Spontaneous Symmetry Breaking in
Scale Invariant Quantum Electrodynamics

by

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Abstract

Quantum Electrodynamics is studied in a scale invariant limit in four dimensions. At sufficiently strong coupling, the theory exhibits spontaneous breaking of both chiral and scale symmetries. The massless bound states corresponding to the pseudoscalar Goldstone boson and the scalar dilaton are observed. The ultraviolet fixed point which governs this phase of the theory requires the mixing of four fermion interactions with the electrodynamic interactions to preserve the fundamental scale symmetry.

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I. Introduction

Dynamical symmetry breaking has important consequences for the application of gauge field theories to particle physics. The structure of chiral symmetry breaking has been extensively studied for a wide variety of gauge theory models. However, detailed dynamical analysis is quite difficult and dynamical calculations have only been performed in simplified models or in lattice versions of the theory. At the classical level, the chiral symmetric limit of a gauge theory is also the limit which exhibits an exact scale invariance in four dimensions. While the scale symmetry is usually broken explicitly by quantum effects, some aspects of the scale symmetry could remain[1]. If this situation occurs, then the spontaneous breaking of the chiral symmetry will be accompanied by the spontaneous breaking of the scale symmetry and its related Goldstone boson, the dilaton. Until now no evidence of spontaneous breaking of scale symmetry has been observed for gauge theories in four dimensions. In this paper we will present exact results which demonstrate the existence of the spontaneous breaking of scale symmetry for a simplified, scale invariant version of quantum electrodynamics. Although the results are presented only for the simplified model, the mechanisms described in this paper should apply to a much broader class of gauge theories which also exhibit dynamical symmetry breaking.

In perturbation theory, quantum electrodynamics has a running coupling constant due to the scale anomaly which breaks explicitly the scale symmetry. However, if we study the theory in the quenched approximation where the internal fermion loops are surpressed, then the theory becomes scale invariant in the limit of vanishing fermion mass[2]. This version of QED has been studied using the methods of lattice gauge theory where the spontaneous breaking of chiral symmetry has been observed when the coupling strength exceeds a certain finite critical value[3]. Analytic methods have been extensively applied to a planar version of this theory, the "ladder" approximation, which should also preserve an exact scale invariance. It is this version of QED which we will analyse for the dynamical breaking of scale symmetry. Actually this theory may also correspond to an exact large N limit of a nonabelian gauge theory where the fermion representations become large along with the order of the gauge group.

II.   The Schwinger-Dyson equation for the fermion self-energy.

A.   Ladder Approximation.

The analytic approach to the study of dynamical symmetry breaking in planar, quenched QED involves the study of the solutions of the Schwinger-Dyson equation for the fermion self-energy, $\Sigma(p)$. In Landau gauge, this equation takes the form

$$\Sigma(p) = m_0 + i(2\pi)^{-4} \cdot \int d^4q \, e^2 \cdot D_{\alpha\beta}(q) \cdot \{\gamma^\alpha [\gamma \cdot (p-q) - \Sigma(p-q)]^{-1} \gamma^\beta\} \,, \quad [2.1]$$

where the photon propagator is given by $D_{\alpha\beta}(q) = (q_\alpha q_\beta/q^2 - g_{\alpha\beta})/q^2$ and m_0 is the bare fermion mass term which breaks both scale and chiral symmetry. In Euclidean space, this equation may be written as

$$\Sigma(p) = m_0 + 3 \cdot e^2 \cdot (2\pi)^{-4} \cdot \int d^4q \, (p-q)^{-2} \cdot \Sigma(q)/(q^2 + \Sigma^2(q))$$

$$= m_0 + (3 \cdot \alpha/4\pi) \cdot \{ \int_0^{p^2} dq^2 \, (q^2/p^2) \cdot \Sigma(q)/(q^2 + \Sigma^2(q)) \qquad [2.2]$$

$$+ \int_{p^2}^\infty dq^2 \, \Sigma(q)/(q^2 + \Sigma^2(q)) \, \}$$

where we have done the angular integration.

Solutions to the homogenous Schwinger-Dyson equation were obtained previously by Johnson, Baker, and Wiley[4]. However, these solutions do not represent spontaneous breaking of chiral or scale symmetry since the vanishing of the bare mass in the continuum limit just represents the fact that the mass operator, $\overline{\psi}\psi$, has an anomalous dimension at finite coupling, α. The dimension of the mass operator becomes $d_{\overline{\psi}\psi} = 2 + \sqrt{1 - 3 \cdot \alpha/\pi}$ which is also represented by the high momentum decrease of the fermion self-energy for these solutions[2], $\Sigma(p) \to (p^2)^{-(3-d_{\overline{\psi}\psi})/2}$ as $p^2 \to \infty$. The situation was clarified by the study by Maskawa and Nakajima[5] in a version of the theory with an explicit ultraviolet cutoff, Λ^2. In the cutoff theory, all the solutions for weak coupling require an explicit fermion bare mass, $m_0 \neq 0$, which vanishes in the continuum limit, $\Lambda^2 \to \infty$; however, the mass operator, $m_0 \overline{\psi}\psi$, stays

finite in this limit and generates the explicit breaking of chiral and scale symmetry. While no solutions to the massless equation exist at weak coupling, there are solutions at sufficiently large values of the coupling constant, α.

