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Fermion Masses in $E_8 \times E_8'$ Superstring Theories

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ABSTRACT

We consider string theories with gauge group $E_8 \times E_8'$ compactified on Calabi-Yau manifolds so that the unified gauge group is $E_6 \times E_8'$. If the fermion content is $N_f \cdot \mathbf{27} + \delta(\mathbf{27} + \overline{\mathbf{27}})$, where $\delta=1$ or 2, and the E_6 is broken by Wilson loops to a gauge group Σ (which contains $SU(3)_c \times SU(2)_L \times U(1)_Y$), then Σ is specified almost uniquely if we require that neither the up nor the down quark mass matrices are identically zero at tree level and that the Higgs doublets arise via the incomplete multiplet mechanism. In particular, Σ must be rank 6. However, if we wish for the neutral mass matrix to be acceptable, we must have the electron mass matrix identically zero at tree level. We comment on a way to possibly have small (but non-zero) neutrino masses in superstring models.

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The $E_8 \times E_8'$ heterotic superstring theory [1] seems to be the most phenomenologically promising of the possible anomaly free [2] superstring theories. The requirement that there be a low energy N=1 supersymmetry is achieved by compactifying the original 10-dimensional $E_8 \times E_8'$ theory on a manifold $M^4 \times K$ where K is a six dimensional Calabi-Yau manifold* [3,4]. This is a Kähler manifold with SU(3) holonomy. In the course of doing this the SU(3) gauge vector field in $E_8 \supset E_6 \times SU(3)$ is set equal to the spin connection of K in such a way as to explicitly break E_8 to E_6. The low energy representations appear as zero modes on K, and are neutral under the other E_8'. If K is chosen as a simply connected manifold, K_{sc}, then, aside from the gravity and gauge multiplets, there are chiral multiplets that comprise an $N_f\,\mathbf{27} + \delta(\mathbf{27} + \overline{\mathbf{27}})$ of E_6. The number of families N_f is determined by the topology of K_{sc} and is given by [3], $N_f = |\chi(K_{sc})/2|$, where $\chi(K_{sc})$ is the Euler characteristic of K_{sc}. The number δ is also determined topologically and is given by the Hodge number $b_{1,1}$. Kähler manifolds always have $b_{1,1} \geq 1$. In this paper we will mainly specialize to $b_{1,1}=1$ except for some comments at the end where we note that in the case $b_{1,1}=2$ our analysis applies almost identically.

The zero modes do not have the proper multiplet structure to act as Higgs to break E_6 all the way down to $SU(3)_c \times U(1)_{EM}$, hence another mechanism must be employed. Such a mechanism exists if K is not simply connected [6,7]. We consider a non-simply connected space which can be written as $K = K_{sc}/G$, where G is a discrete group that acts on K_{sc} freely

*This is true if the added requirement that K be torsion free is imposed. Our results would concievably be different if this were not so [5].

(that is, without fixed points, unless the group element in question is the identity). The space K_{sc}/G is formed from the space K_{sc} by identifying the points x and gx on K_{sc}, where $g \in G$. Most examples that have been considered thus far have been of this form [3,8]. On this space there may be non-trivial E_6 gauge field configurations which, while they cannot be gauge transformed to zero, nonetheless have $F_{\mu\nu}=0$. The possible presence of these gauge fields can break E_6 explicitly by virtue of their contribution to a Wilson loop (which, after all, exhibits gauge invariant information contained in the vacuum value of the E_6 gauge field). Such a Wilson loop is given by $\mathbf{U}_g=P \exp[\, i\int_\Gamma A_m dx^m]$, where the integration is over a non-contractible loop, Γ, in K. Since the group G is finite it follows that $\mathbf{U}_g{}^n=1$ for some integer n. For each element $g \in G$ there is exactly one $\mathbf{U}_g$ in E_6 (which may be the identity). The Wilson loops thus provide a homomorphism of G onto a discrete subgroup, $\overline{G}$, of E_6 which is represented by the $\mathbf{U}_g$. These matrices act similarly to the vacuum values of adjoint Higgs and break E_6.

