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# SYMMETRY IN PHYSICS

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# Symmetry in Physics [1]

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Abstract

We present methods for introducing the concept of symmetry into the introductory physics curriculum.

## 1 Introduction

The concept of symmetry is fundamental to our understanding of the physical world. It is where we can discern true or approximate symmetries, such as those involved in the basic forces of nature, that we profess any real understanding. Where nature displays little or no apparent symmetries, such as in the spectrum of elementary quarks and leptons, we find ourselves most befuddled. Moreover, all thinking in modern theoretical physics is aimed at understanding the possible role of deep mathematical symmetries in nature. The realization of the importance of symmetry to the understanding of the laws of physics is a modern concept, belonging almost entirely to the twentieth century and beginning largely with Einstein and the special theory of relativity.

Why is the concept of symmetry essentially totally absent in the introductory physics course? Symmetry is probably the greatest component of what we mean when we speak of the "beauty of physics", yet the student of physics does not begin to see this underlying motif until rather late in the usual curriculum. Perhaps it is here that we do ourselves the greatest disservice in denying a peak into this structure to the casual physics student. Having delivered the "Symmetry" lecture in the Fermilab Saturday Morning Physics program for the past six years I've found that as a conceptual framework it can be introduced to the introductory (high school) physics students in a substantive and meaningful way. The student must be led to discover the physical manifestations of symmetries after exploring the mathematical concept *without burdensome abstraction.* This may lay the basis for further study of group theory, having provided a concrete realization of the ideas in geometric examples and the ways in which nature is constrained by symmetries. During a unit on conservation of momentum, energy and angular momentum, the underlying origin of these principles as a consequence of the fundamental symmetries of space and time can be demonstrated and the content of Emmy Nöether's famous theorem connecting these can be motivated without attempting a proof (see Section 4).

In this brief article I will outline a set of basic mini-units which can be injected into the standard curriculum at various points without largely disrupting the latter. Pausing to contemplate an elegant symmetry argument in the course of analyzing a tedious physics problem can contribute much to enliven the subject, even for beginners. And, *please* view this as a small beginning, but by no means a conclusion to this subject; you are heartily encouraged to develop it further yourself!

## 2 HOW DO WE THINK ABOUT SYMMETRY?

Mathematicians solve many problems in geometry and topology by turning them into *equivalent algebraic problems.* This approach to understanding symmetry as a subject unto itself begins approximately with the 19th century French mathematician, Galois [1], who in his short, tragic life laid the foundation and fundamental applications of what we call "group theory", a mature branch of modern mathematics (the biography in Scientific American of ref.(1) is highly recommended reading).

We will not develop group theory here in its general form, but rather think concretely about the symmetries of a very simple geometric object... ...the equilateral triangle. This is the simplest nontrivial example and the results for any student the first time through this introductory exercise are often very surprising.

Prepare two transparencies as in Fig.(1) and Fig.(2) each featuring an equilateral triangle, both of the same size. The transparency of Fig.(1) has the three axes of symmetry labeled as I, II and III, while the transparency of Fig.(2) has the vertices labeled as A, B, and C.

Transparency (1) is laid down on the projector table and the students are informed that this is a *reference triangle* which must be considered

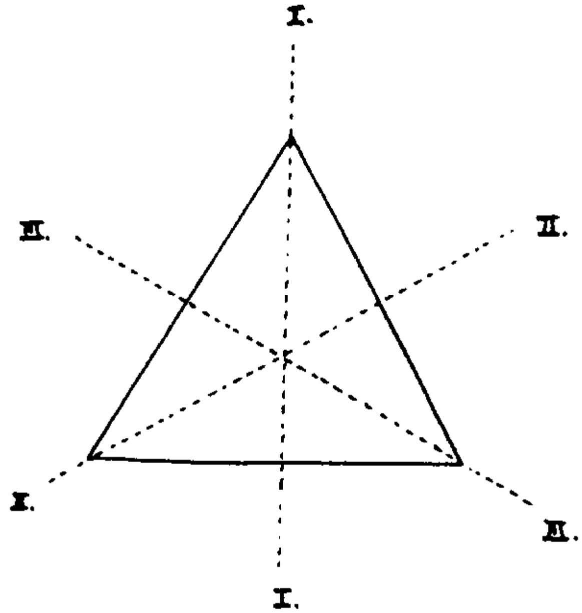


Figure 1: The reference triangle

to be glued in place and has the purpose of serving as a reference grid, or a kind of "coordinate system"; once laid in place we will not move it again. Transparency (2) on the other hand is an *experimental triangle*; we will be overlaying the reference triangle with the experimental triangle. Our problem is to *find all possible* **distinguishable** *ways in which the experimental triangle can be lifted up and brought down on top of the reference triangle.* The vertices of the experimental triangle are labeled to allow us to identify the distinguishable ways in which this can be done.

We begin by overlaying the experimental triangle on the reference triangle with the vertices reading ABC clockwise around the experimental triangle. This will be called the **initial orientation**. Our problem now is to discover a way in which we can pick up the experimental triangle and bring it back down on top of the reference triangle so that the vertices read something other than ABC clockwise. Each such operation is called a **symmetry operation** and our problem is to find all possible distinguishable symmetry operations of the equilateral triangle. How do we proceed?

Some student will no doubt suggest rotating the experimental triangle until the vertices now read CAB clockwise from the top. This certainly corresponds to a symmetry operation, which is a rotation through 120°.

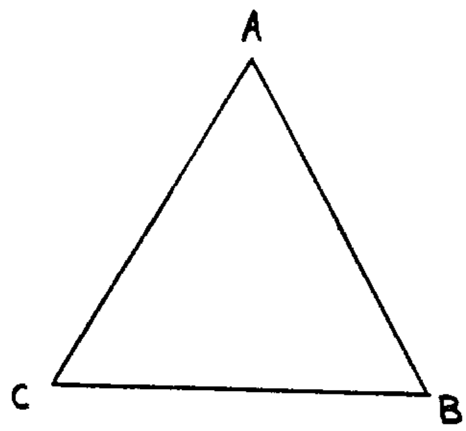


Figure 2: The experimental triangle

We shall designate this first discovery as $\mathbf{Rot}_{120^\circ}$ and it should be written down on the blackboard as such (there will be six such operations and this should be written *second* from the top of a column).

Now return to the initial orientation. What else? It is obvious at this point that a rotation through 240° is another symmetry operation which yields the result BCA. However, it is important to emphasize that we should always return to the initial orientation before performing the next operation (this is a bit like pressing the **CLEAR** button on a pocket calculator before doing the next calculation). Thus we discover a second distinguishable symmetry operation which we designate $\mathbf{Rot}_{240^\circ}$

Q: *Why is this a distinguishable rotation?*
Because the vertices now occur in the sequence BCA, we see that the triangle has been moved to a new orientation, distinguishable from the initial orientation ABC or from the $\mathbf{Rot}_{120^\circ}$ orientation of CAB.

Q: *Why do we distinguish between a symmetry operation and an orientation of the triangle?*
Here there is an important distinction. The symmetry operation takes the triangle from *any* given initial orientation and

> maps it into a new orientation; the position of vertices, A,B, and C, defines a particular, absolute position and orientation of the triangle in space. We are really interested here only in the symmetry operations, but not the absolute positions and orientations in space. That is, we are really only interested in the *relative* positions and orientations that the triangle can be brought to by a symmetry operation starting with any arbitrary initial position and orientation. We only introduce the labeling of vertices to keep track of them, starting always in the initial orientation ABC, with the triangle positioned on the table (we could equally well have labeled sides of the triangle). The symmetry operation can be performed on any equilateral triangle, of any size, drawn with any color ink, in any initial position and orientation. It captures the essence of symmetry, but contains no additional information about any particular triangle or its absolute position and orientation in space.

