the end of Section II: the representation functional is the composition, by functional integration over φ, of two functionals, one as in (2.18) and (2.19), with $F = f_{\Delta}^0 + f_{\Delta}^1$, and a similar one with $F = f_{\Delta}^0 - f_{\Delta}^1$ and $-k$ replacing k.[5] It is also clear that the subtraction which renormalizes the generator is $\frac{1}{2} \operatorname{tr} f_{\Delta}^0 \omega$.

We have not succeeded in finding the explicit form for $\langle \varphi_1 | e^{-iQ_f^{\Delta}} | \varphi_2 \rangle$ when the mass is non-vanishing, owing to the non-commutativity of χ_+^{Δ} and χ_-^{Δ} with $a^2 \Phi^2$.[6] However, in the Appendix we present a heuristic but plausible argument that the renormalizing subtraction should be

$$\tfrac{1}{2} \operatorname{tr} f_{\Delta}^0 \, \omega^a \equiv q_f^{\Delta}, \tag{3.15}$$

where

$$\omega^a(x, y) \equiv \int \frac{dp}{2\pi} e^{-ip(x-y)} \sqrt{p^2 + a^2}$$

$$= -P \frac{a}{\pi(x-y)} K_1(a(x-y)). \tag{3.16}$$

Accepting this, we define the intermediately renormalized generators by

$$\tilde{Q}_f = \lim(Q_f^{\Delta} - q_f^{\Delta}) = \lim(Q_f^{\Delta} - \tfrac{1}{2} \operatorname{tr} f_{\Delta}^0 \, \omega^a), \tag{3.17}$$

which also holds for $a = 0$. As we shall demonstrate presently, conservation requirements fix the finite part of the subtraction, so the final, renormalized generators differ from $\tilde{Q}_f$ by finite terms, given below. Of course, the discussion at this stage makes no reference to dynamics, so considerations of conservation cannot be addressed.

Note that the momentum, Q_1, whose regularized form is

$$P^{\Delta} = \frac{i}{2} \int (\Pi k_{\Delta} \Phi - \Phi k_{\Delta} \Pi), \tag{3.18}$$

[5] The Gaussian functional integral gives

$$U(\varphi_1, \varphi_2) = \int \mathscr{D}\varphi \, U(\varphi_1, \varphi; F^-) \, U(\varphi, \varphi_2; F^+)$$

$$= N_{F^+} N_{F^-} \det^{-1/2} \left[\frac{i}{2\pi} (K_{F^+} + K_{F^-}) \right] \exp \frac{i}{2} \int \{ \varphi_1 K_{F^+} \varphi_1 + \varphi_2 K_{F^-} \varphi_2$$

$$- [\varphi_1 K_{F^-} + \varphi_2 F_{F^-} - i(\varphi_1 + \varphi_2) k](K_{F^+} + K_{F^-})^{-1} [K_{F^+} \varphi_1 + K_{F^-} \varphi_2 + ik(\varphi_1 + \varphi_2)] \},$$

$$F^+ \equiv f_{\Delta}^0 + f_{\Delta}^1, \qquad F^- \equiv f_{\Delta}^0 - f_{\Delta}^1,$$

except when $f^0 = 0$, in which case $U(\varphi_1, \varphi; F^+)$ and $U(\varphi, \varphi_2; F^-)$, and consequently also their composition $U(\varphi_1, \varphi_2)$, become functional delta-functions.

[6] It is also necessary to regulate the mass term. A convenient method is to replace $\int \Phi f^0 \Phi$ by an expression which does not introduce new kernels beyond k_{Δ}. We omit details, since we make no use of them.

requires no subtraction, as is familiar. Also, for the conformal generator, with $f_3^0 \propto x$, a subtraction is not needed because unwanted terms vanish by parity, since Δ may be chosen to have definite parity. [Recall that a restricted conformal transformation is a translation in the inverted coordinate system.]

