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Topologically Massive Chromodynamics In The Perturbative Regime

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ABSTRACT

Topologically massive SU(N) gauge theories are studied by using the loop expansion in Landau gauge. Ward identities for infinitesimal and topologically non-trivial gauge transformations are derived, and checked to one loop order. The renormalized propagators and vertices are shown to be well behaved about zero momentum to arbitrary order in perturbation theory. We also establish that only massive states contribute to the discontinuities of physical amplitudes.

I. INTRODUCTION

There are dynamics possible for gauge theories in an odd number of space-time dimensions which are not open to those in an even number. In three dimensions, a Chern-Simons term can be added to the fundamental action for a gauge field.[1-11] The Chern-Simons term has a coupling which scales like a mass, but unlike the ways in which gauge fields are usually given a mass, no gauge symmetry is broken by its introduction, although parity is.

The Chern-Simons term has topological significance. For a non-abelian gauge group, if the theory is to be invariant under certain large gauge transformations, which are not continuously deformable to the identity, the ratio of the Cherm-Simons mass, m, and the gauge coupling, g^2, must be quantized:[4,5] $4\pi m/g^2$ = an integer.

In this work we study topologically massive SU(N) gauge theories in the loop expansion. Without the Chern-Simons mass, the loop expansion would not get us very far.[1,12] The coupling constant g^2 has dimensions of mass, so for each order in g^2, we obtain a factor of $\sim g^2/\sqrt{p^2}$, where p is a momentum characteristic of whatever process we are considering. Thus perturbation theory cannot be used to compute in the infrared limit, $p\to 0$.

With the Chern-Simons term, however, it seems possible that if we choose $4\pi m/g^2$ to be a very large integer, and if perturbation theory is in fact an expansion in g^2/m, then the infrared behavior should be calculable directly, at least in this region of small g^2/m.

We show that naive expectations are borne out. With the proper choice of gauge, the renormalized propagators and vertices are computable, about zero momentum, as a power series in g^2/m. (We refer to this as their being "infrared finite", and give a precise definition later.) The physical spectrum starts with N^2-1 gluons degenerate in mass, and the only discontinuities of S-matrix elements are from massive states.

These simple conclusions belie a great deal of structure in the theory. While the two and three point functions are infrared finite in Landau gauge, they are not so in any other covariant gauge. Even in Landau gauge, there are individual diagrams which bring in infrared singular terms $\sim g^2/\sqrt{p^2}$. Infrared finiteness happens in Landau gauge because of an infinite number of cancellations, to arbitrarily high order in perturbation theory, between such singular contributions. These cancellations are not proven diagramatically-we do not know how to do this-but as the result of a cancellation theorem. The consequences of this theorem are quite surprising, considering the ease of its proof.

Similarly, to compute on-shell matrix elements, one first calculates off-shell quantities. The discontinuities from (unphysical) massless states only disappear as all legs go on mass shell. This, of course, is typical of gauge theories.[13-17] What is striking here is how the mass shell is approached: the renormalized gluon propagator does have a true massive pole on the real axis, but the factor for wave-function renormalization is imaginary even at the pole.

A priori, it is not obvious that a perturbative analysis should be possible. After all, a customary assumption in perturbation theory is that the (dimensionless) coupling constant can be continuously varied.

This is not possible here, since $g^2/m \sim 1/(\text{integer})$. Nevertheless, we see no pathology in any quantity, in any gauge, which indicates a problem with the loop expansion *per se*. At least as far as topologically massive chromodynamics is concerned, the usual assumptions about perturbation theory appear to be unduly restrictive.

There is one check of consistency that is particularly important. If the renormalized theory is to be invariant under large gauge transformations, a certain Ward identity must hold. This relation is distinct from those implied by invariance under infinitesimal gauge transformations, and so we call it a "topological" Ward identity. This topological Ward identity requires the difference between the renormalized and the bare value of $4\pi m/g^2$ to be an integer. Calculation in Landau gauge for a SU(N) gauge theory shows that this difference is N, to *arbitrary* order in g^2. Consequently, not only does perturbation theory respect the topological Ward identity, but it even knows that the number of colors is an integer.

In Sec. II, we explain what we mean by infrared finiteness, and derive the Ward identities. Two and three point functions are computed to one loop order in Sec. III. Sec. IV presents the cancellation theorem, which leads to a discussion of infrared finiteness to arbitrary loop order in Sec. V. The discontinuities of amplitudes occupy Sec. VI. In appendix A, we discuss some of the physics of an abelian theory with a Chern-Simons mass term, including why it has fractional statistics.[18,19] Appendix B examines the unusual way in which spontaneous symmetry breaking affects the mass spectrum in a gauge theory with a Chern-Simons term. Appendix C contains some computational details necessary to Sec. III.

II. THE QUANTUM THEORY

The lagrangian is a sum of three terms,

$$L = L_0 + L_m + L_{gauge} \quad . \tag{2.1}$$

L_0 is the usual action for a non-abelian gauge field,

$$L_0 = -\frac{1}{2}\,\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) \quad , \tag{2.2}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g\,[A_\mu, A_\nu] \quad . \tag{2.3}$$

L_m is the Chern-Simons term,

$$L_m = -\,im\epsilon^{\mu\nu\lambda}\mathrm{tr}\,(A_\mu\partial_\nu A_\lambda + \frac{2}{3}\,gA_\mu A_\nu A_\lambda) \quad , \tag{2.4}$$

and L_{gauge} includes the gauge-fixing and source terms for covariant gauge,

$$L_{gauge} = -\frac{1}{\xi}\,\mathrm{tr}\,(\partial_\mu A^\mu)^2 + (\partial_\mu\bar{\eta})D^\mu\eta - 2\mathrm{tr}(J_\mu A^\mu) \quad . \tag{2.5}$$

The ghosts of Faddeev and Popov contribute

$$(\partial_\mu\bar{\eta})D^\mu\eta = (\partial_\mu\bar{\eta}^a)(\partial^\mu\eta^a) + gf_{abc}(\partial^\mu\bar{\eta}^a)A^b_\mu\,\eta^c \quad . \tag{2.6}$$

The gauge group is SU(N), with a matrix notation: $A_\mu = A^a_\mu\,\tau^a$, $F_{\mu\nu} = F^a_{\mu\nu}\,\tau^a$. The τ^a are antihermitian matrices in the fundamental representation:

$$\left[\tau^a , \tau^b\right] = f^{abc}\tau^c, \; tr(\tau^a\tau^b) = -\frac{1}{2}\delta^{ab} ;$$

the f^{abc} are the structure constants of SU(N).

The theory is defined in three space-time dimensions, which we take to be Euclidean, of signature (+++). The coupling of the Chern-Simons term is imaginary in Euclidean space-time (the mass m of Eq. (2.4) is real), and real in Minkowski space-time. This is just like the coupling of a θ term in four dimensions.

For an odd number of dimensions, the operation of parity, P, can be defined as a reflection in all axes:

$$x^\mu \underset{P}{\rightarrow} -x^\mu \quad , \; A_\mu \underset{P}{\rightarrow} -A_\mu \quad .$$

The usual gauge field Lagrangian is even under parity, $L_0 \underset{P}{\rightarrow} +L_0$, but the Chern-Simons term is odd, $L_m \underset{P}{\rightarrow} -L_m$. Two reflections give the identity, $P^2=+1$, which is the analogy, in Euclidean space-time, of PT(and CPT) invariance in Minkowski space-time. Under a gauge transformation,

$$A_\mu \rightarrow \Omega^{-1}\left(\frac{1}{g}\,\partial_\mu + A_\mu\right)\Omega \quad . \tag{2.7}$$

L_0 is gauge invariant; L_m is not:

$$\int d^3x\ L_m \rightarrow \int d^3x\ L_m + \frac{im}{g} \int d^3x\ \varepsilon^{\mu\nu\lambda}\ \partial_\mu tr((\partial_\nu \Omega)\Omega^{-1} A_\lambda) \qquad (2.8)$$

$$+ 8\pi^2 \frac{m}{g^2} iw \quad ,$$

where

$$w = \frac{1}{24\pi^2} \int d^3x\ \varepsilon^{\mu\nu\lambda} tr\ \left[\Omega^{-1}(\partial_\mu \Omega)\Omega^{-1}(\partial_\nu \Omega)\Omega^{-1}(\partial_\lambda \Omega)\right]. \qquad (2.9)$$

The set of gauge transformations is divided into global gauge rotations, $\partial_\mu \Omega=0$, and all others, for which we assume that $\Omega(x) \rightarrow 1$ as $x^\mu \rightarrow \infty$. Integrating over global gauge rotations requires the system to have a total color charge equal to zero. In this case, $A_\mu(x)$ falls off faster than $1/|x|$ as $x^\mu \rightarrow \infty$, and the second term on the right hand side of Eq. (2.8), which is a surface integral, vanishes.

The last term in Eq. (2.8) does not vanish in general. The w of Eq. (2.9) is a winding number, which labels the homotopy class of $\Omega(x)$. For continuous $\Omega(x)$, topology tells us that w is an integer. Deser, Jackiw, and Templeton[4,5] observed that even if the Lagrangian is not gauge invariant, the partition function, $\sim\int dA_\mu \exp(-\int d^3x\ L)$, can be, provided that m/g^2 is quantized:

$$4\pi \frac{m}{g^2} = q, \qquad (2.10)$$

$q=0,1,2...$ By convention, m, and so q, are taken to be positive. In the perturbative regime, we assume $q \gg 1$.

It does not matter if L_0 is replaced by bL_0 in Eq. (2.1), since by rescaling A_μ, g, and m, b can always be set to 1, without affecting the quantization condition of Eq. (2.10). The only exception to this is the degenerate case, when b=0. This limit will be of help in Sec. IV in establishing a cancellation theorem about the complete theory.

Quantizing the theory is straightforward. The exact gluon and ghost propagators are, in momentum space,

$$\Delta^{ab}_{\mu\nu}(p) = \delta^{ab}\,\Delta_{\mu\nu}(p) \quad ,$$
$$\tilde{\Delta}^{ab}(p^2) = \delta^{ab}\,\tilde{\Delta}(p^2) \quad . \tag{2.11}$$

From Eqs. (2.2) and (2.5), the bare propagators are

$$\Delta^{\text{bare}}_{\mu\nu}(p) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - m\,\epsilon_{\mu\nu\lambda}\frac{p^\lambda}{p^2} \right) \frac{1}{p^2+m^2} + \xi\,\frac{p_\mu p_\nu}{(p^2)^2} \quad , \tag{2.12}$$

$$\tilde{\Delta}^{\text{bare}}(p^2) = \frac{1}{p^2} \quad . \tag{2.13}$$

Self-energy terms combine with the bare propagators to give the exact ones:

$$\Delta_{\mu\nu}(p)^{-1} = \left(\Delta^{bare}_{\mu\nu}(p)\right)^{-1} + \Pi_{\mu\nu}(p) \quad , \tag{2.14}$$

$$\tilde{\Delta}(p^2) = \frac{1}{\tilde{Z}(p^2)p^2} \quad , \tag{2.15}$$

$$\tilde{Z}(p^2) = 1 + \tilde{\Pi}(p^2) \ . \tag{2.16}$$

The analysis of invariance under infinitesimal gauge transformations proceeds in much the same way as for the massless theory.[16] The longitudinal part of the gluon propagator is not renormalized,

$$p^\mu p^\nu \Delta_{\mu\nu}(p) = \xi \quad ,$$

which means that the gluon self-energy, $\Pi_{\mu\nu}(p)$, is transverse in p:

$$\Pi_{\mu\nu}(p) = (\delta_{\mu\nu}p^2 - p_\mu p_\nu)\Pi_e(p^2) + m\ \varepsilon_{\mu\nu\lambda}p^\lambda\ \Pi_o(p^2) \quad . \tag{2.17}$$

The exact gluon propagator is then

$$\Delta_{\mu\nu}(p) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - m_{ren}(p^2)\varepsilon_{\mu\nu\lambda}\frac{p^\lambda}{p^2}\right) \cdot \frac{1}{Z(p^2)(p^2+m^2_{ren}(p^2))} + \xi\frac{p_\mu p_\nu}{(p^2)^2} \ , \tag{2.18}$$

where

$$Z(p^2) = 1 + \Pi_e(p^2) \quad , \tag{2.19a}$$

$$Z_m(p^2) = 1 + \Pi_o(p^2) \quad , \tag{2.19b}$$

and

$$m_{ren}(p^2) = \frac{Z_m(p^2)}{Z(p^2)}\, m \quad . \tag{2.19c}$$

$m_{ren}(p^2)$ is the renormalized, momentum-dependent "mass."

