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A Study of Correlation Functions for the Delta-Function Bose Gas

Dennis B. Creamer
Center for Studies of Nonlinear Dynamics†
La Jolla Institute
3252 Holiday Ct., Suite 208
La Jolla, CA 92037

H. B. Thacker
Fermi National Accelerator Laboratory
P.O. Box 500
Batavia, IL 60510

and

David Wilkinson
Schlumberger-Doll Research
Old Quarry Rd.
P.O. Box 307
Ridgefield, CT 06877

† Affiliated with the University of California, San Diego

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Abstract

The Gel'fand-Levitan equation for the quantum nonlinear Schrodinger field theory is used to investigate the correlation functions of the delta-function Bose gas. Operator expressions are derived for the field and for nonlocal products of fields in terms of the quantized reflection operators which create and annihilate eigenstates of the Hamiltonian. For the two-point function, an explicit series expression is obtained in which the n^{th} term is determined by well-defined n-body combinatorics in an infinite volume. The inductive properties of this series are discussed and used to express the temperature and chemical potential dependence of the correlation functions entirely in terms of previously known thermodynamic functions. The zero separation limit of the series for the two-point function reproduces the thermodynamics derived by Yang and Yang, while the infinite coupling limit gives the Fredholm determinant result of Schultz and Lenard. The latter is related to the Painleve' V equation by the monodromy arguments of Jimbo, Miwa, Mori, and Sato. The O(1/c) correction to the large coupling limit is calculated from the Gel'fand-Levitan series and expressed in terms of solutions to Painleve' V. The asymptotic behavior of the relevant Painleve' function is discussed and related to the long range behavior of the correlation function.

I. Introduction

The theory of completely integrable quantum systems is a subject of continuing interest in field theory and statistical mechanics.[1] At its present stage of development, the theory provides a general method for constructing the exact eigenstates and eigenvalues of the Hamiltonian or transfer matrix of an integrable model. This is accomplished either by the original wave function ansatz method of Bethe[2] or by the more recently developed quantum inverse method,[1] which reduces Bethe's ansatz to an elegant operator algebra. In general, the eigenstates of the theory must be built on a "pseudovacuum" state which is far from the physical states of interest (e.g. the empty Dirac sea in relativistic fermion theories). The physical states are obtained by filling a large number of negative energy modes, the number of filled modes going to infinity as the infrared (box) and ultraviolet (lattice) cutoffs are removed. Thus, the eigenstates of interest, in particular the physical vacuum, are represented by very complicated Bethe wave functions. Since the number of terms in the inner product of N-body Bethe wave functions grows like $(N!)^2$, the problem of computing physical quantities is formidable.

In this paper we consider the problem of correlation functions in the nonlinear Schrodinger (delta-function Bose gas) model, which is in many ways the simplest of the general class of integrable systems that can be solved by Bethe ansatz or inverse scattering techniques. The infinite coupling ($c=\infty$) limit of this model, known as the impenetrable boson model, has been extensively studied,[3-6] and the calculation of correlation functions has been carried out in closed form in terms of

Painlevé transcendents.[6] Our approach to the general finite coupling case is based on the Gel'fand-Levitan formalism developed earlier by the authors.[7] Some results of this investigation have been reported previously.[8-9] Here we present a detailed discussion of the formalism and results. The main result is a series expansion for the correlation functions of Schrodinger field operators. The n^{th} term in this series may be explicitly calculated from the n^{th} term of the corresponding operator series obtained from the Gel'fand-Levitan equation. (Recently, Korepin[10] has obtained a series expansion for the density-density correlation using a somewhat different approach.) Up to the present time, we have not been able to reduce these results to a closed form expression for the correlation function. However, the term-by-term series calculation is completely well-defined and may be carried out to any desired order. It should serve as a useful guide for subsequent investigations aimed at deriving closed form expressions.

The nonlinear Schrodinger model is described by the Hamiltonian

$$H = \int dx\{\partial_x\phi^* \partial_x\phi + c\phi^*\phi^*\phi\phi\} \tag{1.1}$$

where $\phi(x)$ is a nonrelativistic boson field with canonical commutation relations. The model has a conserved number operator

$$N = \int dx\ \phi^*(x)\phi(x) \tag{1.2}$$

and in the n-particle sector is equivalent to a system of n identical bosons interacting via a two-body δ-function potential of strength 2c.

Although many of our general results for the correlation functions of this model are applicable to the case of arbitrary 2n-point functions, we shall restrict our attention to the 2-point function. At

zero temperature the 2-point correlation function at finite density D is defined as

$$G(x-y) = \lim_{\substack{N,L\to\infty \\ N/L=D}} \langle\psi_{NL}|\phi^*(x)\phi(y)|\psi_{NL}\rangle \tag{1.3}$$

where the state $|\psi_{NL}\rangle$ is the ground state of N particles in a box of length L. In our approach it is actually more convenient to consider the corresponding quantity at finite temperature:

$$G_{\beta,\mu}(x-y) = \lim_{L\to\infty} \frac{\mathrm{Tr}\ \phi^*(x)\phi(y)e^{\beta(\mu N-H)}}{\mathrm{Tr}\ e^{\beta(\mu N-H)}} \tag{1.4}$$

where β is the inverse temperature, μ is the chemical potential, and the trace is to be taken over all states of the system in a box of size L.

Although the evaluation of (1.3) or (1.4) in the general case remains a notorious unsolved problem, there are two special cases for which the result has been known for some time. The first of these is the case x=y first obtained by Yang and Yang[11] by a variational method. Since $\phi^*(x)\phi(x)$ is just the particle density opertaor, we see that $G_{\beta,\mu}(0)$ reduces to $D(\beta,\mu)$, the expression for the particle density D as a function of the temperature and chemical potential. The function $D(\beta,\mu)$ yields all thermodynamic information about the system; in particular the grand partition function $Q = \mathrm{Tr}\ e^{\beta(\mu N-H)}$ may be expressed as

$$\frac{1}{L}\ln Q = \beta\int_{-\infty}^{\mu} d\mu'\ D(\beta,\mu) \tag{1.5}$$

The result of Yang and Yang may be summarized as

$$D(\beta,\mu) = \int \frac{dk}{2\pi}\,\rho(k) \tag{1.6}$$

where the density function $\rho(k,\beta,\mu)$ is given in terms of another function $\varepsilon(k,\beta,\mu)$ by

$$\rho(k) = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln(1+e^{-\beta\varepsilon(k)}) \tag{1.7}$$

The function $\varepsilon(k,\beta,\mu)$, which describes the finite temperature excitation spectrum, satisfies the following nonlinear integral equation

$$\varepsilon(k) = k^2-\mu-\frac{1}{\beta}\int\frac{dq}{2\pi}\Delta(k-q)\ln(1+e^{-\beta\varepsilon(q)}) \tag{1.8}$$

where the kernel Δ is given by

$$\Delta(k-q) = \frac{2c}{(k-q)^2+c^2} \tag{1.9}$$

As will be seen, the β and μ dependence of the full two-point function (1.4) may be expressed in terms of this function $\varepsilon(k,\beta,\mu)$.

The second case for which exact results are known is that of infinitely repulsive coupling $c\to\infty$. In this case the Hamiltonian (1.1) may be diagonalized by means of a Jordan-Wigner transformation and it has been shown by Lenard[4] and Schultz[3] that the two-point function $G_{\beta,\mu}(x-y)$ may be expressed as the first Fredholm minor associated with the kernel

$$K(z,z') = \frac{1}{2}\int dk\,\frac{e^{ik(z-z')}}{1+e^{\beta(k^2-\mu)}} \tag{1.10}$$

acting on the interval $[y,x]$. Both of these known results, i.e. $x=y$

and $c=\infty$, previously obtained by quite distinct methods, arise as simple special cases of our general approach.

The remainder of the paper is organized as follows. In Section II we review the necessary results of the quantum inverse method and derive a normal ordered expansion for $\phi^*(x)\phi(y)$ in terms of the quantized reflection operator. We also describe a method, central to our approach, for handling the thermodynamic limit $L\to\infty$ in (1.4). This enables us to compute directly in an infinite volume without ever introducing a finite box and periodic boundary conditions. In Section III these results are used to obtain an infinite series representation for the two-point function $G_{\beta,\mu}(x-y)$ in which the β and μ dependence is expressed solely in terms of the function $\varepsilon(k,\beta,\mu)$ in (1.8). The form of the result is such that the zero temperature ($\beta\to\infty$) limit may be taken trivially. In the case $x=y$ it is shown that this series representation reproduces the results of Yang and Yang for the thermodynamics of the system. In Section IV we use these results to obtain the first two terms in a large coupling expansion of the zero temperature two-point function (1.3).[8] Section V contains a short discussion.