IV. De Sitter Vacua

The Hamiltonian for our theory is formally given by

$$H = \int dx(\Pi\dot{\Phi} - \mathscr{L}) = \int dx \sqrt{-g}\, T_0^0$$

$$= \tfrac{1}{2} \int dx(\Pi^2 + \Phi'^2 + a^2\Phi^2). \tag{4.1}$$

Owing to the mass term, H is time-dependent and it makes no sense to define the vacuum as the lowest energy eigenvalue: the eigenvalues are time-dependent; equivalently, the time-dependent [functional] Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi(\varphi; t) = \tfrac{1}{2}\int dx\left(-\frac{\delta^2}{\delta\varphi(x)\,\delta\varphi(x)} + \varphi'^2(x) + \frac{m^2}{h^2t^2}\varphi^2(x)\right)\Psi(\varphi; t) \tag{4.2}$$

does not separate in time.

A reasonable definition for a Fock vacuum is that its functional $\Psi(\varphi; t)$ be a Gaussian solution to (4.2) with covariance Ω. Moreover, the Gaussian is further limited by requiring as much invariance as possible against transformations corresponding to the isometries of the background metric.

When the above definition is invoked in flat Minkowski space, the conventional Fock vacuum is regained as the unique translation and Lorentz invariant Gaussian solution to the time-dependent Schrödinger equation.

To implement this program in de Sitter space, we first need to regulate the Hamiltonian. In keeping with the regularization of the isometry generators, we replace in (4.1) and (4.2), $\int dx\Phi'^2 = \int \Phi k^2 \Phi$ by $\int \Phi k_\Delta^2 \Phi$.[7]

$$H^\Delta = \tfrac{1}{2}\int (\Pi^2 + \Phi k_\Delta^2\, \Phi + a^2\Phi^2) \tag{4.3}$$

and the regularized version of (4.2) is

$$i\frac{\partial}{\partial t}\Psi(\varphi; t) = \tfrac{1}{2}\int\left\{-\frac{\delta^2}{\delta\varphi^2} + \varphi\left(k_\Delta^2 + \frac{m^2}{h^2t^2}\right)\varphi\right\}\Psi(\varphi; t). \tag{4.4}$$

[7] One should also regulate $\int dx\Phi^2$; see footnote 6. However, this additional procedure has no effect on our final results; so we omit it.

Making the Gaussian *Ansatz*

$$\Psi(\varphi; t) = N e^{-(1/2)\int \varphi\Omega\varphi}, \tag{4.5}$$

we find that the covariance $\Omega(x, y; t)$ solves

$$-i\frac{\partial}{\partial t}\Omega = -\Omega^2 + k_\Delta^2 + \frac{m^2}{h^2 t^2} \tag{4.6}$$

and the normalization factor $N(t)$ satisfies

$$i\frac{\partial}{\partial t}\ln N = \tfrac{1}{2}\,\mathrm{tr}\,\Omega. \tag{4.7}$$

To obtain solutions to (4.6) and (4.7), we impose the requirement that Ψ be "translation" invariant, even in the presence of the regulator.

$$P^\Delta \Psi = 0 \tag{4.8a}$$

This is possible, because $[H^\Delta, P^\Delta] = 0$. With (3.18), (4.8a) is equivalent to

$$[k_\Delta, \Omega] = 0. \tag{4.8b}$$

When k_Δ commutes with Ω, the latter may be diagonalized with the former, and Ω is a function of the kernel k_Δ. Moreover, since $\Omega(x, y; t)$ is symmetric in (x, y) and $k_\Delta(x, y)$ is anti-symmetric, the former must be an even function of the latter, or equivalently, Ω depends on $|k_\Delta|$.

The solution to (4.6) and (4.7) is now determined as

$$\Omega = -i\frac{\partial}{\partial t}\ln D \tag{4.9}$$

$$N = \det{}^{-1/2} D \tag{4.10}$$

$$D = \left(\frac{1-r^2}{r}\right)^{1/2}\left(\frac{\pi}{|k_\Delta|\;\Theta_\nu'(t|k_\Delta|)}\right)^{1/2}\cos(\Theta_\nu(t|k_\Delta|) - \alpha(|k_\Delta|)). \tag{4.11}$$