Power counting shows that only the gluon self-energy might be ultraviolet divergent. By the form of Eq. (2.17), $\Pi_e(p^2)$ and $\Pi_o(p^2)$ are free of ultraviolet infinities. Any regulator can be used to compute $\Pi_{\mu\nu}(p)$, as long as it respects the symmetries of gauge invariance and parity.

What happens in the infrared is much less obvious. For perturbation theory to make sense, it is clearly necessary for the renormalized propagators and vertices, about zero momentum, to be essentially the same as the bare ones, up to small corrections $\sim g^2/m$. To be precise, we call the propagators "infrared finite" if

$$Z \equiv Z(0), \; Z_m \equiv Z_m(0), \text{ and } \; \tilde{Z} \equiv \tilde{Z}(0), \tag{2.20}$$

are all well-defined, and computable as a power series in g^2/m:

$$Z = 1 + \sum_{n=1}^{\infty} a_n \left(\frac{g^2}{m}\right)^n \quad , \tag{2.21}$$

etc. We shall show that the Z's, which are gauge dependent, are infrared finite only in Landau gauge, $\xi=0$. Notice that the Z(p)'s, as defined, are dimensionless functions, so requiring each Z(0) to obey Eq. (2.21) is a much stronger condition than merely requiring that $\Pi_{\mu\nu}(p)$ and $p^2\tilde{\Pi}(p^2)$ be finite as $p\to 0$.

The exact three-point functions are

$$\Gamma^{abc}_{\mu\nu\lambda}(p,q,r) = igf^{abc}\Gamma_{\mu\nu\lambda}(p,q,r) \quad ,$$
$$\tilde{\Gamma}^{abc}_{\lambda}(p,q;r) = -igf^{abc}\tilde{\Gamma}_{\lambda}(p,q;r) \quad , \tag{2.22}$$

$p+q+r=0$. $\Gamma^{abc}_{\mu\nu\lambda}$ is the proper vertex for the coupling of three gluons, one with momentum p, Lorentz index μ, color index a, and so on. $\tilde{\Gamma}^{abc}_{\lambda}$ is the ghost-ghost-gluon proper vertex, for a gluon with momentum r, Lorentz index λ, and color c, etc. We define [14-17]

$$\tilde{\Gamma}_{\lambda}(p,q;r)=p^{\sigma}\tilde{\Gamma}_{\sigma\lambda}(p,q;r) \quad . \tag{2.23}$$

At tree level,

$$\Gamma^{bare}_{\mu\nu\lambda}(p,q,r) = m\varepsilon_{\mu\nu\lambda} + \delta_{\mu\nu}(p-q)_{\lambda} + \delta_{\nu\lambda}(q-r)_{\mu} + \delta_{\lambda\mu}(r-p)_{\nu} \quad , \tag{2.24}$$

$$\tilde{\Gamma}^{bare}_{\sigma\lambda}(p,q;r) = \delta_{\sigma\lambda} \quad . \tag{2.25}$$

The Ward identity which relates the three-point vertices is [14-17]

$$p^\mu q^\nu \Delta^{tr}_{\sigma\lambda}(r)\Gamma_{\mu\nu\lambda}(p,q,r) = \frac{1}{\tilde{Z}(p^2)}\left(\delta_{\sigma\lambda} - \frac{r^\sigma r^\lambda}{r^2}\right)q^\mu \tilde{\Gamma}_{\lambda\mu}(r,p;q); \qquad (2.26)$$

$\Delta^{tr}_{\sigma\lambda}(r)$ is the transverse part of the gluon propagator.

We shall establish that these vertices are infrared finite in Landau gauge, so for $\xi=0$, we can expand them about zero momentum:

$$\Gamma_{\mu\nu\lambda}(p,q,r) = Z_g(\bar{m}\ \varepsilon_{\mu\nu\lambda} + \delta_{\mu\nu}(p-q)_\lambda + \delta_{\nu\lambda}(q-r)_\mu + \delta_{\lambda\mu}(r-p)_\nu) + \dots , \qquad (2.27)$$

$$\tilde{\Gamma}_{\mu\nu}(p,q;r) = \tilde{Z}_g \delta_{\sigma\lambda} + \dots \qquad (2.28)$$

as p,q, and $r\to 0$. The Ward identity of Eq. (2.26) gives

$$\frac{Z_g}{Z} = \frac{\tilde{Z}_g}{\tilde{Z}} , \qquad (2.29)$$

$$\bar{m} = m_{ren}(0) = \frac{Z_m}{Z}\, m \quad . \qquad (2.30)$$

Eq. (2.29) is the same as in the massless theory; Eq. (2.30) is new, but hardly surprising.

Invariance under infinitesimal gauge transformations can be used to derive relations between higher point Green's functions. For example, the gluon four-point function satisfies the same relation as when $m=0$. The P-odd part of the gluon self-energy, $\Pi_o(p^2)=Z_m(p^2)-1$, is in no way constrained by the infinitesimal Ward identities.

To derive a topological Ward identity, we rescale the fields and couplings so that we obtain a renormalized Lagrangian, L^{ren}, which generates the exact Green's functions, at least about zero momentum. With

$$A_\mu \to \sqrt{Z}\, A_\mu^{ren} \ , \quad \eta \to \sqrt{\tilde{Z}}\, \eta^{ren} \ ,$$

$$g \to \frac{Z_g}{Z^{3/2}}\, g \quad , \quad m \to \frac{Z_m}{Z}\, m \quad , \quad \xi \to Z\, \xi \quad , \quad J_\mu \to Z^{-1/2} J_\mu \quad , \tag{2.31}$$

then

$$L_m^{ren} = -i\ Z_m m\ \varepsilon^{\mu\nu\lambda} \mathrm{tr}\left(A_\mu^{ren} \partial_\nu A_\lambda^{ren} + \frac{2}{3}\frac{Z_g}{Z}\ g\ A_\mu^{ren} A_\nu^{ren} A_\lambda^{ren}\right) \ , \tag{2.32}$$

and similarly for L_0^{ren} and L_{gauge}^{ren}. Under a gauge transformation,

$$A_\mu^{ren} \to \Omega^{-1}\left(\frac{Z}{Z_g}\frac{1}{g}\,\partial_\mu + A_\mu^{ren}\right)\Omega \ . \tag{2.33}$$

The renormalized partition function is invariant under large gauge transformations if

$$4\pi \left(\frac{m}{g^2}\right)_{ren} \equiv 4\pi Z_m \left(\frac{Z}{Z_g}\right)^2 \frac{m}{g^2} = q_{ren} \ . \tag{2.34}$$

q_{ren} is a (positive) integer, but there is no reason why it should be the same integer as the "bare" q of Eq. (2.10). We see that it is only through the topological Ward identity, Eq. (2.34), that Z_m is related to the other renormalization constants of the theory.

Topological Ward identities will often arise in a theory with a multi-valued action. For example, following Schwinger,[20] consider a theory of charged particles interacting with (Dirac) magnetic monopoles, where the monopoles are viewed as fundamental particles, and not as composite entities.[21] Then twice the product of the electric and magnetic charges must be integral, for both the renormalized, $2(eg)_{ren}$, and bare, $2eg$, quantities.[20] This is precisely analogous to the statement that q_{ren} and q need be integers. Similarly, as $2(eg)_{ren} = Z_3$ $2eg$, the photon's wave-function renormalization constant, Z_3, must be a rational number;[20] in the present instance, $Z_m(Z/Z_g)^2$ is a rational number. What is striking about topologically massive chromodynamics is that we can calculate q_{ren} and the Z's directly (see. III). As of yet, it is not possible to compute Z_3 in a field theory of monopoles.

III. ONE LOOP ORDER

Before plunging into calculation, it helps to isolate which diagrams might be infrared singular. We shall discover several cancellations, the reason for which will only become clear in Sec. IV.

There are several ways in which infrared singular terms could arise. One is any diagram involving ghosts, since neither the ghost propagator, nor the ghost-ghost-gluon vertex, are changed by the introduction of the Chern-Simons mass term. For the gluon propagator, Eq. (2.12), problems will arise from the P-odd part, and the piece proportional to the gauge-fixing parameter, ξ. The latter should be worse, since for $p\to 0$, it is $\xi p_\mu p_\nu/(p^2)^2$, versus $-\epsilon_{\mu\nu\lambda}p^\lambda/(mp^2)$ for the P-odd part. In the three gluon vertex, Eq. (2.24), the P-odd piece is more dangerous than the P-even, for a factor of momentum in the numerator of a loop integral will tend to soften the infrared behavior.

The ghost self-energy is simplest. It is independent of ξ,[1,12] receiving no contribution from the P-odd part of the virtual gluon propagator, and so is infrared finite:

$$\tilde{\Pi}(p^2) \sim \frac{g^2}{p^2} \int \frac{k^2p^2-(k\cdot p)^2}{k^2(k+p)^2\ (k^2+m^2)}\, d^3k \qquad (3.1)$$

$$\underset{p\to 0}{\sim} \frac{g^2}{m} \ .$$

Except for the tadpole diagram, the contributions to the gluon self-energy are those of Fig. (1).

The P-odd part of the gluon self-energy, $\Pi_o(p^2)$, is due to Fig. (1.a). For instance, consider the piece of Fig. (1.a) which has a P-odd part at one vertex, with the other vertex and propagators P-even. In Landau gauge,

$$\Pi_{\mu\nu} \sim g^2 m \, \varepsilon_{\mu\nu\lambda} \int \frac{k^\lambda \, d^3k}{(k^2+m^2)((k+p)^2+m^2)} \tag{3.2a}$$

$$\underset{p\to 0}{\sim} m \, \varepsilon_{\mu\nu\lambda} p^\lambda \left(\frac{g^2}{m}\right) \tag{3.2b}$$

When $\xi \neq 0$,

$$\Pi_{\mu\nu} \sim g^2 m \, \varepsilon_{\mu\nu\lambda} p^\lambda \xi^2 \int \frac{d^3k}{k^2 (k+p)^2} \tag{3.2c}$$

$$\sim - m \, \varepsilon_{\mu\nu\lambda} p^\lambda \left(\frac{g^2 N \xi^2}{32 \sqrt{p^2}} \right) . \tag{3.2d}$$

The actual contributions are more complicated than as written in Eqs. (3.2a) and (3.2c), but the differences do not change Eqs. (3.2b) and (3.2d).

Using this type of analysis, it can be shown that $\Pi_o(p^2)$ is infrared finite in Landau gauge, $\Pi_o(0) \sim g^2/m$. Because of the contribution of Eq. (3.2d), $\Pi_o(p^2) = -g^2 N \, \xi^2/(32\sqrt{p^2}) + \ldots$, $\Pi_o(p^2)$ is infrared singular for $\xi \neq 0$. (There is a term linear in ξ in $\Pi_o(p^2)$, but that is infrared finite.)