The solutions to the Schwinger – Dyson equation were analysed in detail for both weak and strong coupling by Fukuda and Kugo[6]. The scale symmetry can be used to generate the general solutions of the form

$$\Sigma(p) = e^t \cdot u(t+t_0), \quad t = \ln(p) \tag{2.3}$$

where $u(x)$ is a universal function which satisfies the differential equation

$$0 = u''(x) + 4 \cdot u'(x) + 3 \cdot u(x) + (\alpha/\alpha_c) \cdot u(x)/(1 + u^2(x)) \tag{2.4}$$

with $\alpha_c = \pi/3$. The infrared boundary condition for a massive solution requires that $0 < \Sigma(0) < \infty$ which implies that $u'(x)/u(x) \to -1$ as $x \to -\infty$. We can require that $e^x \cdot u(x) \to 1$ as $x \to -\infty$, since all other massive solutions for $\Sigma(p)$ can be generated by varying the constant, t_0. With this normalization, the function $u(x)$ is uniquely specified and depends only on the coupling constant, α, in addition to the explicit dependence on x. The infrared mass scale for the fermion is then given by $\Sigma(0) = e^{-t_0}$ for the general solution. The function, $u(x)$, was computed numerically by Fukuda and Kugo[6].

The dependence on the bare mass, m_0, that appears as a parameter of the integral equation now becomes an ultraviolet boundary condition for the differential equation. To make a careful treatment of the ultraviolet behavior, a cutoff is introduced for the fermion momentum integration at $q^2 = \Lambda^2$. The ultraviolet boundary condition becomes

$$2m_0/\Lambda = u'(t_\Lambda+t_0)+3 \cdot u(t_\Lambda+t_0), \quad t_\Lambda = \ln(\Lambda). \tag{2.5}$$

This condition determines the value of the parameter, $t_0 = -\ln(\Sigma(0))$, as a function of the coupling constant, α, the bare mass, m_0, and the cutoff, Λ. In the usual treatment, the bare mass, m_0, is required to vanish in the chiral limit. This limit may be studied by using the asymptotic expansion for $u(x)$, $x \to \infty$. Since $u(x)$ tends quickly to zero we can solve

the linearized equation with the result:

weak coupling $(\alpha < \alpha_c)$: $u(x) \to B \cdot e^{-x \cdot (2 - \sqrt{1 - \alpha/\alpha_c})}$, $x \to \infty$;   [2.6]

strong coupling $(\alpha > \alpha_c)$: $u(x) \to A \cdot e^{-2 \cdot x} \cdot \sin[\sqrt{\alpha/\alpha_c - 1} \, (x + \delta)]$, $x \to \infty$; [2.7]

where the constants B, A, and δ are functions of the coupling constant, α.

At weak coupling, there are no solutions of the mass boundary condition for $m_0 = 0$, and the dependence of the bare mass on the cutoff required to maintain a finite renormalized mass, $\Sigma(0)$, is just that expected from the anomalous dimension of the mass operator, $m_0 \to \text{const} \cdot \Lambda^{-(1 - \sqrt{1 - \alpha/\alpha_c})}$. For strong coupling, the mass boundary condition [2.5] with $m_0 = 0$ admits an infinite number of solutions for the fermion mass scale, $\Sigma(0)$, corresponding to the different solutions of the equation

$$0 = u'(t_\Lambda + t_0) + 3 \cdot u \, (t_\Lambda + t_0)$$

[2.8]

$$\approx A \cdot e^{-(t_\Lambda + 2 \cdot t_0)} \cdot \{ \sin[\sqrt{\alpha/\alpha_c - 1} \, (t_\Lambda + t_0 + \delta)] $$
$$+ \sqrt{\alpha/\alpha_c - 1} \cdot \cos[\sqrt{\alpha/\alpha_c - 1} \, (t_\Lambda + t_0 + \delta)] \}$$

or

$$\theta = \sqrt{\alpha/\alpha_c - 1} \cdot (t_\Lambda + t_0 + \delta) = \sqrt{\alpha/\alpha_c - 1} \cdot [\ln(e^\delta \cdot \Lambda/\Sigma(0))]$$

[2.9]

$$\approx n \cdot \pi - \sqrt{\alpha/\alpha_c - 1} \quad ,$$

where $n-1$ counts the number of nodes of the fermion self-energy, $\Sigma(p)$. Actually, only the solution with no nodes, $n = 1$, corresponds to the vacuum solution since it generates the largest fermion mass scale and, hence, the lowest vacuum energy. In Landau gauge, the vacuum energy for the massless theory[7] is given by

$$W = - (2\pi)^{-4} \cdot \int d^4 p \, \{ 2 \cdot \ln[1 + \Sigma^2(p)/p^2] - 2 \cdot \Sigma^2(p)/(p^2 + \Sigma^2(p)) \} \, , \quad [2.10]$$

where the Schwinger-Dyson equation has been used to eliminate the interaction terms; this expression is a negative definite, monotonic

function of $\Sigma(p)$ which is minimized by the largest mass scale solution.

