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The Critical Dimension for Chaotic Cosmology

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ABSTRACT

Using the ADM formalism for General Relativity the approach to a space-time singularity of a general inhomogeneous universe, in an arbitrary number of dimensions, is studied. The question of whether chaotic behaviour is a generic feature of Einstein's equations, in an arbitrary number of dimensions, is explored. We find that models that contain ten or more spatial dimensions are non-chaotic and their approach toward the initial singularity is monotonic, whereas for those with dimensionality between four and nine their approach is chaotic. A clear geometrical picture is constructed whereby this result can be understood.

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I. Introduction

The nature of the initial singularity has been one of the most outstanding problems in physics. The singularity theorems[1] tell us that at least within the realm of classical (non-quantum) physics an initial singularity is almost unavoidable. Unfortunately, these powerful theorems do not tell us much about the properties of the singularity. In order to learn more about it we have to look for specific solutions of Einstein's equations. The most general homogeneous closed cosmological model containing a singularity is the anisotropic Bianchi IX model. When isotropic, it reduces to the Friedmann-Robertson-Walker (FRW) closed model. Early studies of this model[2] revealed that the nature of the initial singularity was far more complex than anything found in the FRW case. It was a chaotic state and initial conditions were completely unpredictable. The system had an ergodic behaviour. On approach to the initial singularity, the scale factors underwent a sequence of oscillations where periods of expansion and contraction of the scale factors took place in turn (see ref.(3) for a review). The characteristic feature of these oscillations was the fact that both the amplitude and the frequency diverge on approach to the singularity and the system becomes unpredictable. Nevertheless, the volume evolved smoothly as a monotonically decreasing function of time having a zero value at zero time. This oscillatory behaviour was described by a series of "bounces" that changed the model from one Kasner state into another. One of the most striking features of this bizarre behaviour was the fact that it was a generic property of the model. Any Bianchi IX vacuum solution developed a chaotic behaviour on approach to the singularity regardless of its initial values or boundary conditions (on a later hypersurface) i.e. the diverging oscillations appeared for all but a set of measure zero initial conditions. This was all very well, but for all we know the universe need not be homogeneous or even be approximated by a homogeneous model near an initial singularity, so we had to make sure that this property was not a particular feature of these restricted models. However, soon after, it was shown that a "general" inhomogeneous solution existed (or could be constructed) near the singularity, which had the same chaotic behaviour[4].

We should, at this point, clarify what we mean by "general". We will call "general " a solution that contains the maximum number of arbitrary (non-removable) spatial function i.e. all possible gravitational degrees of freedom are present[2]. These functions then determine the initial (boundary) conditions on a given hypersurface. This surprising result lead to the belief that, at least at the classical level, the Big Bang singularity must have possesed some of this properties.

A powerful method can be used to extract more information about these solutions. This is based on a Hamiltonian formulation of General Relativity: by foliating space-time, a canonical Lagrangian can be constructed. The problem of solving Einstein's field equations for the evolution of the universe is exchanged for that of a point particle moving inside a time-dependent potential. This canonical formulation of General Relativity is known as the ADM formalism[5]. The method is very succesful when applied to homogeneous cosmologies where the potential can be easily identified with the spatial-curvature of space. The method will be described later on.

Recently, the idea that our Universe is of a higher spatial dimensionality has been revived. Our old 3+1 space-time is not big enough to accomodate today's gauge theories. Both, supergravity and superstring theories require for their internal consistency and finiteness more than three spatial dimensions. This has become an almost indispensable ingredient in any gauge theory that attempts to explain physics at very high energies. Without trying to justify this assumption from physical first principles and exploiting the mathematical interest that models in arbitrary dimensions have, we will take it to be valid. We will also assume that the space-time is described by a Pseudo-Riemannian manifold of dimension $(n+1)$, with a well defined Lorentzian metric satisfying Einstein's equations, and ask the following question: Is the chaotic behaviour found in Einstein's equations in four-dimensional space-times a generic feature, or is particular to this dimensionality ?