Here, Θ_ν is the real phase of $e^{i(\nu\pi/2)}$ times the Hankel function of order $\nu \equiv \sqrt{\frac{1}{4} - (m^2/h^2)}$.[8]

$$\exp\left(i\frac{\nu\pi}{2}\right)H_\nu^{(1)} = M_\nu e^{i\Theta_\nu}$$
$$M_\nu^2(z)\,\Theta_\nu'(z) = \frac{2}{\pi z}. \tag{4.12}$$

[8] The reason that we introduce the factor $e^{i(\nu\pi/2)}$, which is additional to the conventional formulas for phase and modulus of Hankel functions, is that our Θ_ν remains real as ν becomes imaginary.

The single integration constant α is complex, and may depend on $|k_\Delta|$. Its imaginary part determines $r \equiv \tanh(\operatorname{Im} \alpha)$. Also, Ω is complex, and may be separated into real and imaginary parts.

$$\Omega = \Omega_R + i\Omega_I. \tag{4.13}$$

Various multiplicative constants are adjusted so that $\int \mathscr{D}\varphi \Psi^* \Psi = 1$.

More explicitly, the formulas read

$$N = \det^{-1/2} \left\{ \left(\frac{\pi^2 t}{2} \right)^{1/2} H_\nu^{(1)}(t|k_\Delta|) \right\}$$
$$\times \det^{1/2} \left\{ \frac{r^{-1/2} \cos(\Theta_\nu(t|k_\Delta|) - \theta) + i r^{1/2} \sin(\Theta_\nu(t|k_\Delta|) - \theta)}{\cos(\Theta_\nu(t|k_\Delta|) - \theta) + i \sin(\Theta_\nu(t|k_\Delta|) - \theta)} \right\}. \tag{4.14}$$

A t-independent phase has been dropped, and $\theta \equiv \operatorname{Re} \alpha$.

$$\Omega_R = |k_\Delta| \frac{r\Theta'_\nu(t|k_\Delta|)}{\cos^2(\Theta_\nu(t|k_\Delta|) - \theta) + r^2 \sin^2(\Theta_\nu(t|k_\Delta|) - \theta)} \tag{4.15a}$$

$$\Omega_I = |k_\Delta| \left(\frac{1}{2} \frac{\Theta''_\nu(t|k_\Delta|)}{\Theta'_\nu(t|k_\Delta|)} + \Theta'_\nu(t|k_\Delta|) \frac{(1 - r^2) \tan(\Theta_\nu(t|k_\Delta|) - \theta)}{1 + r^2 \tan^2(\Theta_\nu(t|k_\Delta|) - \theta)} \right). \tag{4.15b}$$

In the massless case, $\nu = \frac{1}{2}$, everything is expressed in terms of elementary functions.

$$D = \left(\frac{1 - r^2}{r} \right)^{1/2} \left(\frac{\pi}{|k_\Delta|} \right)^{1/2} \sin(t|k_\Delta| - \alpha(|k_\Delta|)) \tag{4.16}$$

$$N = e^{-it(1/2)\operatorname{tr}|k_\Delta|} \det^{1/4} \left(\frac{-\pi}{|k_\Delta|} \right)$$
$$\times \det^{-1/2} \left\{ \frac{r^{1/2} \cos(t|k_\Delta| - \theta) + i r^{-1/2} \sin(t|k_\Delta| - \theta)}{\cos(t|k_\Delta| - \theta) + i \sin(t|k_\Delta| - \theta)} \right\} \tag{4.17}$$

$$\Omega_R = |k_\Delta| \frac{r}{r^2 \cos^2(t|k_\Delta| - \theta) + \sin^2(t|k_\Delta| - \theta)} \tag{4.18a}$$

$$\Omega_I = -|k_\Delta| \frac{(1 - r^2) \operatorname{ctn}(t|k_\Delta| - \theta)}{1 + r^2 \operatorname{ctn}^2(t|k_\Delta| - \theta)}. \tag{4.18b}$$