For the P-even part of the gluon self-energy, $\Pi_e(p^2)$, it is clear that the virtual ghost loop, Fig. (1.b), is a problem:

$$\Pi_{\mu\nu}(p) \underset{\text{Fig.(1.b)}}{\sim} + g^2 N \int \frac{k^\mu (k+p)^\nu}{k^2 (k+p)^2} \frac{d^3k}{(2\pi)^3} \quad . \tag{3.3}$$

This appears to contribute an infrared singular term $\sim g^2/\sqrt{p^2}$ to $\Pi_e(p^2)$.

For an arbitrary diagram, we shall refer to that part of it which comes from taking the P-odd piece of each gluon vertex and propagator as the "purely ε-part" of the diagram; the ghosts and their vertices are left unchanged. The purely ε-part of Fig. (1.a) is

$$\Pi_{\mu\nu}(p) \underset{\substack{\text{Fig.(1.a),}\\ \text{purely } \varepsilon\text{-part}}}{\sim} - g^2 N \int \frac{k^\mu (k+p)^\nu}{k^2 (k+p)^2} \frac{m^4}{(k^2+m^2)\ ((k+p)^2+m^2)} \frac{d^3k}{(2\pi)^3} \ . \tag{3.4}$$

Obviously, Eq. (3.3) cancels against Eq. (3.4) as $p \to 0$!

This cancellation is enough to guarantee that $\Pi_e(p^2)$ is infrared finite to one loop order in Landau gauge. This is not true if $\xi \neq 0$. If for each gluon propagator in Fig. (1.a) only the piece $\sim \xi$ is included, the denominator of the loop integral will depend only on k^2 and $(k+p)^2$, and so contribute an infrared singular term to $\Pi_e(p^2)$:

$$\Pi_e(p^2) = - \frac{g^2 N \xi^2}{64\sqrt{p^2}} \left(1 - \frac{m^2}{p^2}\right) + \ . \ . \ . \tag{3.5}$$

There may also be infrared singular terms $\sim \xi$ in $\Pi_e(p^2)$; we did not evaluate them.

The cancellation about zero momentum between Eqs. (3.3) and (3.4) is implicit in the calculations of Deser, Jackiw, and Templeton,[5] although they did not discuss it as such. It turns out to be the key in understanding why the theory is infrared finite in Landau gauge.

Similar cancellations happen for the vertices. About zero momentum in Landau gauge, the purely ε-part of Fig. (2a) cancels against Fig. (2b) (there is no purely ε-part to Fig. (2c), as the four gluon vertex is P-even), as do the purely ε-parts of Figs. (3a) and (3b). This implies that Z_g and $\tilde{Z}_g$ are infrared finite to one loop order.

If ghosts are a problem, why not go to a gauge where they can be ignored? In an axial gauge, $n_\mu A^\mu=0$, ghosts decouple, and there is no P-odd part to the three gluon vertex. The gluon propagator is

$$\Delta^{\rm bare}_{\mu\nu}(p) = \left(\delta_{\mu\nu} - \frac{(n_\mu p_\nu + p_\mu n_\nu)}{n\cdot p} + \frac{p_\mu p_\nu}{(n\cdot p)^2} \right) \frac{1}{p^2+m^2}$$
$$- \left(\varepsilon_{\mu\nu\lambda} p^\lambda + \frac{(p_\mu \varepsilon_{\nu\lambda\kappa} n^\lambda p^\kappa - p_\nu \varepsilon_{\mu\lambda\kappa} n^\lambda p^\kappa)}{n\cdot p} \right) \frac{m}{p^2(p^2+m^2)} \quad . \tag{3.6}$$

Unlike covariant gauges, because of the terms $\sim 1/n\cdot p$ and $\sim 1/(n\cdot p)^2$ in Eq. (3.6), in axial gauge the infrared divergences of individual diagrams must be regulated in some fashion. Given the difficulty of calculating with the propagator of Eq. (3.6), we did not pursue this. After all, as gauge variant quantities, there is no reason why the renormalized propagators and vertices should be infrared finite in a given gauge: what is remarkable is that they are so in Landau gauge. Arguments in axial gauge will help in establishing the cancellation theorem of the next section.

The computation itself requires some diligence; see Appendix C. All results are in Landau gauge:

$$\tilde{\Pi}(p^2) = -\frac{g^2N}{16\pi m}\left\{1-\frac{m^2}{p^2}+\frac{\sqrt{p^2}}{2m}\left[-\pi+\left(\frac{p^2+m^2}{p^2}\right)^2\cdot\right.\right.$$
$$\left.\left.\left(\frac{\pi}{2}+\sin^{-1}\left(\frac{p^2-m^2}{p^2+m^2}\right)\right)\right]\right\} , \tag{3.7}$$

$$\Pi_o(p^2) = \frac{g^2N}{16\pi m}\left\{2+\frac{m^2}{p^2}+\frac{\sqrt{p^2}}{4m^3(p^2)^2}\left[\pi m^2\left((p^2)^2+p^2m^2-m^4\right)\right.\right.$$
$$+\,2(3p^2-m^2)(p^2+m^2)^2\sin^{-1}\left(\frac{p^2-m^2}{p^2+m^2}\right) \tag{3.8}$$
$$\left.\left.-\,6p^2(p^2-2m^2)(p^2+4m^2)\sin^{-1}\left(\sqrt{\frac{p^2}{p^2+4m^2}}\right)\right]\right\} ,$$

$$\Pi_e(p^2) = \frac{g^2N}{32\pi m}\left\{-5-11\frac{m^2}{p^2}+\frac{\sqrt{p^2}}{2m^3(p^2)^2}\left[\pi m^2(2(p^2)^2+\frac{13}{2}p^2m^2\right.\right.$$
$$+\frac{7}{2}m^4)-(p^2-7m^2)(p^2+m^2)^2\sin^{-1}\left(\frac{p^2-m^2}{p^2+m^2}\right)+\left((p^2)^2\right. \tag{3.9}$$
$$\left.\left.\left.-\,13p^2m^2+4m^4\right)\left(p^2+4m^2\right)\sin^{-1}\left(\sqrt{\frac{p^2}{p^2+4m^2}}\right)\right]\right\} .$$

These self-energies were also calculated by Deser, Jackiw, and Templeton.[5] $\tilde{\Pi}(p^2)$ and $\Pi_o(p^2)$ agree with their results; $\Pi_e(p^2)$ does not (Appendix C). From Eqs. (C.7)-(C.9),

$$\tilde{Z} = 1 - \frac{1}{6\pi} \frac{g^2 N}{m} \quad , \tag{3.10}$$

$$Z_m = 1 + \frac{7}{12\pi} \frac{g^2 N}{m} \quad , \tag{3.11}$$

$$Z = 1 - \frac{5}{24\pi} \frac{g^2 N}{m} \quad . \tag{3.12}$$

Further,

$$\tilde{Z}_g = 1 \quad . \tag{3.13}$$

Because the gluon propagator is transverse in Landau gauge, whether or not $m \neq 0$, arguments familiar from four dimensions[15] can be used to conclude that $\tilde{Z}_g = 1$ to any order in perturbation theory. This is true only in the limit of zero momentum -- while $\tilde{Z}_g = \tilde{Z}_g(0) = 1$, $\tilde{Z}_g(p^2) \neq 1$ for $p^2 \neq 0$.

By the Ward identity of Eq. (2.29),

$$Z_g = 1 - \frac{1}{24\pi} \frac{g^2 N}{m} \quad . \tag{3.14}$$

We also verified, directly from the diagrams, that the Ward identity of Eq. (2.30) holds.

The properties of the self-energies for Minkowski values of p^2 (real $p^2 < 0$) will be discussed in Sec. VI. At present we consider only their values at zero momentum.

The renormalized mass is

$$m_{ren}(0) = \frac{Z_m}{Z} m = m \left(1 + \frac{19}{24\pi} \frac{g^2 N}{m} \right) \quad . \tag{3.15}$$

The renormalized mass at zero momentum, $m_{ren}(0)$, enters into the proof of infrared finiteness in Sec. V, but otherwise it is not of especial interest. The gauge-invariant quantity of physical significance is the position of the pole in the renormalized propagator: this is given by $m_{ren}(-m^2)$, Eq. (6.11).

The topological Ward identity of Eq. (2.34) is satisfied:

$$q_{ren} = 4\pi \left(\frac{m}{g^2}\right)_{ren} = q + N \quad . \tag{3.16}$$

Even though perturbation theory includes only small fluctuations about the vacuum, it still respects the invariance under large gauge transformations. The sign of Eq. (3.16) is also interesting-since $q_{ren}>q$, the renormalized value of the dimensionless coupling constant, $\sim(g^2/m)_{ren}$, is <u>less</u> than the bare value.

It is not difficult to argue that, at least in the perturbative regime, $q>>1$, the result of Eq. (3.16) is <u>exact</u>, and valid to <u>any</u> order in perturbation theory. Suppose that the Z's had been calculated to two loop order:

$$Z_m = 1 + \frac{7}{12\pi}\frac{g^2N}{m} + a\left(\frac{g^2N}{\pi m}\right)^2 , \tag{3.17}$$

$$\frac{Z}{Z_g} = \frac{\tilde{Z}}{\tilde{Z}_g} = 1 - \frac{1}{6\pi}\frac{g^2N}{m} + b\left(\frac{g^2N}{\pi m}\right)^2 \quad . \tag{3.18}$$

Plugging into Eq. (2.34), we find

$$q_{ren} = q + N + \frac{16N^2}{q} (a+2b-1/6) \quad . \tag{3.19}$$

q_{ren} can be an integer for arbitrary N, and arbitrary $q>>1$, only if

$$a+2b = 1/6 \quad . \tag{3.20}$$

Beyond–but not at–one loop order, the topological Ward identity acts like a "typical" Ward identity. That is, if we know $Z/Z_g = \tilde{Z}/\tilde{Z}_g$ to $\sim O\{(g^2/m)^n\}$, and Z_m to $\sim O\{(g^2/m)^{n-1}\}$, $n>1$, then the topological Ward identity tells us what the coefficient of Z_m is to $\sim O\{(g^2/m)^n\}$.

The topological Ward identity has a smooth limit at large N. As $N\to\infty$, m and g^2N should be $\sim O(1)$, so take $q=rN$, $q_{ren} = r_{ren}N$. r and r_{ren} are integers, which are large in the perturbative regime, but they are fixed numbers at infinite N. Eq. (3.16) gives $r_{ren} = r+1$.

IV. A CANCELLATION THEOREM

We have seen that in the infrared, the worst problems are due to the purely ε-parts of diagrams. So let us construct a theory in which these are the only diagrams:

$$L_\varepsilon = L' + L'_{gauge} \quad , \tag{4.1}$$

$$L' = - i\, \varepsilon^{\mu\nu\lambda} \mathrm{tr}\left(A'_\mu \partial_\nu A'_\lambda + \frac{2}{3} g' A'_\mu A'_\nu A'_\lambda\right), \tag{4.2}$$

$$L'_{gauge} = - \frac{1}{m\xi} \mathrm{tr}\left(\partial_\mu A'^\mu\right)^2 + \left(\partial^\mu \bar{\eta}\right) D'_\mu \eta - 2\, \mathrm{tr}\left(J'_\mu A'^\mu\right) . \tag{4.3}$$

This is just the original theory, with $L_0=0$. A_μ has been rescaled, $A_\mu = A'_\mu/\sqrt{m}$, and $g' = g/\sqrt{m}$ is a dimensionless coupling constant.

We call L_ε the "ε-theory." Similar models have been considered by Hagen.[9,10] g' is still quantized, $4\pi/(g')^2 = q$.

The gluon propagator is

$$\Delta'^{bare}_{\mu\nu} = - \varepsilon_{\mu\nu\lambda} \frac{p^\lambda}{p^2} + \xi m \frac{p_\mu p_\nu}{(p^2)^2} \quad . \tag{4.4}$$

As $p^2 \to 0$, this propagator has the same singularities as that of Eq. (2.12).