Several attempts to answer this question have been made in the past. There are examples in the literature where homogeneous models have been constructed which approach the singularity in a non-chaotic way. Most of these models have space-times of the form $M^4 \times B$ where both submanifolds are homogeneous and in some instances anisotropic[6].

The case where the topology of the manifold is not that of a product space has only been explored when $n = 4$ where all the possible Lie algebras have been classified. No chaotic behaviour was found in any of these models when the metric was diagonal. The general homogeneous case has not been explored[7].

However the question remains unanswered. The Universe was probably very inhomogeneous initially and it is not possible to extrapolate the results found for homogeneous models to this more general case. Very recently a surprising result was reported in ref.(8) where a "general" solution to Einstein's equations was constructed near the singularity that had no chaotic behaviour for $n > 10$, it contained all the necessary degrees of freedom of the gravitational field to be considered a general solution. Although, uniqueness has not been proven for this solution, its existence is probably sufficient to conclude that if the universe had more than ten spatial dimensions, then it was probably not chaotic in the begining. Worth pointing out is the fact that $n = 3$ is certainly a very special case at it presents the same behaviour in the homogeneous and inhomogeneous cases.

In this paper we will rederive the result found in ref.(8) using a completely different technique, that of Hamiltonian dynamics. We will give a clear geometrical interpretation to the result that $n = 9$ is the dividing line between chaotic and non-chaotic behaviour. A rigorous proof of this statement will be given in section III where we will also argue that the most general models with $3 \leq n \leq 9$ are chaotic near the singularity. With this formalism it is possible to replace the field equations describing the evolution of the universe by a point particle moving inside a time dependent potential, this being the spatial curvature of the model[9]. This allows a qualitative description of the behaviour of the universe. The whole evolution is reduced to a sequence of "bounces" against potential walls with free motion in between. This free motion is described by Kasner solutions while the bouncing law can be derived from the assumption of perfect collisions. The walls of the potential are in general functions of time, their shape is basically preserved as time passes but they move apart as the singularity is approached. We will show that the dividing line when $n = 9$ only reflects the fact that the walls are moving so fast that after bouncing a few times the Universe particle cannot catch up with the walls, the velocity of the walls and the particle aquire their critical value for this n.

We will start by introducing the ADM formalism. The first step consists in splitting the $(n + 1)$ space-time metric $g_{\mu\nu}$ into its space and time components

$$g_{\mu\nu} = \begin{pmatrix} N_i N^i - N^2 & N_i \\ N_i & g_{ij} \end{pmatrix}$$

where μ, ν run from 0 to n, while i, j from 1 to n. N is called the lapse function and N_i the shift functions. It is always possible to choose a gauge where $N_i = 0$. In this gauge the line element is given as

$$ds^2 = -N^2 dt^2 + g_{ij} dx^i dx^j \tag{1}$$

The Einstein-Hilbert action will be the usual one

$$S_{ADM} = \int \sqrt{g}\, NR \quad dx^{(n+1)} \tag{2}$$

Where $g = det(g_{ij})$ and R is the $(n + 1)$-dimensional Ricci scalar. Units are taken so that $16\pi G = c = 1$.

In order to construct a canonical formalism it will prove useful to rewrite the Lagrangian in term of the metric components and its velocities as the canonical variables

$$L_{ADM} = \frac{1}{4N}\sqrt{g}\left(g^{ij}g^{kl} - g^{ik}g^{jl}\right)\dot{g}_{ik}\dot{g}_{jl} - NP \tag{3}$$

with P being the spatial-curvature scalar calculated with g_{ij}. Before varying (3) we would like to mention some invariance properties of the Lagrangian that will be crucial to our argument. The ADM decomposition is formulated entirely in coordinate space. However, this is not always the best frame to use. We will require a formulation in form-space. One of the advantages of this is the fact that the metric can always be put in diagonal form. Let g_{ab} be the metric in form space, then there exist non-singular matrices $\sigma_i^a(\mathbf{x})$ such that

$$g_{ij} = \sigma_i^a g_{ab} \sigma_j^b \tag{4}$$

It is easy to see that (3) is invariant under the metric transformation given in eq.(4). This means that the same equation obtained in coordinate space can be used in form space.