To restrict Ω further, we consider the remaining two isometries (3.6) and (3.7). The generator acting on the Gaussian gives

$$(Q_f^\Delta - q_f^\Delta)\, \Psi = \tfrac{1}{2} \operatorname{tr}(f^0 \Omega - f_\Delta^0\, \omega^a)\, \Psi$$
$$+ \tfrac{1}{2} \int dx\, dy \varphi(x)\, \delta_f \Omega(x, y; t)\, \varphi(y)\, \Psi, \tag{4.19}$$

where

$$\delta_f \Omega = \tfrac{1}{2} \int (-\Omega f^0 \Omega + k_\Delta f^0 k_\Delta + a^2 f^0 + \Omega f^1 k_\Delta - k_\Delta f^1 \Omega). \tag{4.20}$$

In order to control singularities when the regulators are removed, we prefer to consider matrix elements of $Q_f^\Delta - q_f^\Delta$ in the Fock space built on Ψ. Since the wave functional for an arbitrary Fock state is a polynomial in φ multiplying Ψ, the general case is achieved by considering the overlap of (4.19) with $e^{\int J\varphi}\Psi$, where $J(x)$ is a source.

$$\langle J | Q_f^\Delta - q_f^\Delta | \Omega \rangle = \int \mathscr{D}\varphi e^{\int J\varphi} \Psi^* (Q_f^\Delta - q_f^\Delta)\, \Psi = \exp\left(\tfrac{1}{4} \int J \Omega_R^{-1} J \right)$$
$$\times \left[\tfrac{1}{2} \operatorname{tr}(f^0 \Omega - f_\Delta^0\, \omega^a + \tfrac{1}{2} \delta_f \Omega \Omega_R^{-1}) + \tfrac{1}{8} \int J \Omega_R^{-1}\, \delta_f \Omega \Omega_R^{-1} J \right]. \tag{4.21}$$

The first term in brackets, involving the trace, is the diagonal matrix element $\langle \Omega | Q_f^\Delta - q_f^\Delta | \Omega \rangle$. The remaining term, bilinear in J, generates the connected off-diagonal matrix elements.

The off-diagonal matrix elements must vanish whether one wants the vacuum state to be invariant [the generator annihilates it] or merely phase-invariant [the state is an eigenstate of the generators]. However, we cannot set this requirement on the regulated expresssion because the regulated generators are not constants of motion relative to the regulated Hamiltonian, i.e,

$$\frac{\partial}{\partial t} Q_f^\Delta \neq i[Q_f^\Delta, H^\Delta].$$

Therefore, we only demand that $\delta_f \Omega$ vanish when the regulators are removed, $\Delta \to \delta$. In this limit, with

$$\Omega(x, y; t) = \int \frac{dp}{2\pi} e^{ip(x-y)} \Omega(|p|; t)$$

the two isometries imply that

$$t \frac{\partial}{\partial t} \frac{\Omega(|p|; t)}{p} = p \frac{\partial}{\partial p} \frac{\Omega(|p|; t)}{p}, \tag{4.22}$$

which is satisfied provided that α is constant, as we henceforth require. This leaves the state depending on one complex constant, α. It is this parameter dependence that has been identified as characterizing de Sitter invariant vacua [10], but as will be seen here, these states are at best phase-invariant.

We now examine the diagonal part of (4.21), which may also be written as

$$\langle \Omega | Q_f^{\Delta} - q_f^{\Delta} | \Omega \rangle = \tfrac{1}{2} \operatorname{tr}(f^0 \Omega_R - f_{\Delta}^0 \omega^a + \tfrac{1}{2} \delta_f \Omega_R \Omega_R^{-1}) \tag{4.23}$$

because $\operatorname{tr}(f^0 \Omega_I + \frac{1}{2} \delta_f \Omega_I \Omega_R^{-1}) = 0$, as follows from (4.20), when the cyclicity of the trace is used together with the fact that k_{Δ}, Ω_R, and Ω_I all commute. When the regulator is removed, $\delta_f \Omega_R$ vanishes, while the remainder becomes

$$\langle \Omega | \tilde{Q}_f | \Omega \rangle = \tfrac{1}{2} \int dx\, f^0(x) \int \frac{dp}{2\pi} (\Omega_R(|p|; t) - \omega^a(|p|)). \tag{4.24}$$