However, the above result seems unphysical; the fermion mass scale generated by the solution diverges with the cutoff as we pass to the continuum limit for fixed coupling constant,

$$\Sigma(0) = e^{\delta} \cdot \Lambda \cdot \exp\{-\theta/\sqrt{\alpha/\alpha_c - 1}\}$$

$$= e^{\delta + 1} \cdot \Lambda \cdot \exp\{-\pi/\sqrt{\alpha/\alpha_c - 1}\} .$$

[2.11]

The other solutions which could yield lower mass scale for $\Sigma(0)$ are not vacuum solutions and cannot be used to describe the ground state. Hence, we seem to conclude that all the fermion physics occurs at the scale of the cutoff, Λ, and there is no infrared sensible limit for the theory.

This paradox is resolved by Miransky[8] through the observation that $\alpha = \alpha_c$ should be viewed as an ultraviolet fixed point of the theory. A similar observation[9] for the critical coupling, $\eta = \eta_c$, in scale invariant $\eta \varphi^6$ theory in three dimensions led to the nonperturbative solution where the scale symmetry was spontaneously broken. For these solutions to generate a nontrivial infrared limit, the coupling constant must vary as a function of the cutoff, approaching the fixed point in the continuum limit. In QED, the Miransky solution requires that the coupling constant vary according to

$$\alpha/\alpha_c = 1 + \pi^2/\ln^2(\Lambda/\kappa) \rightarrow 1, \quad \Lambda \rightarrow \infty ,$$

[2.12]

where κ is an infrared scale. The fermion mass scale is then given by

$$\Sigma(0) = e^{\delta + 1} \cdot \Lambda \cdot \exp\{-\pi/\sqrt{\alpha/\alpha_c - 1}\} \rightarrow e^{\delta + 1} \cdot \kappa ,$$

[2.13]

which is finite as $\Lambda \rightarrow \infty$. In addition, the pseudoscalar Goldstone boson appears as a massless bound state of the Bethe-Salpeter equation in the ladder approximation. Since a finite value for the fermion self-energy can be maintained in the continuum limit, this solution appears to correspond to a nontrivial example of spontaneously broken chiral symmetry. The cutoff dependence of the bare coupling constant permits

the definition of the beta function,

$$\beta(\alpha) = \Lambda\partial_\Lambda\alpha(\Lambda) = -2\pi^2\cdot\alpha_C/\ln^3(\Lambda/\kappa) = -(2/3)\cdot(\alpha/\alpha_C-1)^{3/2}. \quad [2.14]$$

It is not clear what the effects of this running are for the physical low energy amplitudes as the charge is not renormalized in ladder approximation, even in the massive phase.

B. Self-consistent mixing with four fermion operators

While the above treatment may produce an example of spontaneous chiral symmetry breaking, it does not preserve the full symmetries of the ladder approximation. The ladder diagrams have an exact scale invariance in the massless limit. The spontaneous chiral symmetry breaking also breaks the scale symmetry. If this scale symmetry is also spontaneously broken, then there should exist a dilation in the bound state spectrum. However, there is no evidence for a massless scalar bound state from the study of the ladder diagrams. We will show that this treatment of quenched, planar QED is incomplete as it ignores the mixing of the electromagnetic interactions with relevant four fermion interactions which will preserve the exact scale symmetry.

A similar situation occurs in the $\eta\varphi^6$ field theory, where the spontaneous breaking of scale invariance at the fixed point generates new φ^4 interactions in addition to mass terms. These new interactions are required to preserve the scale symmetry, and the dilaton appears as a massless bound state only if these new interactions are included. These induced φ^4 interactions also generate the running of the physical amplitudes consistent with the nonperturbative beta function computed from the cutoff dependence of the bare coupling constant. The induced four fermion interactions will play the same role at the fixed point in quenched, planar QED.

In perturbation theory, the four fermion operators are irrelevant operators as they have dimension six and do not mix with the relevant operators which have dimension four or less in four dimensions. However, the electromagnetic interactions generate anomalous dimensions for composite operators at finite values of the gauge coupling constant. We have previously noted that the mass operator,

$\bar{\psi}\psi$, has a dimension given by $d_{\bar{\psi}\psi} = 2 + \sqrt{1-\alpha/\alpha_c}$ which is three in perturbation theory but decreases to two at the critical point, $\alpha=\alpha_c=\pi/3$. In planar approximation, the dimension of the four fermion operator, $(\bar{\psi}\psi)^2$, is just twice the dimension of the mass operator, $\bar{\psi}\psi$, and is given by $d_{(\bar{\psi}\psi)^2} = 2 \cdot d_{\bar{\psi}\psi} = 4 + 2 \cdot \sqrt{1-\alpha/\alpha_c}$. In perturbation theory this dimension is six as expected, but as the electromagnetic coupling approaches the critical coupling, the dimension of the four fermion operator approaches four, and it will then mix with the electromagnetic interactions which are also dimension four. This mixing cannot be ignored in the study of the continuum limit of QED. Of course, this mixing must preserve the explicit symmetries of QED. Therefore, the induced interactions must preserve the chiral symmetry of the pure electromagnetic interactions. We will show that the scale symmetry can also be recovered with the inclusion of the induced terms when studied in the planar approximation.