The ultraviolet behavior, however, is very different from that of the original theory. Instead of being super-renormalizable, the ε-theory is only renormalizable, and at least in principle, there can be logarithmic ultraviolet divergences in perturbation theory.

Suppose that from the generating functional, G(J'),

$$\exp\left[-G(J')\right] = \int DA'_\mu(x)\ \exp\left[-\int d^3x\ L_\varepsilon\right] \quad , \tag{4.5}$$

the effective action has been constructed by Legendre transformation:

$$A'_\mu(x) = \frac{\delta}{\delta J'_\mu(x)}\, G(J') \quad , \tag{4.6}$$

$$S_{eff}(A'_\mu) = G(J') - \int d^3x\ J'_\mu A'^\mu \ . \tag{4.7}$$

In general, $S_{eff}\,(A'_\mu)$ will be a very complicated functional of $A'_\mu(x)$. It can only depend on gauge-invariant operators such as $tr(F'_{\mu\nu})^2$, but its dependence on them can be non-local, to arbitrarily high order.

An exception is Landau gauge, where $S_{eff}(A'_\mu)$ is very simple. L' is always odd under parity, $L' \underset{P}{\to} -L'$. As $\xi\to 0$, the term $\sim tr(\partial_\mu A'^\mu)^2$ in L'_{gauge} can be taken to vanish. When this happens, $\bar\eta, \eta$, and J'_μ can be defined to transform under parity in such a way that L'_{gauge} is also P-odd, $L'_{gauge} \underset{P}{\to} -L'_{gauge}$, $\xi=0$.

Hence in Landau gauge, we have the unusual circumstance of a theory in which all fields, and so their bare propagators, as well as all bare vertices, are odd under parity. The effective action is constructed in the usual fashion by tying together these bare propagators and vertices. But if each and every propagator and vertex is P-odd, then order by order in perturbation theory, there is simply no way that any P-even term can enter into $S_{eff}(A'_\mu)$: being odd under parity is a symmetry of the ε-theory, respected both by the bare and effective actions.

This conclusion is only possible because <u>all</u> of the bare propagators and vertices are P-odd. If some fields in the theory were P-even, the bare action would have P-even parts to it, so that although it might be P-odd overall, this would be violated by loop effects, through P-even terms in the effective action. Such examples can easily be constructed with (interacting) scalar fields, for with scalars, it is inevitable that if some fields are P-odd, others will be P-even. Of course, if the bare action has both P-even and P-odd terms to begin with, so will the effective action. As we shall see below, this happens in the ε-theory outside of Landau gauge, $\xi \neq 0$.

Indeed, it is so extraordinary to have an (interacting) field theory in which all propagators and vertices are P-odd, that the only other examples we are aware of are essentially direct generalizations of the ε-theory in Landau gauge. These are gauge theories in an odd number of space-time dimensions, with no matter fields, for which the action is entirely a Chern-Simons term; generalizations that involve supersymmetry[3] and gravity[4,5] are also possible. As for the ε-theory, we expect that in a gauge which does not introduce P-even terms, such as Landau, that the effective action of these theories is P-odd, like the bare one. Eq. (4.8) also generalizes to these theories in an obvious manner.

Returning to the ε-theory, how can we construct an effective action which is P-odd? To be odd under parity, $S_{eff}(A'_\mu)$ must involve an odd number of A'_μ's and ∂_μ's; to be Euclidean invariant, these need to be contracted with objects like with the anti-symmetric tensor, $\varepsilon^{\mu\nu\lambda}$. Thus we might expect $S_{eff}(A'_\mu)$ to be constructed from $\int d^3x\, L'$ times P-even functions such as $tr(F'_{\mu\nu})^2$. Remember, however, that L' transforms non-trivially under gauge transformations, Eq. (2.8), so any term such as $\int d^3x\, L'\, \mathrm{tr}(F'_{\mu\nu})^2$, etc., cannot result in a gauge-invariant partition function. The only way that $S_{eff}(A'_\mu)$ can be P-odd, and $\exp\left\{-S_{eff}(A'_\mu)\right\}$ gauge invariant, is if $S_{eff}(A'_\mu)$ has exactly the same form as the original action:

$$S_{eff}(A'_\mu) \underset{\xi=0}{=} \int d^3x\, -iZ'\varepsilon^{\mu\nu\lambda}\mathrm{tr}(A'_\mu\partial_\nu A'_\lambda + \frac{2}{3}\frac{Z_{g'}}{Z'}\, g'A'_\mu A'_\nu A'_\lambda) \quad , \tag{4.8}$$

where Z' and $Z_{g'}$ are wave-function and vertex renormalization constants. The Ward identity of Eq. (2.26) holds, and implies

$$\frac{Z_{g'}}{Z'} = \frac{\tilde{Z}_{g'}}{\tilde{Z}'} \quad , \tag{4.9}$$

with $\tilde{Z}'$ and $\tilde{Z}_{g'}$ the renormalization constants for the ghost and its vertex. In the ε-theory, there is no constant analogous to Z_m.

To determine $S_{eff}(A'_\mu)$ in Landau gauge, we need only to find Z' and $Z_{g'}$. To do so, we consider the ε-theory in axial gauge, $n_\mu A'^\mu=0$. It is apparent that in axial gauge, the ε-theory is a free field theory! (It is less obvious why $m/(g')^2$ is still quantized, but it is.[6]) This means that in axial gauge, all renormalization group functions must vanish. This includes the β-function for g', $\beta(g')$, the anomalous dimension of A'_μ, $\gamma_{A'_\mu}(g')$, and the anomalous dimensions of composite operators, such as that for $\mathrm{tr}(F'_{\mu\nu})^2$, $\gamma_{\mathrm{tr}(F'_{\mu\nu})^2}(g')$:

$$\beta(g') = \gamma_{A'_\mu}(g') = \gamma_{\mathrm{tr}(F'_{\mu\nu})^2}(g') = \ldots = 0 \tag{4.10}$$

for all g'.

The renormalization group functions of a gauge theory are gauge invariant only at a fixed point. Since every value of g' is a fixed point, $\beta(g') = 0$, Eq. (4.10) is valid in any gauge. For $\beta(g') = \gamma_{A'_\mu}(g') = 0$ to be true, $Z_{g'}$ and Z' must be finite functions of g'.

Let us return to Landau gauge. To all orders in g', $\tilde{Z}_{g'}=1$.[15] By the Ward identity of Eq. (4.9), this implies that $\tilde{Z}'$ is also a finite function of g'. Explicit calculation to two loop order shows that

$$Z' = \tilde{Z}' = 1 \; ; \tag{4.11}$$

to $\sim O\left[(g')^4\right]$, there are not even finite terms $\neq 1$ in the Z' 's.

Our cancellation theorem is the statement that, in Landau gauge, the effective action of the ε-theory is given by Eq. (4.8), and that the Z' 's are finite. (We suspect that Eq. (4.11) holds to all orders in g', but have not proven this. To establish infrared finiteness in Sec. V, the Z' 's of the ε-theory do not have to be = 1, but merely finite.) All of the cancellations between the purely ε-parts of diagrams in Sec. III are examples of this theorem.

For an arbitrary n-point function between gluons in the original theory, the theorem guarantees that when $\xi=0$, to <u>any</u> order in g^2, the leading infrared divergences from the purely ε-parts of diagrams must cancel against each other.

What happens in the ε-theory for covariant gauges other than Landau? The term $\sim \mathrm{tr}(\partial_\mu A'^\mu)^2$ in L'_{gauge} does not vanish if $\xi \neq 0$, so the bare action has both P-even and P-odd terms. Consequently, the effective action will include P-even terms such as $\mathrm{tr}(F'_{\mu\nu})^2$ and the like. Eq. (3.5) is the simplest example of such a contribution.

By Eq. (4.10), any gluon renormalization constant is finite. $\tilde{Z}'$ and $\tilde{Z}_{g'}$ might be infinite for $\xi \neq 0$, as long as $\tilde{Z}'/\tilde{Z}_{g'}$ is finite. The detailed form of $S_{eff}(A'_\mu)$ embodies one simple property. To recover Eq. (4.8) in Landau gauge, any gluon n-point function need vanish as a power of ξ, when $\xi \to 0$, if $n \geq 4$.

V. INFRARED FINITENESS

Henceforth we restrict ourselves to Landau gauge.

We start by considering the infrared singular terms for the P-even part of the gluon self-energy, $\Pi_e(p^2)$. To two loop order, there are no diagrams whose purely ε-part contributes to $\Pi_e(p^2)$. There are still infrared singular terms, such as that of Fig. (4.a):

$$\Pi_{\mu\nu}(p) \underset{\text{Fig.}(4.a)}{\sim} g^2 \int \frac{k^\mu (k+p)^\nu}{(k^2)^2 (k+p)^2} (k^2 \tilde{\Pi}(k^2)) d^3k$$
$$\underset{p\to 0}{\sim} g^2 \tilde{\Pi}(0) \int \frac{k^\mu (k+p)^\nu}{k^2 (k+p)^2} d^3k \quad . \tag{5.1}$$

$\tilde{\Pi}(0) \sim g^2/m$, so this seems to give a piece $\sim g^2/\sqrt{p^2} \cdot (g^2/m)$ in $\Pi_e(p^2)$. The purely ε-part of Fig. (4.a) vanishes, since to one loop order, the ghost self-energy does not depend on the P-odd part of the gluon propagator (Sec. III).

The diagram of Fig. (4.a) can be viewed as a self-energy insertion on one of the ghost legs of Fig. (1.b). The other diagrams which are in danger of contributing $\sim g^2/\sqrt{p^2} \cdot (g^2/m)$ to $\Pi_e(p^2)$ include a diagram which is like a vertex renormalization for each vertex of Fig. (1.b), and similarly for Fig. (1.a).

There is a convenient way of organizing these contributions, which is a kind of "infrared renormalization." Let ΔL stand for the connections to the bare propagators and vertices, computed about zero momentum. For now, we include only the one-loop terms in ΔL, so $\Delta L \sim O(g^2/m)$. We rewrite the bare Lagrangian, L, as

$$L = L + \Delta L - \Delta L \equiv L^{ren} - \Delta L \quad . \tag{5.2}$$

L^{ren} is the renormalized Lagrangian of Sec. II, which generates the renormalized propagators and vertices, about zero momentum, to one loop order.

Our strategy is transparent. To calculate to $\sim O(g^4)$, in two loop diagrams we take $L^{ren} \sim L$, so Fig. (4.a) is unchanged. $\Delta L \sim O(g^2/m)$ contributes through insertions in one loop diagrams, like that of Fig. (4.b). Evidently, Figs. (4.a) and (4.b) cancel about zero momentum. Because the Z's contain terms of $\sim O(g^2/m)$, one-loop diagrams must be recalculated, using L^{ren} instead of L. For Fig. (1.b),

$$\Pi_{\mu\nu}(p) \underset{\substack{\text{Fig.(1.b),} \\ L^{ren}}}{\sim} + g^2 N \left(\frac{\tilde{Z}_g}{\tilde{Z}} \right)^2 \int \frac{k^\mu (k+p)^\nu}{k^2 (k+p)^2} \frac{d^3k}{(2\pi)^3} \quad . \tag{5.3}$$

The purely ε-part of Fig. (1.a) gives:

$$\Pi_{\mu\nu}(p) \underset{\substack{\text{Fig. (1.a),} \\ \text{purely } \varepsilon\text{-part,} \\ L^{ren}}}{\sim} - g^2 N \left(\frac{Z_g}{Z} \right)^2 \int \frac{k^\mu (k+p)^\nu}{k^2 (k+p)^2} \frac{m^4_{ren}}{(k^2+m^2_{ren})((k+p)^2+m^2_{ren})} \frac{d^3k}{(2\pi)^3} \quad , \tag{5.4}$$

$m_{ren} = m_{ren}(0)$. By the Ward identity $Z_g/Z = \tilde{Z}_g/\tilde{Z}$, Eqs. (5.3) and (5.4) cancel against each other about zero momentum. This is enough to show that there are no terms $\sim g^2/\sqrt{p^2} \cdot (g^2/m)$ in $\Pi_e(p^2)$.

