LBL 17639
UCB PTH-84/12
Fermilab PUB-84/38-T

April 1984

CONSISTENT AND COVARIANT ANOMALIES IN GAUGE AND GRAVITATIONAL THEORIES

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Abstract

The gauge structure of anomalies and the related currents is analysed in detail. We construct the covariant forms for both the currents and the anomalies for general gauge theories in even-dimensional space-time. The results are then extended to determine the structure of gravitational anomalies. These can always be interpreted as anomalies for local Lorentz transformations.

This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contracts DE-AC03-76SF00098 and DE-AC02-76-CHO-3000 and in part by the National Science Foundation under research grant PHY-81-18547

1. Introduction

The gauge principle is used as the fundamental basis for present theories of all known forces, from electromagnetism to gravitation. Anomalies [1-5] result when gauge invariance cannot be maintained in the quantum theory. A complete understanding of anomalies is essential for the full application of these theories to physical problems.

The anomaly is usually defined as the gauge variation of the connected vacuum functional in the presence of external gauge fields. When an anomaly occurs, this variation does not vanish and the vacuum functional is not gauge invariant. The gauge currents are no longer covariantly conserved but have the anomalies as their divergence. As a consequence of its definition the anomaly satisfies certain consistency conditions [6] which restricts its functional form. For the non-singlet, non-abelian, chiral anomaly, the consistency conditions imply that the anomaly cannot have a covariant expression. Similarly, the anomaly implies that the non-singlet gauge currents cannot have covariant transformation properties.

However, a number of authors [7-9] have recently presented explicit calculations of the non singlet anomaly and have obtained covariant results. The same situation occurs for the case of gravitational anomalies. In the work by Alvarez-Gaumé and Witten [10] they are presented in covariant form, while the gravitational consistency conditions would imply that they should have a non-covariant form.

In this paper, we clarify the situation by showing that both the covariant and the non-covariant anomalies can be correct forms for the covariant divergence of different currents. For the gravitational anomalies, the two forms correspond to different energy momentum tensors.

We shall use the term "consistent" anomaly to refer to the covariant divergence of the current J_μ obtained by varying the vacuum functional with respect to the external gauge potential. The "covariant" anomaly is obtained by modifying the current by adding to it a local function of the gauge potential. The resulting current $\tilde{J}_\mu$ is determined so as to be covariant under local gauge transformations, which implies that its covariant divergence is also covariant. The consistent anomaly has fundamental significance, since it reflects directly the gauge dependence of the vacuum functional. The related covariant anomaly, on the other hand, is distinguished by its simple gauge transformation properties and the the covariant current may have significance when used to construct gauge invariant couplings to other fields. As shown in this paper, it is always possible to construct the covariant forms of the current and of the anomaly from the knowledge of the consistent anomaly. Hence the anomaly cancellation conditions are the same for either form. We note that our ability to modify the form of the anomaly by changing the definition of the local currents is different from the ambiguity in the form of the anomaly arising from the addition of local functions of the gauge potential to the vacuum functional [3].

Let us illustrate the situation by the case of non-abelian gauge anomalies in two space-time dimensions. The consistent anomaly is known to be[F1]

$$\mathcal{D}_\mu J^\mu = c\ \partial_\mu A_\lambda\, \varepsilon^{\lambda\mu} \qquad (1.1)$$

where c is a certain constant and a matrix notation has been used for both the current and the gauge potential. The right hand side satisfies the consistency condition [6] but is non-covariant. The current J^μ also

transforms non-covariantly. We now define a new current

$$\tilde{J}^\mu = J^\mu + c\, A_\lambda\, \varepsilon^{\lambda\mu} . \qquad (1.2)$$

Its covariant divergence is

$$\begin{aligned}\mathcal{D}_\mu \tilde{J}^\mu &= c\ \partial_\mu A_\lambda\, \varepsilon^{\lambda\mu} + \partial_\mu (c A_\lambda\, \varepsilon^{\lambda\mu}) \\ &+ c\,[A_\mu, A_\lambda]\, \varepsilon^{\lambda\mu} = c\, F_{\mu\lambda}\, \varepsilon^{\lambda\mu} ,\end{aligned} \qquad (1.3)$$

where

$$F_{\mu\lambda} = \partial_\mu A_\lambda - \partial_\lambda A_\mu + [A_\mu, A_\lambda] \qquad (1.4)$$

is the Yang-Mills field strength. The right hand side of (1.3) is now covariant. The current $\tilde{J}^\mu$ may also be shown to be covariant, but it cannot be obtained from the variation of a vacuum functional with respect to the gauge field A_μ, since the covariant anomaly does not satisfy the consistency condition. Observe that the linearized right hand side of (1.3) is twice the right hand side of (1.1) (this factor becomes $1 + \nu/2$ in ν dimensions and may be considered as a Bose symmetry factor for the linearized anomaly). We emphasize the care which is needed in interpreting the linearized calculations.