When passing from K_{sc} to K the value of N_f changes to $N_f'=|\chi(K)/2| =|\chi(K_{sc})/(2 \dim(G))|$, but the fields no longer comprise **27**'s of E_6, since E_6 has been broken to a subgroup* Σ. The number of degrees of freedom comprising left handed families is nonetheless $27|\chi(K)/2|$ [3,7], each family falling into a reducible representation, **R**, of Σ. Of the zero mode degrees of freedom appearing on K_{sc} only certain of them survive to remain in the spectrum when we make the mathematical transition from K_{sc} to K [7]. If Ψ(x) is a field on K_{sc} (possibly with E_6 indices) then as Ψ is transported from x to x+dx, Ψ will change by a gauge transform [9]: $\Psi(x) \to \exp[-i\, A_m dx^m]\Psi(x+dx)$. As we move from the point x to the point gx on K_{sc}, we pick up the full factor $\mathbf{U}_g{}^\dagger$; however, for Ψ(x) to also be a mode

*Σ is, of course, required to contain $SU(3)_c \times SU(2)_L \times U(1)_Y$.

on K, the result of transporting from x to gx must equal $\Psi(x)$. Thus, we must have $\Psi(x)=\mathbf{U}_g{}^{\dagger}\Psi(gx)$. In the case where $\delta=1$ we can easily answer the following question [7]: Which components of the $\overline{\mathbf{27}}$ are in the spectrum on K? Since* for this $\overline{\mathbf{27}}$, $\Psi(x)=\Psi(gx)$, the requirement that a component of the $\overline{\mathbf{27}}$ be in the spectrum on K is that it be invariant under U_g for all $g\in G$. These modes will fall into a (generally) reducible representation $\overline{\mathbf{r}}$ of Σ. Thus the zero mode representation content of the theory is [7] (aside from gravitational and gauge modes) $N_f'\mathbf{R}+(\mathbf{r}+\overline{\mathbf{r}})$. These "incomplete multiplets" $(\mathbf{r}+\overline{\mathbf{r}})$ have been suggested as candidates for the Higgs multiplets** that break Σ to $SU(3)_c\times U(1)_{EM}$ [7,11,12]. We adopt this view in this paper and explore some of its consequences. In the following we will impose the condition that both the up and down quark mass matrices are not identically zero at tree level. From this we determine that the subgroup Σ of E_6 left unbroken by the Wilson loops is almost uniquely determined and that the group $\overline{G}$ is restricted to be Z_n (in particular, $\overline{G}$ cannot be non-abelian).

We first consider the case in which $\overline{G}$ is abelian (and thus Σ is rank 6). The smallest group that Σ can be is $SU(3)_c \times SU(2)_L \times U(1)_{8L} \times U(1)_{3R} \times U(1)_{8R}$ where the labels on the U(1) factors refer to the diagonal generators fo E_6 as expressed in the maximal embedding $E_6\supset SU(3)_c \times SU(3)_L \times SU(3)_R$. Under $E_6\supset SU(3)_c \times SU(2)_L \times U(1)_{8L} \times U(1)_{3R} \times U(1)_{8R}$ the **27** of E_6 decomposes as

*This is a property of Calabi-Yau spaces in which $b_{1,1}=1$. It is related to the fact that the Kähler form is invariant under $x\rightarrow gx$ [10].

**This presumes discrete symmetries that distinguish these Higgs multiplets from the other multiplets in the spectrum that have the same quantum numbers under Σ. Such symmetries are probably needed to keep the proton from decaying too fast [7]. They depend upon the manifold K_{sc} and the group G [7]. Each case must be treated separately (and with respect!).

$$\mathbf{27} = A_1(\mathbf{1},\mathbf{2},-1,1,1) + B_{-1}(\mathbf{1},\mathbf{2},-1,-1,1) + C_{-1}(\mathbf{1},\mathbf{2},-1,0,-2)$$
$$+ D_2(\mathbf{1},\mathbf{1},2,1,1) + E_0(\mathbf{1},\mathbf{1},2,-1,1) + F_0(\mathbf{1},\mathbf{1},2,0,-2)$$
$$+ G_{1/3}(\mathbf{3},\mathbf{2},1,0,0) + H_{-2/3}(\mathbf{3},\mathbf{1},-2,0,0) + I_{-4/3}(\bar{\mathbf{3}},\mathbf{1},0,-1,-1)$$
$$+ J_{2/3}(\bar{\mathbf{3}},\mathbf{1},0,1,-1) + K_{2/3}(\bar{\mathbf{3}},\mathbf{1},0,0,2)$$

where the subscripts denote the weak hypercharges of the respective components.

