Fermi National Accelerator Laboratory

FERMILAB-Pub-86/48-A
March 1986

The Effect of Inflation on Anisotropic Cosmologies

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ABSTRACT

We study the effects of anisotropic cosmologies on inflation. By properly formulating the field equations it is possible to show that any model that undergoes sufficient inflation will become isotropic on scales grater than the horizon today. Furthermore, we shall show that it takes a very long time for anisotropies to become visible in the observable part of the Universe. It is interesting to note that the time scale will be independent of the Bianchi Model and of the initial anisotropy.

March 1986

Introduction

In order to explain the homogeneity and flatness of the presently observed Universe, it is usually assumed that this has undergone a period of exponential expansion [1]. In most scenarios the expansion of the Universe is described within the framework of the homogeneous and isotropic Friedman-Robertson-Walker cosmology (FRW). The reasons for this are purely technical. The simplicity of the field equations and the existence of analytical solutions in most of the cases has justified this over simplification for the geometry of space-time. However, there are no compelling physical reasons to assume the former before the inflationary period. To drop the assumption of homogeneity would make the problem intractable, however the isotropy of the space is something that can be relaxed. Several authors[2] have studied particular cases of anisotropic models and found that the scenario predicted by the FRW model is essentially unchanged even when large anisotropies where present before the inflationary period.

In this paper we will assume the universe is homogeneous but not necessarily isotropic. It will then be described by one of the Bianchi Models[3]. It has been shown elsewhere[4] that under very general conditions all Bianchi cosmologies (except maybe Bianchi IX) with a cosmological constant and an energy-momentum tensor satisfying the strong and dominant energy conditions, will unavoidably enter a phase of exponential expansion. With the help of this result we will show that if the number of e-folds the Universe expands during its exponential phase is given by N then it will take a time of the order $t \simeq e^{2N}\sqrt{\Lambda}$, where Λ is the cosmological constant, for anisotropy to have any effect on the observable universe. One remarkable result is the independence of this result from the type or magnitude of the initial anisotropy. We should point out that this holds even for models that do not contain the FRW as a special case (only Bianchi I, V, VII_0 and IX contain FRW models).

The Field Equations

We shall write Einstein's field equations in the form

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \tag{1}$$

with

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu + pg_{\mu\nu} + \Lambda g_{\mu\nu} + \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial_\alpha\phi\partial^\alpha\phi) \tag{2}$$

Units are such that $8\pi G = \hbar = c = 1$ and the signature is $(-,+,+,+)$. We shall assume that the fluid is at rest in the comoving coordinate system so that $u_\mu = \delta^0_\mu$ and the velocity can be normalized to give $u_\mu u^\mu = -1$. We can see that the energy momentum tensor has contributions from a perfect fluid for which we will assume the existence of an equation of state of the form $p = \gamma\rho$ and from a homogeneous massles scalar field ϕ with potential $V(\phi)$. In particular, we could identify the cosmological constant Λ with $V(0)$.

As mentioned earlier only homogeneous and anisotropic models will be studied here. In particular those which have space-like surfaces of homogeneity (i.e. a G_3 acting simply transitively on a V_3). This type of cosmological models have been widely studied and classified some time ago [3]. They essentially fall into one of 9 classes of equivalence (these are not disjoint classes), the so called Bianchi Models (type VI and VII are really one parameter families of models labeled by a parameter h).

In the past it has proven useful to classify the nine Bianchi types into two disjoint groups depending on the different properties of the isometry groups (the Lie groups). We will call them class A and class B. For models of class A the metric and field equations can be written in a compact notation[3]. For the other models the field equations have to be given independently.

In order to highlight the important features of the field equations we are going to write them in the following form

$$\frac{\ddot{X}}{X} + \frac{\dot{X}}{X}\frac{\dot{Y}}{Y} + \frac{\dot{X}}{X}\frac{\dot{Z}}{Z} = F_1(X,Y,Z) + \bar{T} \tag{3.1}$$

$$\frac{\ddot{Y}}{Y} + \frac{\dot{Y}}{Y}\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y}\frac{\dot{Z}}{Z} = F_2(X,Y,Z) + \bar{T} \tag{3.2}$$

$$\frac{\ddot{Z}}{Z} + \frac{\dot{Z}}{Z}\frac{\dot{X}}{X} + \frac{\dot{Z}}{Z}\frac{\dot{X}}{X} = F_3(X,Y,Z) + \bar{T} \tag{3.3}$$

where $\dot{X} \equiv \frac{dX}{dt}$, etc., with t the proper time.