Are there other symmetry operations to consider? A student may suggest a rotation by $-120°$. But we now see that this takes the triangle to BCA (from the initial orientation, of course) and therefore this is not a new operation, i.e. it is not *distinguishable.* We may thus write the equation: $\mathbf{Rot}_{-120^\circ} = \mathbf{Rot}_{240^\circ}$ . Again, we do not care about the *path* that takes us from one orientation to another; we only care about the new orientation *relative to* the initial orientation. That's what defines a symmetry operation. Thus $\mathbf{Rot}_{-120^\circ}$ and $\mathbf{Rot}_{240^\circ}$ are the same.

At this point a student may suggest a rotation through 360°. Is this a symmetry operation? We see that it maps the triangle from the initial orientation ABC back to the initial orientation ABC. Consequently, it *is* a symmetry operation, but a very special one. For one, it is equivalent to doing nothing at all. As such we shall refer to it as the "do nothing operation", or *the identity operation.* We shall denote it at the top of our blackboard list by the boldface **1**. Secondly, note that the identity element is a symmetry operation of any object; even an amoeba has the symmetry of the identity symmetry. Thirdly, we note that a rotation through 360° is equivalent to a rotation through any *integer multiple* of 360°, e.g. 720°, $-360°$, etc. All are equivalent to the "do nothing" operation, **1**.

Q: *Then isn't our* $\mathbf{Rot}_{120^\circ}$ *equivalent to* $\mathbf{Rot}_{120^\circ+360^\circ\times N}$ *where* $N$ *is an integer?*
Yes.
What is the analogous statement for $\mathbf{Rot}_{240^\circ}$?

We now have three symmetry operations; are there more? In fact, the student will generally suggest performing a *reflection* about one of the three axes of the reference triangle. We begin with the initial orientation and consider "skewering" the experimental triangle (as if we had a barbecue skewer) along one of the axes of symmetry indicated on our reference triangle. For example, skewering along axis I, we then pick up the triangle and flip it and we arrive at the new orientation, ACB. We denote this symmetry operation as a *reflection about axis I* or as $\mathbf{Ref}_I$. Similarly, we return to the initial orientation and consider the other two symmetry operations, (a) the reflection about axis II, or $\mathbf{Ref}_{II}$ which yields the vertex position BAC and (b) the reflection about axis III, or $\mathbf{Ref}_{III}$ which yields the vertex position CBA. Thus we now have a list on the blackboard of six of the symmetry operations which has the form:

Table I. The six symmetry operations of the equilateral triangle.

| Notation | Operation | Vertices |
|---|---|---|
| $\mathbf{1}$ | "do nothing" or identity | ABC |
| $\mathbf{Rot}_{120^\circ}$ | rotate by 120° | CAB |
| $\mathbf{Rot}_{240^\circ}$ | rotate by 240° | BCA |
| $\mathbf{Ref}_I$ | reflect about axis I | ACB |
| $\mathbf{Ref}_{II}$ | reflect about axis II | BAC |
| $\mathbf{Ref}_{III}$ | reflect about axis III | CBA |

Are there any other symmetry operations? At this point many students recognize that we have discovered essentially the six permutations of three objects, i.e. the six permutations of the three vertices of the triangle. That raises an interesting question:

Q: *Are the symmetries of all such objects, such as squares, pentagons hexagons, cubes, etc. given by the permutations of their vertices?*

> In fact the answer is *no*. It doesn't work that way for the square as we can easily see. Suppose we have a square with vertices labeled ABCD. A true symmetry operation of the square is a rotation through 90° and gives DABC, which is also a permutation of the vertices. However, is there a symmetry operation which can give the vertex ordering BACD, which is certainly a valid permutation of ABCD? (Think in terms of an experimental square on a transparency; what would we have to do to the transparency to get BACD starting from ABCD?) Clearly, this is not a symmetry operation of the entire square because we would have to *twist* the experimental square to get the vertices into this position, but then the sides would not overlay properly! Thus, while all symmetry operations of geometric objects are indeed permutations, not all permutations are symmetry operations of geometric objects. We have actually discovered our first example of a subgroup; the square is a subgroup of the group of permutations of four objects. The equilateral triangle is simpler and it does have only six symmetry operations, the ones we've listed above, which are equivalent (isomorphic) to the permutations of three objects.

Thus far our exercise has been almost trivial, but now we make the great observation of Galois and his colleagues. We now ask, can we obtain additional symmetry operations by combining together two of the operations previously obtained? That is, let us take two of our six operations, say $\mathbf{Rot}_{120^\circ}$ and $\mathbf{Ref}_{II}$, and first perform one of them on the experimental triangle (try $\mathbf{Rot}_{120^\circ}$) and *without returning to the initial orientation* perform the other operation ($\mathbf{Ref}_{II}$). We see that if we begin in the initial orientation that $\mathbf{Rot}_{120^\circ}$ leads to CAB and then following with $\mathbf{Ref}_{II}$ we obtain the orientation ACB. But ACB is not a new orientation of the triangle, and it corresponds to $\mathbf{Ref}_{I}$ as seen by our table. We have therefore discovered an interesting result: *first performing* $\mathbf{Rot}_{120^\circ}$ *and following it by* $\mathbf{Ref}_{II}$ *yields the* **result** $\mathbf{Ref}_{I}$.

Let us write an equation for this result:

$$\mathbf{Rot}_{120^\circ} \otimes \mathbf{Ref}_{II} = \mathbf{Ref}_{I}. \tag{2.1}$$

Here we have introduced a symbol, $\otimes$, which represents the action of combining the symmetry operations in the order indicated (without returning to the initial orientation in between). It is easily seen that the

$\otimes$ combination of any pair of our symmetry operations (which we also refer to as "elements") produces another of the elements. We say that our set of elements is **closed** under the operation $\otimes$. Thus, in a sense the combining of two symmetry operations is something like *multiplication of numbers.* In this sense the "do nothing operation" is the *identity*:

$$\mathbf{1} \otimes X = X \otimes \mathbf{1} = X. \tag{2.2}$$

> Q: *Why do we call this "multiplication" rather than "addition"?*
> The answer is really one of convention. Multiplication and addition have very similar mathematical properties; the identity element in addition is 0, while in multiplication it is 1. The inverse of 4 under addition is -4, while under multiplication it is $\frac{1}{4}$. Hence, the positive and negative integers **close** under addition while the rationals close under multiplication. Note however that there is an important difference between addition and multiplication: 0 has no multiplicative inverse, i.e. infinity makes no sense mathematically. Denoting the combination of symmetry operation by "multiplication" is also a consequence of the fact that matrices can *represent* the group elements and matrix multiplication can *represent* the $\otimes$ operation as we shall see subsequently.

Thus we have made a very important observation: *the symmetry operations form an algebraic system with an operation consisting of performing successive operations.* This algebraic system is called a **group**. The symmetry operations are the analogues of the rational numbers under this group multiplication. We refer to the symmetry operations as **group elements** or simply as **elements**. We present the complete multiplication table of the symmetry group of the equilateral triangle, designated as $S_3$, in Table(II). The entries in this table should be verified by performing several of the cases with the experimental triangle and reference triangle transparencies. Table(II) is to be read like a highway mileage map; if we choose to perform the product $X \otimes Y$ we first find the row labeled on the left by $X$, then the column labeled on the top by $Y$ and we look up the corresponding entry in that row and column for the result.