From the weak hypercharges we can determine which components of the superfield, **27**, contain which fermions. Thus, the superfield $G_{1/3}$ is $[u,d]_L$, $I_{-4/3}$ is $u_R{}^c$, $H_{-2/3}$ is D_L (the D is sometimes also refered to as g*), D_2 is $e_R{}^c$, and A_1 is $[E, N]_R{}^c$. There is an ambiguity at this stage for $J_{2/3}$ and $K_{2/3}$, and for B_{-1} and C_{-1} depending upon mixings between "light" and "heavy" fields in the mass matrices. Both $J_{2/3}$ and $K_{2/3}$ are candidate $d_R{}^c$ and $D_R{}^c$. Similarly, B_{-1} and C_{-1} are candidate $(\nu,e)_L$ and $(N,E)_L$. E_0 and F_0 are neutral singlets (candidate $\nu_R{}^c$ and $N_R{}^c$).

As a consequence of this, the Higgs supermultiplet that gives rise to an up quark mass matrix must transform as the conjugate of $G_{1/3}\, I_{-4/3} \sim (\mathbf{1},\mathbf{2},1,-1,-1)$. Thus, the superpotential coupling giving rise to an $\mathbf{M}_u$ would be $G_{1/3}\, I_{-4/3}\, \widetilde{A}_1$, where the tilde denotes a Higgs. For the down quark mass matrix there are two choices. Case I: $G_{1/3}\, J_{2/3} \sim (\mathbf{1},\mathbf{2},1,1,-1)$ where $\mathbf{M}_d$ arises from $G_{1/3}\, J_{2/3}\, \widetilde{B}_{-1}$. Case II: $G_{1/3}\, K_{2/3} \sim (\mathbf{1},\mathbf{2},1,0,2)$ where $\mathbf{M}_d$ arises from $G_{1/3}\, K_{2/3}\, \widetilde{C}_{-1}$.

We parameterize the matrix $\mathbf{U}_g$ as

$$\mathbf{U}_g = \mathbf{1} \times \begin{bmatrix} \alpha & & \\ & \alpha & \\ & & \alpha^{-2} \end{bmatrix} \times \begin{bmatrix} \beta\delta & & \\ & \beta\delta^{-1} & \\ & & \beta^{-2} \end{bmatrix}$$

*In ref. [13], for example.

in the $SU(3)_c \times SU(3)_L \times SU(3)_R$ basis. In case I we wish for $\widetilde{A}_1$ and $\widetilde{B}_{-1}$ to be invariant under U_g; and, in case II we want $\widetilde{A}_1$ and $\widetilde{C}_{-1}$ to be invariant (we will see later that the case in which all three $\widetilde{A}_1$, $\widetilde{B}_{-1}$ and $\widetilde{C}_{-1}$ are invariant leads to unbroken $SU(3)_c \times SU(3)_L \times SU(3)_R$). Under U_g:

$$
\begin{array}{lll}
\widetilde{A}_1 \to \alpha^{-1}\delta\beta\, \widetilde{A}_1\,; & \widetilde{E}_0 \to \alpha^2\delta^{-1}\beta\, \widetilde{E}_0\,; & \widetilde{I}_{-4/3} \to \delta^{-1}\beta^{-1}\widetilde{I}_{-4/3}\,; \\
\widetilde{B}_{-1} \to \alpha^{-1}\delta^{-1}\beta\, \widetilde{B}_{-1}\,; & \widetilde{F}_0 \to \alpha^2\beta^{-2}\, \widetilde{F}_0\,; & \widetilde{J}_{2/3} \to \delta\beta^{-1}\widetilde{J}_{2/3}\,; \\
\widetilde{C}_{-1} \to \alpha^{-1}\beta^{-2}\, \widetilde{C}_{-1}\,; & \widetilde{G}_{1/3} \to \alpha\, \widetilde{G}_{1/3}\,; & \widetilde{K}_{2/3} \to \beta^2\, \widetilde{K}_{2/3}\,; \\
\widetilde{D}_2 \to \alpha^2\delta\beta\, \widetilde{D}_2\,; & \widetilde{H}_{-2/3} \to \alpha^{-2}\, \widetilde{H}_{-2/3}\,. &
\end{array}
$$