The fermion lagrangian used for this study includes both the electromagnetic and the chirally invariant four fermion interactions,

$$L_f = \bar{\psi}\{i\gamma\cdot\partial - e\gamma\cdot A - \mu_0\}\psi + (1/2)\cdot G_0\cdot[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2] , \qquad [2.15]$$

where we have also introduced a fermion mass term to provide a soft breaking of the chiral and scale symmetries. To be consistent with the planar approximation for the electromagnetic interactions, we must keep only the planar diagrams involving the four fermion interactions. The new diagrams are similar to those of the large N, chirally invariant Gross-Neveu model[10] except that the bubble diagrams now must include all of the radiative corrections of planar QED. The fermion self-energy may be calculated from the modified Schwinger-Dyson equation displayed in Fig.1 which now includes the fermion bubble diagram,

Figure 1. The Schwinger-Dyson equation.

where the full fermion propagator is used inside the diagrams. In the

presence of spontaneous chiral symmetry breaking, the four fermion interactions contribute a self-consistent bare mass term to the Schwinger-Dyson equation. The four fermion interactions also contribute to the fermion scattering amplitudes where we must include the sum over the bubble contributions to both the scalar and the pseudoscalar channels. For instance, the contributions to the fermion-antifermion scattering amplitudes are shown in Fig. 2.

Figure 2. The fermion-antifermion scattering amplitude.

It is the presence of these diagrams which will explain the physical running of the coupling constant and which will be responsible for our ability to preserve both scale and chiral symmetry. The Goldstone bosons of the scale and chiral symmetries will appear as poles in the induced diagrams due to the vanishing of the bubble denominators in the symmetry limit.

The Schwinger-Dyson equation may be solved in exactly the same manner that was previously used to study the pure ladder diagrams at strong coupling, $\alpha > \alpha_c$. The only difference with the previous calculation is the inclusion of a contribution to the bare mass term from the induced interactions,

$$m_0 = \mu_0 - G_0 \cdot \langle \overline{\psi}\psi \rangle_0. \qquad [2.16]$$

The vacuum expectation value of the scalar density can be computed from the fermion self-energy,

$$\langle \overline{\psi}\psi \rangle_0 = -(2\pi)^{-4} \cdot \int d^4q \ 4 \cdot \Sigma(q)/(q^2 + \Sigma^2(q))$$

$$[2.17]$$

$$= -(2\pi^2)^{-1} \cdot \int^t_\Lambda dt \ e^{3t} \cdot u(t+t_0)/(1+u^2(t+t_0)).$$

Since this integral is linearly divergent, even with the improved ultraviolet behavior of $u(x)$, the integral is dominanted by the

contributions at large t where we may use the asymptotic solution for u(x) to secure

$$\langle \bar{\psi}\psi \rangle_0 \approx - (2\pi^2)^{-1} \cdot \int^{t_\Lambda} dt\, e^{3t} \cdot u(t+t_0)$$

$$= - (2\pi^2)^{-1} \cdot \int^{t_\Lambda} dt\, e^{3t} \cdot A \cdot e^{-2(t+t_0)} \cdot \sin[\sqrt{\alpha/\alpha_C - 1}\,(t+t_0+\delta)]$$

[2.18]

$$= - (2\pi^2)^{-1} \cdot A \cdot e^{(t_\Lambda - 2t_0)} \cdot \int_0^\infty dy\, e^{-y} \cdot \sin[\sqrt{\alpha/\alpha_C - 1}\,(t_\Lambda + t_0 + \delta - y)]$$

$$\approx - (2\pi^2)^{-1} \cdot A \cdot e^{(t_\Lambda - 2t_0)} \cdot \{ \sin[\sqrt{\alpha/\alpha_C - 1}\,(t_\Lambda + t_0 + \delta)]$$
$$- \sqrt{\alpha/\alpha_C - 1} \cdot \cos[\sqrt{\alpha/\alpha_C - 1}\,(t_\Lambda + t_0 + \delta)]\}.$$

This last approximation is valid for α near the critical coupling. We may now combine this result with the original QED boundary condition for the bare mass parameter to obtain the full gap equation including the four fermion couplings.

$$m_0 = \mu_0 - G_0 \cdot \langle \bar{\psi}\psi \rangle_0$$

[2.19a]

$$= \mu_0 + G_0 \cdot (2\pi^2)^{-1} \cdot A \cdot \Lambda \cdot e^{-2t_0} \cdot \{\sin[\sqrt{\alpha/\alpha_C - 1}\,(t_\Lambda + t_0 + \delta)]$$
$$- \sqrt{\alpha/\alpha_C - 1} \cdot \cos[\sqrt{\alpha/\alpha_C - 1}\,(t_\Lambda + t_0 + \delta)]\},$$

$$m_0 = (1/2) \cdot \Lambda \cdot [u'(t_\Lambda + t_0) + 3 \cdot u(t_\Lambda + t_0)]$$

[2.19b]

$$= (1/2) \cdot \Lambda \cdot A \cdot e^{-2(t_\Lambda + t_0)} \cdot \{\sin[\sqrt{\alpha/\alpha_C - 1}\,(t_\Lambda + t_0 + \delta)]$$
$$+ \sqrt{\alpha/\alpha_C - 1} \cdot \cos[\sqrt{\alpha/\alpha_C - 1}\,(t_\Lambda + t_0 + \delta)]\}$$