It will prove useful to introduce Misner's decomposition for the metric[10]

$$g_{ab} = e^{2\alpha}(e^{2\beta})_{ab} \equiv e^{2\alpha}\tilde{g}_{ab}$$

with β_{ab} a traceless $n \times n$ matrix and $g = e^{2n\alpha}$. We shall assume that α is only a function of time, then the Lagrangian finally takes the following form

$$L_{ADM} = \frac{1}{4\tilde{N}}\dot{\tilde{g}}_{ac}\dot{\tilde{g}}_{bd}\tilde{g}^{ab}\tilde{g}^{cd} - \frac{n(n-1)}{\tilde{N}}\dot{\alpha}^2 + \tilde{N}e^{2n\alpha}P \tag{5}$$

with $\tilde{N} = e^{-n\alpha}N$. From its variation with respect to $\tilde{N}$ we get the zero-zero Einstein equation

$$\frac{1}{4\tilde{N}^2}\dot{\tilde{g}}_{ac}\dot{\tilde{g}}_{bd}\tilde{g}^{ab}\tilde{g}^{cd} - \frac{n(n-1)}{\tilde{N}^2}\dot{\alpha}^2 - e^{2n\alpha}P = 0 \tag{6}$$

At this point we will fix completely the gauge by taking $\tilde{N} = \dot{\alpha}$ and the σ^a to make the metric diagonal. In this case the line element takes the form

$$ds^2 = -e^{2n\alpha}d\alpha^2 + e^{2\alpha}(e^{2\beta})_{ab} \ \ \sigma^a\sigma^b \tag{7}$$

with

$$\left(e^{2\beta}\right) = diag\left(e^{2\beta^1}, \ ..., e^{2\beta^n}\right) \ \ ; \quad \sigma^a = \sigma^a_i dx^i \tag{8}$$

The β 's are general functions of the space-time coordinates, subject to the constraint

$$\sum_{a=1}^{n} \beta^a = 0 \tag{9}$$

In this particular gauge eq. (6) becomes

$$\sum_{a=1}^{n} \left(\frac{d\beta^a}{d\alpha}\right)^2 - e^{2n\alpha}P = n(n-1) \tag{10}$$

This equation is formally identical to that of a point particle moving inside a time dependent potential of the form $-e^{2n\alpha}P$. The analysis of the motion will be done in the next section.

II. The Potential Picture

As we know from the study of the three-dimensional mixmaster model, the analogy between the evolution of the Universe and that of a point particle moving inside a time dependent potential is both intuitive and useful. We saw in the last section that the potential for the problem was given essentially by the spatial curvature of space. This curvature is a well defined object in terms of the metric components [11],

$$-e^{2n\alpha}P = e^{2(n-1)\alpha}\left[\frac{1}{2}\sum_{\substack{a,b,c \\ a\neq b\neq c}} \left[\left(C^a_{bc}e^{\beta^a-\beta^b-\beta^c}\right)^2 + C_{aab}C^a_{bc}e^{\beta^a-2\beta^b-\beta^c}\right] + \sum_a D^a e^{-2\beta^a}\right] \tag{11}$$

where both $C^a_{bc}(\mathbf{x})$ and $D^a(\mathbf{x})$ are functions of the spatial coordinates. The $C^a_{bc}(\mathbf{x})$ are defined in terms of the σ^a by the relation

$$d\sigma^a = -\frac{1}{2}C^a_{bc}(\mathbf{x})\sigma^b\sigma^c$$

The explicit form of the $C^a_{bc}(\mathbf{x})$ and $D^a(\mathbf{x})$ will be of no importance for the subsequent calculations. No further assumptions for β are necessary, however we should point out that the solution with $\frac{d\beta^a}{d\alpha} = const.$ and $\alpha = \frac{1}{n}log(t)$ corresponds to the usual Kasner solution.