In general, the p integral diverges; however, for precisely one value of the parameter, i.e., $r = 1$, the p integration is finite. This is obvious in the massless case (4.18), where for $r = 1$, $\Omega_R(|p|; t) = |p|$, and the integrand vanishes because the subtraction compensates precisely. In the massive case, the compensation is not complete: at $r = 1$, $\Omega_R(|p|; t) = |p|\, \Theta'_\nu(t|p|)$. Nevertheless, the large $|p|$ behavior of this matches the first two terms of $\omega^a(|p|)$; hence, the integrand is $\mathcal{O}(p^{-3})$ and the p integration converges.

In fact, it is necessary to adjust the subtraction by finite terms so that the compensation is exact at non-zero mass as well. Observe that the integrand is $|p|$ times a function of $t|p|$; hence, the p integration produces a result proportional to t^{-2}. Since f^0 is proportional to t, the final result for the diagonal matrix element is proportional to t^{-1}. On the other hand, matrix elements of a constant of motion between states that solve the time-dependent Schrödinger equation are time-independent even when the constant possesses explicit time-dependence. Thus, our regularization and renormalization does not respect conservation of the generator. This defect is remedied by effecting a complete subtraction through the definition

$$:Q_f: = \lim(Q_f^{\Delta} - \tfrac{1}{2} \operatorname{tr} f^0 |k_{\Delta}|\, \Theta'_\nu), \tag{4.25}$$

which differs from $\tilde{Q}_f$ by finite terms.

In conclusion, therefore, we have shown that the vacuum with $r = 1$, the so-called "Euclidean" vacuum, is completely invariant against de Sitter transformations.

$$:Q_f: |\Omega\rangle|_{r=1} = 0. \tag{4.26}$$

The other vacua satisfy

$$:Q_f: |\Omega\rangle = (r-1) \int dx\, f^0(x) \int_0^{\infty} \frac{dp}{2\pi}\, p\Theta'_\nu(tp) \frac{1 - (r+1)\sin^2(\Theta_\nu(tp) - \theta)}{1 + (r^2 - 1)\sin^2(\Theta_\nu(tp) - \theta)} |\Omega\rangle. \tag{4.27a}$$

As already mentioned, the eigenvalue is non-zero only for the second Killing vector, corresponding to dilatations, because f^0 vanishes for translations while for spatial conformal transformations f^0 is odd in x, hence, the integral over all x may be set to zero. For dilatations, $:Q_2: \equiv D$, we have

$$D|\Omega\rangle = (r-1)\frac{L}{t}\int_0^\infty \frac{dk}{2\pi} k\Theta_\nu'(k) \frac{1-(r+1)\sin^2(\Theta_\nu(k)-\theta)}{1+(r^2-1)\sin^2(\Theta_\nu(k)-\theta)} |\Omega\rangle, \quad (4.27b)$$

where L is the [infinite] length of space. The eigenvalue is not only infrared divergent, but also ultraviolet divergent owing to the k integral. Moreover, it is time-dependent. Thus, these vacua are only phase-invariant, and the 1-cocycle which occurs is infinite.

It may appear puzzling that two generators of a non-Abelian group annihilate all vacua, but a third does not. The resolution lies in the infinite eigenvalue of (4.27). The situation is analogous to the Poincaré group in flat spacetime, where the Lorentz generator when commuted with the momentum gives the Hamiltonian. However, the former two annihilate the ground state, which is Lorentz and translation invariant, while the Hamiltonian possesses an infinite eigenvalue—the zero point energy. Physically what is being said is that one cannot translate or boost an infinitely heavy object. Similar remarks apply to our theory in de Sitter space.

It should be appreciated that we cannot redefine the generators so that some other vacuum, with $r \neq 1$, becomes invariant at the expense of phase-invariance of the Euclidean vacuum. The point is that only finite redefinition is permitted at this stage, but the eigenvalue in (4.27) is infinite and it cannot be removed.

The $r = 1$ vacuum has been previously preferred: it is the one that naturally arises in a Euclidean formulation [9], it allows a sensible definition of energy [12], and it is relevant to the inflationary program [6]. Now we see that also it is the unique, completely de Sitter invariant state.