Near the critical coupling, the parameters of the asymptotic expansion become $A \to \tilde{A}/\sqrt{\alpha/\alpha_C - 1}$ with $\tilde{A} \approx 1.2$ and $\delta \approx .55$. The gap equation becomes

$$\mu_0 \cdot \Lambda = (1/2) \cdot \tilde{A} \cdot e^{-2 \cdot t_0} \cdot \{ [(1 - G_0 \cdot \Lambda^2/\pi^2)/\sqrt{\alpha/\alpha_C - 1}\,]$$
$$\cdot \sin[\sqrt{\alpha/\alpha_C - 1}\,(t_\Lambda + t_0 + \delta)] + [1 + G_0 \cdot \Lambda^2/\pi^2\,] \qquad [2.20]$$
$$\cdot \cos[\sqrt{\alpha/\alpha_C - 1}\,(t_\Lambda + t_0 + \delta)]\}.$$

This equation must be solved for the fermion mass scale, $\Sigma(0) = e^{-t_0}$. As before the angle, $\theta = \sqrt{\alpha/\alpha_c - 1} \cdot (t_\Lambda + t_0 + \delta)$, must be between 0 and π to give the no node vacuum solution with the lowest vacuum energy. Other solutions for larger angles correspond to smaller fermion mass scales but higher vacuum energies. Using the Schwinger-Dyson equation to eliminate both interaction terms, the vacuum energy becomes

$$W = - (2\pi)^{-4} \cdot \int d^4p \; \{2 \cdot \ln[1 + \Sigma^2(p)/p^2] - 2 \cdot \Sigma^2(p)/(p^2 + \Sigma^2(p))$$
$$+ 2 \cdot \mu_0 \cdot \Sigma(p)/(p^2 + \Sigma^2(p)) \; \}, \qquad [2.21]$$

where we now keep the dependence on the bare mass parameter, μ_0. As before, the vacuum energy is negative definite function of $\Sigma(p)$ which is minimized by the solution with the largest mass scale.

It is convenient to introduce renormalized values for the four fermion coupling $G = G_0 \cdot \Lambda^2/\pi^2$ and for the mass parameter $\mu = \mu_0 \cdot \Lambda$. Note that the cutoff dependence implied for G_0 and μ_0 is just that expected from the anomalous dimension of the four fermion and mass operators at $\alpha = \alpha_c$. The gap equation becomes

$$\mu = (1/2) \cdot \tilde{A} \cdot e^{2 \cdot \delta} \cdot \Lambda^2 \cdot e^{-2 \cdot \theta/\sqrt{\alpha/\alpha_c - 1}}$$
$$\cdot \{ \; [(1-G)/\sqrt{\alpha/\alpha_c - 1} \;] \cdot \sin\theta + [1+G] \cdot \cos\theta \}, \qquad [2.22]$$

$$\Sigma(0) = e^\delta \cdot \Lambda \cdot e^{-\theta/\sqrt{\alpha/\alpha_c - 1}} . \qquad [2.23]$$

There always exists one solution for θ, or equivalently for $\Sigma(0)$, in the region $0 < \theta < \pi$, and this is the ground state solution. As in the pure case where $G = 0$, a nontrivial continuum limit requires that the gauge coupling constant must approach the critical value, $\alpha \to \alpha_c$ as $\Lambda \to \infty$. However, the particular limit differs from the case of Miransky when $\theta \neq \pi$, and we have

$$\alpha/\alpha_c = 1 + \theta^2/\ln^2(\Lambda/\kappa), \quad \Lambda \to \infty , \qquad [2.24]$$

where κ is the infrared scale. The value of θ depends on the value of the induced coupling G. We will find that the scale invariant theory corresponds to the ultraviolet fixed point with $G = 1$ and $\alpha \to \alpha_c$. For this value of G, $\theta \to \pi/2$ in the symmetry limit. The beta function for α

is also modified as

$$\beta(\alpha) = -(2 \cdot \alpha_c/\theta) \cdot (\alpha/\alpha_c - 1)^{3/2} = -(2\pi/3\theta) \cdot (\alpha/\alpha_c - 1)^{3/2}. \qquad [2.25]$$

III.   The Goldstone poles.

We now wish to examine the symmetry structure of these solutions. If there is spontaneous symmetry breaking, Goldstone bosons should appear as poles in the fermion-antifermion scattering amplitudes. For $G \neq 0$, there will be no massless bound states of the pure ladder diagrams as the induced bare mass term, m_0, will not vanish and will appear as an explicit chiral symmetry breaking for these diagrams. The Goldstone poles will appear in the induced diagrams and come from the zeros of the bubble denominators. To demonstrate this mechanism we must be able to compute the bubble diagrams including all of the electromagnetic radiative corrections. Fortunately, we are able to do this computation analytically in zero momentum limit.