In this paper we discuss various aspects of the gauge structure of anomalies and their currents. In Chapter 2 we study the gauge dependence of the currents and their anomalies and apply conventional methods to construct the covariant currents and anomalies for four-dimensional gauge theories. In Chapter 3 we discuss the structure of the consistent anomaly in arbitrary even space-time dimensions and give also the explicit expressions for the covariant currents and the covariant anomalies. This is done by using the compact notation of exterior differential forms and the techniques described in Refs. [11-16]. The needed results are collected and, in part, rederived in Appendix A.

Our results are generalized to include gravitational anomalies in Chapter 4. A theory with spinor fields in curved space must be formulated so that it is covariant under general coordinate transformations (which we shall call Einstein transformations) as well as under local Lorentz transformations. Local Lorentz invariance of the connected vacuum functional is usually assumed and the gravitational anomalies are taken to be anomalies of the Einstein transformations. In Chapter 5 we shall formulate the consistency conditions for the combined Einstein and Lorentz anomalies [13] and we shall find the form of these anomalies. We also show that the Einstein anomalies can always be transformed into Lorentz anomalies (and vice versa) by adding local corrections to the vacuum functional. Hence it is always possible to define the vacuum functional so that all gravitational anomalies are indeed violations of local Lorentz invariance alone. This appears to us a preferred canonical form for the gravitational anomalies. The treatment of gravitational anomalies in Chapters 4 and 5 relates their structure to that of gauge anomalies.

Throughout this paper the anomalies will be expressed in terms of symmetric invariant polynomials which shall not be further specified. The particular polynomial appropriate to each situations depends on the spin of the particles propagating in the loops of the vacuum functional and can be determined by an explicit perturbation calculation, as done in the paper by Alvarez-Gaumé and Witten [10] for the gravitational anomaly. The correct polynomial can also be determined directly from the appropriate index theorem. This approach will be

discussed in a subsequent paper by Alvarez, Singer and Zumino.[17]

2. Gauge Structure of Currents and Anomalies in Four Dimensions

The consistent anomaly is determined by the gauge dependence of the vacuum functional defined in presence of external gauge fields $A_\mu^a(x)$. The vacuum functional W[A] may be considered as a non-local function of these gauge fields. Under infinitesimal gauge transformations the gauge potentials transform according to

$$T_\Lambda A_\mu{}^a = (D_\mu \Lambda)^a = (\partial_\mu \Lambda + [A_\mu, \Lambda])^a$$
$$T_\Lambda F_{\mu\nu}{}^a = ([F_{\mu\nu}, \Lambda])^a \quad , \tag{2.1}$$

where Λ^a is the infinitesimal gauge parameter. The gauge dependence of the vacuum functional defines the anomaly

$$\begin{aligned} T_\Lambda W[A] &= \int dx \frac{\delta W}{\delta A_\mu^a} T_\Lambda A_\mu^a \\ &= \int dx\, J^\mu{}_a(x)\,(D_\mu \Lambda)^a \\ &= -\int dx\, D_\mu J^\mu{}_a(x)\,\Lambda^a(x) \\ &= \int dx\, \Lambda^a(x)\, G_a(A) \quad , \end{aligned} \tag{2.2}$$

where $G_a(A)$ is the anomaly and the current $J^\mu{}_a(x)$ is defined as the functional derivative of the vacuum functional.

The consistency condition follows from considering the commutator of two gauge transformations on the vacuum functional ,

$$(T_\Lambda T_{\Lambda'} - T_{\Lambda'} T_\Lambda)\, W[A] = T_{[\Lambda,\Lambda']} W[A] . \tag{2.3}$$

Using (2.2) this implies

$$\int dx\,(\Lambda'^a T_\Lambda G_a - \Lambda^a T_{\Lambda'} G_a) = \int dx\, [\Lambda, \Lambda']^a G_a . \tag{2.4}$$

The consistent anomaly must obey this consistency condition (2.4).