Thus, in case I we require that $\alpha^{-1}\delta\beta = 1$ and $\alpha^{-1}\delta^{-1}\beta = 1$. In case II we require that $\alpha^{-1}\delta\beta = 1$ and $\alpha^{-1}\beta^{-2} = 1$. Hence, we find:

case I:

$$
\mathbf{U}_g = \mathbf{1} \times \begin{bmatrix} \alpha & & \\ & \alpha & \\ & & \alpha^{-2} \end{bmatrix} \times \begin{bmatrix} \alpha & & \\ & \alpha & \\ & & \alpha^{-2} \end{bmatrix}
$$

case II:

$$
\mathbf{U}_g = \mathbf{1} \times \begin{bmatrix} \alpha & & \\ & \alpha & \\ & & \alpha^{-2} \end{bmatrix} \times \begin{bmatrix} \alpha & & \\ & \alpha^{-2} & \\ & & \alpha \end{bmatrix}
$$

In each case the requirement that $\widetilde{A}_1$ and only one of $\widetilde{B}_{-1}$ or $\widetilde{C}_{-1}$ are in the spectrum requires that $\alpha^3 \neq 1$. The only other fields that are in the spectrum are $\widetilde{F}_0$ in case I and $\widetilde{E}_0$ in case II.

From the forms of the matrices for case I and case II we see that $\overline{G}$ is restricted to $\overline{G} = Z_n$, $n \geq 4$. To what group, Σ, does E_6 break down to in each of these cases? The answer is different depending upon whether $n = 4$ or $n \geq 5$. From the forms of the matrices for case I and case II it is clear that Σ

must be at least as large as*
$SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{8L} \times U(1)_{8R}$. We can determine whether this is indeed the full Σ by counting the number of massless E_6 gauge bosons** . To do this we need to know how the **78** of E_6 transforms under $\mathbf{U}_g$. The $SU(3)_c \times SU(2)_L \times U(1)_{8L} \times U(1)_{3R} \times U(1)_{8R}$ decomposition fo the **78** is :

$$\begin{aligned} \mathbf{78} = &(\mathbf{8},\mathbf{1},0,0,0) + (\mathbf{1},\mathbf{3},0,0,0) + 3\,(\mathbf{1},\mathbf{1},0,0,0) + [\,(\mathbf{1},\mathbf{1},0,2,0) \\ &+ (\mathbf{1},\mathbf{2},3,0,0) + (\mathbf{1},\mathbf{1},0,1,3) + (\mathbf{1},\mathbf{1},0,-1,3) + (\mathbf{3},\mathbf{2},-1,-1,-1) \\ &+ (\mathbf{3},\mathbf{2},-1,1,-1) + (\mathbf{3},\mathbf{2},-1,0,2) + (\mathbf{3},\mathbf{1},2,-1,-1) + (\mathbf{3},\mathbf{1},2,1,-1) \\ &+ (\mathbf{3},\mathbf{1},2,0,2) + \text{comp. conj.}] \end{aligned}$$

In case I, under $\mathbf{U}_g$, this transforms into

$$\begin{aligned} &(\mathbf{8},\mathbf{1},0,0,0) + (\mathbf{1},\mathbf{3},0,0,0) + 3\cdot(\mathbf{1},\mathbf{1},0,0,0) + [\,(\mathbf{1},\mathbf{1},0,2,0) \\ &+ \alpha^3(\mathbf{1},\mathbf{2},3,0,0) + \alpha^3(\mathbf{1},\mathbf{1},0,1,3) + \alpha^3(\mathbf{1},\mathbf{1},0,-1,3) + \alpha^{-2}(\mathbf{3},\mathbf{2},-1,-1,-1) \\ &+ \alpha^{-2}(\mathbf{3},\mathbf{2},-1,1,-1) + \alpha(\mathbf{3},\mathbf{2},-1,0,2) + \alpha(\mathbf{3},\mathbf{1},2,-1,-1) + \alpha(\mathbf{3},\mathbf{1},2,1,-1) \\ &+ \alpha^4(\mathbf{3},\mathbf{1},2,0,2) + \text{comp. conj.}] \end{aligned}$$

thus, if $n \geq 5$, there are 16 massless gauge bosons and Σ is thus $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{8L} \times U(1)_{8R}$. A similar result is true for case II for $n \geq 5$ with $SU(2)_{R'}$ replacing $SU(2)_R$. If $n = 4$, then there are 22 massless gauge bosons in both cases I and II. For $n = 4$, Σ is $SU(4)_c \times SU(2)_L \times SU(2)_R \times U(1)_{35}$ in case I ($U(1)_{35}$ appears in $E_6 \supset SU(6) \times SU(2)_R$ where $SU(6) \supset SU(4)_c \times SU(2)_L \times U(1)_{35}$), and