Table II. $S_3$ Group multiplication table

| | $\mathbf{1}$ | $\mathbf{Rot}_{120°}$ | $\mathbf{Rot}_{240°}$ | $\mathbf{Ref}_I$ | $\mathbf{Ref}_{II}$ | $\mathbf{Ref}_{III}$ |
|---|---|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{1}$ | $\mathbf{Rot}_{120°}$ | $\mathbf{Rot}_{240°}$ | $\mathbf{Ref}_I$ | $\mathbf{Ref}_{II}$ | $\mathbf{Ref}_{III}$ |
| $\mathbf{Rot}_{120°}$ | $\mathbf{Rot}_{120°}$ | $\mathbf{Rot}_{240°}$ | $\mathbf{1}$ | $\mathbf{Ref}_{III}$ | $\mathbf{Ref}_I$ | $\mathbf{Ref}_{II}$ |
| $\mathbf{Rot}_{240°}$ | $\mathbf{Rot}_{240°}$ | $\mathbf{1}$ | $\mathbf{Rot}_{120°}$ | $\mathbf{Ref}_{II}$ | $\mathbf{Ref}_{III}$ | $\mathbf{Ref}_I$ |
| $\mathbf{Ref}_I$ | $\mathbf{Ref}_I$ | $\mathbf{Ref}_{II}$ | $\mathbf{Ref}_{III}$ | $\mathbf{1}$ | $\mathbf{Rot}_{120°}$ | $\mathbf{Rot}_{240°}$ |
| $\mathbf{Ref}_{II}$ | $\mathbf{Ref}_{II}$ | $\mathbf{Ref}_{III}$ | $\mathbf{Ref}_I$ | $\mathbf{Rot}_{240°}$ | $\mathbf{1}$ | $\mathbf{Rot}_{120°}$ |
| $\mathbf{Ref}_{III}$ | $\mathbf{Ref}_{III}$ | $\mathbf{Ref}_I$ | $\mathbf{Ref}_{II}$ | $\mathbf{Rot}_{120°}$ | $\mathbf{Rot}_{240°}$ | $\mathbf{1}$ |

There are several important properties of this group multiplication table that are shared by all groups:

- A group is a set of elements and a composition law, $\otimes$, such that the product of any two elements yields another element in the set.

- Every group has an identity element satisfying eq.(2.2).

- Each element of the group has a unique inverse element. That is, given an element $X$ there exists one and only one element, $Y$ (which may even be $X$ itself), such that $X \otimes Y = Y \otimes X = \mathbf{1}$.

- Group multiplication is *associative*. That is, given $X$, $Y$ and $Z$ we have $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$. In words, first perform $Y$ and follow by $Z$ and remember the result (call it $W$). Now return to the initial orientation and first do $X$ and follow by $W$. This result will be the same as having first done $X$ followed by $Y$ then followed by $Z$. This is the meaning of associativity and you should carefully think it through to make sure you understand it.

- Each element of the group occurs once and only once in each row and each column of the multiplication table. This can actually be proved as a theorem from the preceding statements. This is a powerful constraint on the mathematical structure of the group; essentially the group multiplication table forms a kind of "magic square".

- **Group multiplication is not necessarily commutative**[2]. That is, $X \otimes Y$ need not equal $Y \otimes X$!!!

This last result, namely that group multiplication is not commutative, is really quite remarkable. Here we have discovered a simple system of six elements with a multiplication law and the system is not even commutative. For example, (*this should definitely be shown explicitly with the triangles*) consider first performing $\mathbf{Rot}_{240^\circ}$ and following by $\mathbf{Ref}_{II}$, that is, calculate $\mathbf{Rot}_{240^\circ} \otimes \mathbf{Ref}_{II}$. You should obtain the result $\mathbf{Ref}_{III}$. On the other hand, consider first performing $\mathbf{Ref}_{II}$ followed by $\mathbf{Rot}_{240^\circ}$, i.e. compute $\mathbf{Ref}_{II} \otimes \mathbf{Rot}_{240^\circ}$. The result is $\mathbf{Ref}_{I}$. Summarizing:

$$\begin{aligned} \mathbf{Rot}_{240^\circ} \otimes \mathbf{Ref}_{II} &= \mathbf{Ref}_{III} \\ \mathbf{Ref}_{II} \otimes \mathbf{Rot}_{240^\circ} &= \mathbf{Ref}_{I} \end{aligned}$$

Thus, although ordinary multiplication is commutative, e.g. $3 \cdot 4 = 4 \cdot 3$, group multiplication need not be. When a group has commutative multiplication it is said to be an **abelian group**, after the mathematician Abel. The general group, such as the equilateral triangle group, is noncommutative, or **nonabelian**.

We finish this discussion with an important example as to how group mathematics underlies the structure of our physical world. One may wonder how noncommutative mathematics can have anything at all to do with nature, or physics. A simple demonstration will show this.

> Take a textbook and hold it in front of you with the binding down as though you were going to open it up on a table. Now extend your right arm parallel to your chest and parallel to the floor (like a right turn signal) and let this be the positive x–axis. Now extend your arm straight out in front of you; let this define the positive y–axis. We wish to rotate the textbook by 90° about the positive x-axis and follow this by a

[2] This discussion should be considered for introduction into a mathematics class as a unit for this reason. The concept of the commutative property of ordinary addition or multiplication is almost vacuous without showing a counterexample, namely a system in which it doesn't hold! Our equilateral triangle symmetry group affords such a simple example. Unfortunately, there are no simple examples of nonassociative systems, even though they do exist.

rotation through 90° about the y–axis. The rotations should always be performed in the sense of a right hand screwdriver. Perform the two successive rotations and note the book's position. Now return the book to the initial orientation and perform first the rotation about the y–axis followed by a rotation about the x–axis. You will find that the book ends up in two different positions. The symmetry group consisting of rotations through 90° is a noncommutative group. The continuous group consisting of all rotations of objects in three dimensions (the full symmetry of a sphere) is thus noncommutative. It is known as **O(3)** and it governs the physics of angular momentum and spin.

The subject of group theory is an entire branch of mathematics in which many people have specialized and undertake ongoing research. The continuous groups, possessing an infinite number of operations that vary continuously with "angle" parameters, like the rotations of a sphere about a given axis through any angle, were first completely classified early in the 20th century by Cartan. Remarkably, only very recently have all possible discrete symmetry groups been classified. This job was made difficult by the existence of certain "sporadic" groups, such as the "monster group" with $\approx 8 \times 10^{53}$ elements. The classification of the discrete groups constitutes one of the longest and least comprehensible theorems in mathematics (2).

The application of group theory to physics is a rich and fundamentally important subject, significantly different than pure mathematical research into groups and their properties. While mathematicians may struggle to classify the discrete groups, such as the monster, nature embodies only a small subset of all possible mathematical symmetry groups. It is remarkable, however, that as we probe deeper into the shortest distances and most elemental properties of matter we seem to discover evidence of ever more sophisticated symmetry groups at work. Nature seems to read books on group theory!

## Exercises and food for thought:

1. Construct the symmetry group of the square and its associated multiplication table. Verify the properties discussed above for the

general group.

2. Construct the permutation group of four objects and its associated multiplication table. Is it isomorphic to the square's group?

3. A subgroup of a group is a subset of elements which themselves form a group. Clearly, each subgroup must contain the identity. Can you identify some subgroups of the equilateral triangle group? What is the largest subgroup of the equilateral triangle symmetry group (not counting the entire group itself [each set is a subset of itself]; technically we want the largest *proper* subgroup).

4. If the square in the preceding problem is squashed into a rectangle identify the surviving subgroup (see the preceding problem) which describes the remaining symmetry. (As a preliminary exercise consider an isosceles triangle in which vertex A is lifted along axis I in Fig.(1); what is the resulting symmetry group? Is it a subgroup?)

5. If an infinite floor is tiled with equilateral triangles we have a *lattice*. Is the symmetry group of the equilateral triangle also a symmetry group of the lattice? (*Yes, but it is only a subgroup; It is called the "point group" of the lattice*). Is the full symmetry group of the lattice equivalent to the group of a single triangle? (*No. It includes in addition the set of translations along the axes which bring the entire lattice down on top of itself*). Does a rotation commute with a translation? This problem illustrates how groups enter solid state physics in which they are of paramount importance.

## 3 Physical Situations Involving Symmetry

In the previous lecture we introduced the formal notion of the symmetry group. Specifically what has it to do with physics? We present here a number of problems which illustrate the power of symmetry arguments in the solution of physics problems. Our last examples, (5) and (6), that of the modes of oscillation of a system of three coupled masses, goes the farthest in illustrating the power of group theory (and group representation theory).