The consistent anomaly also determines the gauge dependence of the basic non-abelian current $J^\mu{}_a$. Naively, this current would be expected to transform covariantly under gauge transformations. The effect of the anomalies can be determined by evaluating in two ways the commutator of a gauge variation and the variation which defines the current

$$(\delta_B T_\Lambda - T_\Lambda \delta_B)\, W[A] \quad , \tag{2.5}$$

where δ is defined by

$$\delta_B A_\mu{}^a = B_\mu{}^a \tag{2.6}$$

and

$$\delta W[A] = \int dx \frac{\delta W}{\delta A_\mu{}^a} \delta A_\mu{}^a = \int dx\, J^\mu{}_a(x) B_\mu{}^a(x) . \tag{2.7}$$

The commutator may be evaluated directly

$$\delta_B T_\Lambda - T_\Lambda \delta_B = \delta_{[B,\Lambda]} \quad . \tag{2.8}$$

Applying this operator to the vacuum functional we obtain

$$\begin{aligned} &(\delta_B T_\Lambda - T_\Lambda \delta_B)\, W[A] \\ &\quad = \int dx \left\{ (\delta_B G_a)\Lambda^a - (T_\Lambda J^\mu{}_a) B_\mu{}^a \right\} \\ &\quad = \int dx\, J^\mu{}_a \left([B, \Lambda]\right)^a . \end{aligned} \tag{2.9}$$

This gives immediately the gauge transformation properties of the non-abelian current

$$\int dx \left(T_\Lambda J^\mu{}_a\right) B_\mu{}^a = -\int dx \left(\left[\Lambda, J^\mu\right]\right)_a B_\mu{}^a + \int dx \left(\delta_B G_a\right) \Lambda^a . \quad (2.10)$$

The first term on the right hand side of (2.10) gives the usual transformation property of the current while the second term is dictated by the consistent anomaly. The basic current $J^\mu{}_a$ will only be covariant if the anomaly vanishes.

We shall now demonstrate the existence of a covariant non-abelian current $\tilde{J}_\mu{}^a$ and compute its covariant divergence. This result was obtained independently by Paranjape and Goldstone [18] and can also be inferred from some work by Niemi and Semenoff[19]. In subsequent chapters we shall generalize these results to gauge and gravitational anomalies in higher dimensional space times.

To construct the covariant non-abelian current we must find a local polynomial in the gauge potential, $X^\mu{}_a(A)$, with an anomalous gauge transformation property opposite to that of the basic current

$$\int dx \left(T_\Lambda X^\mu{}_a\right) B_\mu{}^a = -\int dx \left(\left[\Lambda, X^\mu\right]\right)_a B_\mu{}^a - \int dx \left(\delta_B G_a\right) \Lambda^a . \quad (2.11)$$

The covariant current is then given by

$$\tilde{J}^\mu{}_a = J^\mu{}_a + X^\mu{}_a(A) \quad , \qquad (2.12)$$

since (2.10 and (2.11) imply

$$T_\Lambda \tilde{J}^\mu{}_a = -\left(\left[\Lambda, \tilde{J}^\mu\right]\right)_a \quad . \qquad (2.13)$$

It is not obvious that an appropriate local expression $X^\mu{}_a(A)$ can always be found.

In four dimensions, the consistent non-abelian anomaly for spin one-half fermions is well known [3-4]:

$$G^a(x) = -\frac{1}{24\pi^2} \varepsilon^{\mu\nu\rho\sigma} \operatorname{Tr}\left\{\lambda^a \partial_\mu \left(A_\nu \partial_\rho A_\sigma + \frac{1}{2} A_\nu A_\rho A_\sigma\right)\right\}, \quad (2.14)$$

where Tr is the trace over Fermi multiplets and λ^a is the gauge coupling matrix. The equation (2.11) for $X^\mu{}_a$ becomes

$$\begin{aligned} \int dx \left\{T_\Lambda X^\mu{}_a + \left(\left[\Lambda, X^\mu\right]\right)_a\right\} B_\mu{}^a &= -\int dx \left(\delta_B G_a\right) \Lambda^a \\ = \frac{i}{48\pi^2} \int dx\, \varepsilon^{\mu\nu\rho\sigma}\, \partial_\mu \Lambda^a\, B_\nu{}^b \,&\cdot \qquad\qquad (2.15) \\ \cdot \operatorname{Tr}\Big\{\left(\lambda_a \lambda_b + \lambda_b \lambda_a\right) F_{\rho\sigma} & \\ - \lambda_a \lambda_b A_\rho A_\sigma - \lambda_b \lambda_a A_\rho A_\sigma - \lambda_a A_\rho \lambda_b A_\sigma \Big\} &. \end{aligned}$$