II. Review of the Quantum Inverse Formalism

This Section contains the essential results of the quantum inverse method for the nonlinear Schrodinger model, together with our prescription for handling the thermodynamic limit. The quantum inverse method[1] for the Hamiltonian (1.1) is implemented through the linear Zakharov-Shabat eigenvalue problem

$$(i\frac{\partial}{\partial x} + \frac{1}{2}\xi)\psi_1 = -\sqrt{c}\psi_2\phi$$

$$(i\frac{\partial}{\partial x} - \frac{1}{2}\xi)\psi_2 = \sqrt{c}\phi^*\psi_1 \tag{2.2}$$

The scattering data operators $a(\xi)$ and $b(\xi)$ are defined in terms of the Jost solutions $\psi(x,\xi)$ or $\chi(x,\xi)$ with the properties[12-14]

$$\begin{pmatrix}1\\0\end{pmatrix}e^{i\xi x/2} \xrightarrow[x\to-\infty]{} \begin{pmatrix}\psi_1(x,\xi)\\ \psi_2(x,\xi)\end{pmatrix} \xrightarrow[x\to\infty]{} \begin{pmatrix}a(\xi)e^{i\xi x/2}\\ b(\xi)e^{-i\xi x/2}\end{pmatrix} \tag{2.3a}$$

$$\begin{pmatrix}-b^*(\xi)e^{i\xi x/2}\\ a(\xi)e^{-i\xi x/2}\end{pmatrix} \xrightarrow[x\to-\infty]{} \begin{pmatrix}\chi_1(x,\xi)\\ \chi_2(x,\xi)\end{pmatrix} \xrightarrow[x\to\infty]{} \begin{pmatrix}0 & e^{-i\xi x/2}\\ 1 & \end{pmatrix} \tag{2.3b}$$

The quantized reflection operator $R(\xi)$ is given by

$$R(\xi) = \frac{i}{\sqrt{c}}\, b(\xi)\, a^{-1}(\xi) \tag{2.4}$$

and satisfies the simple commutation relations

$$[H,R^*(\xi)] = \xi^2 R^*(\xi) \tag{2.5}$$

$$R(\xi)R(\xi') = S(\xi'-\xi)R(\xi')R(\xi) \tag{2.6a}$$

$$R(\xi)R^*(\xi') = S(\xi-\xi')R^*(\xi')R(\xi) + 2\pi\delta(\xi-\xi') \tag{2.6b}$$

where H is the Hamiltonian, and S is the two-body S-matrix

$$S(\xi-\xi') = \frac{\xi-\xi'-ic}{\xi-\xi'+ic} \tag{2.7}$$

From these relations we see that the states $|k_1,\ldots,k_N\rangle = R^*(k_1)\ldots R^*(k_N)|0\rangle$ are eigenstates of the Hamiltonian with energy $\sum_{n=1}^{N} k_n^2$. These states are identical with those obtained by means of Bethe's ansatz.

The inverse transformation from the reflection operators R(k) back to the Heisenberg field $\phi(x)$ is accomplished by means of the quantized version of the Gel'fand-Levitan equation.[7] By using the analytic properties in ξ of the Jost solution $\chi(x,\xi)$ it was shown in Ref. 7 that the components χ_1 and χ_2 may be expressed as expansions in the operators $R(\xi)$ and $R^*(\xi)$. The asymptotic behavior

$$\chi(x,\xi)e^{i\xi x/2} \xrightarrow[\xi\to\infty]{} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{\xi}\begin{pmatrix} \sqrt{c}\phi(x) \\ ic\int_x^\infty j_0(x')dx' \end{pmatrix} + O(1/\xi^2) \tag{2.8}$$

yields corresponding series for the field operator $\phi(x)$ and the charge-density operator $j_0(x) = \phi^*(x)\phi(x)$. Some properties of the $\phi(x)$ series were studied in Refs. 7 and 15, while $j_0(x)$ was used in Ref. 16 to derive the equilibrium thermodynamics of this model at finite temperature and density.

The Gel'fand-Levitan series for the field $\phi(x)$ may be expressed in the form

$$\phi(x) = \sum_{N=0}^{\infty} \int \prod_{i=1}^{N} \left(\frac{dp_i}{2\pi}\right) \prod_{j=0}^{N} \left(\frac{dk_j}{2\pi}\right) g_N(p,k;x)\, R^*(p_1)\ldots R^*(p_N) R(k_N)\ldots R(k_0). \tag{2.9}$$

where

$$g_N(p,k;x) = \frac{(-c)^N e^{i(\sum_0^N k_i - \sum_1^N p_i)x}}{\prod_{m=1}^{N} (p_m - k_m - i\varepsilon)(p_m - k_{m-1} - i\varepsilon)} \tag{2.10}$$

Another interesting representation for $\phi(x)$ can be derived by exploiting the connection between the R operators and the Bethe's ansatz eigenstates. This representation is discussed in Appendix C.

We can also obtain a representation similar to (2.9) for the charge density operator $j_0(x)=\phi^*(x)\phi(x)$. For the study of correlation functions, we are interested in the normal ordered series for the more general bilocal operator $\phi^*(x)\phi(y)$. Using the normal ordered series for $\phi(y)$ and $\phi^*(x)$, the product $\phi^*(x)\phi(y)$ can be normal-ordered by use of the algebra (2.6). This procedure is simply accomplished by use of the analytic properties of the Jost solutions. In what follows we shall suppose that $x>y$. If we write the series analogous to (2.9) for $\phi^*(x)$, we can normal order the product by moving $\phi(y)$ past the $R(k)$'s in each term. For example, the first non-trivial term comes from

$$[\phi^*(x)]^{(2)} = c^2 \int dp_0 dp_1 dk_1 \, \frac{R^*(p_0)R^*(p_1)R(k_1)e^{i(k_1-p_0-p_1)x}}{(p_0-k_1-i\varepsilon)(p_1-k_1-i\varepsilon)} \tag{2.11}$$

Then,

$$[\phi^*(x)]^{(2)}\phi(y) = \int dp_1 dp_1 dk_1 \frac{R^*(p_0)R^*(p_1)\{\phi(y)R(k_1)+[R(k_1),\phi(y)]\}}{(p_0-k_1-i\varepsilon)(p_1-k_1-i\varepsilon)} \times e^{i(k_1-p_0-p_1)x} \tag{2.12}$$

The first term in the brackets gives a normal ordered series while the second involves the commutator

$$[R(k),\phi(y)] = -(a^*(k))^{-1}\psi_2^*(y,k)\gamma_2^*(y,k) \tag{2.13}$$

where

$$\gamma_2(x,k) = \chi_1^*(x,k)-i\sqrt{c}\, R^*(k)\chi_2(x,k) \tag{2.14}$$

on the real k axis. The commutation relation (2.13) is the hermitian conjugate of the one derived in Ref. 7. In that reference it was also shown that $\gamma_2(x,\zeta)$ is analytic in the lower half ζ plane. As $(a^*(\xi))^{-1}$ and $\psi_2^*(x,\zeta)$ are analytic in the upper half ζ plane, this implies that the commutator $[R(k),\phi(y)]$ is analytic in the upper half plane. By using the large k behavior of the Jost solutions, the asymptotic behavior of the commutator (2.13) is found to be

$$e^{iky}[R(k),\phi(y)] \underset{k\to\infty}{\sim} O(1/k) \tag{2.15}$$

Thus the second term in Eq.(2.12) involves the integral

$$\int \frac{dk_1\ e^{ik_1(x-y)}\{e^{ik_1y}[R(k_1),\phi(y)]\}}{(p_0-k_1-i\varepsilon)(p_1-k_1-i\varepsilon)} \tag{2.16}$$

which, for x>y, can be evaluated by closing the contour in the upper

half plane. This gives zero by analyticity. The same procedure can be used term by term in the series for $\phi^*(x)$ yielding a normal ordered double series for the bilocal product which is obtained simply by inserting the series for $\phi(y)$ between the R^*'s and the R's of each term in the series for $\phi^*(x)$. Thus we have a representation of the form

$$\phi^*(x)\phi(y) = \sum_{N=0}^{\infty} \int \prod_{0}^{N} \frac{dp_i}{2\pi} \prod_{0}^{N} \frac{dk_i}{2\pi} F_N(k,p;x,y)\, R^*(p_0)\ldots R^*(p_N) R(k_N)\ldots R(k_0) \tag{2.17a}$$

in which the F_N are expressed as sums of products of the g_N in (2.10).

$$F_N(p,k;x,y) = \sum_{n=0}^{N} g_n^*(k_0\ldots k_{n-1};p_0\ldots p_n;x)\; g_{N-n}(p_{n+1}\ldots p_N;k_n\ldots k_N;y) \tag{2.17b}$$

By similar arguments we may obtain normal ordered series expressions for arbitrary multilocal products of field operators. Although we will only discuss the two-point function here, many of our results generalize to the 2n-point functions for arbitrary n.