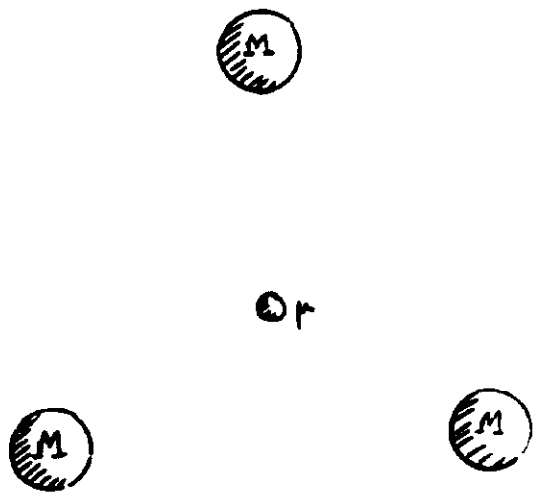


Figure 3: Symmetric arrangement of four masses, all at rest

**Example 1.**

In Fig.(3) we have a symmetric arrangement of three masses following the equilateral triangle with a single mass situated at the center. What is the force of gravity exerted upon the center mass due the other three? Of course, we must be careful in stipulating that the system really is fully symmetric. Thus, each particle is at rest and has no internal degree of freedom (such as angular momentum, or quadrupole moment of mass) which violates the symmetry; or at least we seek an approximate solution in the limit in which such complications can be ignored.

Therefore, it is obvious that the force, by symmetry, vanishes for the particle in the center. This involves the symmetry considerations of the preceding section as well. Suppose that by adding up the individual force vectors due to the three masses at the vertices we had obtained the nonvanishing result shown in Fig.(4). Now this answer is clearly wrong, but why is it wrong from the point of view of the symmetry group? Consider performing a typical group operation on this result, e.g., perform the operation $\mathbf{Ref}_I$. This maps the system into itself, but it maps the answer into a new one shown in Fig.(5). The same system cannot produce two different results, so we have shown that the result is wrong. Actually, consideration of $\mathbf{Ref}_I$ alone does not eliminate a result lying on the axis I, however we may then consider, e.g., $\mathbf{Rot}_{120^\circ}$ to dispose

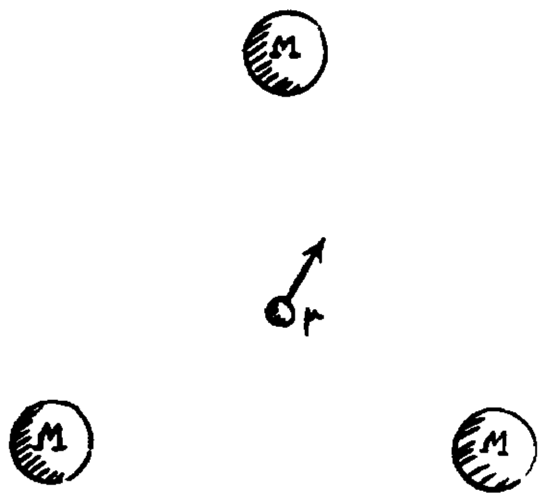

Figure 4: Hypothetical result for the force exerted on center mass

of that case. Clearly, the only result which is invariant under any of the group operations is *zero*.

This trivial example illustrates a very important aspect of nature: If a physical configuration possesses a given symmetry then the dynamics of the system will possess the symmetry[3]. Thus, our result in this case does not upon the kind of force law involved so long as the symmetry constraint is in effect. The central particle could be a pion surrounded by (spinless) nuclei interacting through the strong nuclear force and the conclusion would be the same: the force must vanish by symmetry!

**Example 2.**

In Figure(6) we have yet another configuration which this time does not possess the full symmetry of the equilateral triangle. Now what is

[3]There is a caveat here of great importance: the system must be *stable* in the symmetric configuration. Many systems, such as ferromagnets, though they are described by rotationally invariant equations of motion, are unstable in rotationally invariant states. These systems undergo "spontaneous symmetry breaking" at low temperatures. The ferromagnet develops nonzero magnetization below the Curie temperature. The problem shown above actually exhibits this phenomenon. If the small mass at the origin in the above problem was slightly displaced away from the center it would then experience a nonzero force pulling it farther *away* from the center and the small symmetry breaking fluctuation is amplified. Replacing all the masses by positive charges eliminates the instability.

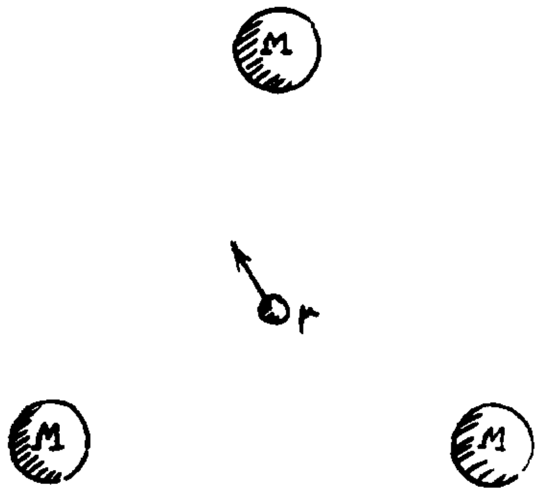


Figure 5: Action of $\mathbf{Ref}_I$ on answer obtained in Fig.4

the force exerted upon the central mass?

Of course, there is a residual symmetry here. This system is described by the subgroup of the full equilateral triangle symmetry group consisting of the elements **1** and $\mathbf{Ref}_I$. Thus, the resulting force vector must lie along the axis I. Symmetry does not tell us what the sign of the force is. For gravity the force is away from the center while for electromagnetism it depends in an obvious way upon the choice of charges. This typifies the situation scientists often face in understanding a new phenomenon. A symmetry may be present which goes along way toward controlling the physics, while some unknown underlying dynamics may be present which determines the quantitative outcome.

In the present case it is instructive to consider the small displacement from the center, $a$, of the mass $\mu$. The general inverse square law gives:

$$F = \frac{\alpha}{(r-a)^2} - \frac{\beta}{(r + a\cos\theta)^2} \tag{3.1}$$

where $\alpha$ and $\beta$ are given by the precise form of the force law (i.e. gravity versus electromagnetism, etc.). Here $\theta$ is 60° as is seen by the the geometry of the situation. Symmetry tells us that $F = 0$ when $a = 0$, hence that $\alpha = \beta$. For $\frac{a}{r} \ll 1$ we may consider the first terms in the series

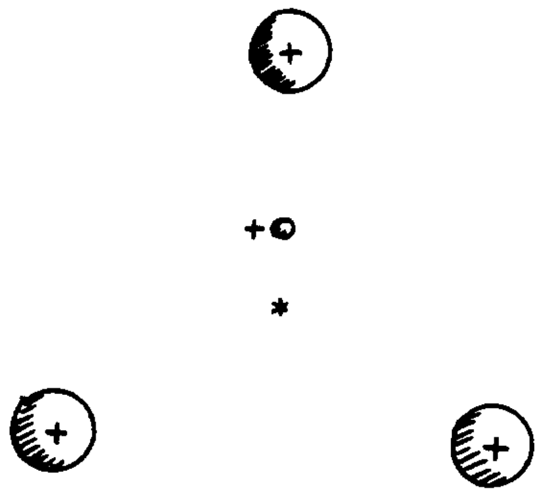

Figure 6: Asymmetric arrangement of masses, all at rest

expansion of eq.(3.1) and thus:

$$F \approx \frac{3a\alpha}{r^3} \tag{3.2}$$

using $\cos\theta = \frac{1}{2}$. For gravity $\alpha = GM\mu$ and the force is up, away from the center and the configuration is unstable. If the vertices correspond to charges, Q, and center to charge $-q$, then $\alpha = kqQ$ and the force is toward the center and the central mass is in a stable potential.