The extension to higher orders is direct. At n loop order, the most infrared singular term in $\Pi_e(p^2)$ can be no worse than $\sim g^2/\sqrt{p^2} \cdot (g^2/m)^{n-1}$. There are several ways these terms could arise.

The first is from the purely ε-parts of n-loop diagrams. These vanish unless n is odd, as for Figs. (1.a) and (1.b). Calculating with $L^{ren} \sim L$, the cancellation theorem to n loop order tells us that the sum of the purely ε-parts of these diagrams vanish about zero momentum.

Secondly, there are infrared renormalizations of the purely ε-parts of diagrams to n' loop order, n'<n. These are diagrams computed to n' loop order with L^{ren}; n' must be odd, and the Z's of L^{ren} include terms up to $\sim O\left((g^2/m)^{n-n'}\right)$. An example is Eqs. (5.3) and (5.4). The sum of these terms vanish by the Ward identities, and the cancellation theorem to n' loop order.

Finally, there are contributions which can be viewed, diagrammatically, as self-energy and vertex insertions into the purely ε-parts of diagrams at n" loop order, n" odd and <n. These diagrams will have parts that are not purely ε-like, arising from the self-energy and vertex insertions, such as Fig. (4.a). These diagrams cancel about zero momentum against insertions of ΔL, computed to $\sim O\left((g^2/m)^{n-n''}\right)$, into the purely ε-parts of n" loop diagrams: e.g., Figs. (4.a) and (4.b).

This shows that order by order by order in g^2, all terms $\sim g^2/\sqrt{p^2} \cdot (g^2/m)^{n-1}$ in $\Pi_e(p^2)$ cancel about zero momentum.

The possible infrared singularities of the P-odd part of the gluon self-energy, $\Pi_o(p^2)$, the ghost self-energy, $\tilde{\Pi}(p^2)$, and the three-gluon vertex are all similar. For instance, at two loop order each of them has diagrams whose purely ε-part contributes, in a schematic form,

$$\sim \left(\frac{g^2}{m}\right)^2 \int \frac{1}{k_1^{\,2} k_2^{\,2} (k_1+k_2+p)^2} \frac{m^2}{(k_1^{\,2}+m^2)} d^3k_1 d^3k_2$$
$$\underset{p\to 0}{\sim} \left(\frac{g^2}{m}\right)^2 \ln\left(\frac{m^2}{p^2}\right) . \tag{5.5}$$

To arbitrary order, the worst terms are $\sim(g^2/m)^n \ln(m^2/p^2)$, where n need be even. Because these terms arise from the purely ε-parts of diagrams, they have a direct interpretation in terms of diagrams in the ε-theory of Sec. IV. A term $\sim(g')^{2n}\ln(\Lambda^2/p^2)$ in the ε-theory (Λ = an ultraviolet cutoff) corresponds to one $\sim(g^2/m)^n\ln(m^2/p^2)$ for the purely ε-part of a diagram in the original theory. We know from Sec. IV that to any order in $(g')^2$, there are no ultraviolet logarithms in Z', $\tilde{Z}'$, and $Z_{g'}$; this implies that to any order in g^2, there are no infrared logarithms in $\Pi_o(p^2)$, $\tilde{\Pi}(p^2)$, and the three gluon vertex, respectively.

There is one point which we have overlooked. Besides terms $\sim g^2/\sqrt{p^2}\cdot(g^2/m)^{n-1}$ in $\Pi_e(p^2)$, there are also infrared logarithms possible, $\sim(g^2/m)^n \ln(m^2/p^2)$, for even n. By the Ward identity of Eq. (2.26), these infrared logarithms must cancel, since they do so in Z_g and $\tilde{Z}$ (remember $\tilde{Z}_g$=1 in Landau gauge). Similarly, by its Ward identity, the four gluon vertex must also be infrared finite.

Having made no pretense of rigor, this concludes our proof that the renormalized propagators and and vertices are infrared finite in Landau gauge.

Our process of infrared renormalization is similar to ultraviolet renormalization in a renormalizable field theory, but the analogy is not exact. Consider, for example, a proper n-point function of gluons, $\Gamma^{(n)}$. We suppress the color and Lorentz indices, and take the n-1 independent momenta, and their dot products, etc., to be of the same order, $\sim\sqrt{p^2}$. To one loop order, the purely ε-parts of diagrams give

$$\Gamma^{(n)} \underset{p^2\to 0}{\sim} \frac{g^n}{(\sqrt{p^2})^{n-3}} .$$

$\left(\Gamma^{(2)}\right.$ is $\sim g^2\sqrt{p^2}$ since $\left.\Gamma^{(2)}\sim\Pi_{\mu\nu}(p).\right)$ By the cancellation theorem, the purely ε-parts of diagrams cancel about zero momentum, and so $\Gamma^{(n)}$ is really only singular as

$$\Gamma^{(n)} \underset{p^2\to 0}{\sim} \frac{g^n}{m(\sqrt{p^2})^{n-4}} .$$

$\Gamma^{(n)}$ is finite as $p^2\to 0$ for $n\leq 4$, but it seems improbable that this will be so if $n\geq 5$.

This is unlike ultraviolet renormalization, where once the ultraviolet infinites are removed from the propagators and vertices, they will not show up in higher n-point functions. With our infrared renormalization, the propagators and vertices are infrared finite, but higher point functions are not.

This is not a significant matter, though. $\Gamma^{(n)}$ is only singular as all of its external momenta $\to 0$. Suppose we insert a $\Gamma^{(n)}$, $n\geq 5$, as part of a diagram for a propagator or vertex to some high order in g^2. Then the point at which $\Gamma^{(n)}$ is singular will be a set of measure zero for the loop integrals, and can be ignored. Thus the infrared singularities of the $\Gamma^{(n)}$ for $n\geq 5$ does not contradict our proof of infrared finiteness for $n\leq 4$.

In any case, our real interest in the $\Gamma^{(n)}$, $n>4$, is when all of its external legs go on the mass shell. This is the subject to which we turn next.

VI. THE DISCONTINUITIES OF AMPLITUDES

The bare equation for the propagation of a gluon is

$$(\partial^2\delta_{\mu\nu}+im\varepsilon_{\mu\nu\lambda}\partial^\lambda)A^{a,\nu}(x) = 0 \qquad (6.1)$$

in Landau gauge. Expanding $A^a_\mu(x)$ in plane waves,

$$A^a_\mu(x) = e^a_\mu \exp(ip\cdot x) + c.c. \quad , \qquad (6.2)$$

the polarization vector, e^a_μ, is transverse to p_μ, $p^\mu e^a_\mu = 0$. Under gauge transformations,

$$A^a_\mu \to A^a_\mu + \partial_\mu\Lambda^a + gf^{abc}A^b_\mu\Lambda^c \quad . \qquad (6.3)$$

We neglect the last term in Eq. (6.3), on the grounds that it generates perturbative corrections to asymptotic states. For the gauge transformed A^a_μ (x) to remain in Landau gauge, $\partial^2\Lambda^a(x)=0$. A solution is

$$\Lambda^a = -ic^a\exp(ip^o\cdot x) + c.c. \quad , \qquad (6.4)$$

with p^o_μ a null vector, $(p^o_\mu)^2 = 0$. Thus, if the gluon's momentum is null, $p^2 = 0$, by setting $p^o_\mu = p_\mu$, e^a_μ is defined only up to the transformation

$$e^a_\mu \to e^a_\mu + c^a p_\mu \quad . \qquad (6.5)$$

For massless gluons, the part of e^a_μ parallel to p_μ can be eliminated by Eq. (6.5), with the remainder perpendicular to p_μ: $e^a_\mu = e^a p^\perp_\mu$, where $p^\perp \cdot p = 0$, but $\varepsilon^{\mu\nu\lambda} p_\nu p^\perp_\lambda \neq 0$. This shows that for each color index, a massless gluon in these dimensions has one (physical) degree of freedom,[5] versus two in four dimensions.

For $m \neq 0$, consider first the massive pole in the propagator, at $p^2=-m^2$. The polarization vector satisfies

$$e^a_\mu = \frac{1}{m} \varepsilon_{\mu\nu\lambda} e^{a,\nu} p^\lambda \quad ,$$
$$p^\mu e^a_\mu = 0 \quad . \tag{6.6}$$

For example, in the rest frame

$$p^\mu = (-im,0,0) \quad ;$$

we take the first coordinate to be time, and the other two, space. The solution for e^a_μ is

$$e^a_\mu = \frac{e^a}{\sqrt{2}} (0,1,i) \quad ,$$

so e^a_μ is a right-handed (spatial) vector for $m>0$. For the opposite sign of the Chern-Simons mass, $m<0$, e^a_μ is left-handed in the rest frame. Outside of the rest frame, e^a_μ has both time and spatial components; the latter are a definite mixture of left and right handed terms, depending upon the sign of m.

When $m \neq 0$, if $p^2 = 0$,

$$m\, \varepsilon^{\mu\nu\lambda} e^a_\nu p_\lambda = 0 \quad , \; p^\mu e^a_\mu = 0 \quad . \tag{6.7}$$

The solution is $e^a_\mu = -c^a p_\mu$, but by the residual gauge freedom, we can set $c^a = 0$. Consequently, while the bare gluon propagator does have poles at zero momentum, there are no physical degrees of freedom associated with the massless modes.

This result is not that surprising. Unlike a Higgs mechanism, the introduction of a Chern-Simons mass term does not alter the number of physical degrees of freedom for the gauge field. On the mass shell, there is one degree of freedom per color index for a massless gluon, so when $m \neq 0$, this single degree of freedom goes into the massive mode, leaving only gauge variant parts for the massless pole.

Physical amplitudes are obtained in the usual fashion. For example, to obtain n-particle T-matrix elements $T^{(n)}$, one starts with the proper n-point function, $\Gamma^{(n)ab\ldots}_{\mu\nu\ldots}(p,q\ldots)$. Each leg is put on the mass shell, $p^2=q^2=\ldots=-m^2$, and dotted with a suitable polarization vector,

$$T^{(n)} = e^a_\mu(p) e^b_\nu(q) \ldots \Gamma^{(n)ab\ldots}_{\mu\nu\ldots}(p,q\ldots) \quad .$$

If it can be shown that the massless modes do not contribute, then the Cutkosky rules imply that the only discontinuities of $T^{(n)}$ for Minkowski values of the momenta, p^2 real and <0, are those of massive particles.

The contribution of the massless modes to the discontinuities of physical amplitudes cancel as a consequence of gauge invariance, in essentially the same way as they do in ordinary gauge theories with spontaneous symmetry breaking.[13-17] To show that the discontinuities from intermediate states with a single massless mode vanish, we start with a n-point amplitude in which all of the legs except one are on the mass shell, $\tilde{T}^{(n)a}_{\mu}(p)$; p is the momentum of that one leg, etc. The infinitesimal Ward identities can be used to show that $p^{\mu}\tilde{T}^{(n)a}_{\mu}(p) = p^2$ times a function which is regular at $p^2 = 0$.[17] For the massless mode, $e^a_{\mu} = -c^a p_{\mu}$, so $e^a_{\mu}(p)\tilde{T}^{(n)a}_{\mu}(p) \to 0$ as $p^2 \to 0$, which establishes what we desire. The extension to intermediate states with more than one massless mode, for which the contribution of ghosts must be added, can be carried out, following, e.g., Ref. 17.

The massless modes do not contribute to the discontinuities of physical amplitudes, but they do for quantities that are gauge variant. This is illustrated by the self energies to one loop order, Eqs. (C.12)-(C.14)-they all have branch cuts which start at zero momentum.