*This is the group in case I. In case II it is $SU(3)_c \times SU(2)_L \times SU(2)_{R'} \times U(1)_{8L} \times U(1)_{8R'}$ where $SU(2)_R$ and $SU(2)_{R'}$ differ only through exchange of a multiplet of weak hypercharge 2/3 with one of hypercharge -4/3 and a multiplet of hypercharge 2 with one of hypercharge 0 in the **27**. They give rise to phenomenologically equivalent left-right symmetric embeddings.

**The method of Dynkin weights [12,14] is more powerful than that presented here and leads to the same conclusions.

$SU(2)_{R'}$ replaces $SU(2)_R$ in case II. If $\alpha^3 = 1$ (so that $\tilde{A}_1$, $\tilde{B}_{-1}$, and $\tilde{C}_{-1}$ are all in the spectrum), then

$$\mathbf{U}_g = \mathbf{1} \times \begin{bmatrix} \alpha & & \\ & \alpha & \\ & & \alpha \end{bmatrix} \times \begin{bmatrix} \alpha & & \\ & \alpha & \\ & & \alpha \end{bmatrix}$$

and the unbroken group Σ is $SU(3)_c \times SU(3)_L \times SU(3)_R$. In the analysis of Dine et al [13] this is an unacceptable choice for Σ since it leads to a value for $\sin^2(\theta_w)$ that is too large.

In case I, the only other member of the incomplete multiplet that is in the spectrum beside $\tilde{A}_1$ and $\tilde{B}_{-1}$ is the neutral singlet $\tilde{F}_0$. The remaining terms that may be in the superpotential which couple $\tilde{A}_1$, $\tilde{B}_{-1}$ and $\tilde{F}_0$ to fermion bilinears are (along with the mass terms that they may give rise to):

$C_{-1}D_2\tilde{B}_{-1}$	$\rightarrow$	$e_L e_R{}^c$ mass term	(1)
$C_{-1}E_0\tilde{A}_1$	$\rightarrow$	$\nu_L E_0$ mass term	(2)
$K_{2/3}H_{-2/3}\tilde{F}_0$	$\rightarrow$	$D_L D_R{}^c$ mass term	(3)
$A_1B_{-1}\tilde{F}_0$	$\rightarrow$	$E_L E_R{}^c$ and $N_L N_R{}^c$ mass terms	(4)
$B_{-1}F_0\tilde{A}_1$	$\rightarrow$	$N_L F_0$ mass term	(5)
$A_1F_0\tilde{B}_{-1}$	$\rightarrow$	$N_R{}^c F_0$ mass term	(6)

Note that term (1) gives a mass to the e and term (4) gives a mass to the E and that there is no mixing at tree level between the e and the E. Similarly, the d and the D do not mix at tree level. (Note that there may be $B_{-1}\tilde{F}_0\tilde{A}_1$ and $A_1\tilde{F}_0\tilde{B}_{-1}$ terms also.)

The Yukawa couplings of some of these terms are related since Σ is larger than $SU(3)_c \times SU(2)_L \times U(1)_{8L} \times U(1)_{3R} \times U(1)_{8R}$. For example, in case I, the couplings of $B_{-1}F_0\tilde{A}_1$ and $A_1F_0\tilde{B}_{-1}$ are equal since both (A_1, B_{-1}) and $(\tilde{A}_1, \tilde{B}_{-1})$ form $SU(2)_R$ doublets. Similarly, the couplings of $C_{-1}D_2\tilde{B}_{-1}$ and $C_{-1}E_0\tilde{A}_1$ are equal. Thus, the presence of a mass for the electron requires a Dirac mass for the neutrino. A discrete symmetry banishing such a neutrino mass does likewise to the electron's mass. A similar result holds in case II.