To analyze the evolution of the point particle we require the evolution equation for the time-depenent walls. The equipotential surfaces for the three different types of walls are given by

$$\beta^a - \beta^b - \beta^c + (n-1)\alpha = const. \quad (a \neq b \neq c) \tag{12.a}$$

$$-\beta^a + (n-1)\alpha = const. \tag{12.b}$$

$$\frac{1}{2}\beta^a - \beta^b - \frac{1}{2}\beta^c + (n-1)\alpha = const. \quad (a \neq b \neq c) \tag{12.c}$$

As mentioned before the β's have to satisfy eq.(9). From this we can see that the potential walls expand as the singularity is approached. In order to study the motion between bounces, an approximation has to be made; since the potential in eq.(11) is exponential, we can safely ignore it when the universe-particle is far from the "walls", i.e. we assume free motion between bounces. In general none of the C^a_{bc} or D^a functions vanish, so all possible walls should be present. This is the major difference with the homogeneous cases where the symmetries imposed on the spatial hypersurfaces forces some of the C^a_{bc} and D^a to vanish, so effectively some of the walls are not present. However, if there is a non-zero measure region in β-space along which, the universe-particle never hits any walls, then there is no chaotic motion near the singularity. The universe-particle will, after a (perhaps large) number of bounces go into this region, becoming a free particle. We shall now derive a condition for the universe-particle not to bounce indefinitely.

The velocities of the different walls satisfy the following equations,

$$-\frac{d\beta^a_{wall}}{d\alpha} + \frac{d\beta^b_{wall}}{d\alpha} + \frac{d\beta^c_{wall}}{d\alpha} = n - 1 \quad (a \neq b \neq c) \tag{13.a}$$

$$\frac{d\beta^a_{wall}}{d\alpha} = n - 1 \tag{13.b}$$

$$-\frac{1}{2}\frac{d\beta^a_{wall}}{d\alpha} + \frac{d\beta^b_{wall}}{d\alpha} + \frac{1}{2}\frac{d\beta^c_{wall}}{d\alpha} = n - 1 \quad (a \neq b \neq c) \tag{13.c}$$

From eqs. (9) and (13) we get the direction of the velocity vectors for the three kinds of walls to be

$$\mathbf{n}_{abc} = \frac{1}{\sqrt{3-\frac{1}{n}}}((0,..,0,-1,0,..,0,1,0,..,0,1,0,..,0) - \frac{1}{n}(1,1,..,1,..,1))$$

$$\mathbf{n}_a = \frac{1}{\sqrt{1-\frac{1}{n}}}((0,..,0,1,0,..,0,0,..,0,0) - \frac{1}{n}(1,1,..,1,..,1))$$

$$\tilde{\mathbf{n}}_{abc} = \frac{1}{\sqrt{\frac{3}{2}-\frac{1}{n}}}((0,..,0,-\frac{1}{2},0,..,0,1,0,..,0,\frac{1}{2},0,..,0) - \frac{1}{n}(1,1,..,1,..,1))$$

In the expression for $\mathbf{n}_{abc}$ the overall factor is due to proper normalisation, the first piece in the parenthesis can be read off (13.a) directly, and the -1, 1 and 1 appear as the a'th, b'th and c'th entries. The second piece is necessary to ensure that $\mathbf{n}_{abc}$ lies in the hyperplane

$$\sum_{A=1}^{n} n^A_{abc} = 0$$

The expressions for $\mathbf{n}_a$ and $\tilde{\mathbf{n}}_{abc}$ are derived in a similar manner. From (13) we can also read off the speed of the walls, we find

$$V_{abc} = \frac{n-1}{\sqrt{3-\frac{1}{n}}}$$

$$V_a = \frac{n-1}{\sqrt{1-\frac{1}{n}}}$$

$$\tilde{V}_{abc} = \frac{n-1}{\sqrt{\frac{3}{2}-\frac{1}{n}}}$$

for the three cases. Let us introduce the velocity of the universe-particle by $\frac{d\beta_a}{d\alpha} \equiv \mathbf{v} = (v^1, v^2, ..., v^n)$. In terms of this eqs. (9) and (10) become (in the region where the potential is negligible)

$$\sum_{A=1}^{n} v^A = 0 \tag{14}$$

and

$$\sum_{A=1}^{n} (v^A)^2 = n(n-1) \tag{15}$$

The motion of the universe-particle as well as that of the walls is restricted to the hyperplane defined by eq.(14), therefore from here on, when we speak about walls we mean the intersection with this hyperplane. Obviously the inner products of vectors defined in this hyperplane coincide with that of the big n-dimensional space.