This raises an obvious question-if $\Pi_e(p^2)$ and $\Pi_o(p^2)$ each have such branch cuts, how can the renormalized propagator have a simple pole at $p^2 = -m^2$? To answer this, we observe that a physical amplitude is formed from the gluon self-energy, $\Pi^{ab}_{\mu\nu}(p)$, by contracting each leg with the proper polarization vector, and setting $p^2 = -m^2$:

$$(e^a_\mu)^* \Pi^{ab}_{\mu\nu}(p)e^b_\nu\Big|_{p^2=-m^2} = -(e^*_a \cdot e^a)\; m^2\Big(\Pi_e(-m^2)-\Pi_o(-m^2)\Big) \quad , \tag{6.8}$$

where Eq. (6.6) has been used. As a gauge-invariant quantity, the discontinuity of Eq. (6.8) must start with the exchange of two massive gluons. Kinematically, this is impossible at $p^2 = -m^2$, hence

$$\mathrm{Im}\Big(\Pi_e(-m^2) - \Pi_o(-m^2)\Big) = 0 \quad . \tag{6.9}$$

The renormalized mass is given by Eq. (2.19c), so Eq. (6.9) ensures that $m_{ren}(p^2)$ is real at $p^2=-m^2$ to one loop order, although $m_{ren}(p^2)$ is complex for $0>p^2>-m^2$ and $-m^2>p^2$.

The results of Sec. III obey Eq. (6.9), Eqs. (C.18) and (C.19):

$$\mathrm{Im}\Big(\Pi_e(-m^2)\Big) = \mathrm{Im}\Big(\Pi_o(-m^2)\Big) = \frac{g^2N}{64m} \quad . \tag{6.10}$$

It is worth mentioning that Eq. (6.9) is a Ward identity which must hold in any covariant gauge. An example is the ξ^2 terms in $\Pi_o(p^2)$ and $\Pi_e(p^2)$- at $p^2=-m^2$, these terms cancel, Eqs. (3.2d) and (3.5).

By Eq. (6.10), the wave-function renormalization constant is complex on the mass shell, $\mathrm{Im}\{Z(-m^2)\}\neq 0$. This phenomenon is only possible if the gauge theory is non-abelian and has a Chern-Simons mass term, for without the Chern-Simons mass, the gluon has only a single self-energy, Π, which satisfies $\mathrm{Im}(\Pi) = 0$ on the mass shell. Even so, that $\mathrm{Im}(Z(-m^2)) \neq 0$ here appears to be just a curiosity, since $\mathrm{Im}(Z(-m^2))$ cannot be measured directly in any physical process.

On the mass shell, the renormalized mass is (Eqs. (C.18) and (C.19))

$$m_{phys} \equiv m_{ren}(-m^2) = m\left(1 + \frac{g^2 N}{32\pi m}(27\ln 3 - 4)\right) \tag{6.11}$$

to one loop order. This m_{phys} determines the gauge-invariant position of the pole in the renormalized propagator, and so is properly termed the physical mass.

Our arguments about the discontinuities of physical amplitudes apply only to one loop order, but they can easily be extended to arbitrary order. To higher order, it is necessary to take into account the shift in the physical mass from its bare value, and that $Z(-m^2_{phys})$ is complex. Eq. (6.8), evaluated at $p^2=-m^2_{phys}$, will ensure that the massive pole in the gluon propagator remains a simple pole. Thus the renormalized on shell equation for a gluon differs from the bare one, Eq. (6.1), merely by the replacement of m with m_{phys}; $Z(-m^2_{phys})$ factors out. The remaining steps go through unchanged.

Our results can also be used to show that the correlation functions of gauge invariant operators fall off exponentially over large distances in Euclidean space-time. This is best shown by example: we compute the two-point function of $tr(F_{\mu\nu})^2$ to ~O(1). At leading order, we can take only the abelian piece of the operator, $tr(F_{\mu\nu})^2 \sim tr(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$, and the bare gluon propagators in the one-loop diagram. The result is

$$\langle 0|\mathrm{tr}\left(F^2_{\mu\nu}(-p)\right)\mathrm{tr}\left(F^2_{\mu\nu}(p)\right)|0\rangle$$

$$= 8(N^2-1)\int \frac{(k\cdot(k+p)-m^2)^2+k^2(k+p)^2-m^4}{(k^2+m^2)((k+p)^2+m^2)}\frac{d^3k}{(2\pi)^3} \quad , \tag{6.12}$$

independent of ξ. Terms involving $k^2(k+p)^2$ in the denominator of the loop integral have cancelled against identical terms in the numerator. This can be understood by computing the two-point function as an infinite sum of form factors,

$$\langle 0|\mathrm{tr}(F_{\mu\nu})^2\mathrm{tr}(F_{\mu\nu})^2|0\rangle = \sum_n |\langle n|\mathrm{tr}(F_{\mu\nu})^2|0\rangle|^2 \;; \tag{6.13}$$

$\sum_n$ represents the sum over intermediate states. Gauge invariance for the operator and its form factor exclude any massless states from the $\sum_n$. To ~0(1), only (massive) two-particle intermediate states contribute, Eq. (6.12).

Our results show that it is possible to answer detailed questions about the physics of topologically massive chromodynamics. Further studies are presently underway.

APPENDIX A: THE ABELIAN THEORY

In this appendix we consider some elementary aspects of an abelian gauge theory with a Chern-Simons mass term. After solving two problems in statics, we discuss how charged particles can be said to exhibit fractional statistics[18,19] over large distances.

The Lagrangian is

$$L = \frac{1}{4} F_{\mu\nu}{}^2 + \frac{m}{2} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + J^\mu A_\mu \quad , \tag{A.1}$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The space-time is Minkowski in this appendix, with signature (−++). Also: $x^\mu = (x^0, \vec{x})$, $\varepsilon_{0ij} = +\varepsilon_{ij}$, $i,j = 1,2$. In two space dimensions, the curl of two vectors is a scalar: $\vec{a}\times\vec{b} = \varepsilon_{ij} a^i b^j$.

The action transforms by a surface term under a gauge transformation[5], so m is not quantized.

We assume that all matter fields are heavy: if their mass is m_M, their charge e, $e^2/m_M \ll 1$. Proving infrared finiteness in the quantum theory is trivial, so we can take the photon to interact with a fixed external source J^μ.

The field-strength tensor $F_{\mu\nu}$ is composed of an electric field $\vec{E}$ and a pseudo-scalar magnetic field B:

$$\vec{E} = \vec{\partial} A^0 + \partial_0 \vec{A} \quad , \quad B = \vec{\partial}\times\vec{A} \quad . \tag{A.2}$$

There are three equations of motion:

$$\vec{\partial}\cdot\vec{E} + mB = J^0 \quad , \tag{A.3}$$

$$\vec{\partial}\times B - \partial_0\vec{E} + m\vec{E}\times = \vec{J} \quad , \tag{A.4}$$

and one Bianchi identity,

$$\partial_0 B - \vec{\partial}\times\vec{E} = 0 \quad . \tag{A.5}$$

For static charge distributions,

$$B = \frac{1}{\vec{\partial}^2-m^2}\,(\vec{\partial}\times\vec{J}-mJ^0) \quad , \tag{A.6}$$

$$\vec{E} = \vec{\partial}\,\frac{1}{\vec{\partial}^2-m^2}\left(J^0 - \frac{m}{\vec{\partial}^2}\,\vec{\partial}\times\vec{J}\right) \quad ; \tag{A.7}$$

the $\vec{E}$ and B fields fall off exponentially, ~exp(-mr), over distances far ($r \gg m^{-1}$) from any charge distribution (by convention, m>0). To solve static problems, it helps to recognize that

$$\vec{\partial}^2(B+mA^0) = \vec{\partial}\times\vec{J} \quad . \tag{A.8}$$

For the distributions we consider, this implies that $B+mA^0$ is constant away from sources, so

$$-m\vec{E} = \vec{\partial}B \tag{A.9}$$

in source-free regions. Eq. (A.9) is reminiscent of the self-duality condition in four dimensions.

We turn now to our two examples.

Point Charge:[2]

$$J^0 = e\delta^2(\vec{x}) \quad , \vec{J} = 0 \tag{A.10}$$

For $m = 0$, $\vec{E} = e\hat{r}/(2\pi r)$. When $m \neq 0$, one matches $A^0 \sim \ln(r)$, valid for $r \ll m^{-1}$, onto the solution of the free, massive wave equation which falls off at spatial infinity:

$$A^0 = -\frac{e}{2\pi} K_0(mr) \quad , \tag{A.11}$$

K_0 the modified Bessel function of zeroth order. In this instance, $B = -mA^0$, so

$$B \sim -\frac{em}{2\pi} \ln(rm) \quad , r \ll m^{-1} \quad , \tag{A.12}$$

$$B \sim \frac{e}{2} \sqrt{\frac{m}{2\pi r}} \exp(-mr) \quad , r \gg m^{-1} \quad . \tag{A.13}$$

This shows that static charges induce magnetic flux. Indeed, consider a distribution of charge arbitrary except that it is bounded in size. Integrating Gauss' law, Eq. (A.3), over a region A whose contour C is everywhere far from any charge, since $\vec{E} \sim \exp(-mr) \sim 0$ on C, we obtain a relation[2,5] between the total flux, $\Phi = \int_A B \, d^2\vec{x}$, and the total charge, $e_{tot} = \int_A J^0 \, d^2\vec{x}$:

$$\Phi = e_{tot}/m \quad . \tag{A.14}$$

The Chern-Simons mass transforms a particle with charge e into a flux "tube", of width $\sim m^{-1}$, and strength e/m.

Line Charge

$$J^0 = e\delta(x) \ , \ \vec{J} = ev\ \delta(x)\hat{y} \quad , \tag{A.15}$$

which is a wire with current ev along $\hat{y}$. For m=0,

$$B = \frac{e}{2}\ v\ \theta'(x) \quad , \quad \vec{E} = \frac{e}{2}\ \theta'(x)\hat{x} \quad , \tag{A.16}$$

$\theta'(x) = \pm 1$ for $x \gtrless 0$. When $m \neq 0$,

$$B = \frac{e}{2}\ \{1 + v\theta'(x)\}\exp(-m|x|) \quad ,$$
$$\vec{E} = B\theta'(x)\hat{x} \quad . \tag{A.17}$$

It is amusing to note that if the charges move at the speed of light, v = 1, by Eq. (A.17) there are fields on only one side of the wire.

The solution for the line charge shows that the Chern-Simons mass produces a separate part of the electric field, $\vec{E} \sim \vec{v}\times$, from moving charges. This is like the Hall effect.[8]

For two charged particles, the interaction energy between them, $E_{int.}$, vanishes over large distances in the static limit:

$$E_{int.}(r) \underset{r>>m^{-1}}{\sim} \frac{e_1 e_2}{2\pi m} \frac{\vec{v}\times\vec{r}}{r^2} \quad . \tag{A.18}$$

The charges of the particles are e_1 and e_2; their relative separation and velocity are $\vec{r}$ and $\vec{v}$, respectively.

While $E_{int.}(r)\to 0$ as $r \to \infty$, quantum mechanical effects can still produce correlations between particles over large distances. Suppose that we fix particle 1, and rotate particle 2 infinitesimally slowly around 1 by 2π. The wave function of 2 changes by $\exp\left(ie_2\oint\vec{A}\cdot d\vec{l}\right) = \exp(ie_2\Phi_1) = \exp(ie_1e_2/m)$. The wave function of 1 changes by an equal amount, since 2 is itself a source of magnetic flux, and so the total two-body wave function changes by $\exp(2ie_1e_2/m)$. Interchange of 1 and 2 is like a rotation of the relative wave function by π, so under interchange, the two-body wave function changes by $\exp(ie_1e_2/m)$.