V. Further Properties

A. Conformal Symmetries

The de Sitter spacetime is conformally flat; see (3.3). Hence, the conformal Killing equations

$$f_{\mu;\nu} + f_{\nu;\mu} = g_{\mu\nu} g^{\alpha\beta} f_{\alpha;\beta} \quad (5.1)$$

possess the same solutions as in flat space: all conformal Killing vectors are parametrized by two arbitrary functions.

$$\begin{aligned} f^0 &= \mathscr{F}(x+t) - \mathscr{G}(x-t) \\ f^1 &= \mathscr{F}(x+t) + \mathscr{G}(x-t). \end{aligned} \quad (5.2)$$

The three Killing vectors (3.5)–(3.7) correspond to $\mathscr{F}(z) = \mathscr{G}(z) = \frac{1}{2}, \frac{1}{2}z, \frac{1}{4}z^2$.

For a massless field, the conformal Killing vectors give a further infinity of time-independent generators, because the conserved energy-momentum tensor is traceless. [In the de Sitter space quantum theory, the two-dimensional trace anomaly $-(1/24\pi) R$ does not invalidate the above statement, since the energy-momentum tensor may be "improved" by a term proportional to $g_{\mu\nu} R$, which removes the anomalous trace, yet remains conserved for constant R.] However, it is known that in the quantum theory, the infinite conformal Lie algebra is realized with a center (see (2.25)), and the group is represented projectively. Consequently, no state can be invariant against all transformations: the commutators of generators do not close; therefore all generators cannot annihilate a state.

An exception occurs for the $SO(2, 2) = SO(2, 1) \otimes SO(2, 1)$ subgroup of restricted conformal transformations which arise from Killing vectors that are at most quadratic in their argument. For these, the center vanishes, as is seen from (2.25), and one may inquire how the de Sitter vacua respond to the corresponding transformations.

Our previous analysis answers this question for the diagonal $SO(2, 1)$ subgroup, which is spanned by the three Killing vectors (3.5)–(3.7), but we still need to analyze the effect of the remaining three transformations corresponding to conformal Killing vectors $\mathscr{F}(z) = -\mathscr{G}(z) = \frac{1}{2}, \frac{1}{2}z, \frac{1}{4}z^2$. Respectively, these give rise to a time translation,

$$f_4^\mu = (1, 0), \tag{5.3}$$

a Lorentz transformation

$$f_5^\mu = (x, t), \tag{5.4}$$

and a restricted temporal conformal transformation,

$$f_6^\mu = (\tfrac{1}{2}(t^2 + x^2), xt). \tag{5.5}$$

The regulated charges are subtracted as before, and the action on Ψ is as in (4.19) and (4.20), while the matrix elements are as in (4.21) and (4.23). All three vectors lead to the same result. For the off-diagonal part to vanish in the limit $\Delta \to \delta$, i.e., for lim $\delta_f \Omega$ to be zero, one must have

$$\Omega^2(|p|; t) = p^2, \tag{5.6}$$

which is true only for the Euclidean, $r = 1$, vacuum. Moreover, for that state the subtraction completely removes the diagonal part (4.23), so that the Euclidean vacuum of the massless theory is strictly $SO(2, 2)$ invariant, while the vacua at $r \neq 1$ are not even phase-invariant.

Of course, this result is not surprising: owing to conformal invariance, the background metric is invisible and the situation must be as in flat spacetime, where a unique Poincaré invariant vacuum exists.

B. Passage to Minkowski Space

Let us discuss now how the family of phase-invariant de Sitter vacua goes over to Minkowski states in the limit of flat space, i.e., for small h, in the metric (3.1).