Having derived a series expression for the operator $\phi^*(x)\phi(y)$ in terms of the eigenmode operators R and R^*, we now wish to evaluate the 2-point function $G_{\beta,\mu}(x-y)$ by calculating the traces in Eq.(1.4). In order to do this, we must carefully define our prescription for treating the infrared divergent terms which arise in these traces in the thermodynamic limit $L \to \infty$. The basic problem here is that, in an infinite volume, matrix elements of the form

$$\langle p_1,\ldots,p_N|\phi^*(x)\phi(y)|k_1,\ldots,k_N\rangle \tag{2.18}$$

have singular terms proportional to $(p_i-k_i)^{-1}$ which prevent us from immediately setting $p_i=k_i$ as is required for calculating the trace. Imagine calculating the matrix element (2.18) as an integral over N-body coordinate space (Bethe's ansatz) wave functions,

$$\begin{aligned}&\langle p_1\ldots p_N|\phi^*(x)\phi(y)|k_1\ldots k_N\rangle \\ &\quad = \int dz_2\ldots dz_N\ \psi_p^*(x,z_2\ldots z_N)\psi_k(y,z_2\ldots z_N)\end{aligned} \tag{2.19}$$

where ψ_p and ψ_k are the wave functions for the states $|p_1\ldots p_N\rangle$ and $|k_1\ldots k_N\rangle$. If the system is in a finite box, the z-integrations are bounded and we may immediately set $p_i=k_i$. However, in an infinite volume there will be unbounded z integrations of the form

$$\int_{-\infty}^{\infty} dz\ e^{i(k_i-p_i)z}\ \sim\ i(p_i-k_i)^{-1}. \tag{2.20}$$

Since much of our Gel'fand-Levitan formalism is valid only in the infinite volume case, we must find the correct prescription for handling the singularities arising from unbounded z integrations. Our criterion for determining the correct prescription is that it should give the same answer that would be obtained by putting the system in a box and taking the limit $L\to\infty$ at the end of the calculation. As we shall see, this requirement leads uniquely to the use of an infinitesimal boost as a means of cutting off the infrared singularities represented by vanishing denominators of the form (2.20). This method was first applied to thermodynamic calculations in the course of a detailed graphical derivation of the Yang and Yang equation of state.[17] However, the reason

for its validity is quite general and easily seen if we keep in mind the structure of the Bethe's ansatz wave functions in (2.19). First we note that dividing by the partition function in Eq.(1.4) simply removes the disconnected parts of the trace in the numerator, so we only need to concern ourselves with connected parts of the matrix element (2.19). Now consider the divergent terms of the form (2.20) which arise from the integration over z_N with the other z_i's held fixed. Of all the terms in the product of wave functions in (2.19), the only terms which diverge as $p_N \to k_N$ are those for which the z_N integration is unbounded. Leaving out inessential factors, this gives

$$e^{-i\theta_1}\int_{-\infty}^{z} dz_N e^{i(k_N-p_N)z_N} + e^{i\theta_2}\int_{z'}^{\infty} dz_N e^{i(k_N-p_N)z_N} \tag{2.21}$$

where z and z' are the smallest and largest of the coordinates $x,y,z_1,\ldots,z_{N-1}$. The phases θ_1 and θ_2 are sums of two-body phase shifts which depend on the relative ordering of the k's and p's and on which set of states we are using (e.g. in states or out states). However, the relative phase of the two terms is always

$$e^{i(\theta_1+\theta_2)} = \prod_{i=1}^{N-1} S(k_N-k_i)S(p_i-p_N). \tag{2.22}$$

If the system were enclosed in a finite box of length L, the limits of integration $\pm\infty$ in (2.21) would be replaced by $\pm L/2$, and the terms proportional to L would include a factor

$$(e^{-i\theta_1} - e^{i\theta_2}) = e^{-i\theta_1}[1 - e^{i(\theta_1+\theta_2)}]. \tag{2.23}$$

Thus, when we set $p_i=k_i$, $i=1,\ldots,N$ to take the trace, this factor vanishes,

$$[e^{i(\theta_1+\theta_2)} - 1] = 0 \tag{2.24}$$

i.e. the terms which might have diverged as $L \to \infty$ actually vanish. In order to reproduce this result without introducing a box, we can simply let the p_i's be slightly different from the k_i's in such a way that all momentum differences are unchanged, i.e.

$$p_N-p_i = k_N-k_i, \quad i=1,\ldots,N-1 \tag{2.25}$$

so that the factor (2.24) remains zero. This requirement dictates that the state $|p_1\ldots p_N\rangle$ is related to the state $|k_1\ldots k_N\rangle$ by an infinitesimal Galilean boost, i.e. $p_i=k_i+q$, where q is a small momentum which plays the role of an infrared cutoff. By these considerations we are led to the formula which will actually be used to calculate correlation functions from the Gel'fand-Levitan formalism,

$$G_{\beta,\mu}(x-y) = \lim_{q\to 0} \mathrm{Tr}\{e^{-iqK}\phi^*(x)\phi(y)e^{-\beta(H-\mu N)}\} \tag{2.26}$$

where K is the generator of Galilean boosts,

$$K = \int x\ \phi^*(x)\phi(x)dx\ . \tag{2.27}$$

Note that it is not necessary to divide by a partition function to cancel out disconnected pieces in (2.26). This is a convenient property of the infinitesimal boost cutoff which results from the fact that a disconnected contribution will always be proportional to a "vacuum"

subgraph (i.e. a graph not connected to the operator $\phi^*(x)\phi(y)$) which will vanish by momentum conservation, since $\sum p_i - \sum k_i = nq \neq 0$ where n is the number of particles in the vacuum subgraph. Equivalently, we might have included a denominator $Tr\{e^{-iqK}e^{-\beta(H-\mu N)}\}$ in (2.26), but this is just unity since only the zero particle state contributes.

III. Correlation Functions

This section contains the essential points of the computation of correlation functions in our quantum inverse approach. In order to present the structure of the computation as simply as possible, the most technical material is relegated to the Appendices. Our starting assertion is that the finite temperature two-point function $G_{\beta,\mu}(x-y)$ may be computed from the representation

$$G_{\beta,\mu}(x-y) = \lim_{q\to 0} \frac{\mathrm{Tr}\ e^{-iqK}\ \phi^*(x)\phi(y)e^{-\beta(H-\mu N)}}{\mathrm{Tr}\ e^{-iqK}\ e^{-\beta(H-\mu N)}} \tag{3.1}$$

where the boost operator e^{-iqK} is defined in (2.27) and $\phi^*(x)\phi(y)$ is expressed in terms of the fundamental operators $R(k)$ and $R^*(p)$ by (2.17). Performing this substitution we obtain

$$G_{\beta,\mu}(x-y) = \lim_{q\to 0} \sum_{N=0}^{\infty} \int \Big(\prod_{j=0}^{N} \frac{dk_j}{2\pi}\frac{dp_j}{2\pi} \Big) F_N(p,k;x,y)\Lambda^{(N)}(p,k;q) \tag{3.2}$$

where $\Lambda^{(N)}$ is the thermal trace of a product of R operators

$$\Lambda^{(N)}(p,k;q) = \frac{\mathrm{Tr}\ e^{-iqK}\ R^*(p_0)\ldots R^*(p_N)R(k_N)\ldots R(k_0)e^{-\beta(H-\mu N)}}{\mathrm{Tr}\ e^{-iqK}e^{-\beta(H-\mu N)}} \tag{3.3}$$

When formulated in this way, the calculation is in principle no more complicated than our previous derivation[16] of the thermodynamics of the system, the latter, as pointed out in Section I, being equivalent to computing the zero separation correlation function $G_{\beta,\mu}(0)$. [Indeed it is easy to see that as $y \to x$ the operator expression (2.17) reduces to

the expression for $\phi^*(x)\phi(x)$ considered in this earlier work.] The fundamental result which is needed is an expression for the quantity $\Lambda^{(N)}$ defined in (3.3). In Ref. 16 it was shown, by using the cyclic property of the trace and the algebra of the R operators, that this quantity may be written as a multiple fugacity series, with one summation for each pair of $R^*(p)$ and $R(k)$:

$$\Lambda^{(N)}(p,k;q) = \sum_{n_0\ldots n_N=1}^{\infty} \prod_{i=0}^{N} (-1)^{n_i+1} e^{n_i\beta(\mu-k_i^2)} \times\langle 0|R(k_N+n_Nq)\ldots R(k_0+n_0q)R^*(p_0)\ldots R^*(p_N)|0\rangle\times\{1 + O(q)\} \quad (3.4)$$