For a given velocity $\mathbf{v}$ of the universe-particle the speed at which it moves towards a wall, is $\mathbf{n}_w \cdot \mathbf{v}$ where $\mathbf{n}_w$ represents any of the three direction vectors above. Therefore the condition that the universe-particle does not hit this wall simply becomes $\mathbf{n}_w \cdot \mathbf{v} < V_w$, and the requirement that the universe does not hit any walls becomes $\mathbf{max}_w(\mathbf{n}_w \cdot \mathbf{v} - V_w) < 0$. Since we are only asking for the existence of some $\mathbf{v}$ satisfying this, then the "no-bounce" condition can be formulated as follows: If

$$\mathbf{min}_{\mathbf{v}}(\mathbf{max}_w(\mathbf{n}_w \cdot \mathbf{v} - V_w)) < 0 \tag{16}$$

then a region in velocity space exists within which the universe-particle never bounces and eventually moves freely having a non-chaotic behaviour near the singularity. Finally, we shall express the velocities in terms of the Kasner indices by the relation $p^a = \frac{1}{n}(v^a + 1)$. In terms of this eqs. (14) and (15) become

$$\sum_{A=1}^{n} p^A = 1 \tag{17}$$

$$\sum_{A=1}^{n} (p^A)^2 = 1 \tag{18}$$

and the condition (16) when calculated for the three different walls reads:

$$\min_{\mathbf{p}}(\max_{abc}(-p^a + p^b + p^c - 1)) < 0 \quad (a \neq b \neq c) \tag{19.a}$$

$$\min_{\mathbf{p}}(\max_{a}(p^a - 1)) < 0 \tag{19.b}$$

$$\min_{\mathbf{p}}(\max_{abc}(-\frac{1}{2}p^a + p^b + \frac{1}{2}p^c - 1)) < 0 \quad (a \neq b \neq c) \tag{19.c}$$

Eq.(19.b) is always satisfied and it is easy to show that if we assume, the first is satisfied too, then the third follows and viceversa. Then, we only need to assume one of the conditions holds, as the other two contain no new information. We will choose eq. (19.a) to build the argument of the next section where we will show this condition can only be satisfied if $n \geq 10$. This will prove the existence of a region in velocity space for which the universe-particle can never reach any wall. This result will also imply that for $n \leq 10$ there is at least one wall that can not be neglected.

It will prove useful to rewrite (19.a) in a different way,

$$\max_{\mathbf{p}}(\min_{abc}\alpha^{abc}) > 0 \tag{20}$$

where $\alpha^{abc} = 1 + p^a - p^b - p^c$.

III. The Critical Dimension

In this section we shall show (i) that for spatial dimension $n \geq 10$ is it possible to find $\mathbf{p}$'s with all the α^{abc}'s positive, and (ii) that for $n \leq 9$ and any given $\mathbf{p}$ at least one of the α^{abc}'s are negative. This proves that for $n \leq 9$ the general solution to Einstein's equations is chaotic, while for $n \geq 10$ non-chaotic solutions exist. We shall specifically show that for $n \geq 10$ a whole region of parameter space ($\mathbf{p}$-space) can be found in which the solution is non-chaotic. The existence of such a region suggests that the general solution for $n \geq 10$ is never chaotic, since one might expect that the universe after a finite number of bounces will end up in the region of parameter space corresponding to non-chaotic behaviour. When first in this region the universe will never bounce again, i.e. the universe remains non-chaotic for arbitrary small times. However for $n \leq 9$ the universe will bounce indefinitely due to the presence of a negative α^{abc}, i.e. the universe is chaotic for small times.