In Minkowski space, the general Gaussian solution of the regulated Schrödinger equation in Minkowski time t_M can be written in the form (4.9), (4.10), where now the kernel D is given by [assuming translation invariance]

$$D_M = \left(\frac{1-r_M^2}{r_M}\right)^{1/2} \left(\frac{\pi}{\omega_\Delta^m}\right)^{1/2} \cos(t_M \omega_\Delta^m - \alpha_M)$$
$$\omega_\Delta^m \equiv \sqrt{k_\Delta^2 + m^2}. \tag{5.7}$$

The complex integration constant α_M may in general depend on $|k_\Delta|$; only after imposing Lorentz invariance do we get $r_M \equiv \tanh(\operatorname{Im} \alpha_M) = 1$, and the unique conventional Fock vacuum is recovered.

For a phase-invariant de Sitter state we know that

$$D = \left(\frac{1-r^2}{r}\right)^{1/2} \left(\frac{\pi}{|k_\Delta|\, \Theta'_\nu(\tilde{t}|k_\Delta|)}\right)^{1/2} \cos(\Theta_\nu(\tilde{t}|k_\Delta|) - \alpha). \tag{5.8}$$

We restore here the tilde notation of Eq. (3.2) to distinguish between the usual time [t of (3.1)] and the conformal time [$\tilde{t}$ of (3.2) and (3.3)]. For small h, $\tilde{t}$ diverges as $1/h$ and ν becomes imaginary.

Let us first consider the Euclidean vacuum, for which $r = 1$ and $\Omega = -i(\partial/\partial\tilde{t}) \ln D = |k_\Delta|\, \Theta'_\nu(\tilde{t}|k_\Delta|)$. In this case, one finds [14] $\Omega \sim_{h \to 0} \omega_\Delta^m$, so that in the limit the unique Lorentz invariant Minkowski ground state is selected.

In the general case, D and hence Ω do not have a limit. In fact, as h approaches zero, the asymptotic behavior of Θ_ν [14] gives

$$D \underset{h \to 0}{\sim} \left(\frac{1-r^2}{r}\right)^{1/2} \left(\frac{\pi}{\omega_\Delta^m}\right)^{1/2} \cos\left(\frac{1}{h}\omega_\Delta^m + \frac{m}{2h} \ln\left(\frac{\omega_\Delta^m - m}{\omega_\Delta^m + m}\right) - \alpha\right). \tag{5.9}$$

However, since (5.9) has the same form as (5.7), we can compare the two expressions for large values of Minkowski time, $t_M \sim 1/h$, and we identify

$$\alpha_M = \alpha - \frac{m}{2h} \ln\left(\frac{\omega_\Delta^m - m}{\omega_\Delta^m + m}\right). \tag{5.10}$$

In this way it can be said that the family of phase-invariant de Sitter states produces in the flat limit a one-parameter family of states in Minkowski space which are not in general Lorentz invariant, because α_M in (5.10) depends on k.

In the massive theory, there is an exception to this behavior, given by those de Sitter states for which $\operatorname{Im} \alpha$ diverges as h vanishes. [Although α is $\tilde{t}$ and $|k_\Delta|$ independent, it can be a function of the dimensionless parameter m/h.] In this case, Eq. (5.10) shows that r_M is driven to unity, so that these de Sitter states approach the unique Lorentz invariant Minkowski vacuum, and the arbitrary parameter α decouples.

This difference between massive and massless theory can be physically understood by looking at the behavior of the two-point function $G(x, y)$ [13].

Besides the usual short distance singularity as x approaches y, in de Sitter space, for $r \neq 1$, $G(x, y)$ possesses also a singularity at $x = y'$, where y' is a point diametrically opposite to y on the de Sitter hyperboloid (the antipodal point). In other words, an observer at x feels both the effect of a "charge" at y and of the "image charge" at y' (even if y' is outside his horizon). In the massive case, as $h \to 0$, the image charge produces an exponentially vanishing effect $O(e^{-m/h})$ upon x and the resulting vacuum state becomes insensitive to the choice of r, and r_M is driven to 1. In the massless case, the long range force due to the image charge at y' is always felt at x and as $h \to 0$ the vacuum state always acquires the same α-parameter dependence which was originally present in de Sitter space, so that $\alpha_M = \alpha$.