We note that the Gell-Mann-Ramond-Slansky mechanism [15] for giving the neutrino a small mass will not work in the present context. However, in the more general case in which we allow both the scalar partners of the left-handed doublet neutrino and those of the neutral singlet fermions (the E_0 and F_0) to acquire vacuum values, a variant of this mechanism can possibly work. The neutral fermion mass matrix will then involve some of the gauginos of Σ. In the model of case I, for example, if we allow $\langle E_0\rangle_{scalar}\neq 0$ then the neutral mass matrix is a 6×6 matrix (ignoring family indices) involving ν_L, N_L, $N_R{}^c$, E_0, $\widetilde{F}_0$ and the gaugino (for example, the gaugino corresponding to the unbroken diagonal generator T_{3N} of $SU(2)_N$,where $SU(2)_N$ is located in the lower 2×2 block of $SU(3)_R$ and thus commutes with weak hypercharge). This matrix yields only small and large eigenvalues (compared to M_w, assuming that the singlet vacuum values are large) if $\langle\widetilde{F}_0\rangle \gg \langle E_0\rangle_{scalar} \gg M_w$. In this case E and e mixing now occurs, as does d and D mixing. For a reasonably acceptable maximum value for the neutrino mass (10-100 GeV) we require that $\langle\widetilde{F}_0\rangle/\langle E_0\rangle_{scalar}\approx 10^{10}$ or larger. For $\langle E_0\rangle_{scalar}/M_w\approx 100$, this would require that $\langle\widetilde{F}_0\rangle\approx 10^{14}$ GeV. Such large values were ruled out by Dine et al[13]; however, Barr [16] has recently shown that these values may be permissable. Other models might fare better, since the example we just presented requires a rather unnatural fine tuning to achieve $\langle\widetilde{F}_0\rangle/\langle E_0\rangle_{scalar}\approx 10^{10}$.

Up to this point we have assumed that $\overline{G}$ is abelian and thus that Σ is rank 6 [7]. Let us now assume that $\overline{G}$ may be non-abelian (and thus that Σ

may be rank 5). In this case, in the $SU(3)_c \times SU(3)_L \times SU(3)_R$ basis, the matrix $\mathbf{U}_g$ takes the form

$$\mathbf{U}_g = \mathbf{1} \times \begin{bmatrix} \alpha & & \\ & \alpha & \\ & & \alpha^{-2} \end{bmatrix} \times \begin{bmatrix} \mu & \\ & \mathbf{V}_g \end{bmatrix}$$

with $\mathbf{V}_g$ a 2×2 matrix such that $\det(\mathbf{V}_g) = \mu^{-1}$. This is the most general form for a non-abelian $\overline{G}$ (other values for $\mathbf{U}_g$ would break weak hypercharge). Let us examine the decomposition of the **27** under $E_6 \supset SU(3)_c \times SU(2)_L \times SU(2)_N \times U(1)_{8L} \times U(1)_{8N}$:

$$\begin{aligned}\mathbf{27} = {} & A_{-1}(\mathbf{1},\mathbf{2},\mathbf{2},-1,1) + B_1(\mathbf{1},\mathbf{2},\mathbf{1},-1,-2) + C_0(\mathbf{1},\mathbf{1},\mathbf{2},2,1) \\ & + D_2(\mathbf{1},\mathbf{1},\mathbf{1},2,-2) + E_{1/3}(\mathbf{3},\mathbf{2},\mathbf{1},1,0) + F_{-2/3}(\mathbf{3},\mathbf{1},\mathbf{1},-2,0) \\ & + G_{2/3}(\overline{\mathbf{3}},\mathbf{1},\mathbf{2},0,-1) + H_{-4/3}(\overline{\mathbf{3}},\mathbf{1},\mathbf{1},0,2)\end{aligned}$$

To do our analysis we need to know that $E_{1/3}$ is $[u,d]_L$, $H_{-4/3}$ is $u_R{}^c$, and that $G_{2/3}$ contains $d_R{}^c$ (and $D_R{}^c$). If $\overline{G}$ is non-abelian then $\mathbf{V}_g$ completely breaks $SU(2)_N$ and thus A_{-1} (and also C_0) is broken up into two fields (which are mixed by $\mathbf{V}_g$). The up quark mass matrix comes from $E_{1/3}H_{-4/3}\widetilde{B}_1$; and thus the invariance of $\widetilde{B}_1$ under $\mathbf{U}_g$ requires that $\mu = \alpha$. In order to give the down quark a mass we must require that a linear combination of the components of $\widetilde{A}_{-1}$ (call them Λ_1 and Λ_{-1} corresponding to $T_{3N} = \pm 1$ respectively) be invariant under $\mathbf{U}_g$. We write this linear combination as $\rho\, \Lambda_1 + \eta\, \Lambda_{-1}$. Then, under $\mathbf{U}_g$:

$$(\rho, \eta) \begin{bmatrix} \Lambda_1 \\ \Lambda_{-1} \end{bmatrix} \rightarrow \alpha^{-1}(\rho, \eta)\, \mathbf{V}_g \begin{bmatrix} \Lambda_1 \\ \Lambda_{-1} \end{bmatrix}$$