For all the α^{abc}'s to be positive it is enough to require the smallest, α_{min} to be positive. The smallest $\alpha^{abc} = 1 + p^a - p^b - p^c$ results when p^a is as small as possible and p^b and p^c are as large as possible. Regarded as a function of the p's we shall search for maxima of α_{min} and check if any of these corresponds to positive values of α_{min}. Clearly it is sufficient to check the region I of $\mathbf{p}$-space where $1 \geq p^1 \geq p^2 \geq \ldots \geq p^n \geq \frac{2}{n} - 1$ (notice that the smallest value any p can take is $\frac{2}{n} - 1$). It is necessary to check not only the interior of I but also the boundary of I, which corresponds to regions with some of the p^a's equal. To do this we consider subregions S of I defined as the regions of $\mathbf{p}$-space where $p > q^1 > \ldots > q^{i_S} > r$ and p appears k times, q^i appears l_i times, r appears m times and $k + m + \sum l_i = n$ (this means that $p^1 = p^2 = \ldots = p^k = p,\ p^{k+1} = \ldots = p^{k+l_1} = q^1$ etc.). In this notation we wish to extremize $\alpha_{min} = 1 + r - (2 - \delta_{1k})p - \delta_{1k}q^1$ under the constraints $\sum p^i = \sum (p^i)^2 = 1$, i.e. $c_1 \equiv kp + mr + \sum l_i q^i - 1 = 0$ and $c_2 \equiv kp^2 + mr^2 + \sum l_i (q^i)^2 - 1 = 0$. Using the theory for Lagrangian multipliers we extremize

$$\alpha_{min} = 1 + r - (2 - \delta_{1k})p - \delta_{1k}q^1 + \lambda c_1 + \mu c_2 \tag{21}$$

After some tedious calculation we find that extremas correspond to

$$p = -\frac{\lambda}{2\mu} + \frac{1}{2\mu k}(2 - \delta_{1k}),\ q^1 = -\frac{\lambda}{2\mu} + \frac{1}{2\mu l_1}\delta_{k1},\ r = -\frac{\lambda}{2\mu} - \frac{1}{2\mu m},\ q^i = -\frac{\lambda}{2\mu}$$

for $i = 2, \ldots, i_S$ and

$$\lambda = \frac{1}{n}\left(1 \pm \sqrt{\frac{n}{n-1}}\sqrt{\frac{1}{k}(4 - 3\delta_{k1}) + \frac{1}{l_1}\delta_{k1} + \frac{1}{m} - \frac{1}{n}}\right)$$

$$\mu = \mp\frac{1}{2}\sqrt{\frac{n}{n-1}}\sqrt{\frac{1}{k}(4 - 3\delta_{k1}) + \frac{1}{l_1}\delta_{k1} + \frac{1}{m} - \frac{1}{n}}$$

in which case

$$\alpha_{min} = \frac{n-1}{n} \pm \sqrt{\frac{n-1}{n}} \sqrt{\frac{1}{k}(4 - 3\delta_{k1}) + \frac{1}{l_1}\delta_{k1} + \frac{1}{m} - \frac{1}{n}} \tag{22}$$

Since we are considering the region with $p > q^1 > ... > q^{i_s} > r$ we can ignore the upper sign. We also see that the maximum appears when $\delta_{k1} = 0$, then $\alpha_{min} = \frac{n-1}{n} - \sqrt{\frac{n-1}{n}}\sqrt{\frac{4}{k} + \frac{1}{m} - \frac{1}{n}}$. This is largest for k as large as possible, that is when $k = n - m$ (so all $l_i = 0$). At its maximum we then find

$$\alpha_{min} = \frac{n-1}{n} - \sqrt{\frac{n-1}{n}} \sqrt{\frac{4}{n-m} + \frac{1}{m} - \frac{1}{n}} \tag{23}$$

This is maximal when $m = \frac{n}{3}$ (of course m must be an integer, so if n is not a multiple of 3 then m equal to the integer closest to $\frac{n}{3}$ maximizes α_{min}).