This phase is of little consequence if the particles are not identical.[2] If the particles are identical, $e_1 = e_2 = e$, let us choose a gauge in which the vector potential is essentially zero everywhere, except around the two particles. In this gauge, the two-body wave function must be defined so that upon interchange of 1 and 2, there is an additional factor of $\exp(ie^2/m)$ which multiplies the usual ± 1.

It is in this gauge variant sense that charged particles exhibit fractional statistics. Wilczek[18] first observed that flux tubes with arbitrary flux have fractional statistics. It is known[19] that charged particles coupled to an abelian gauge field with a Chern-Simons mass, but no term $\sim \frac{1}{4} F_{\mu\nu}^2$, do as well, so it is not surprising to find fractional statistics in the full theory, Eq. (A.1). What we find of interest is that the full theory provides, physically, such a direct example of Wilczek's original insight, since any charged particle acts like a flux tube over large distances.

The effect only occurs for particles separated by distances $\gg m^{-1}$. Over distances $\lesssim m^{-1}$, charged particles do generate magnetic fields, but their mutual electric fields are not negligible, and there is no simple expression obtained as they encircle. In particular, it is sensible to speak of the charged particles as being, fundamentally, either bosons or fermions - the equal time (anti-) commutation relations between the charged fields follow from their properties at short distances, and remain those of (fermions) bosons. Further, it is only the charged fields, and not the photon itself, which have fractional statistics: e.g., the contribution of the photon field to the generator of angular momentum is standard (appendix, Ref. 9).

The ratio e^2/m, which fixes how fractional the statistics of identical charged particles are at large distances, is an arbitrary number. We remark that the e^2 and m which enter here are renormalized, and not bare, quantities; they are obtained from the renormalized photon propagator about zero momentum. In this way, the fractional nature of the statistics, $\approx e^2/m$, is itself renormalized.

Does a non-abelian gauge theory with a Chern-Simons mass term exhibit fractional statistics? To answer this, we first need to understand how to measure the total color charge in a non-abelian system.

Let J^μ be an external source of color, for either gluons or matter fields. As before, we choose a region A whose boundary C is everywhere far ($\gg m^{-1}$) from where $J^\mu \neq 0$.

The obvious definition of the total charge,

$$g_{tot} = \int_A J^0 d^2\vec{x} \quad , \tag{A.19}$$

is a color matrix, but otherwise it is not very physical. The color current is only covariantly conserved, $D_\mu J^\mu = 0$, so g_{tot} is generally time dependent; g_{tot} is also gauge dependent.

To avoid these problems, we define the "global" color charge, Q:

$$Q = 2 \oint_C \left(\vec{E}\cdot - \frac{m}{2}\vec{A}\times\right)\vec{n}\, dl \quad ; \tag{A.20}$$

$\vec{n}$ is the normal to C. Using Gauss' law, and that $\vec{E}\sim 0$ on C,

$$Q = \int_A \left\{J^0 - g\left[\vec{A},\ \cdot\vec{E}\right] - mg\ \vec{A}\times\vec{A}\right\} d^2\vec{x} \tag{A.21}$$

The last two terms in Eq. (A.21) represent the corrections to g_{tot} which are necessary in a non-abelian theory.

Why is Q superior to g_{tot}? Unlike g_{tot}, Q is independent of time. This is because the vector K^μ,

$$K^\mu = \partial_\nu F^{\nu\mu} - m\ \varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda \tag{A.22}$$

has zero divergence, $\partial_\mu K^\mu = 0$. Neglecting surface terms at spatial infinity, $Q = \int_A K^0\ d^2\vec{x}$, $\partial_0 Q = 0$. Secondly, Q is invariant under arbitrary local gauge transformations Ω, as long as color fields at spatial infinity are unchanged by Ω: if $\Omega = \exp(\Lambda)$, by Eq. (A.20) Q is invariant if $\Lambda(x^0,\vec{x})\to 0$ as $\vec{x}\to\infty$.

These properties of Q are not accidental. In $A^0 = 0$ gauge, Q is precisely the charge associated with global rotations of color.

Now let J^μ represent two identical, colored point particles, 1 and 2. They are far enough apart so that we can integrate over regions $\gg m^{-1}$ in size around each without crossing the other. Their local color charges, as in Eq. (A.19), are g_1 and g_2, and their global color charges, as in Eq. (A.20), are Q_1 and Q_2. As particle 2 is wrapped around 1 by 2π, the two-body wave function changes by $\sim\exp(ig_1Q_2/m)\exp(ig_2Q_1/m)$. This factor is not invariant under local gauge transformations, as would be a term like $\exp\,(2iQ_1Q_2/m)$. Consequently, identical colored particles do affect each other over large distances, but this has no (relatively!) simple interpretation as a sort of fractional statistics.

APPENDIX B: SPONTANEOUS SYMMETRY BREAKING

When spontaneous symmetry breaking occurs, the presence of a Chern-Simons term for the gauge fields alters the mass spectrum in a striking way. We illustrate the effect with an abelian gauge field, but it also happens if the gauge field is non-abelian.

We take as our Lagrangian

$$L = \frac{1}{4} F_{\mu\nu}^2 + \frac{im}{2} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$
$$+ |D_\mu \phi|^2 - \mu^2 |\phi|^2 + \frac{\lambda}{2} (|\phi|^2)^2 + L_{gauge} \quad , \tag{B.1}$$

$D_\mu = \partial_\mu + ieA_\mu$; μ^2, λ and $m>0$. We return in this appendix to Euclidean space-time. ϕ is a complex scalar field:

$$\phi = \frac{1}{\sqrt{2}} (\phi_0 + \phi_1 + i\, \phi_2) \quad , \tag{B.2}$$

for real ϕ_0, ϕ_1, and ϕ_2, with

$$\phi_0{}^2 = \frac{2\mu^2}{\lambda} \; , \; \langle\phi_1\rangle = \langle\phi_2\rangle = 0 \quad . \tag{B.3}$$

The gauge field is redefined as

$$B_\mu = A_\mu + \frac{1}{e\phi_0} \partial_\mu \phi_2 \quad . \tag{B.4}$$

By a suitable choice of L_{gauge} ('t Hooft gauge), the bare inverse propagator for B_μ becomes

$$\Delta^{-1}_{\mu\nu} = (p^2+m_s^2)\delta_{\mu\nu} + (\xi^{-1}-1)p_\mu p_\nu + m\,\varepsilon_{\mu\nu\lambda}p^\lambda \quad , \tag{B.5}$$

$$m_s^2 = e^2\phi_0{}^2 \quad . \tag{B.6}$$

The bare propagator for B_μ is found to be

$$\Delta_{\mu\nu} = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{(p^2+\xi m_s^2)} - \frac{m}{(p^2+m_s^2)}\,\varepsilon_{\mu\nu\lambda}p^\lambda\right)\frac{p^2+m_s^2}{D(p^2)} + \xi\, p_\mu p_\nu \frac{(p^2+m^2+m_s^2)}{D(p^2)(p^2+\xi m_s^2)} \quad , \tag{B.7}$$

where

$$D(p^2) = (p^2+m_s^2)^2 + m^2p^2 = (p^2+m_+^2)(p^2+m_-^2) \quad , \tag{B.8}$$

and

$$m_\pm^2 = m_s^2 + \frac{m^2}{2} \pm \frac{n^2}{2} \quad , \qquad n^2 = \sqrt{m^2(m^2+4m_s^2)} \quad . \tag{B.9}$$

What is remarkable about $\Delta_{\mu\nu}$ is that it has <u>two</u> distinct poles which are physically significant, one at $p^2 = -m_+^2$, and one at $p^2 = -m_-^2$. The piece of $\Delta_{\mu\nu}\sim p_\mu p_\nu$ also has a pole at $p^2 = -\xi m_s^2$, but this is a gauge-variant excitation - e.g., it decouples for $\xi\to\infty$, as expected in 't Hooft gauge.

About the two poles in $\Delta_{\mu\nu}$,

$$\Delta_{\mu\nu} \underset{p^2\to -m_+^2}{\sim} \left(\frac{m^2+n^2}{2n^2}\right) \frac{\delta_{\mu\nu}}{p^2+m_+^2} + \dots \quad , \tag{B.10}$$

$$\Delta_{\mu\nu} \underset{p^2\to -m_-^2}{\sim} \left(\frac{n^2-m^2}{2n^2}\right) \frac{\delta_{\mu\nu}}{p^2+m_-^2} + \dots \quad , \tag{B.11}$$

This shows that each of the two poles, at $p^2 = -m_+^2$ and $p^2 = -m_-^2$, contribute to the $\delta_{\mu\nu}$ part of $\Delta_{\mu\nu}$ with positive residue, so both are physical, gauge invariant excitations for m and $m_s \neq 0$.

In the limit that $m_s \ll m$,

$$m_+^2 \sim m^2 \quad , \quad m_-^2 \sim \frac{m_s^4}{m^2} \quad , \tag{B.12}$$

and $\Delta_{\mu\nu}$ behaves as

$$\Delta_{\mu\nu} \underset{\substack{p^2\to -m_+^2 \\ m_s \ll m}}{\sim} \frac{\delta_{\mu\nu}}{p^2+m^2} + \dots \quad , \tag{B.13}$$

$$\Delta_{\mu\nu} \underset{\substack{p^2\to -m_-^2 \\ m_s \ll m}}{\sim} \left(\frac{m_s^2}{m^2}\right) \frac{\delta_{\mu\nu}}{p^2+m_-^2} + \dots \quad . \tag{B.14}$$

When $m_s \to 0$, $m_- \to 0$, but from Eq. (B.14), the state at $p^2 = -m_-^2$ decouples from the $\delta_{\mu\nu}$ piece of $\Delta_{\mu\nu}$. Turning off the symmetry breaking, $\phi_0 = m_s = 0$, removes one of the two physical poles in $\Delta_{\mu\nu}$; this agrees with our analysis in the text.

To see why $\Delta_{\mu\nu}$ has two physical poles for m and $m_s \neq 0$, we consider the (bare) on-shell equation for B^μ, as in Sec. VI:

$$\left((-\partial^2+m_s^2)\delta_{\mu\nu}-im\ \varepsilon_{\mu\nu\lambda}\partial^\lambda\right) B^\nu(x) = 0 \quad , \tag{B.15}$$

$\partial_\mu B^\mu(x) = 0$. With

$$B_\mu(x) = e_\mu \exp(ip\cdot x) + c.c. \quad , \tag{B.16}$$

$p^\mu e_\mu = 0$, and

$$\left(p^2+m_s^2\right) e_\mu + m\ \varepsilon_{\mu\nu\lambda} e^\nu p^\lambda = 0 \quad . \tag{B.17}$$

We solve for e_μ in the rest frame:

$$p^\mu = (-i\tilde{m}\ ,\ 0\ ,\ 0) \quad ,$$

$$e_\mu = (0\ ,\ e_1,\ e_2)\ .$$

Without the Chern-Simons mass, m = 0, $\tilde{m} = m_s$, and e_1 and e_2 are arbitrary. This is what usually happens with spontaneous symmetry breaking in three dimensions - at $p^2 = -m_s^2$, the B_μ field has two degrees of freedom, one from A_μ, and one from the scalar ϕ_2.

With the Chern-Simons mass, and $m_s \neq 0$, there are two solutions to Eq. (B.17), $\tilde{m} = m_\pm$. Their polarization vectors satisfy

$$e_2^\pm = i \frac{m^2 \pm n^2}{2mm_\pm} e_1^\pm \quad . \tag{B.18}$$

B_μ must still have two degrees of freedom on the mass shell, but instead of one mass, with a two-component polarization vector, when m and $m_s \neq 0$ B_μ is on shell at two distinct masses, though the polarization vector of each has only one (independent) component.