Note that (3.4) is not an exact formula, but rather contains corrections of order q. A sufficient condition for it to be correct to use (3.4) with the terms of order q neglected when evaluating (3.2) is that the function $F_N(k,p)$ have the property that $F_N(k_i,p_j=k_{P_j}+n_{P_j}q)$ is finite as $q\to 0$ for every permutation P of $0,1,2,\ldots,N$ and every choice of the n_i's. Unfortunately, this requirement is not satisfied by the $F_N(k,p)$ defined in (2.17b). However, in the following we shall be able to prove to all orders in N that the condition is satisfied if we replace $F_N(k,p)$ by a symmetrized quantity $\overline{F}_N(k,p)$ which is obtained from $F_N(k,p)$ by first symmetrizing the integrand over $k_0,\ldots,k_N$ and over $p_0,\ldots,p_N$ and then using the relation (2.6a) to recover the original ordering of the R's and R^*'s in each term. For example

$$\overline{F}_1(k_0,k_1,p_0,p_1) = \tfrac{1}{4}\{F_1(k_0,k_1,p_0,p_1) + S_{10}F_1(k_1,k_0,p_0,p_1) + S^{01}F_1(k_0,k_1,p_1,p_0) + S_{10}S^{01}F(k_1,k_0,p_1,p_0)\} \quad (3.5)$$

where we have used the notation

$$S_{ij} = S(k_i - k_j)$$
$$S^{ij} = S(p_i - p_j) \tag{3.6}$$

It is easy to see that the symmetrized function $\bar{F}_N(k,p)$ is of the general form

$$\bar{F}_N(p,k) = \prod_{\substack{i,j=0 \\ i>j}}^{N} \left(\frac{k_{ij}}{k_{ij}+ic}\right)\left(\frac{p_{ji}}{p_{ji}+ic}\right) \frac{H_N(p,k)}{\prod_{i,j=0}^{N}(p_i - k_j - i\varepsilon)} \tag{3.7}$$

where

$$k_{ij} = k_i - k_j \tag{3.8a}$$
$$p_{ij} = p_i - p_j \tag{3.8b}$$

and $H_N(k,p)$ is a symmetric function of $k_0,\ldots,k_N$ and of $p_0,\ldots,p_N$. If we replace $F_N(k,p)$ by $\bar{F}_N(k,p)$ in (3.2) then the contribution to the correlation function is the same for each of the (N+1)! contractions in the vacuum matrix element $\langle 0|R(k_N+n_Nq)\ldots R(k_0+n_0q)R^*(p_0)\ldots R^*(p_N)|0\rangle$ so that provided the limit $q \to 0$ exists we obtain

$$G_{\beta,\mu}(x-y) = \sum_{N=0}^{\infty} \int \frac{dk_0}{2\pi}\ldots\frac{dk_N}{2\pi} \sum_{n_0\ldots n_N=1}^{\infty} f_N(k,n;x-y) \prod_{i=0}^{N} e^{n_i\beta(\mu-k_i^2)} \tag{3.9}$$

where

$$f_N(k,n) = (N+1)!\ \lim_{q\to 0} \bar{F}_N(p,k)\big|_{p=k+nq} \tag{3.10}$$

(For notational convenience we have supressed the x-y argument of f_N.) From (3.7) we easily conclude that $f_N(k,n)$ is symmetric under combined permutations of the k_i and n_i, again provided the limit $q \to 0$ exists. We now prove the existence of this limit. Possible poles as $q \to 0$ are due to poles in $\bar{F}_N(k,p)$ at $p_i=k_j$, $i,j=0,1,\ldots,N$ which in turn are due to the poles in the original functions $g_N(k,p)$ in (2.10). Because of the symmetry property of H_N in (3.7) it is sufficient to consider the nature of the pole at $p_N=k_N$. In Appendix A it is shown that if we regard $\bar{F}_N(k,p)$ as a function of p_N with $k_0,\ldots,k_N$ and $p_0,\ldots,p_{N-1}$ fixed, then the residue of the pole at $p_N=k_N$ is given by

$$\frac{-i}{(N+1)^2}\left\{\prod_{i=0}^{N} S_{Ni}S^{iN} - 1\right\}\bar{F}_{N-1}(k_0\ldots k_{N-1},p_0\ldots p_{N-1}) \tag{3.11}$$

where we have used the notation (3.6), and p_N should be set to k_N since we are considering the residue at $p_N=k_N$. The important point is that the quantity in curly brackets is of order q when $p_i=k_i+n_iq$, $i=0,1,\ldots,N-1$. Thus if we make the inductive hypothesis that the limit (3.10) exists for $N=M-1$ then it exists for $N=M$ since the pole in $\bar{F}_M(k,n)$ at $p_M=k_M$ does not lead to a pole in q as $q\to 0$. Since the limit (3.10) exists trivially for $N=0$, this proves the existence of the limit for all N. Furthermore, from the above we see that the residue of the pole in $f_N(k,n)$ at $n_N=0$ is given by

$$\frac{1}{N+1} \sum_{j=0}^{N} n_j \, \Delta(k_N - k_j) \, f_{N-1}(k_0 \ldots k_{N-1}, n_0 \ldots n_{N-1}) \tag{3.12}$$

where

$$\Delta(k) = -i\frac{\partial}{\partial k} \ln S(k) = \frac{2c}{k^2+c^2} \tag{3.13}$$

is the kernel defined previously in (1.9). Let us split the function $f_N(k,n,x-y)$ into a part $f_N^{pole}(k,n,x-y)$ which contains a pole in one or more of the n_i and a part $\tilde{f}_N(k,x-y)$ which is free of such poles

$$f_N(k,n) = \tilde{f}_N(k) + f_N^{pole}(k,n) \tag{3.14}$$

[A more precise definition of $\tilde{f}_N$ is the following: the n-dependence of the function f_N is of the general form $f_N = P_N/(n_0 \ldots n_N)$ where P_N is a homogeneous multinomial of degree N+1 in the n_i; the quantity $\tilde{f}_N$ is the coefficient of $n_0 \ldots n_N$ in this multinomial.]

By repeated application of the basic induction formula (3.12) it is clear that the pole terms in (3.14) may be related to non-pole terms of lower order. Suppose for the moment that the pole terms in (3.14) were absent, as is the case in the limit $c \to \infty$ when $\Delta(k_i - k_j)$ vanishes. Then the fugacity sums in (3.9) could be done trivially to obtain

$$G_{\beta,\mu}(x-y) = \sum_{N=0}^{\infty} \int \Big(\prod_{j=0}^{N} \rho_0(k_j) \frac{dk_j}{2\pi} \Big) \, \tilde{f}_N(k,x-y) \tag{3.15}$$

where

$$\rho_0(k) = \frac{1}{1+e^{\beta(k^2-\mu)}} \tag{3.16}$$

is a Fermi-Dirac distribution. In Appendix B it is shown that the combined effect of the pole terms in (3.14), when summed to all orders, is just to replace the "bare" functions $\rho_0(k)$ by "dressed" functions $\tilde{\rho}(k)$ given by

$$\tilde{\rho}(k) = \frac{1}{1+e^{\beta\varepsilon(k)}} \tag{3.17}$$

where $\varepsilon(k)$ is the excitation energy function of Yang and Yang defined in (1.8). Thus we have

$$G_{\beta,\mu}(x-y) = \sum_{N=0}^{\infty} \int \Big(\prod_{j=0}^{N} \tilde{\rho}(k_j) \frac{dk_j}{2\pi} \Big) \tilde{f}_N(k,x-y) \tag{3.18}$$

Eq. (3.18) is a fundamental result of our analysis. It expresses the full Green's function at finite temperature and density in terms of known functions $\tilde{\rho}(k,\beta,\mu)$ with temperature and chemical potential independent coefficients $\tilde{f}_N(k,x-y)$ which are obtained from N-body combinatorics on the system defined in an infinite volume. A similar formula obtains for any 2n-point function. It is convenient to think of the functions $\tilde{\rho}(k)$ as defining the measure over which the k-integrations are done. In the zero temperature limit $\beta\to\infty$ this measure becomes very simple, since $\tilde{\rho}(k)$ becomes a simple step function,

$$\tilde{\rho}(k) = 1 \qquad |k|<k_F$$
$$= 0 \qquad |k|>k_F \tag{3.19}$$

where k_F is the Fermi momentum defined by $\epsilon(k_F)=0$. Thus at zero temperature we have

$$G(x-y) = \sum_{N=0}^{\infty} \int_{-k_F}^{k_F} \frac{dk_0}{2\pi}\ldots\frac{dk_N}{2\pi}\, \tilde{f}_N(k,x-y) \tag{3.20}$$

If desired, the Fermi momentum k_F may be expressed in terms of the density D by the implicit equation $D=G(0)$.