**Example 3.**

One of the most conspicuous atomic transitions is the $2P \to 1S$ "dipole transition" in, e.g. atomic Hydrogen. The $1S$ orbital is the groundstate quantum mechanical motion of an electron in the Hydrogen atom and is the shape of a perfectly rotationally symmetric "cloud" if the atom is in free space. The $2P$ levels are not rotationally symmetric and "point in a direction" like a vector, and form, therefore, a "triply degenerate" state, i.e. the electron can at any instant be in one of three independent P–orbitals, $2P_x$, $2P_y$, or $2P_z$, corresponding to the three independent directions in space. The three 2P states are degenerate, i.e. have the same energy, because of rotational symmetry. If the electron is in the $2P_z$ state we can just rotate the atom at no cost in energy (or just rotate our reference frame) and thus put the electron into a $2P_y$ or $2P_x$

state. There are also higher orbitals such as the $D$, $F$ etc. with more orientational information than the $S$ or $P$ (these are like tensors).

If the atom is placed in a strong magnetic field we have broken the rotational symmetry. Suppose the magnetic field points in the z–direction. Then we still have rotational symmetry in the perpendicular x–y plane. Therefore, the $2P_x$ and $2P_y$ orbitals will remain degenerate, but the $2P_z$ orbital will develop a slightly different energy. This energy splitting is proportional to the magnetic field strength and the observed transition photons from $2P_z \rightarrow 1S$ and $2P_x$ or $2P_y \rightarrow 1S$ will have slightly different energies. This is known as the **Zeeman effect** and is one of the principal methods for determining the presence and strength of magnetic fields in the sun and other astronomical objects.

Incidently, putting atoms into crystal lattices also breaks up the degeneracy of atomic levels due to the interatomic forces and their symmetries which follow the symmetries of the crystal lattice. Can you think of a way to get the $2P_x$, $2P_y$ and $2P_z$ to each have different energies? If the hydrogen atom could be placed in a perfect cubic crystal would the $2P$ levels be degenerate? (*Ans: yes*) What about a non–cubic lattice?

**Example 4.**

Figure(7) shows yet another configuration of large masses, $M$, arranged on the vertices of a symmetric hexagon, however the topmost vertex has a mass $m \neq M$. Given this each vertex is a distance $a$ from the center, find the force experienced by the center mass $\mu$ (this should be done in less than one minute).

It is clear that the resulting force must be directed along the vertical axis of symmetry and thus can depend only upon the uppermost and lowermost masses. Since the force vanishes when $m = M$, it must depend only upon the difference $m - M$ (gravity depends only linearly upon the "pullers"). Also, when $m \gg M$ the force must be in the $\vec{up}$ direction. Hence, *without any computation at all,* we arrive at the answer:

$$F = \frac{G\mu(m - M)}{a^2}\vec{up} \tag{3.3}$$

These are only a small handful of simple, illustrative problems. I urge you to develop more of them (particularly *clever* ones). The following

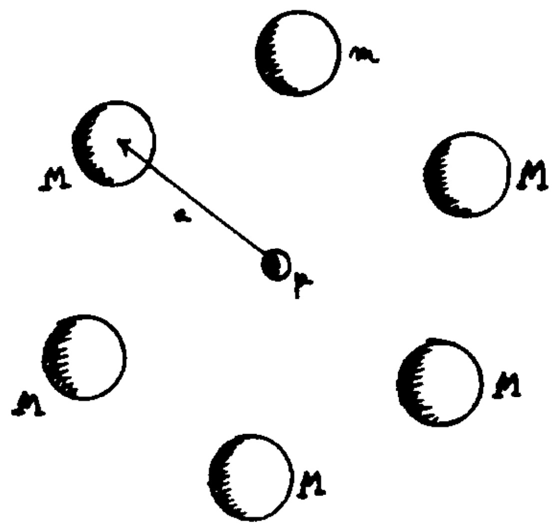

Figure 7: Hexagonal arrangement of masses; the topmost mass is different. What is the force on $\mu$?

discussion amplifies the theory of symmetry groups to get at a very powerful relationship between physics and symmetry through the properties of *group representations.*

**Example 5.**

We consider the question, "Can we find any standard mathematical objects which close under multiplication and satisfy the same rules of combination as any given symmetry group?" Such a set of objects forms what is known as a **representation** of the group. The problem of classifying all of the representations of symmetry groups is an extremely important and very rich subject and forms what is known as **representation theory**. It also has a great many physical applications.

For any group there always exists a **trivial representation** which consists of letting each element be represented by the number 1. Thus we write:

$$\begin{array}{lcl} \mathbf{1} & \rightarrow & 1 \\ \mathbf{Rot}_{120^\circ} & \rightarrow & 1 \\ \mathbf{Rot}_{240^\circ} & \rightarrow & 1 \\ \mathbf{Ref}_{I} & \rightarrow & 1 \\ \mathbf{Ref}_{II} & \rightarrow & 1 \\ \mathbf{Ref}_{III} & \rightarrow & 1 \end{array} \tag{3.4}$$

This forms a representation in the sense that the product of any two elements as given by Table(II) produces a result consistent with the product of any of the representative elements. Thus $\mathbf{Rot}_{240^\circ} \otimes \mathbf{Ref}_{II} = \mathbf{Ref}_{III}$ is consistent with $1 \times 1 = 1$; but of course this is trivially the case. The trivial representation thus contains no information about the group.

Are there any other representations using only numbers? It should be fairly clear to you upon thought that the absolute magnitude of each number representing a symmetry element must be unity (for example, since $\mathbf{Ref}_I \otimes \mathbf{Ref}_I = \mathbf{1}$ the number representing $\mathbf{Ref}_I$, call it $\alpha$, must satisfy $\alpha^2 = 1$; also the rotations map unit vectors into unit vectors. Indeed, if we consider *only the three element subgroup* consisting of $\mathbf{1}$, $\mathbf{Rot}_{120^\circ}$ and $\mathbf{Rot}_{240^\circ}$, we could always write down a *complex representation*:

$$\begin{array}{lcl} \mathbf{1} & \rightarrow & 1 \\ \mathbf{Rot}_{120^\circ} & \rightarrow & \exp(2\pi i/3) \\ \mathbf{Rot}_{240^\circ} & \rightarrow & \exp(4\pi i/3) \end{array} \tag{3.5}$$

however, when we include the reflections this ceases to be a possible representation of the full equilateral triangle group).

There is a representation of the full equilateral triangle group which is nontrivial, yet involves only 1 and $-1$. It is:

$$\begin{array}{lcr} \mathbf{1} & \rightarrow & 1 \\ \mathbf{Rot}_{120^\circ} & \rightarrow & 1 \\ \mathbf{Rot}_{240^\circ} & \rightarrow & 1 \\ \mathbf{Ref}_{I} & \rightarrow & -1 \\ \mathbf{Ref}_{II} & \rightarrow & -1 \\ \mathbf{Ref}_{III} & \rightarrow & -1 \end{array} \tag{3.6}$$

This representaion recognizes the difference between those operations which can be done without lifting the experimental triangle (the rotations) and those that require lifting and flipping (reflections). Another way of saying it is that rotations are direct operations which we may perform upon the system while reflections require a mirror or a **parity transformation**. We shall call this the **parity representation.** We see that the correspondence with the multiplication table(I) may be checked by considering several examples:

$$\begin{array}{lll} \mathbf{Rot}_{120^\circ} \otimes \mathbf{Rot}_{240^\circ} = \mathbf{1} & \rightarrow & 1 \cdot 1 = 1 \\ \mathbf{Ref}_I \otimes \mathbf{Rot}_{240^\circ} = \mathbf{Ref}_{III} & \rightarrow & (-1) \cdot 1 = (-1) \\ \mathbf{Ref}_{II} \otimes \mathbf{Ref}_{III} = \mathbf{Rot}_{120^\circ} & \rightarrow & (-1) \cdot (-1) = 1. \end{array} \tag{3.7}$$

Both the trivial representation and the parity representation are examples of **unfaithful representations** because the same representative (1 or $-1$) corresponds to two or more elements (e.g. $-1$ corresponds to $\mathbf{Ref}_I$, $\mathbf{Ref}_{II}$ and $\mathbf{Ref}_{III}$). Are there any faithful representations of the equilateral triangle symmetry group?