Thus, the fixed vector (ρ, η) must satisfy the eigenvalue condition $(\rho, \eta)\mathbf{V}_g = \alpha\,(\rho, \eta)$ for all elements of G. If we combine this condition with the conditon that $\mathbf{V}_g$ must be unitary we can completely determine $\mathbf{V}_g$ in terms of α, ρ and η. We find,

$$\mathbf{V}_g = \begin{bmatrix} \alpha + |\zeta|^2 \alpha^{-2} & \zeta\,(\alpha - \alpha^{-2}) \\ \zeta^*(\alpha - \alpha^{-2}) & \alpha|\zeta|^2 + \alpha^{-2} \end{bmatrix} \times [1/(1 + |\zeta|^2)] \ ,$$

where $\zeta = \rho/\eta$. Note that $\mathbf{V}_g$ can be written as $\mathbf{V}_g = \alpha\,\mathbf{A} + \alpha^{-2}\,\mathbf{B}$, where **A** and **B** depend only on ζ. We easily find that, $\mathbf{AB} = \mathbf{BA} = 0$, $\mathbf{A}^2 = (1 + |\zeta|^2)\mathbf{A}$, and $\mathbf{B}^2 = (1 + |\zeta|^2)\mathbf{B}$. (By using this information we easily see that if $\alpha^n=1$ then $\mathbf{V}_g{}^n=1$.) We can now show that all the $\mathbf{V}_g$ commute with each other: If $\mathbf{V}_g = \alpha\,\mathbf{A} + \alpha^{-2}\,\mathbf{B}$ and $\mathbf{V}_{g'} = \alpha'\,\mathbf{A} + \alpha'^{-2}\,\mathbf{B}$, then $\mathbf{V}_g\mathbf{V}_{g'} = \alpha\alpha'\,\mathbf{A}^2 + \alpha^{-2}\,\alpha'^{-2}\,\mathbf{B}^2$, which is symmetric in α and α'. Thus, $\mathbf{V}_g$ and $\mathbf{V}_{g'}$ commute, and hence $\overline{\mathrm{G}}$ is necessarily abelian. The eigenvalues of $\mathbf{V}_g$ are α and α^{-2}; thus, we recover the results that we arrived at previously where we assumed that $\overline{\mathrm{G}}$ was abelian from the outset .

Our restriction to models with $b_{1,1}=1$ has allowed us to obtain conditions on Σ and $\overline{\mathrm{G}}$ without reference to G. In fact, our entire discussion is also valid for the case $b_{1,1}=2$, subject to our other assumptions. The reason for this is that certainly one of the $\overline{27}$ representations satisfies $\Psi(x)=\Psi(gx)$, as this is the property of Calabi-Yau spaces mentioned earlier [7]. However, the remaining $\overline{27}$ must also satisfy this condition since $b_{1,1}$ does not change when we make the transition from K_{sc} to K_{sc}/G (in contrast to the number of

27's), and since it must satisfy $\Psi(x)=\Psi(gx)$ in the special case where $\mathbf{U}_g=1$ for all g.

The requirement that neither $\mathbf{M}_u$ nor $\mathbf{M}_d$ be identically zero at tree level is not necessary if it is possible that the zero mass matrix can be generated as a calculable correction at one loop order or higher. One interpretation of our results might be that (assuming the Higgs to arise via the incomplete multiplet mechanism) this situation is a necessity. We are currently studying whether or not this can happen in the context of $E_8 \times E_8'$ superstring unification. Another interpretation might be that it is necessary that the vacuum values of at least some scalar neutrinos be non-zero. As we have mentioned above, this might be needed to give the light neutrinos acceptable masses in models in which discrete symmeries fail to achieve this aim.

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