Thus the computation of the two-point function is reduced to the evaluation of the quantities $\tilde{f}_N(k,x-y)$ in (3.14). In the zero separation limit x=y, it is possible to express these quantities in simple closed form. It is easy to see that the factor $(\Sigma p-\Sigma k)$ in the operator expansion for $\phi^*(x)\phi(x)$ leads to an overall factor Σn_i in the quantity $f_N(k,n)$ defined in (3.10). From this we see that the residue of the pole in $f_N(k,n)$ at $n_i=0$ is of the form

$$\Big(\sum_{\substack{j=0\\ j\neq i}}^{N} n_i\Big)\big(\tilde{f}_{Ni}(k) + f_{Ni}^{pole}(k,n)\big) \tag{3.21}$$

where $\tilde{f}_{Ni}$ is free of poles in the n_j and f_{Ni}^{pole} has poles in one or more of the n_j, $j\neq i$. The non-pole piece $\tilde{f}_N(k)$ of (3.14) is given in terms of these $\tilde{f}_{Ni}$ by

$$\tilde{f}_N(k) = \sum_{i=0}^{N} \tilde{f}_{Ni}(k) \tag{3.22}$$

Using (3.12) we may show that the $\tilde{f}_{Nj}$ satisfy the induction property

$$\tilde{f}_{Ni}(k_0 \ldots k_N) = \frac{1}{N+1} \sum_{\substack{j=0 \\ j \neq i}}^{N} \Delta(k_i - k_j)\, \tilde{f}_{N-1,j}(k_0 \ldots k_{i-1}, k_{i+1} \ldots k_N) \tag{3.23}$$

Since, when x=y, we have $\tilde{f}_{00}=1$, we may solve this recursion relation to obtain, for example

$$\tilde{f}_{NN} = \frac{1}{N+1} \operatorname*{Sym}_{0 \leq j \leq N-1} \Delta_{01}\Delta_{12} \ldots \Delta_{N-1,N} \tag{3.24}$$

where $\Delta_{ij}=\Delta(k_i-k_j)$, and the right hand side is to be symmetrized over $k_0 \ldots k_{N-1}$. The $\tilde{f}_{Ni}$ for $i \neq N$ may be obtained by permutation. Substituting (3.23) into (3.21) we obtain

$$\tilde{f}_N = \operatorname*{Sym}_{0 \leq j \leq N} \Delta_{01}\Delta_{12} \ldots \Delta_{N-1,N} \tag{3.25}$$

where the right hand side is to be symmetrized over $k_0 \ldots k_N$. Substituting into the general result (3.18) we find that $D(\beta,\mu)=G_{\beta,\mu}(0)$ is given by

$$D(\beta,\mu) = \int \frac{dk}{2\pi}\, \rho(k) \tag{3.26}$$

where $\rho(k)$ satisfies the linear integral equation

$$\rho(k) = \tilde{\rho}(k) + \int\frac{dq}{2\pi}\,\tilde{\rho}(k)\Delta(k-q)\rho(q) \tag{3.27}$$

which is the result obtained first by Yang and Yang.[11] It is easy to see that equation (3.27) is equivalent to equations (1.7) and (1.8) of the Introduction.

IV. Large c Expansion

In this Section we will use the previously developed formalism to compute the first two terms in the strong coupling expansion of the zero temperature two-point function,[8,18]

$$G(x-y) = \sum_{k=0}^{\infty} \frac{G^{(k)}(x-y)}{c^k} \tag{4.1}$$

Although we have been able to derive these results directly from the representation (2.9)-(2.10) of the field operator, they are more easily derived using a different representation which is more closely related to the form of Bethe's ansatz. This representation is discussed in Appendix C. We find (c.f. Eq.(C.6)) that $\phi(x)$ can be expressed in the form (2.9) with $g_N(x)$ given by

$$g_N(x) = (-1)^N e^{ik_0 x} \int dz_1 \ldots dz_N \theta(x<z_1<z_2<\ldots<z_N)$$
$$\times \prod_{i=1}^{N} \{e^{i(k_i-p_i)z_i} (1 - S_{i0} \prod_{j=1}^{i-1} S_{ij} S^{ji})\} \tag{4.2}$$

In position space the series has the form

$$\phi(x) = R(x) - \int dz_1 \theta(x<z)\{R(1;1x)-R(1;x1)\}$$
$$+ \int dz_1 dz_2 \theta(x<z_1<z_2)\{R(12;21x) - R(12;2x1) - R(21;1x2) + R(21;x12)\}$$
$$+ \ldots \tag{4.3}$$

where $R(x)$ is the Fourier transform

$$R(x) = \int \frac{dk}{2\pi} R(k)\, e^{ikx} \quad , \tag{4.4}$$

and, for example, R(12;21x) is defined by

$$R(12;21x) = R^*(z_1)R^*(z_2)R(z_2)R(z_1)R(x) \quad , \tag{4.5}$$

As observed in Appendix C, a direct proof that the representation (4.2) is equivalent to the Gel'fand-Levitan representation (2.10) involves showing that the symmetrized integrand $\bar{g}_N(x)$ is the same in the two cases.

Using the position space form (4.3) and the reordering theorem discussed in Section 2, it is easy to show that the operator $\phi^*(x)\phi(y)$ is given by

$$\phi^*(x)\phi(y) = \sum_{N=0}^{\infty} \Gamma_N(x,y) \tag{4.6}$$

where, for example, $\Gamma_2(x,y)$ is given by

$$\begin{aligned}\Gamma_2(x,y) = \int dz_1 dz_2 \{ &\theta(y<z_1<z_2<x)[R(x12;21y)+R(x21;y12)-R(x12;2y1)-R(x21;1y2)] \\ &+\theta(y<z_1<x<z_2)[R(x12;21y)+R(2x1;y12)-R(x12;2y1)-R(2x1;1y2)] \\ &+\theta(y<x<z_1<z_2)[R(x12;21y+R(21x;y12)-R(2x1;1y2)-R(1x2;2y1)]\}\end{aligned} \tag{4.7}$$

and we have used the notation

$$R(x12;21y) = R^*(x)R^*(z_1)R^*(z_2)R(z_2)R(z_1)R(y) \quad , \tag{4.8}$$

etc. In momentum space (4.7) becomes

$$\Gamma_N(x,y) = \int \prod_{i=0}^{N} \left(\frac{dp_i}{2\pi}\frac{dk_i}{2\pi}\right) F_N(p,k;x,y)\, R^*(p_0)\ldots R^*(p_N)R(k_N)\ldots R(k_0) \quad (4.9a)$$

where

$$F_N(p,k;x,y) = \sum_{M=0}^{N} F_N^M(p,k;x,y) \quad (4.9b)$$

and F_N^M is given by

$$F_N^M(p,k;x,y) = (-1)^N e^{i(k_0 y - p_0 x)} \int dz_1 \ldots dz_N \prod_{i=1}^{N} e^{i(k_i - p_i)z_i}$$

$$\times\theta(y<z_1<\ldots<z_{N-M}<x<z_{N-M+1}<\ldots<z_N) \prod_{i=1}^{N-M} \left(1-S_{i0}\prod_{j=1}^{i-1} S_{ij}S^{ji}\right)$$

$$\times\prod_{i=N-M+1}^{N} \left(1-\prod_{j=0}^{i-1} S_{ij}S^{ji}\right) \quad (4.10)$$

The general structure of the term F_N^M is that N-M of the N z-integrations are "trapped" between y and x, while the remaining M integrations are unbounded. Associated with the i-th integration there is a factor

$$\left(1-S_{i0}\prod_{j=1}^{i-1} S_{ij}S^{ji}\right)$$

containing a product of an odd number of S-matrices if the i-th z-integration is trapped, or

$$\left(1-\prod_{j=0}^{i-1} S_{ij}S^{ji}\right)$$

containing a product of an even number of S-matrices if the i-th integration is untrapped.

An immediate consequence of the above for the consideration of a large c expansion is that each untrapped integration leads to a factor 1/c. Thus, the zeroth order term $G^{(0)}$ receives contributions from only the M=0 term, while the first correction term $G^{(1)}$ requires only the M=0 and M=1 terms. Since the forward singularities arise only from momentum denominators coming from the untrapped integrations, we see that the M=0 term contains no forward singularities, while the M=1 term contains only a single pole. This greatly simplifies the analysis.

Let us first consider the contribution of the M=0 term. The product of S-matrix factors in F_N^0 may be written

$$\frac{\left\{\prod_{i=1}^{N}[(k_{i0}+ic)\prod_{j=1}^{i-1}(k_{ij}+ic)(p_{ji}+ic) - c.c.]\right\}\prod_{i=1}^{N}(p_{0i}+ic)}{\prod_{i=1}^{N}\prod_{j=0}^{i-1}[(k_{ij}+ic)(p_{ji}+ic)]} \tag{4.11}$$

where for convenience we have multiplied both numerator and denominator by a factor $\prod_{i=1}^{N}(p_{0i}+ic)$. Since the combination

$$\frac{R^*(p_0)\ldots R^*(p_N)R(k_N)\ldots R(k_0)}{\prod_{i=1}^{N}\prod_{j=0}^{i-1}[(k_{ij}+ic)(p_{ji}+ic)]} \tag{4.12}$$

is antisymmetric under interchange of any pair of k's or p's, and the M=0 term is free of forward singularities (so that we may use the trace theorem (3.4) directly), it is sufficient to sum the numerator of (4.11) over all permutations of $k_0\ldots k_N$ (with a minus sign for each odd permutation) and then set p_i equal to k_i for i=0,1...N. As $c\to\infty$ it is clear that the denominator reduces to

$$(ic)^{N(N+1)}[1+O(1/c^2)] \tag{4.13}$$

while even before the antisymmetrization and contraction, the curly bracket in the numerator is of the form

$$2^N(ic)^{N^2}[1+O(1/c^2)] \tag{4.14}$$

Thus, to order 1/c the contribution of the M=0 term to the 2-point function (3.18) is

$$\sum_{N=0}^{\infty}(-2)^N\int\prod_{i=0}^{N}[\tilde{\rho}(k_i)\frac{dk_i}{2\pi}](1-\frac{i}{c}\sum_{i=1}^{N}k_{0N})$$

$$\times\int dz_1\ldots dz_N\theta(y<z_1<z_2<\ldots<z_N<x)\ D_N \tag{4.15}$$

where D_N is an $(N+1)\times(N+1)$ determinant of exponentials, e.g.