In fact, there is one more representation that is faithful, but it cannot be given in terms of numbers. It requires $2 \times 2$ matrices and the multiplication operation is now matrix multiplication. The matrix representation may be written as:

$$\begin{array}{lll} \mathbf{1} & \rightarrow & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \mathbf{Rot}_{120^\circ} & \rightarrow & \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \\ \mathbf{Rot}_{240^\circ} & \rightarrow & \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \\ \mathbf{Ref}_I & \rightarrow & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ \mathbf{Ref}_{II} & \rightarrow & \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \\ \mathbf{Ref}_{III} & \rightarrow & \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \end{array} \tag{3.8}$$

Again, we see that the representation gives results consistent with Table(II) for a few sample cases:

$$\mathbf{Rot}_{120^\circ} \otimes \mathbf{Rot}_{240^\circ} = \mathbf{1}$$

$$\rightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{Ref}_I \otimes \mathbf{Rot}_{240^\circ} = \mathbf{Ref}_{III}$$

$$\rightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \tag{3.9}$$

$$\mathbf{Ref}_{II} \otimes \mathbf{Ref}_{III} = \mathbf{Rot}_{120^\circ}$$

$$\rightarrow \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} .$$

The reader is invited to check other cases. Of course, this kind of representation is possible because matrix multiplication is itself *noncommutative.* In fact, the set of all matrices of a given order, e.g. $2 \times 2$, with *nonvanishing determinant* form a group with respect to matrix multiplication (why must we stipulate nonvanishing determinant?).

Are there any higher matrix representations? Of course, we can always combine the two numerical representations (these are $1 \times 1$ matrices) and the $2 \times 2$ case to form higher dimensional matrix representations. For example, consider the set of $3 \times 3$ matices consisting of a 1 in the upper left-hand corner and a $2 \times 2$ matrix from our set of matrix representations in the lower right-hand corner, with zeros everywhere else. This *is* a representation, but it contains no new information not already contained in the cases examined above. It is known as a **reducible** representation because it is a set of matrices that are **block diagonal**, and each block contains one of the three basic representation described above. *The three representations we've discussed above are the only* **irreducible** *representations of the equilateral triangle symmetry group.* In general it is not trivial to decide if a given representation is reducible. This is because we may take a group of block diagonal matrices and multiply on the left by some matrix, $S$, and on the right by $S^{-1}$ and we still preserve the group multiplication table, but the resulting matrices no longer *appear* to be block diagonal. Nonetheless, such a representation is equivalent to the block diagonal reducible one and is itself reducible. So the general problem is to determine whether a given representation is equivalent to a block diagonal one by right multiplication by some $S$ and left multiplication by $S^{-1}$. If no such $S$ exists, then the representation is irreducible and therefore interesting. This problem is the solved and constitutes the central subject of group representation theory. We refer the reader to any good book on group theory for a discussion of representation theory.

**Example 6.**

We consider now a physical system as shown in Figure(8) which consists of three equal masses attached together by springs. This may be viewed as a kind of molecule and we will assume that it can move only in the 2-dimensional plane for simplicity (such a system could be fabricated out of air-pucks and springs for use on an air table as a demonstration). The system possesses in its equilibrium rest state the symmetry of the equilateral triangle; it will be governed by the symmetry group of the triangle in a very interesting way. The system can undergo several kinds of motion, consisting of uniform center–of–mass translations, rotations and internal oscillations. Indeed, this is a property shared by all molecular systems and the heat capacity of a gas essentially counts the various states of motion of the systems comprising the gas. First, we may count the number of independent motions of the system, e.g. how many numbers must be given to specify the exact state of the system at any time? These are called **degrees of freedom** of the system. Since we have 3 masses and each mass requires 2 coordinates (an x– and a y– coordinate) we thus have $2 \times 3 = 6$ degrees of freedom. The tying together of the masses by springs does not change the counting because the springs are free to stretch and given the 6 coordinates we can calculate the length of

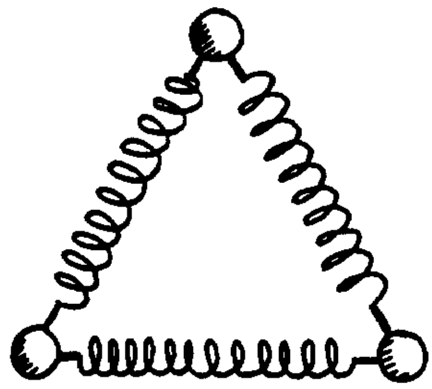

Figure 8: "Molecule" consisting of masses and springs.

each spring (if, on the other hand, the masses were attached together by 3 rigid rods which are not free to stretch we would have $6 - 3 = 3$ degrees of freedom).

Describing the motion by giving the 6 coordinates does not tell us much about the system's motion as a whole. Therefore, we wish to describe the system in terms of basic motions that we can separate qualitatively. In fact, this separation also reduces the mathematical complexity of analyzing the motion of the system. These are called the **normal modes** of the system's motion.

First, suppose the system is initially at rest in its equilibrium position. Clearly, two modes of motion are just given by uniform translation of the center of mass in the x– or y– directions. Thus, we may dispose of 2 degrees of freedom (d.o.f.) and analyze the remaining 4 d.o.f. by holding the center of mass of the system fixed in space.

With the center of mass fixed we may consider a uniform oscillation of the system as a whole as shown in Fig.(9). This is known as a **breather mode** since the system simply expands and contracts but remains always in the fully symmetrical shape. Thus, if we act upon the breather mode with one of our symmetry operations, e.g. simply perform $\mathbf{Rot}_{120^\circ}$ or $\mathbf{Ref}_{II}$ on the system at some arbitrary instant of time, we see that we

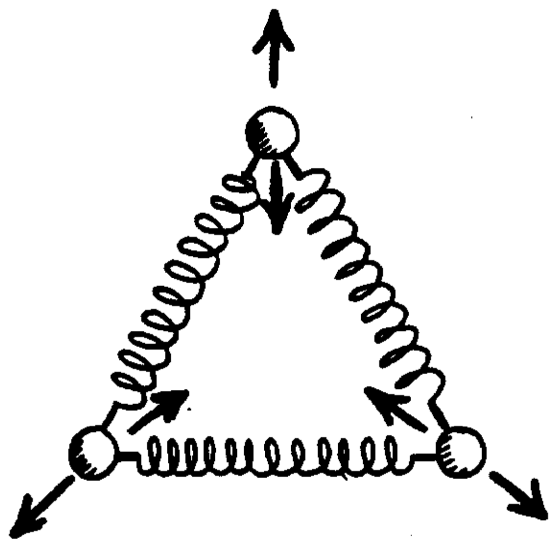

Figure 9: The breather mode

remain in exactly the same state of motion. *Therefore, the breather mode corresponds to the trivial representation of the group.* It has a characteristic frequency, $\omega_0$ given by the mass, M, and the spring constant, K. Next, we may consider a uniform rotation about the fixed center of mass of the system, Fig.(10). Let us consider clockwise rotation with angular frequency $\omega_r$. If we act at some instant upon the system with the symmetry group elements, $\mathbf{1}$, $\mathbf{Rot}_{120^\circ}$ and $\mathbf{Rot}_{240^\circ}$, we find that the motion will remain the same. On the other hand, acting with the elements $\mathbf{Ref}_I$, $\mathbf{Ref}_{II}$ and $\mathbf{Ref}_{III}$ we see that the motion becomes counterclockwise, i.e. these operations map the frequency into $-\omega_r$. *Therefore, uniform rotation corresponds to the parity representation of the group.* Therefore, we are left with 2 remaining d.o.f. Consider the motion shown in Fig.(11). Here one of the vertices moves out along an axis of symmetry while the other vertices are attracted toward the axis. These three motions, corresponding to the three vertices, are not *independent*, i.e. if we add together all three motions with equal strength we obtain no motion at all ($\alpha + \beta + \gamma = 0$). However, if we act upon any of the motions we obtain one of the others. For example, $\mathbf{Rot}_{120^\circ}(\alpha) = \beta$, or $\mathbf{Ref}_{II}(\alpha) = \gamma$. How do we rewrite these three modes in terms of two so that they close under the action of the symmetry elements? An answer is the following:

$$\textit{mode } 1 = \alpha - \beta \qquad \textit{mode } 2 = \alpha + \beta - 2\gamma \tag{3.10}$$