We first must compute the full vertex functions for the fermion matrix elements containing the scalar and pseudoscalar operators. At zero momentum transfer, the bare vertex function may be computed from the bare mass derivative of the fermion self-energy function as

$$\Gamma^0_S(p,p) = \partial_{m_0} \Sigma(p) = e^t \cdot u'(t+t_0)/(\partial m_0/\partial t_0) , \qquad [3.1]$$

where $(\partial m_0/\partial t_0)$ may be computed from the mass boundary condition, Eq.[2.19b]. The vertex function, expressed as a function of $t = \ln(p)$, also satisfies a differential equation

$$0 = \Gamma^0_S{}''(t) + 2 \cdot \Gamma^0_S{}'(t) + (\alpha/\alpha_c) \cdot [(1-u^2(t))/(1+u^2(t))^2] \cdot \Gamma^0_S(t) \qquad [3.2]$$

along with the boundary condition

$$2 = \partial_t \Gamma^0_S (t_\Lambda) + 2 \cdot \Gamma^0_S(t_\Lambda) \qquad [3.3]$$

which is similar to the boundary condition of Eq. [2.5]. The solution is of course the same as Eq. [3.1]. We define a renormalized vertex function by

requiring that $\Gamma^R_s(p,p) \rightarrow 1$ as $p \rightarrow 0$ so that

$$\Gamma^R_s(p,p) = \Gamma^R_s(t) = -e^{(t+t_0)} \cdot u^{\bullet}(t+t_0) = Z_s \cdot \Gamma^0_s(t) \qquad [3.4]$$

and

$$Z_s = -(1/2) \cdot e^{(t_\Lambda + t_0)} \cdot [u^{\bullet\bullet}(t_\Lambda + t_0) + 3 \cdot u^{\bullet}(t_\Lambda + t_0)]$$

$$[3.5]$$

$$= (1/2) \cdot \tilde{A} \cdot e^{-(t_\Lambda + t_0)} \cdot \{(2/\sqrt{\alpha/\alpha_c - 1}) \cdot \sin[\sqrt{\alpha/\alpha_c - 1}\,(t_\Lambda + t_0 + \delta)]$$
$$+ \cos[\sqrt{\alpha/\alpha_c - 1}\,(t_\Lambda + t_0 + \delta)]\}.$$

The pseudoscalar vertex function, $\Gamma^0_p(p,p)$, is given by the chiral Ward identity as

$$\Gamma^0_p(p,p) = \Sigma(p)/m_0 = e^t \cdot u(t+t_0)/m_0 \,, \qquad [3.6]$$

but may also be computed from the differential equation

$$0 = \Gamma^0_p{}^{\bullet\bullet}(t) + 2 \cdot \Gamma^0_p{}^{\bullet}(t) + (\alpha/\alpha_c) \cdot [1/(1+u^2(t))] \cdot \Gamma^0_p(t) \qquad [3.7]$$

with the same boundary condition

$$2 = \partial_t \Gamma^0_p(t_\Lambda) + 2 \cdot \Gamma^0_p(t_\Lambda). \qquad [3.8]$$

The solution is proportional to the fermion self-energy function. The renormalized vertex, defined so that $\Gamma^R_p(0,0) = 1$, is given by

$$\Gamma^R_p(p,p) = \Gamma^R_p(t) = e^{(t+t_0)} \cdot u(t+t_0) = Z_p \cdot \Gamma^0_p(t) \qquad [3.9]$$

and

$$Z_p = (1/2) \cdot e^{(t_\Lambda + t_0)} \cdot [u^{\bullet}(t_\Lambda + t_0) + 3 \cdot u(t_\Lambda + t_0)]$$

$$[3.10]$$

$$= (1/2) \cdot \tilde{A} \cdot e^{-(t_\Lambda + t_0)} \{(1/\sqrt{\alpha/\alpha_c - 1}) \cdot \sin[\sqrt{\alpha/\alpha_c - 1}\,(t_\Lambda + t_0 + \delta)]$$
$$+ \cos[\sqrt{\alpha/\alpha_c - 1}\,(t_\Lambda + t_0 + \delta)]\}.$$

We may now use these vertex functions to calculate the bare bubble integrals which appear in the denominators of the induced diagrams. The radiative corrections may all be associated with one bare vertex function with the result that the scalar bubble function at zero momentum is given by

$$B^0{}_S(0) = - i \cdot (2\pi)^{-4} \cdot \int d^4p \ tr\{(\gamma \cdot p - \Sigma(p))^{-1} \cdot \Gamma^0{}_S(p,p) \cdot (\gamma \cdot p - \Sigma(p))^{-1}\}$$

$$= - (2\pi)^{-4} \cdot \int d^4p \ 4 \cdot [(p^2 - \Sigma^2(p))/(p^2 + \Sigma^2(p))^2] \cdot \Gamma^0{}_S(p,p) \qquad [3.11]$$

$$= - (2 \cdot \pi^2)^{-1} \cdot \int^{t_\Lambda} dt \ e^{2 \cdot t} \ [(1 - u^2(t+t_0))/(1 + u^2(t+t_0))^2] \cdot \Gamma^0{}_S(t)$$