$$D_2 = \det\begin{bmatrix} e^{ik_0(y-x)} & e^{ik_0(z_1-x)} & e^{ik_0(z_2-x)} \\ e^{ik_1(y-z_1)} & 1 & e^{ik_1(z_2-z_1)} \\ e^{ik_2(y-z_2)} & e^{ik_2(z_1-z_2)} & 1 \end{bmatrix} \tag{4.16}$$

Since the integrand is symmetric under interchange of k_i,z_i with k_j,z_j for $1\leq j<i\leq N$ we may rewrite (4.15) as

$$\sum_{N=0}^{\infty}\frac{(-2)^N}{N!}\prod_{i=0}^{N}\int\tilde{\rho}(k_i)\frac{dk_i}{2\pi}\prod_{i=1}^{N}\int_y^x dz_i\Big(1-\frac{i}{c}\sum_{k=1}^{N}k_{0N}\Big)\ D_N \tag{4.17}$$

For the M=1 term let us first perform the "untrapped" z_N integration. Since the integrand is explicity of order 1/c we simply pick out the leading term and then treat the R(k) and $R^*(p)$ as fermion operators.

To this order it is easy to see that the integrand $F_N^{1\prime}$ for $N\geq 1$ is given by

$$-\frac{(-2)^N}{c}e^{ik_0y}e^{-i(p_0+p_N-k_N)x}\int dz_1\ldots dz_{N-1}\prod_{i=1}^{N-1}e^{i(k_i-p_i)z_i}$$
$$\times\theta(y<z_1<\ldots<z_{N-1}<x)\left[\frac{\sum_{i=0}^{N}(p_i-k_i)}{p_N-k_N}-(N+1)\right] \qquad (4.18)$$

The second term in the square bracket does not contribute because it gives a symmetric expression in p_0 and p_N multiplying an antisymmetric fermion operator product. The first term in the square bracket is finite as $q\to 0$ when the p_j are set to $k_{P_j}+n_{P_j}q$ for any permutation P and any choice of the n_j. Thus, it is sufficient to sum over permutations of the k_i only. Clearly only those permutations which map k_N into itself survive in the limit $q\to 0$. Picking out the non-pole contribution according to the prescription (3.14) we obtain the contribution

$$-\frac{(-2)^N}{c}\int\prod_{i=0}^{N}\tilde{\rho}(k_i)\frac{dk_i}{2\pi}\Big]\int dz_1\ldots dz_{N-1}\theta(y<z_1<\ldots<z_{N-1}<x)\,D_{N-1} \qquad (4.19)$$

where D_{N-1} is the determinant typified by (4.16). Replacing the ordered integrations by unordered ones as before, separating out the trivial k_N integration, and summing over N, we obtain the M=1 contribution to order 1/c as

$$\frac{2}{c}\int\frac{dk}{2\pi}\,\tilde{\rho}(k)\sum_{N=0}^{\infty}\frac{(-2)^N}{N!}\int\prod_{i=0}^{N}\left[\frac{dk_i}{2\pi}\,\tilde{\rho}(k_i)\right]\int dz_1\ldots dz_N\;D_N \tag{4.20}$$

where we have relabelled the sum to run from N=0 to ∞.

The total two-point function through order 1/c is the sum of the M=0 and M=1 contributions (4.17) and (4.20). The c-dependence of these quantities is partly explicit and partly implicit in the quantities $\tilde{\rho}(k)$. In the zero temperature limit, $\tilde{\rho}(k)$ becomes a step function with support between $-k_F$ and k_F, where k_F is the Fermi pseudomomentum. The Fermi pseudomomentum is uniquely related to the density D by the equation D=G(0) which to this order in $\frac{1}{c}$ reads

$$D = \left(1+\frac{2k_F}{\pi c}\right)\frac{k_F}{\pi} \tag{4.21}$$

Here we will treat k_F rather than D as the independent variable, so that the zero temperature two-point function to order 1/c reads

$$G(x-y) = \sum_{N=0}^{\infty}\frac{(-i)^N}{N!}\int_{-k_F}^{k_F}\frac{dk_0}{2\pi}\cdots\frac{dk_N}{2\pi}\left(1+\frac{2k_F}{\pi c}-\frac{i}{c}\sum_{j=1}^{N}k_{0j}\right)$$

$$\times\int dz_1\ldots dz_N\;D_N \tag{4.22}$$

If we define the scaled variable

$$t = k_F(x-y) \tag{4.23}$$

it is possible to express these results in terms of an integral kernel

$$K(u,v) = \frac{\sin(u-v)}{u-v} \tag{4.24}$$

acting on the interval [0,t]. Let us define quantities $R(t,\lambda)$ and $D_1(t,\lambda)$ to be the usual resolvent kernel and first Fredholm minor, but with their arguments evaluated at the end points of the integration region, i.e.

$$R(t,\lambda) = \lambda K(0,t) + \lambda^2 \int_0^t dz\, K(0,z)\, K(z,t) + \ldots \tag{4.25a}$$

$$D_1(t,\lambda) = \lambda K(0,t) - \frac{\lambda^2}{2!}\int_0^t dz\, \det\begin{bmatrix} K(0,t) & K(0,z) \\ K(z,t) & K(z,z) \end{bmatrix} + \ldots \tag{4.25b}$$

With these definitions we easily recover the result of Schultz[3] and Lenard[4] for the infinite c case

$$G^{(0)}(t) = \frac{k_F}{2}\, D_1(t,\lambda=\tfrac{2}{\pi}) \tag{4.26}$$

After some manipulation it turns out that the first correction term $G^{(1)}$ may also be expressed in terms of the quantities (4.25)[8]

$$\frac{G^{(1)}}{G^{(0)}} = \frac{2k_F}{\pi}\Big[1+\Big(\frac{\partial^2 \ln R}{\partial t \partial \lambda} - \frac{\partial \ln R}{\partial t}\frac{\partial \ln R}{\partial \lambda}\Big) - \Big(\frac{\partial^2 \ln D_1}{\partial t \partial \lambda} - \frac{\partial \ln D_1}{\partial t}\frac{\partial \ln D_1}{\partial \lambda}\Big)\Big]_{\lambda=\frac{2}{\pi}} \tag{4.27}$$

The work of Jimbo, Miwa, Mori, and Sato[6] has shown that the quantities (4.25) may be expressed in terms of Painlevé transcendents. Let us define a function $\Phi(t,\lambda)$ by the differential equation and boundary

condition

$$\Phi'' = [(\Phi')^2-1]\cot\Phi + (1-\Phi')/t \tag{4.28a}$$

$$\Phi \sim t - \lambda t^2 \quad \text{as} \quad t \to 0 \tag{4.28b}$$

where the prime denotes differentiation with respect to t. [The function $\Phi(t,\lambda)$ is related to $y(t,\lambda)$ of Ref. 8, Eq.(7.98), by $y=e^{-2i\Phi}$; in terms of y the equation (4.28a) is a Painlevé equation of the fifth kind]. Then the resolvent $R(t,\lambda)$ and Fredholm minor $D_1(t,\lambda)$ may be expressed as

$$R = \frac{1-\Phi'}{2\sin\Phi} \tag{4.29a}$$

$$\frac{\partial \ln D_1}{\partial t} = \frac{t(\Phi'^2-1)}{4\sin^2\Phi} + \cot\Phi - \frac{1}{t} \tag{4.29b}$$

Since $D_1(t,\lambda) = \lambda$ at t=0, these equations completely specify $R(t,\lambda)$ and $D_1(t,\lambda)$, and hence the functions $G^{(0)}$ and $G^{(1)}$ in terms of the differential equation and boundary condition (4.28).