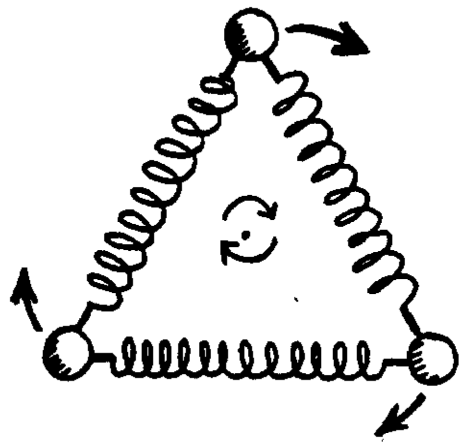

Figure 10: Uniform rotation

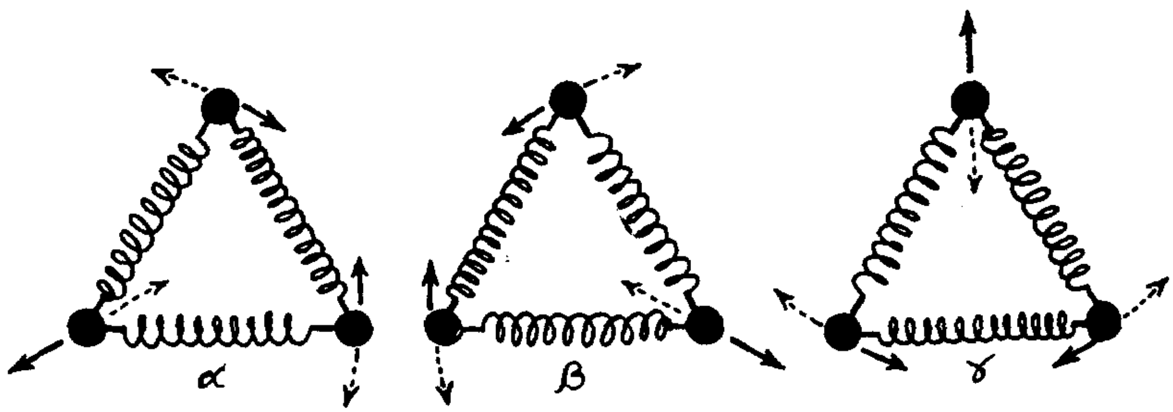


Figure 11: The doublet modes

Now under the action of any symmetry element these two modes go into linear combinations of themselves. *Thus, these two modes correspond to the doublet representation of the symmetry group.* Because they are related to each other by symmetry elements, they *must have the same frequency of oscillation.* Of course, this frequency will not be the same as that of the breather mode. The normal modes forming a representation of the group are said to be **degenerate** in their oscillation frequency.

**Example 7.**

In the mid 1960's the strongly interacting particles could be placed into multiplets of a continuous symmetry group,$SU(3)$ [3]. One of the representations of $SU(3)$ has eight components, and is known as an octet (the $SU(3)$ symmetry elements can be represented as $8 \times 8$ matrices in an irreducible way; the eight members of the octet mix amongst themselves under an $SU(3)$ transformation). The eight spin–0 mesons fit into one multiplet, the eight spin–$\frac{1}{2}$ baryons into another, and so on (see Fig.(12)). There is also a *10* component representation into which the spin–$\frac{3}{2}$ baryonic resonances fit (in fact, one, the $\Omega^-$, was missing at the time $SU(3)$ was discovered and it was correctly predicted by the theory). The particles in the multiplets were not degenerate indicating that the $SU(3)$ symmetry was not exact, but the pattern was clearly established.

The puzzle was that the smallest representation of SU(3), namely a triplet consisting of three spin–$\frac{1}{2}$ particles with charges ($\frac{2}{3}$, $-\frac{1}{3}$, $-\frac{1}{3}$) were not seen directly (in these units the electron charge is -1). These particles are known respectively as the "up" quark, the "down" quark, and the "strange quark".

Today, however, we have compelling evidence that the quarks do exist but are permanently confined within the particles we see in the laboratory. All of the mesons are composed of quark and anti–quark, while each baryon contains three quarks. $SU(3)$ symmetry is not exact because the strange quark mass is much larger than the up or down quark masses (this is analogous to breaking the equilateral triangle symmetry by making one of the vertex masses much different than the others; the doublet modes would cease to be degenerate). Incidently, there is another totally distinct $SU(3)$ symmetry associated with the force holding the quarks together inside of the mesons and baryons. This latter $SU(3)$ is called the "color symmetry" and it is an exact symmetry (it is also a *gauge group*, which is a subject beyond the level of our present discussion).

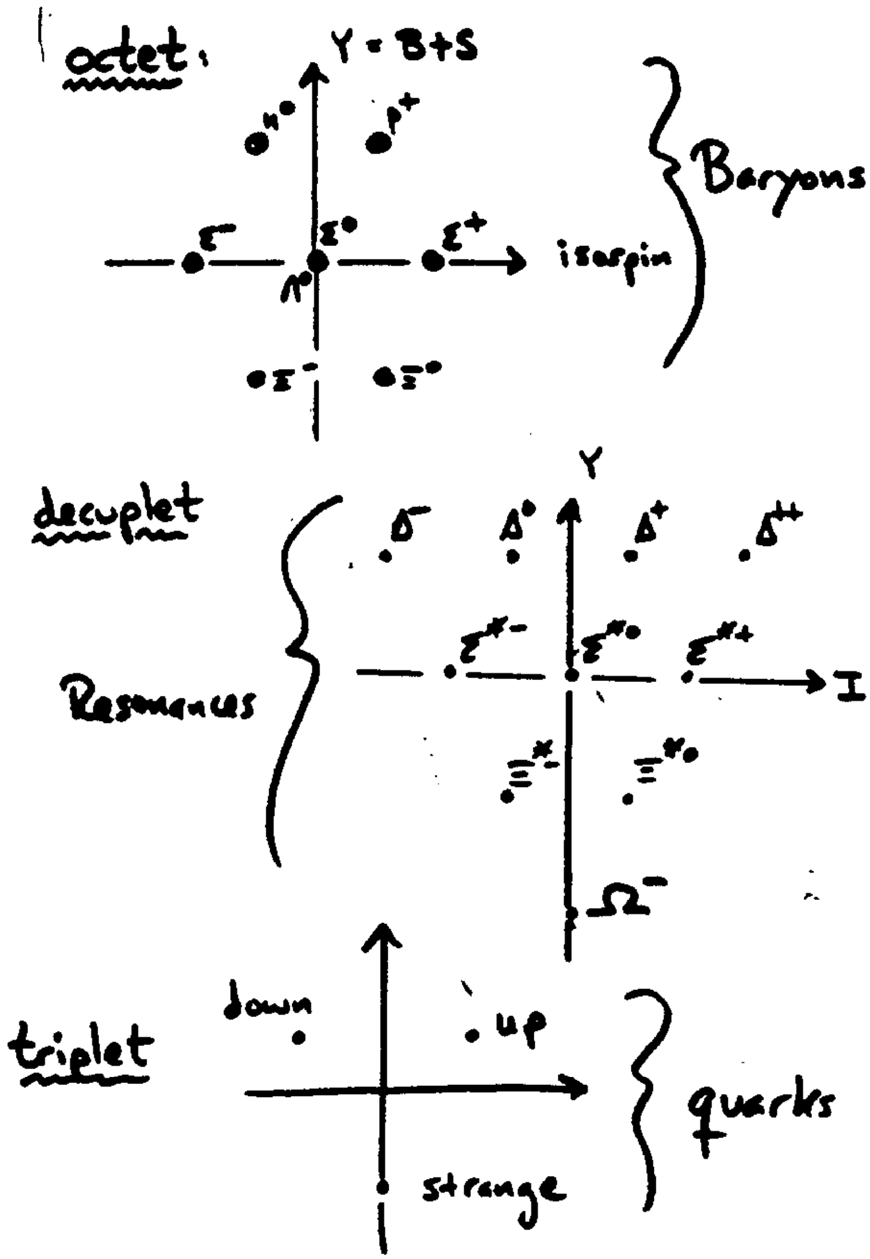


Figure 12: $SU(3)$ baryon octet and resonance decuplet

## 4 SYMMETRIES OF THE LAWS OF PHYSICS AND NÖETHER'S THEOREM

What are the symmetries of the basic laws of physics? What do mean by a symmetry in a law of physics? Just as in the case of the equilateral triangle a symmetry operation is an action we perform upon a geometric configuration such that it remains unchanged, we mean now an operation we can perform upon an experiment such that the outcome of the experiment is unchanged. However, the operation must hold for *any conceiveable experiment* if it is to qualify as a symmetry of the *laws of physics.*