This integral has an apparent quadratic divergence from the factor $e^{2 \cdot t}$ which is reduced to a linear divergence by the behavior of the vertex function for large t. However, the integral is still dominated by the momenta near the cutoff where we may accurately use the asymptotic expansion to obtain

$$B^0{}_S(0) = - (2\pi^2)^{-1} \cdot \int^{t_\Lambda} dt \ e^{2 \cdot t} \ (-1/Z_S) \cdot e^{(t+t_0)} \cdot A \cdot e^{-2 \cdot (t+t_0)}$$
$$\cdot \{-2 \cdot \sin[\sqrt{\alpha/\alpha_C - 1} \ (t+t_0+\delta)] + \sqrt{\alpha/\alpha_C - 1} \cdot \cos[\sqrt{\alpha/\alpha_C - 1} \ (t+t_0+\delta)]\}$$
$$[3.12]$$
$$\approx - (2\pi^2)^{-1} \cdot (1/Z_S) \cdot \tilde{A} \cdot e^{(t_\Lambda - t_0)} \cdot \{(2/\sqrt{\alpha/\alpha_C - 1})$$
$$\cdot \sin[\sqrt{\alpha/\alpha_C - 1}(t_\Lambda + t_0 + \delta)] - 3 \cdot \cos[\sqrt{\alpha/\alpha_C - 1}(t_\Lambda + t_0 + \delta)]\} \ ,$$

where this result becomes exact as $\alpha \to \alpha_C$. Substituting the expression [3.5] for Z_S we obtain the bare scalar bubble function

$$B^0{}_S(0) = - (1/\pi^2) \cdot \Lambda^2 \cdot \{(2/\sqrt{\alpha/\alpha_C - 1}) \cdot \sin \theta - 3 \cdot \cos \theta\}/$$
$$\{(2/\sqrt{\alpha/\alpha_C - 1}) \cdot \sin \theta + \cos \theta\} \ , \qquad [3.13]$$

where $\theta = \sqrt{\alpha/\alpha_C - 1} \cdot (t_\Lambda + t_0 + \delta)$ as before. Alternatively, we could also have derived this result from our expression for the vacuum expectation value of the scalar density as $B^0{}_S(0) = \partial_{m_0} \langle \bar{\psi}\psi \rangle_0$. The same methods as above can also be applied to compute the bare pseudoscalar bubble function at zero momentum with the result

$$B^0{}_p(0) = -(1/\pi^2) \cdot \Lambda^2 \cdot \{(1/\sqrt{\alpha/\alpha_C-1}) \cdot \sin\theta - \cos\theta\}/$$
$$\{(1/\sqrt{\alpha/\alpha_C-1}) \cdot \sin\theta + \cos\theta\}. \qquad [3.14]$$

Since we are interested in the pole structure at zero momentum, it is convenient to use renormalized vertex functions for the numerators of the induced amplitudes and define renormalized denominator functions. The induced amplitudes have the form

$$S_{ff}(p,p;p',p') = S_{ff}(ladder) - [\Gamma^R{}_s(p,p)] \cdot [\Gamma^R{}_s(p',p')]/D^R{}_s(0)$$
$$- [\Gamma^R{}_p(p,p)i\gamma_5] \cdot [\Gamma^R{}_p(p',p')i\gamma_5]/D^R{}_p(0). \qquad [3.15]$$

The renormalized denominator functions are given by

$$D^R{}_s(0) = Z^2{}_s/G_0 + Z^2{}_s \cdot B^0{}_s(0)$$

$$[3.16]$$

$$= (1/4\pi^2) \cdot \tilde{A}^2 \cdot e^{2\cdot\delta} \cdot \Lambda^2 \cdot e^{-2\cdot\theta/\sqrt{\alpha/\alpha_C-1}}$$
$$\cdot \{[2 \cdot (1/G-1)/\sqrt{\alpha/\alpha_C-1}] \cdot \sin\theta + [1/G+3] \cdot \cos\theta\}$$
$$\cdot \{[2/\sqrt{\alpha/\alpha_C-1}] \cdot \sin\theta + \cos\theta\}$$

and

$$D^R{}_p(0) = Z^2{}_p/G_0 + Z^2{}_p \cdot B^0{}_p(0)$$

$$[3.17]$$

$$= (1/4\pi^2) \cdot \tilde{A}^2 \cdot e^{2\cdot\delta} \cdot \Lambda^2 \cdot e^{-2\cdot\theta/\sqrt{\alpha/\alpha_C-1}}$$
$$\cdot \{[(1/G-1)/\sqrt{\alpha/\alpha_C-1}] \cdot \sin\theta + [1/G+1] \cdot \cos\theta\}$$
$$\cdot \{[1/\sqrt{\alpha/\alpha_C-1}] \cdot \sin\theta + \cos\theta\}.$$

Recalling our expression for the gap equation (cf. Eqs. [2.22], [2.23]),

$$\mu = (1/2) \cdot \tilde{A} \cdot e^{2\cdot\delta} \cdot \Lambda^2 \cdot e^{-2\cdot\theta/\sqrt{\alpha/\alpha_C-1}}$$
$$\cdot \{ [(1-G)/\sqrt{\alpha/\alpha_C-1}] \cdot \sin\theta + [1+G] \cdot \cos\theta \},$$

$$\Sigma(0) = e^\delta \cdot \Lambda \cdot e^{-\theta/\sqrt{\alpha/\alpha_C-1}},$$

it is clear that the pseudoscalar denominator and the gap equation have a common factor which vanishes in the chiral limit, $\mu_0 \to 0$. However, the scalar denominator will not vanish in the same limit for arbitrary values of the renormalized four fermion coupling, G. Only for the fixed point value, G = 1, will the gap equation and the scalar denominator have a common zero. For this fixed point value of the coupling, quenched, planar quantum electrodynamics preserves the scale symmetry, and the scalar denominator will vanish in the symmetry limit.