There is one more equation, namely the $\binom{0}{0}$ equation, however it will not be used to construct the argument so we will omit it. X,Y,Z represent the scale factors along the principal directions. $\bar{T}$ is going to contain all the information of the fluids, scalar field and the cosmological constant. The specific form is given by $\bar{T} = \rho + \frac{1}{2}\dot{\phi}^2 + \Lambda$. Using the continuity equation is easy to get the form of the energy-density and the dynamical equation for the scalar field,

$$\rho = \frac{\rho_0}{(XYZ)^{(1+\gamma)}} \tag{4.1}$$

$$\ddot{\phi} + \left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}\right)\dot{\phi} = -\frac{dV}{d\phi} \tag{4.2}$$

with ρ_0 an integration constant closely related to the initial entropy (in any open or flat model this identification is not important).

For Class A model we have

$$F_1(X,Y,Z) \equiv \frac{1}{2(XYZ)^2}\left[(N_2Y^2 - N_3Z^2)^2 - (N_1X^2)^2\right] \tag{5}$$

$F_2(X,Y,Z)$ and $F_3(X,Y,Z)$ can be obtained by a cyclic permutation of the elements in the numerator of equation (5). The N_i define the different Bianchi types and are given in table 1.

Bianchi type	N_1	N_2	N_3
I	0	0	0
II	1	0	0
VI_0	1	-1	1
VII_0	1	1	0
$VIII$	1	1	-1
IX	1	1	1

Table 1

For Bianchi V and $VI_h(h \neq 0)$ we have

$$F_1(X,Y,Z) \equiv \frac{-2(a_0^2 + q_0^2)}{X^2} + \frac{2b^2}{Y^4Z^2} \tag{6.1}$$

$$F_2(X,Y,Z) \equiv \frac{-2(a_0^2 + a_0q_0)}{X^2} - \frac{2b^2}{Y^4Z^2} \tag{6.2}$$

$$F_3(X,Y,Z) \equiv \frac{-2(a_0^2 - a_0q_0)}{X^2} \tag{6.3}$$

with a_0 a positive constant. If we take $q_0 = b = 0$ we get Bianchi type V, for q_0, b non-zero constants we get Bianchi type $VI_h(h \neq 0)$ and finally if we take $h = -1$ we get Bianchi type III.

For Bianchi types IV and $VII_h(h \neq 0)$ the equations are slightly more complicated. However they essentially have the same form as those mentioned above. For a full description of types IV and VII see ref.(5) and (6) respectively. Every argument used in this paper is applicable to these two models, but for sake of clarity we will not treat them in a separate way.

The only Bianchi model representing a closed Universe is type IX, it generalizes the closed FRW, Bianchi I and VII_0 are flat while V and VII_h are open, generalizing their FRW counterparts.

The Inflationary Phase

Granted that the model is going to enter an inflationary phase [4] we will now show that is going to take a very long time for anisotropy to act back on the observable Universe once inflation has come to an end. Leaving effectively a Universe that is almost undistinguishable from a perfect FRW. This is a general feature of Bianchi cosmologies.

The argument we shall use is applicable to almost all Bianchi Models (except maybe type IX, we shall address this point later). This generalizes and extends previous attempt [2] where either the anisotropies were treated as small perturbation on a FRW background or the anisotropic models contained FRW as a special case. In these models it was found that inflation was a remarkably efficient method of isotropization. However, one could ask whether this feature is not a consequence of the fact that even without inflation these models would have become FRW-like anyway. We shall show that the answer to this question is certainly negative.

We shall build our argument using only eq. (3) and concentrating on a model of class A (the same argument can be used for all the other models). The effect of a successful inflation on the Universe is twofold, on the one hand it makes the Universe (scale factors) expand exponentially, on the other hand it generates an effective reheating converting the vacuum energy into radiation through coupling of ϕ to other fields. If we denote by N the number of e-folds the Universe expands during inflation, then this parameter is going to determine the time-scale for anisotropies to act back on the observable Universe. We shall assume for concreteness that the universe inflates due to the presence of a cosmological constant Λ, identified in some suitable model with the initial value of some scalar field potential. After inflation the effect of the scalar field is negligible and the cosmological constant becomes zero. When translated into the scale factors it means we can relate the values immediately before and immediatly after inflation by,

$$(X(t), Y(t), Z(t))e^N(\tilde{X}(t), \tilde{Y}(t), \tilde{Z}(t)) \quad ; \quad t \geq t_0 \tag{7}$$