APPENDIX

As discussed in Section II, the subtraction q_f^{Δ} which allows defining the renormalized generators, $\tilde{Q}_f = \lim_{\Delta \to \delta} (Q_f^{\Delta} - q_f^{\Delta})$, is found by examining the representation functional for the finite transformation, $\langle \varphi_1 | e^{-i\tau Q_f^{\Delta}} | \varphi_2 \rangle$, as the regulator Δ is removed. Q_f^{Δ} is given by (3.14); since it is quadratic in the dynamical variables Π and Φ, the representation functional must be a Gaussian in φ_1, φ_2.

$$\langle \varphi_1 | e^{-i\tau Q_f^{\Delta}} | \varphi_2 \rangle = N \exp \left\{ -\tfrac{1}{2} \int (\varphi_1 A \varphi_1 - 2\varphi_1 B \varphi_2 + \varphi_2 C \varphi_2) \right\}. \tag{A.1}$$

The Schrödinger-like equation (2.4a), produces a set of differential equations for the normalizing constant N, and for the kernels A, B, C, which are also contrained by unitarity.

$$\begin{aligned} C(x, y; \tau) &= A^*(x, y; -\tau) \\ B(x, y; \tau) &= B^*(y, x; -\tau) \\ N(\tau) &= N^*(-\tau). \end{aligned} \tag{A.2}$$

We have not succeeded in finding explicit solutions for these equations. However, from the discussion of Section II we know that the infinities which appear when the regulator is removed are confined to the normalization constant N.

The equation which determines N involves A.

$$i\partial_\tau (\ln N) = \tfrac{1}{2} \operatorname{tr}(f^0 A). \tag{A.3}$$

In the limit $\Delta \to \delta$ the divergent part of $\ln N$, which is linear in τ, can be extracted from the τ-independent divergence in $\lim_{\Delta \to \delta} \operatorname{tr}(f^0 A)$.

Our heuristic procedure for finding q_f^{Δ} consists of removing the regulators first, determining $A(x, y; \tau)$, and extracting the infinite part as $x \to y$, after τ has been continued to imaginary values $\tau \to -i\tau$.

When $\Delta = \delta$, A satisfies

$$i\partial_\tau A = A f^0 A + k f^1 A - A f^1 k - k f^0 k - a^2 f^0, \tag{A.4}$$

with the boundary condition that $A \propto \tau^{-1}$ as $\tau \to 0$. Since only the dilatation generator requires a subtraction, we take $f^\mu = (t, x)$. In this case (A.4) is solved by

$$A(x, y; \tau) = \frac{2}{it^2} \int_0^\infty \frac{dp}{2\pi}\, p \cos \frac{p}{t}(x-y) \left\{ \Theta'_\nu(p) \operatorname{ctn}(\Theta_\nu(p) - \Theta_\nu(pe^{-\tau})) - \frac{1}{2} \frac{\Theta''_\nu(p)}{\Theta'_\nu(p)} \right\}. \tag{A.5}$$

As before, Θ_ν is the phase of the Hankel function of order $\nu = \sqrt{\frac{1}{4} - (m^2/h^2)}$; see (4.12).[9]

For $x \to y$, an ultraviolet divergence occurs in $A(x, y; -i\tau)$. The large p behavior of the bracketed quantity in (A.5), with $\tau \to -i\tau$, is $i(1 + (m^2/(2h^2p^2)))$. Hence we conclude that A behaves as

$$\int \frac{dp}{2\pi} e^{-ip(x-y)} \left(|p| + \frac{a^2}{2|p|} \right)$$

for $x \sim y$. This then is the renormalizing subtraction—a result which differs from (3.15) only by finite terms.

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[9] The other kernels can also be found: C is determined by unitary, see (A.2), while B is given by

$$B(x, y; \tau) = \frac{2}{it^2} \int_0^\infty \frac{dp}{2\pi}\, p \cos \frac{p}{t}(xe^{\tau/2} - ye^{-\tau/2}) \frac{\sqrt{\Theta'_\nu(pe^{\tau/2})\, \Theta'_\nu(pe^{-\tau/2})}}{\sin(\Theta_\nu(pe^{\tau/2}) - \Theta_\nu(pe^{-\tau/2}))}.$$