We are now ready to draw a number of conclusions. For $n \leq 8$, the maximum of α_{min} is negative. This means that given any velocity $\mathbf{p}$ of the universe-particle, at least one of the $\alpha^{abc}(\mathbf{p})$'s corresponding to this $\mathbf{p}$ is negative. In our geometrical picture this means that the universe-particle is bound to bounce off some wall corresponding to one of the negative α's. After the bounce the universe-particle has some velocity $\mathbf{p}'$, again at least one of the $\alpha^{abc}(\mathbf{p}')$'s are negative, so a bounce is inevitable, and so on. This shows that for $n \leq 8$ the universe-particle wil l continue bouncing indefinitely. For $n = 9$, $\alpha_{min} = 0$ in its maximum, but for all other $\mathbf{p}$'s it is negative, so except for $\mathbf{p}$ in a set of measure zero, we expect the same chaotic behaviour for $n = 9$ as for $n \leq 8$. For $n \geq 10$ we get $\alpha_{min} > 0$ in its maximum and therefore $\alpha_{min} > 0$ in a whole region around its maximum ensuring that the universe-particle moves freely and never bounces. (Also notice that for n large α_{min} tends to 1 in its maximum and to $1 - \sqrt{3}$ in its minimum, so for even large n there is always a region in $\mathbf{p}$-space with at least one of the α's negative.) The only points (in $\mathbf{p}$-space) not included in the preceeding analysis are the "endpoints" of I: (i) $p^1 = 1,\ p^2 = ... = p^n = 0$ with $\alpha_{min} = 0$, and (ii) $p^1 = ... = p^{n-1} = \frac{2}{n},\ p^n = \frac{2}{n} - 1$ with $\alpha_{min} = -\frac{2}{n}$. However these do not affect our conclusions.

We have argued above that in $n+1$ dimensions with $n \geq 10$ we can find a non-zero measure region in **p**-space for which all the α^{abc}'s are positive. This ensures that solutions with **p** in this region are general in the sense defined in the introduction and are well behaved on approach to the singularity. For $n \leq 9$ we have shown that except for **p** in a set of measure zero, at least one of the $\alpha^{abc}(\mathbf{p})$'s are negative ensuring that the general solution for $n \leq 9$ is chaotic.

Of course the results of this section can be obtained directly from Einstein's equations in their normal form. However, to do this it is necessary to assume that the metric components scale like some power of time, $g_{aa} \sim t^{p^a}$, where the exponents can be functions of the spatial coordinates and satisfy the two Kasner relations (17) and (18). In this case the requirement that the spatial curvature terms (which corresponds to the potential terms in our picture) be negligible translates into the condition that $t^2 R^a_b \sim t^{2\alpha^{abc}}$ vanishes for small t. This means that the α^{abc}'s must be positive, in agreement with (20). The main difference between this approach and the approach described in section II is that in the latter we need not make any assumptions about the functional dependence of the metric components. The first approach was used in ref.(8) to argue that when $n \geq 10$ the universe is non-chaotic.

IV. Conclusions

Using the Hamiltonian formulation of General Relativity, we have studied the behaviour of a general inhomogeneous solution to Einstein's Equations. The question posed in the Introduction: Is a Universe described by General Relativity necessarily chaotic near an initial singularity ? The question has several answers. i) If the Universe has only three spatial dimensions then the answer is yes, chaotic behaviour is a generic feature of this Universe[4]. ii) If the spatial-dimensionality of the universe lies in the range $4 \leq n \leq 9$ and the spatial hypersurfaces are homogeneous, then probably the answer is no. However, for the most general inhomogeneous model with $4 \leq n \leq 9$, the answer is yes, for $n \geq 10$, the answer is no.

We have constructed a potential picture where most of the above results can be understood in a geometrical way. The problem of solving Einstein's equations is replaced by that of a point-particle moving inside a time dependent potential; chaotic behaviour is just a reflection of the way in which the point-particle bounces off the walls of this potential. The free motion of the particle (far away from the walls) is described by a Kasner model, while the bouncing law is derived from the assumption that the collisions are perfectly elastic. Our construction permits us to derive a condition for which the particle velocity vector is such that it can never reach any wall. In this case no chaotic behaviour is present for $n \geq 10$. Even though we have been able to construct a geometrical description of the behaviour of the universe near the initial singularity, we cannot provide a deep explanation to why the model picks up the particular dimensionality of ten as the critical one for chaotic behaviour.

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