The initial conditions are determined by demanding that $(\tilde{X}(t_0), \tilde{Y}(t_0), \tilde{Z}(t_0))$ take the values of the respective scale factors immediately before inflation. After inflation the field equations look like

$$\frac{\ddot{\tilde{X}}}{\tilde{X}} + \frac{\dot{\tilde{X}}}{\tilde{X}}\frac{\dot{\tilde{Y}}}{\tilde{Y}} + \frac{\dot{\tilde{X}}}{\tilde{X}}\frac{\dot{\tilde{Z}}}{\tilde{Z}} = \frac{F_1(\tilde{X}, \tilde{Y}, \tilde{Z})}{e^{2N}} + \frac{\Lambda(\tilde{X}(t_0)\tilde{Y}(t_0)\tilde{Z}(t_0))^{\frac{4}{3}}}{(\tilde{X}(t)\tilde{Y}(t)\tilde{Z}(t))^{\frac{4}{3}}} \tag{8}$$

Where the last term appears as a consequence of the conversion of vacuum energy into radiation. This equation becomes more transparent if we introduce new variables $(x, y, z) \equiv \frac{1}{e^N\sqrt{2\Lambda}}(\tilde{X}, \tilde{Y}, \tilde{Z})$ and $\tau \equiv \frac{1}{2e^{2N}\sqrt{\Lambda}}t$ then eq.(8) becomes

$$\frac{\ddot{x}}{x} + \frac{\dot{x}}{x}\frac{\dot{y}}{y} + \frac{\dot{x}}{x}\frac{\dot{z}}{z} = F_1(x, y, z) + \frac{1}{(xyz)^{\frac{4}{3}}} \tag{9}$$

with initial conditions given by $(x, y, z) = \frac{1}{e^N\sqrt{2\Lambda}}(\tilde{X}(t_0), \tilde{Y}(t_0), \tilde{Z}(t_0))$ at $\tau = \frac{1}{2e^{2N}\sqrt{\Lambda}}t_0$ (i.e. $x(\tau = 0) = 0$. Equation (9) shows that at $\tau \simeq 1$ we may expect the anisotropic term to become important; when translated to our proper time this corresponds to $t \simeq e^{2N}\sqrt{\Lambda}$.

If we now look back at the equation for the ϕ field we see that the effect of anisotropy is not of great consequence. The way the scalar dynamics are coupled to the expanding Universe is trough a "friction term" proportional to the volumetric expasion rate . For this models it goes like $\frac{d(log(XYZ))}{dt}$. This term is such that

$$\left(\frac{\dot{X}}{X}+\frac{\dot{Y}}{Y}+\frac{\dot{Z}}{Z}\right)\geq\sqrt{3\Lambda} \tag{10}$$

Then the only effect of anisotropy before inflation is to increase the friction coefficient and make inflation more efficient (the field will roll slower over the potential). This was first noticed by Steigman and Turner[2]. This result shows that after a succesful inflation any Bianchi model "turns into" a FRW and will stay that way for a long time.

The only problem with a Bianchi IX model, as it is with a closed FRW model, is one of timescales. If it is possible to enter the inflationary phase well before the model recollapses then all our arguments still hold true. However, this requires a comparision of two time scales that are strongly dependent on initial conditions. We require a knowledge of the initial anisotropy as well as the initial energy density in the form of radiation. We would like to point out that the uncertainty in type IX has nothing to do with our argument breaking down for closed models, but rather with initial conditions. In the other cases the initial conditions are irrelevant since these will only alter the time it takes for the inflationary phase to start, and our Universe is probably insensitive to those timescales.

Conclusions

We have shown that any homogeneous model belonging to one of the Bianchi classes that undergoes sufficient inflation is going to become isotropic to a very high degree of accuracy. Furthermore, the time it takes to become anisotropic after inflation is very long. In all models we found the timescale to be of order $e^{2N}\sqrt{\Lambda}$ where N is the number of e-folds the Universe expands during the inflationary phase and Λ is the cosmological constant. It seems remarkable that this timescale is independent of both the Bianchi type and the initial conditions set on the anisotropy, curvature and radiation content.

We would like to thank M.S. Turner and D. Lindley for several encouraging conversations. This work was supported by the Department of Energy and NASA.

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