Let us imagine an effort to map out the symmetries of the laws of physics. We suppose that we have a vast region of perfectly empty space and an enormous amount of time. For example, we go into a void in the Universe that measures $\approx 10^5$ parsecs on each side with a laboratory (fig.(13)). The laboratory moves through the void, carrying out various experiments. For example, the laboratory measures the quantities:

### Physical Quantities:

1. The masses of the electron, proton, mesons, W–bosons, etc.
2. The electric charges of these particles.
3. The speed of light.
4. Planck's constant.
5. The lifetimes of various particles and nuclear levels.
6. Newton's universal constant of gravitation.

These quantities are measured to enormous precision and the values are plotted against various configurations of the laboratory:

### Configurations:

1. The position in space of the laboratory,

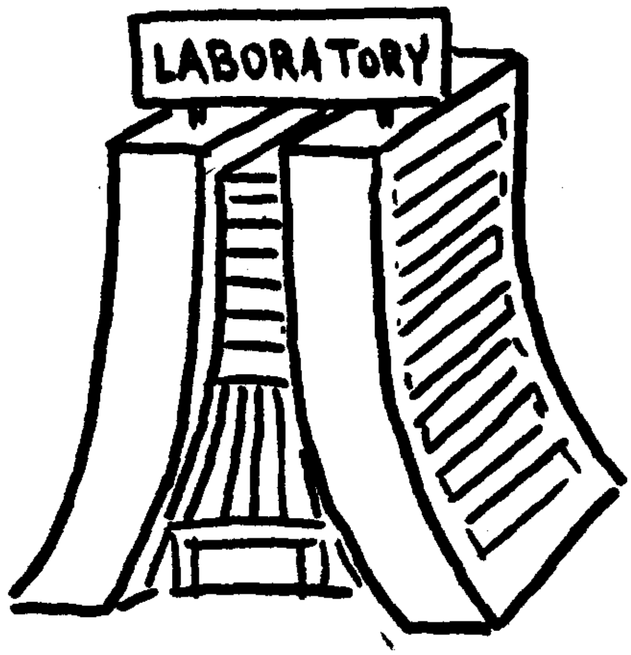


Figure 13: Laboratory for measurement of symmetries in physics.

2. The orientation in space of the laboratory,

3. The time of the measurement,

4. The velocity of the laboratory.

After many billions of years the results of the study are completed. They may be summarized simply: *The physical quantities measured by the laboratory have no dependence upon the configuration of the laboratory.* In other words, changing the position, orientation, time, or velocity of the laboratory does not influence the outcome of experiments conducted in the laboratory! *These are symmetries of the laws of physics.*

In fact, such an experiment can be performed. For example, here at Fermilab we might try to determine if some basic physical quantity, such as the lifetime of the charged pi–meson, depends upon the orientation of the lab. One simply performs the experiment at different times of the day and looks for correlations between time of day and measured lifetime; the rotation of the earth takes care of the reorientation of the laboratory (of course, we must be careful about systematic errors in such an experiment; maybe the power company switches to a different generator in the evening which somehow contaminates our pion beam and gives us a fake signal. What are other potential systematic errors and how do you avoid them in the design of an experiment?). Such an experiment is remniscent of the famous Michelson–Morley experiment which failed to show any dependence of the speed of light upon the absolute state of motion of the earthbound laboratory through an "ether–filled" space. As the consequence of symmetry is simplicity, this experiment, once properly interpreted by the special theory of relativity, washed away the concept of the ether from physics.

There are other compelling indications from astrophysics that our Universe is the same everywhere and for all time. The physical processes occuring in distant stars and more distant galaxies produce the same spectral lines as in laboratory measurements on earth. Such measurements reach back 10 billion years to the early Universe where even Quasi-stellar objects reveal spectral lines of Hydrogen equivalent to those we see today[4]. Also, the measurements are independent of direction is

[4]Of course, these lines are redshifted due to the general recession of these objects, but the redshift is a universal multiplicative effect and the relative frequency ratios of lines are not affected

space. In fact, it is hard to imagine such a homogeneous and isotropic Universe if the laws of physics themselves are not independent of space, time, direction and state of motion.

But can we be absolutely sure that these symmetries are exact and that lurking well below the sensitivity of our experiments there are not small violations of translational invariance or perhaps the fundamental constants change very slowly as the Universe expands? The answer is no. We may prefer the esthetic simplicity of the belief in absolute symmetry, but we can be no more sure than our best experiments can determine. Yet these space–time symmetries are so nearly exact that we can proceed to ***understand*** nature by insisting that they are truly exact. The reult is a completely self–consistent picture of physical laws.

Changing the position, orientation, time, or velocity of the laboratory does not influence the outcome of experiments conducted in the laboratory! In fact, these symmetries form the ***Poincaré Group*** of space–time symmetries of the laws of nature — the basic space–time symmetry group of the laws of physics. They hold over cosmological as well as microscopic and subnuclear scales. What are the physical consequences of these symmetries? This connection is given by one of the most important ***theorems*** in theoretical physics, known as **Nõether's Theorem** [4]:

**For every continuous symmetry of the laws of physics there exists a corresponding conservation law.**

Since the translational symmetry operations can act in any one of three directions in space we find that there is a conserved quantity known as **momentum** which forms a three component vector. Since temporal symmetry operations act in one direction of time we find that there exists a conserved quantity known as **energy** which forms a scalar in Newtonian physics. Relativity unites space and time and in so doing melds energy and momentum together into one quantity called a 4–vector.

Rotational symmetry operations can be performed in any of three independent directions and thus there exists a vector quantity known as **angular momentum**. In relativity the rotations combine together with the three independent Lorentz transformations and angular momentum becomes associated with a six component *tensor*.

Thus Nõether's theorem gives us the remarkable connection between symmetry and dynamical, physical conserved quantities:

$$\begin{array}{lcr} \text{momentum} & \longleftrightarrow & \text{space translations} \\ \text{energy} & \longleftrightarrow & \text{time translations} \\ \text{angular momentum} & \longleftrightarrow & \text{rotations} \end{array} \qquad (4.1)$$

Furthermore, even electric charge, baryon number, quark color and other conserved quantities are associated with symmetries in a deeper and more abstract manner.

As the arts and music have moved in this century farther away from symmetry, indeed adopting *antisymmetry* as a structural element, it is remarkable that symmetry has been increasingly understood by physicists as fundamental to the formulation of the laws of physics. How many times have we glimpsed an equilateral triangle's simplicity yet missed it's inner complexity and logical beauty? So too it is with nature. Perhaps her deepest secrets lie hidden before our very eyes!

## Footnotes and References


[1] *The Short Life of Evariste Galois*, T. Rothman *Scientific American* April (1982) 136

[2] *The Enormous Theorem*, D. Gorenstein *Scientific American*, Dec, (1985) 104

[3] *The Eight-Fold Way*, M. Gell–Mann and Y. Neeman, *Benjamin, New York* (1964)

[4] see e.g. *Women in Mathematics*, L.M. Osen, *MIT Press* (1974) 141