Of particular interest is the large distance behavior of the correlation function G(x-y). To obtain this we need to know the large t behavior of the differential equation (4.28). A discussion of this behavior, based on a combination of analytical and numerical work is contained in Appendix D. Using these results one may show that the long distance behavior of the infinite c 2-point function is given by

$$G^{(0)} = \frac{k_F}{\pi}\rho_\infty \frac{1}{\sqrt{t}}\left(1 - \frac{1+4\cos 2t}{32t^2} - \frac{3\sin 2t}{16t^3} + \ldots\right) \tag{4.30}$$

where ρ_∞ is a numerical constant given by

$$\rho_\infty = 0.924182203782$$

As shown by Vaidya and Tracy,[5] this constant is related to Glaisher's constant A by

$$\rho_\infty = \pi\, e^{1/2}\, 2^{-1/3}\, A^{-6} \tag{4.31}$$

For the first correction term $G^{(1)}$, we obtain, using (D.31), the long distance behavior

$$\frac{G^{(1)}}{G^{(0)}} = \frac{2k_F}{\pi}\left[\ln t + \gamma + 3\ln 2 - \frac{1}{2} + \frac{\sin 2t}{4t} + O\left(\frac{\ln t}{t^2}\right)\right] \tag{4.32}$$

Combining this with the $c=\infty$ result $G^{(0)}$, and normalizing the result to the value $D=G(0)$, which to this order is given by (4.21), we obtain the long distance behavior of the 2-point function through order $1/c$:

$$\frac{G(t)}{G(0)} = \rho_\infty\left[1 + \frac{2k_F}{\pi c}\left(\gamma + 3\ln 2 - \frac{3}{2}\right)\right]$$

$$\times\, t^{-\frac{1}{2}+\frac{2k_F}{\pi c}}\left[1 + \frac{k_F}{\pi c}\frac{\sin 2t}{2t} + O\left(\frac{\ln t}{t^2}\right)\right] \tag{4.33}$$

Note that we have interpreted a $\ln t$ term in $G^{(1)}$ as the first order expansion of an asymptotic power $t^{-\nu}$ with

$$\nu = \frac{1}{2} - \frac{2k_F}{\pi c} - O(1/c^2) \tag{4.34}$$

This result is in agreement with the results of Popov[19] and of Haldane[20]

who have used an effective long range theory to calculate the exact value of ν for general coupling c.

V. Discussion

To summarize our main result, the finite temperature two-point correlation function (3.1) is given by the series (3.18), where $\tilde{\rho}(k)$ is given by (3.17), (1.8), and (1.9), and the functions $\tilde{f}_N(k,x-y)$ are given by (3.14), (3.10), (3.5) (and its generalization to arbitrary N, c.f. Eq. (C.2)), (2.10), and (2.17b). We have also proven the existence of the limit $q \to 0$ in Eq. (3.10). This assures that there are no infrared divergence problems in computing the series term by term. At no time in the calculation do we have to put the system in a box or impose periodic boundary conditions. For the case of zero separation, x=y, the series (3.18) reproduces the thermodynamics derived by Yang and Yang,[11] and is in fact identical to the series generated by iteration of the integral equation (3.27). For $c=\infty$, (3.18) reduces to the series expansion of a first Fredholm minor determinant, the Schultz-Lenard result.[3-4] Our result for the $O(1/c)$ term in the large c expansion[8] was also obtained from the series (3.18). Thus, nearly all of the known results for the two-point function can be derived from this formula. [The known result for the asymptotic correlation exponent $\nu = 1/\{2[\rho(k_F)]^2\}$ has only been obtained to first order in the 1/c expansion, Eq. (4.34). It would be interesting to derive this result from (3.18) for arbitrary c.]

Unfortunately, a direct term-by-term analysis of the series (3.18) has so far not yielded a closed-form expression for the correlation function at arbitrary values of c. Explicit calculation of the first few terms in the series is straightforward, but the results have so far not been particularly illuminating. The complication of summing over

permutations of the p's and k's has prevented any further simplification of the series. In this respect the problem is not unlike those which arise in a direct computation of Bethe's ansatz matrix elements, e.g. Eq. (2.19). The potential advantage of the Gel'fand-Levitan approach is that it formulates the connection between the N-particle and the (N-1)-particle calculation. At the operator level this connection results from the fact that successive terms of the Gel'fand-Levitan series come from the iteration of a linear integral equation. The sum over permutations involved in computing matrix elements obscures this connection, although we have managed to obtain an induction formula for the symmetrized integrands of the field operator $\phi(x)$, Eq. (C.15)-(C.16). For the symmetrized integrands of the nonlocal product $\phi^*(x)\phi(y)$, we have not been able to derive a simple formula relating the N-particle integrand to the (N-1)-particle integrand. It is interesting that both the Yang-Yang and Schultz-Lenard integral equations as well as the monodromy arguments of Jimbo, et al. follow from such an induction property for the special cases x=y and $c=\infty$. Perhaps by a rearrangement or partial resummation of the series (3.18), some integral equation or generalized monodromy structure might be obtained for the general case. The results we have discussed here should provide a useful framework for further study of the correlation functions.

Appendix A

In this Appendix we prove the basic induction formula (3.11) for the symmetrized integrand of the operator expression $\phi^*(x)\phi(y)$. Let us write the unsymmetrized integrand in the form

$$F_N(k_0\cdots k_N, p_0\cdots p_N) = \frac{P_N(k_0\cdots k_N, p_0\cdots p_N)}{\prod_{i,j=0}^{N}(p_i - k_j)} \tag{A.1}$$

Similarly for the symmetrized integrand we write

$$\overline{F}_N(k_0\cdots k_N, p_0\cdots p_N) = \frac{\overline{P}_N(k_0\cdots k_N, p_0\cdots p_N)}{\prod_{i,j=0}^{N}(p_i - k_j)} \tag{A.2}$$

In terms of $\overline{P}_N$ the result (3.11) takes the form

$$\overline{P}_N(p_N = k_N) = \frac{-i}{(N+1)^2}\Big(\prod_{i=0}^{N-1} S_{Ni} S^{iN} - 1\Big)$$

$$\times\Big(\prod_{j=0}^{N-1} k_{Nj} p_{jN}\Big)\, \overline{P}_{N-1}(k_0\cdots k_{N-1}, p_0\cdots p_{N-1}) \tag{A.3}$$

Let us introduce a notation for the symmetrization process described in (3.5):

$$\overline{F}_N = \sum_{j=0}^{N} F_N \tag{A.4}$$

Since the denominator in (A.1) and (A.2) is symmetric in the k_i and the p_i we then have

$$\overline{P}_N = \sum_{j=0}^{N} P_N \tag{A.5}$$

With these preliminaries we will now prove the result (A.3). For notational simplicity, we will give the result only for the case N=2. Let us first consider the case x=y=0. Then the unsymmetrized quantity P_2 is given by

$$P_2 = (-c)^2\Big\{\sum_{i=0}^{2}(p_i-k_i)\Big\}(p_0-k_2)(p_0-k_1)(p_1-k_2)(p_2-k_0) \tag{A.6}$$

Let us perform the symmetrization by first averaging over the $3^2=9$ positions of k_2 and p_2, keeping the relative position of k_0 and k_1 and of p_0 and p_1 fixed. It is clear that when $p_2=k_2$ only 5 of these 9 permutations contribute and we obtain:

$$\begin{aligned}\overline{P}_2(p_2=k_2) = \frac{(-c)^2}{9}\sum_{j=0}^{1}\sum_{i=0}^{2}(p_i-k_i)\\ \times\Big\{&(p_0-k_2)(p_0-k_1)(p_1-k_2)(p_2-k_0)\\ &+ (p_0-k_1)(p_0-k_2)(p_1-k_1)(p_2-k_0)S_{21}\\ &+ (p_0-k_1)(p_0-k_2)(p_2-k_1)(p_1-k_0)S_{21}S^{12}\\ &+ (p_0-k_1)(p_0-k_0)(p_2-k_1)(p_1-k_2)S_{21}S^{12}S_{20}\end{aligned}$$

$$+ (p_2-k_1)(p_2-k_0)(p_0-k_1)(p_1-k_2)S_{21}S^{12}S_{20}S^{02}\} \qquad (A.7a)$$

Without loss of generality let us momentarily set $p_2=k_2=0$. Combining the above terms then yields

$$\begin{aligned}\overline{P}_2(p_2=k_2) = \frac{(-c)^2}{9} \sum_{j=0}^{1} \Big(\prod_{i=0}^{1} (p_i-k_i)\Big)(p_0-k_1) \\ \times\{-k_0p_0p_1(1+S_{21}) + k_0k_1p_0(1+S^{12})S_{21} \\ -k_1p_0p_1(1+S_{20})S_{21}S^{12} + k_0k_1p_1(1+S^{02})S_{21}S^{12}S_{20}\}\end{aligned} \qquad (A.7b)$$

Noting the identities

$$\begin{aligned}1+S_{ij} &= -\frac{i}{c}\, k_{ji}(S_{ij}-1) \\ 1+S^{ji} &= \frac{i}{c}\, p_{ji}(S^{ji}-1)\end{aligned} \qquad (A.8)$$

and using Galilean invariance to restore an arbitrary value for p_2 and k_2 we obtain

$$\begin{aligned}\overline{P}_2(p_2=k_2) = \frac{(-c)^2}{9} \sum_{j=0}^{1} \Big(\prod_{i=0}^{1} (p_i-k_i)\Big)(p_0-k_1) \\ \times\{\tfrac{i}{c}k_{02}k_{12}p_{02}p_{12}(S_{21}S^{12}S_{20}S^{02}-1)\}\end{aligned} \qquad (A.9)$$

Since the curly bracket is symmetric in k_0 and k_1 and in p_0 and p_1 we may rearrange (A.9) to give

$$\overline{P}_2(p_2=k_2) = \frac{-i}{9}(S_{21}S^{12}S_{20}S^{02}-1)k_{20}k_{21}p_{02}p_{12}$$

$$\times\{\sum_{j=0}^{1}(-c)(\prod_{j=0}^{1}(p_i-k_i))(p_0-k_1)\} \qquad (A.10)$$

But the expression in curly brackets is by definition $\overline{P}_1(k_0,k_1,p_0,p_1)$ so we have proven (A.3) for N=2 and x=y. The generalization to arbitrary N follows in a straightforward manner.