Why? Remember that the Chern-Simons mass is P-odd, so the mass spectrum should reflect this handedness. This is not possible if B_μ is on shell at one mass point with two independent components for e_μ. So, B_μ "splits" into two on shell masses, m_+ and m_-. The polarization vector of each, $e_\mu^\pm$, is a definite mixture of right and left-handed terms for m>0. When the sign of the Chern-Simons mass is flipped, this mixture changes: $e_2^\pm/e_1^\pm \to -e_2^\pm/e_1^\pm$ as $m \to -m$.

APPENDIX C: THE SELF ENERGIES TO ONE LOOP ORDER

We present here some of the details of the calculation of the self energies to leading order in Sec. III.

The integrals are

$$\tilde{\Pi}(p^2) = \frac{g^2 N}{p^2} \int \frac{(k\cdot p)^2 - k^2 p^2}{k^2 (k+p)^2 (k^2+m^2)} \frac{d^3k}{(2\pi)^3} \quad , \qquad (C.1)$$

$$\Pi_o(p^2) = \frac{g^2 N}{p^2} \int \frac{P_o}{Q} \frac{d^3k}{(2\pi)^3} \; , \qquad (C.2)$$

$$\Pi_e(p^2) = -\frac{g^2 N}{4p^2} \left\{ \int \frac{P_e}{Q} \frac{d^3k}{(2\pi)^3} + \frac{2m}{\pi} \right\} \quad , \qquad (C.3)$$

where

$$Q = k^2(k^2+m^2)\;(k+p)^2\left[(k+p)^2+m^2\right] \quad , \qquad (C.4)$$

$$P_o = \left[k^2p^2-(k\cdot p)^2\right](5k^2+5k\cdot p+4p^2+2m^2) \quad , \qquad (C.5)$$

$$\begin{aligned} P_e = {} & 6k^6 + 18k^4 k\cdot p + 20k^4p^2 + 22k^2(k\cdot p)p^2 \\ & - 12(k\cdot p)^3 + 9k^2p^4 - 7(k\cdot p)^2p^2 \qquad (C.6) \\ & + m^2(2k^4 + 4k^2k\cdot p + k^2p^2 + (k\cdot p)^2). \end{aligned}$$

Dimensional regularization was used to compute these integrals; e.g., the last term in Eq. (C.3) is due to the tadpole diagram. While generally care must be taken in applying dimensional regularization to theories that involve the antisymmetric tensor $\varepsilon_{\mu\nu\lambda}$, we do not need to concern ourselves with such subtleties. Unlike a renormalizable theory, in a super-renormalizable theory such as this, any ambiguities in going from 3 to 3+ε dimensions will vanish smoothly as $\varepsilon \to 0$.

The resulting self energies are given in Eqs. (3.7)-(3.9).

About zero momentum,

$$\tilde{\Pi}(p^2) \underset{p^2 << m^2}{\sim} \frac{g^2 N}{m} \left(-\frac{1}{6\pi} + \frac{\sqrt{p^2}}{32m} - \frac{p^2}{30\pi m^2} + \ldots \right) , \tag{C.7}$$

$$\Pi_o(p^2) \underset{p^2 << m^2}{\sim} \frac{g^2 N}{m} \left(\frac{7}{12\pi} - \frac{\sqrt{p^2}}{16m} + \frac{37p^2}{240\pi m^2} + \ldots \right) , \tag{C.8}$$

$$\Pi_e(p^2) \underset{p^2 << m^2}{\sim} \frac{g^2 N}{m} \left(-\frac{5}{24\pi} - \frac{\sqrt{p^2}}{128m} + \frac{13p^2}{480\pi m^2} + \ldots \right) . \tag{C.9}$$

For large momenta,

$$\tilde{\Pi}(p^2) \underset{p^2 >> m^2}{\sim} \frac{g^2}{\sqrt{p^2}} \left(-\frac{1}{16} + \frac{m}{6\pi\sqrt{p^2}} - \frac{m^2}{32p^2} + \ldots \right) , \tag{C.10}$$

$$\Pi_o(p^2) \underset{p^2 >> m^2}{\sim} \frac{g^2 N}{\sqrt{p^2}} \left(\frac{13}{32} - \frac{4m}{3\pi\sqrt{p^2}} - \frac{m^2}{32p^2} + \ldots \right) , \tag{C.11}$$

$$\Pi_e(p^2) \underset{p^2 >> m^2}{\sim} \frac{g^2 N}{\sqrt{p^2}} \left(-\frac{11}{64} + \frac{m}{3\pi\sqrt{p^2}} + \frac{15m^2}{64p^2} + \ldots \right) . \tag{C.12}$$

The first terms on the right hand side of Eqs. (C.10) and (C.12) agree with the one loop results in the massless theory,[1,12] as they should.

The discontinuities of these amplitudes can be extracted directly. To continue to Minkowski momenta, we take

$$p^2 = \exp(-i\pi)s,$$

with s a positive, real number. Using

$$\mathrm{Re}\,\sin^{-1}\left(\frac{p^2-m^2}{p^2+m^2}\right) = -\frac{\pi}{2} + \pi\,\theta(s-m^2) \quad , \tag{C.13}$$

$$\mathrm{Re}\,\sin^{-1}\left(\sqrt{\frac{p^2}{p^2+4m^2}}\right) = \frac{\pi}{2}\,\theta(s-4m^2) \quad , \tag{C.14}$$

$\theta(s) = 0$ or 1 for $s<0$ or >0, we find

$$\mathrm{Im}\,\tilde{\Pi} = -\frac{g^2N}{32m}\frac{\sqrt{s}}{m}\left(1-\left(1-\frac{m^2}{s}\right)^2\theta(s-m^2)\right) \quad , \tag{C.15}$$

$$\mathrm{Im}\,\Pi_o = \frac{g^2N}{64m}\frac{\sqrt{s}}{m}\left(4 - \frac{3s}{m^2} + \frac{2(3s+m^2)(s-m^2)^2}{s^2m^2}\,\theta(s-m^2) - \frac{3(s+2m^2)(s-4m^2)}{sm^2}\,\theta(s-4m^2)\right) \tag{C.16}$$

$$\mathrm{Im}\,\Pi_e = \frac{g^2N}{128m}\frac{\sqrt{s}}{m}\left(1 + \frac{s}{m^2} - 2\,\frac{(s+7m^2)(s-m^2)^2}{s^2m^2}\,\theta(s-m^2) + \frac{(s^2+13sm^2+4m^2)(s-4m^2)}{s^2m^2}\,\theta(s-4m^2)\right) \quad . \tag{C.17}$$

On the mass shell,

$$\Pi_o(-m^2) = \frac{g^2N}{16\pi m}\left(1+ \frac{27}{4}\ \ln 3\right) + i\ \frac{g^2N}{64m} \quad , \tag{C.18}$$

$$\Pi_e(-m^2) = \frac{g^2N}{16\pi m}\left(3- \frac{27}{4}\ \ln 3\right) + i\ \frac{g^2N}{64m} \quad . \tag{C.19}$$

Using the analyticity of the self energies in the cut p^2 plane, they can be written in a dispersive form, as an integral over their imaginary parts along the cut. This is the form that Deser, Jackiw, and Templeton chose.[5] $\tilde{\Pi}$, eq. (C.12), agrees with their result, as does that for Π_o, eq. (C.13), up to an overall difference in sign for Π_o. Our result for Π_e, eq. (C.13), does not agree with theirs. However, our Π_e has the correct limit at large momenta, eq. (C.12), and satisfies the proper Ward identity on the mass shell, eqs. (6.10), (C.18), and (C.19). The Π_e of Ref. 5 does not satisfy this Ward identity; it was this that lead us to the labor of recomputing the self energies in the first place.

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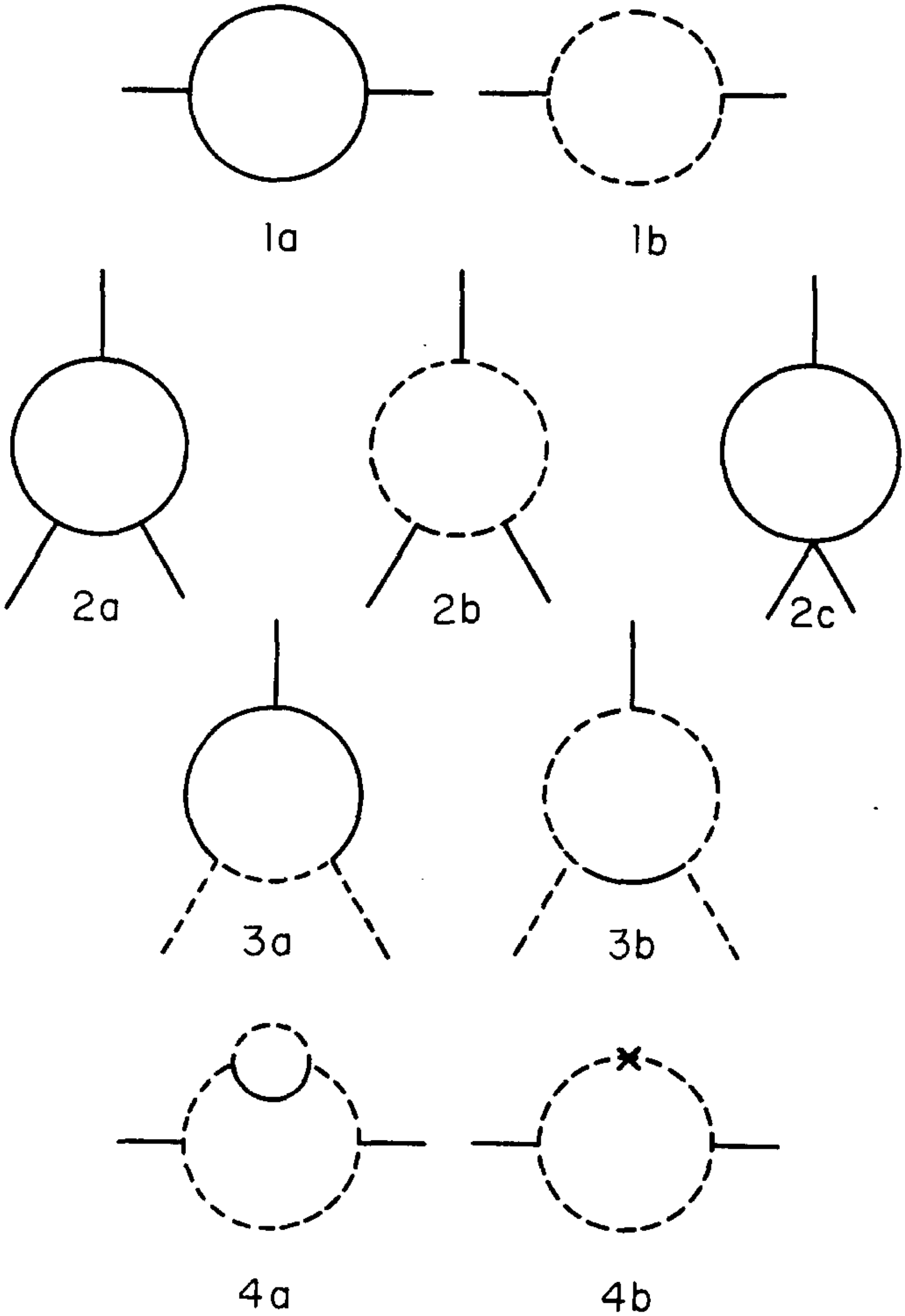
FIGURE CAPTIONS

Fig. 1: Contributions to the gluon self-energy at one-loop order. Solid lines denote gluons; dotted lines, ghosts.

Fig. 2: One-loop corrections to the three-gluon vertex.

Fig. 3: One-loop corrections to the ghost-ghost-gluon vertex.

Fig. 4: Two contributions $\sim O(g^4)$ to $\Pi_e(p^2)$. In Fig. (4.b), the cross denotes a term for ghost wave function renormalization, $\sim O(g^2/m)$, from ΔL.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