Near the ultraviolet fixed point $[\alpha \to \alpha_c^+$ and $G \to 1]$, we can use the expressions for the renormalized scalar denominator Eq. [3.16], the gap equation [2.22], and the fermion mass scale Eq. [2.23] to compute the asymptotic behavior for the beta functions of both the gauge coupling constant Eq. [2.25] and the four fermion coupling constant as

$$\beta_\alpha(\alpha,G) = -(2\cdot\pi/3)\cdot(\alpha/\alpha_c-1)^{3/2}/\arctan(2\cdot\sqrt{\alpha/\alpha_c-1}/(G-1))$$

$$[3.18]$$

$$\beta_G(\alpha,G) = -(G-1)\cdot(\alpha/\alpha_c-1)^{1/2}/\arctan(2\cdot\sqrt{\alpha/\alpha_c-1}/(G-1)) .$$

The angle, $\theta = \arctan(2\cdot\sqrt{\alpha/\alpha_c-1}/(G-1))$, is defined in the range $0<\theta<\pi$. These beta functions are obviously nonperturbative and reflect the approach to the fixed point of our explicit solutions.

At the scale invariant fixed point for the four fermion coupling [G=1], we may still introduce the soft symmetry breaking by the fermion mass parameter, $\mu=\mu_0\cdot\Lambda$. The above equations simplify with the result:

gap equation: $\mu = (1/2)\cdot\tilde{A}\cdot e^{2\cdot\delta}\cdot\Lambda^2\cdot e^{-2\cdot\theta/\sqrt{\alpha/\alpha_c-1}}\cdot\{2\cdot\cos\theta\}$, $\qquad$ [3.19]

fermion mass scale: $\Sigma(0) = e^\delta \cdot \Lambda \cdot e^{-\theta/\sqrt{\alpha/\alpha_c-1}}$, $\qquad$ [3.20]

scalar denominator: $D^R_s(0) = (1/4\pi^2) \cdot \tilde{A}^2 \cdot e^{2\cdot\delta} \cdot \Lambda^2 \cdot e^{-2\cdot\theta/\sqrt{\alpha/\alpha_c-1}}$

$\qquad\qquad \cdot\{4\cdot\cos\theta\}\cdot\{[2/\sqrt{\alpha/\alpha_c-1}]\cdot\sin\theta + \cos\theta\}$, $\qquad$ [3.21]

pseudoscalar
$\qquad$ denominator: $D^R_p(0) = (1/4\pi^2) \cdot \tilde{A}^2 \cdot e^{2\cdot\delta} \cdot \Lambda^2 \cdot e^{-2\cdot\theta/\sqrt{\alpha/\alpha_c-1}}$

$\qquad\qquad \cdot \{2\cdot\cos\theta\}\cdot\{[1/\sqrt{\alpha/\alpha_c-1}]\cdot\sin\theta + \cos\theta\}$. $\qquad$ [3.22]

It is now clear that the gap equation, the scalar denominator, and the pseudoscalar denominator all have a common factor of $\cos\theta$ which will vanish in the chiral limit, $\theta \to \pi/2$ as $\mu \to 0$. With the proper strength of the induced terms, quenched, planar QED preserves both chiral and scale symmetry.

We can also study the explicit breaking of scale and chiral symmetry as we have kept the dependence on the lagrangian mass parameter, μ. Expanding to first order in the explict symmetry breaking, we can derive the exact low energy limits,

$$D^R_S(0) = -4 \cdot \langle \mu_0 \bar{\psi}\psi \rangle_0 / \Sigma^2(0),$$

$$[3.23]$$

$$D^R_p(0) = - \langle \mu_0 \bar{\psi}\psi \rangle_0 / \Sigma^2(0).$$

$$[3.24]$$

Since we have normalized the numerator factors to one at zero momentum, the above limits can be translated into a low energy theorem for the fermion-antifermion scattering amplitudes.

IV. Conclusions.

We have presented a novel solution to quenched, planar QED where both chiral symmetry and scale symmetry remain exact but are spontaneously broken in the true ground state. The theory is governed by a nonperturbative, ultraviolet fixed point where $G \to 1$ and $\alpha \to \alpha_c^+$. A remarkable feature of this solution is the relevance of four fermion operators which have dimension four at the fixed point instead of their perturbative values of dimension six. Another remarkable feature is the fact that the theory may be studied by analytic methods. Of course, it would be interesting to determine what features of our solution are preserved beyond our rather severe truncation of QED. Perhaps lattice methods[3] can be used to study the nontrivial aspects of the theory and strong coupling. Our solution may also have implications beyond QED where non-abelian gauge theories may display the effects of approximate but spontaneously broken scale invariance[1].

Acknowledgements

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