Let us now consider the case $x\neq y$. This is actually a simple extension of the x=y result. The only difference is that P_N is now split up into a sum of N+1 terms $P_N(k_0\ldots k_N,p_0\ldots p_N;m)$, $0\leq m\leq N$, in each of which $p_0\ldots p_m$ and $k_0\ldots k_{m-1}$ are associated with $\phi^*(x)$ and $p_{m+1}\ldots p_N$ and $k_m\ldots k_N$ are associated with $\phi(y)$. In the m-th term the factor $\Sigma(p_i-k_i)$ of $\phi^*(0)\phi(0)$ is replaced by

$$(p_m-k_m)\exp-i\{(\sum_{0}^{m}p_i-\sum_{0}^{m-1}k_i)x+(\sum_{m+1}^{N}p_i-\sum_{m}^{N}k_i)y\} \qquad (A.11)$$

Let $\overline{P}_N(k_0\ldots k_N,p_0\ldots p_N;m)$ denote the symmetrized version of $P_N(k_0\ldots k_N,p_0\ldots p_N;m)$, and let $\overline{P}_N(k_0\ldots k_N,p_0\ldots p_N;m,N)$, $m=0,1\ldots N-1$ denote those terms of $\overline{P}_N(k_0\ldots k_N,p_0\ldots p_N)$ in which m of $k_0\ldots k_{N-1}$ and m+1 of $p_0\ldots p_{N-1}$ are associated with $\phi^*(x)$ and the remainder associated with $\phi(y)$. Then by a simple extension of the x=y result we find

$$\overline{P}_N(p_N=k_N;m,N) = \frac{-i}{(N+1)^2}\prod_{i=0}^{N-1}(S_{Ni}S^{iN-1})$$

$$\times \prod_{i=0}^{N-1} k_{Ni}p_{iN}\,\overline{P}_{N-1}(k_0\cdots k_{N-1},p_0\cdots p_{N-1};m) \qquad (A.12)$$

[The essential point is that in order to get a non-zero result when $p_N=k_N$, p_N and k_N must both be associated with $\phi^*(x)$ or both with $\phi(y)$. In either case the exponential factor $e^{-i(p_N-k_N)x}$ or $e^{-i(p_N-k_N)y}$ is unity.] Summing (A.12) over m from 0 to N-1 then gives the desired result (A.3) for $x\neq y$.

Appendix B

The purpose of this Appendix is to prove that the effect of the pole terms f_N^{pole} in (3.14) is just to replace $\rho_0(k)$ by $\tilde{\rho}(k)=1/(1+e^{\beta\varepsilon(k)})$ in (3.15). From (3.9), (3.12), and (3.14) we see that the full two point function $G_{\beta,\mu}(x-y)$ is given by the following graphical prescription:

(1) Draw all possible tree graphs with one base vertex for each disconnected part. A base vertex has no incoming lines and an arbitrary number of outgoing lines, while all other vertices have one incoming line and an arbitrary number (≥ 0) of outgoing lines. For each such graph let (M+1) denote the number of base vertices (i.e. the number of disconnected parts) and (N+1) the total number of vertices. Label the vertices as $k_0 \ldots k_N$.

(2) Assign a factor $\prod_{i=0}^{N} (-1)^{n_i} e^{n_i\beta(\mu-k_i^2)}$ to each vertex.

(3) For each line joining k_i to k_j include a factor $\frac{n_i}{n_j}\times\Delta(k_i-k_j)$ where Δ is as in (1.9), and where vertex k_i is nearer to, and k_j further from, the base vertex.

(4) Sum over the n_i and integrate over the k_i $\left[\sum_{n_i=1}^{\infty} \text{ and } \int_{-\infty}^{\infty} \frac{dk_i}{2\pi}\right]$

(5) Include a weight factor $\frac{1}{(N+1)!}\tilde{f}_M(k_{i_0}\ldots k_{i_M})$, where $k_{i_0}\ldots k_{i_M}$ are the base vertices, and sum over all topologically distinct diagrams and all distinct labellings of the vertices.

More conveniently the last instruction may be replaced by

(5') Fix the labelling of the base vertices to be $k_0\ldots k_M$ and sum over all topologically distinct diagrams with weight factor $\tilde{f}_M(k_0\ldots k_M)$.

It is now clear that the two-point function is of the form

$$G_{\beta,\mu}(x-y) = \sum_{M=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{2\pi}_0 \ldots \frac{dk}{2\pi}_M \tilde{f}_M(k_0 \ldots k_M; x-y)\ \tilde{\rho}(k_0) \ldots \tilde{\rho}(k_M) \qquad (B.1)$$

where $\tilde{\rho}(k_0)$ is given by the rules (1)-(4) together with:

(5$\tilde{\rho}$) Sum over topologically distinct graphs with M=0 (i.e. connected directed tree graphs) with base vertex k_0, and suppress the integral dk_0.

To show that $\tilde{\rho}(k)$ is given by (3.17) let us define a quantity $\sigma(k_0)$ by the rules (1)-(4) together with:

(5σ) Sum over topologically distinct graphs with M=0 and only one line, say (k_0,k_1), joining the base vertex k_0 to the rest of the diagram. Replace the factor for this line by $\frac{1}{\beta n_1}\Delta$ and suppress the integral $\frac{dk}{2\pi}_0$ and the sum over n_0 [n_0 does not appear].

In terms of this $\sigma(k)$ let us define $\varepsilon(k)$ by

$$\varepsilon(k) = k^2-\mu+\sigma(k) \qquad (B.2)$$

Then it is a simple graphical exercise to verify that $\tilde{\rho}(k)$ is given by

$$\tilde{\rho}(k) = \frac{1}{1+e^{\beta\varepsilon(k)}} \qquad (B.3)$$

and that $\varepsilon(k)$ satisfies the nonlinear integral equation

$$\varepsilon(k) = k^2 - \mu - \frac{1}{\beta}\int \frac{dq}{2\pi}\, \Delta(k-q)\, \ln(1+e^{-\beta\varepsilon(q)}) \qquad (B.4)$$

The result (B.1) with $\tilde{\rho}(k,\beta,\mu)$ given by (B.3) and (B.4), shows that the two-point function $G_{\beta,\mu}(x-y)$ may be obtained from the non-pole pieces $\tilde{f}_N(k,x-y)$ of the quantity $f_N(k,n,x-y)$ in (3.14). It is also possible to express $G_{\beta,\mu}(x-y)$ in terms of the quantity $f_N^*(k,x-y)$ obtained from $f_N(k,n,x-y)$ by setting all the n_i's equal to unity:

$$f_N^*(k,x-y) = f_N(k,n_i=1,x-y) \tag{B.5}$$

That this is possible is obvious a priori, since $f_N^*(k,x-y)$ contains the non-pole piece $\tilde{f}_N(k,x-y)$ and the pole piece can be expressed in terms of non-pole pieces of lower order. What is not immediately obvious is that when this is done the two-point function can be expressed in a factorized form similar to (B.1):

$$G_{\beta,\mu}(x-y) = \sum_{M=0}^{\infty} \int \prod_{i=0}^{M} \left(\frac{dk_i}{2\pi}\, \rho^*(k_i)\right) f_M^*(k,x-y) \tag{B.6}$$

In the remainder of this Appendix we will show that (B.6) is correct with the function $\rho^*(k,\beta,\mu)$ given in terms of $\tilde{\rho}(k,\beta,\mu)$ by

$$\rho^*(k) = \tilde{\rho}(k)\, \exp\left(-\int \frac{dk'}{2\pi} \Delta(k-k')\, \tilde{\rho}(k')\right) \tag{B.7}$$

To see this, let us define $f_N^*(k,x-y)$ by (B.5) and introduce a quantity

$$G(x-y) = \sum_{M=0}^{\infty} \int \prod_{i=0}^{M} \left(\frac{dk_i}{2\pi}\, r(k_i)\right) f_M^*(k,x-y) \quad , \tag{B.8}$$

where $r(k)$ is, for the moment, an arbitrary function. The strategy is to