From the three possible terms for the polynomial $X^\mu{}_a$ we find the unique result

$$\begin{aligned} X^\mu{}_a = \frac{1}{48\pi^2} \varepsilon^{\mu\nu\rho\sigma} &\cdot \\ \cdot \operatorname{Tr}\left\{\lambda_a \left(A_\nu F_{\rho\sigma} + F_{\rho\sigma} A_\nu - A_\nu A_\rho A_\sigma\right)\right\} &. \quad (2.16) \end{aligned}$$

By applying a group transformation to (2.16) we can reproduce (2.15).

We may now compute the covariant anomaly $\tilde{G}_a$, by a direct evaluation of the covariant divergence of the current

$$\tilde{G}_a = -\mathcal{D}_\mu \tilde{J}^\mu{}_a = G_a - \mathcal{D}_\mu X^\mu{}_a$$
$$= -\frac{1}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} \mathrm{Tr}\left\{\lambda_a F_{\mu\nu} F_{\rho\sigma}\right\}. \qquad (2.17)$$

We observe that the covariant anomaly may be expressed solely in terms of a product of field strengths as expected by covariance. The linearized form of the consistent anomaly (2.14) and of the covariant anomaly (2.17) are the same except that the covariant anomaly is three times larger.

We emphasize the need for a complete specification of the structure of the anomalous currents before the gauge anomalies can be properly interpreted. The consistent anomaly is directly related to the gauge dependence of the vacuum functional. It is appropriate for the study of anomaly cancellation between fermion multiplets but also for the derivation of physical consequences of anomalous non-dynamical currents such as the flavor chiral currents in QCD [6] [20,21]. The covariant current, on the other hand, has a simple gauge structure and may have physical significance when coupled to other external non-gauge fields. Since the covariant anomaly is directly related to the consistent anomaly, it may also be used to study anomaly cancellation. In the above discussion we have focussed on the ambiguities in defining appropriate non-abelian currents. There is also the ambiguity in defining the vacuum functional, as one is always free to modify the vacuum functional by adding local polynomials in the gauge fields. This freedom is exploited when we use the functional for gauging dynamically different anomaly free subgroups.

3. Chiral Anomalies in Higher Dimensions

Following the notation of Refs. [11,12] we now describe the Yang-Mills field strength by means of the Lie-algebra valued 2-form

$$F = dA + A^2 \qquad (3.1)$$

where d denotes exterior differentiation, and

$$A = A_\mu dx^\mu \qquad (3.2)$$

is the gauge potential 1-form. Explicitly

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \qquad (3.3)$$

(the differentials dx^μ anticommute). Let

$$\mathcal{P}(F_1, F_2, \cdots F_n) \qquad (3.4)$$

be a symmetric invariant polynomial of degree n in the Lie-algebra valued variables $F_1, \ldots F_n$. For compact notation, if some of the F_i's are equal, say $F_4 = F_5 = \ldots F_n = F$, we shall write (3.4) as

$$\mathcal{P}(F_1, F_2, F_3, F^{n-3}). \qquad (3.5)$$

Using the Bianchi identities for the field strength,

$$dF = FA - AF \; , \qquad (3.6)$$

one shows easily that

$$d\,\mathcal{P}(F^n) = 0 \; . \qquad (3.7)$$

Actually one can write

$$\mathcal{P}(F^n) = d\,\omega_{2n-1}(A, F) \; , \qquad (3.8)$$

where the (2n-1)-form is given by

$$\omega_{2n-1}(A,F) = n\int_0^1 dt\, P(A, F_t^{\,n-1}) \qquad (3.9)$$

with

$$F_t = t\,dA + t^2A^2 = tF + (t^2-t)A^2 . \qquad (3.10)$$

The consistent non-singlet anomaly is obtained as follows. Introduce an odd (anticommuting) Lie-algebra valued element v and an infinitesimal gauge transformation $\mathcal{J}$

$$\begin{aligned} \mathcal{J}A &= -Dv = -dv - \{A, v\} \\ \mathcal{J}F &= Fv - vF \\ \mathcal{J}v &= -v^2 \end{aligned} \qquad (3.11)$$

which satisfies

$$\mathcal{J}^2 = d\mathcal{J} + \mathcal{J}d = d^2 = 0 . \qquad (3.12)$$

$\mathcal{J}$ is the generator of a Becchi-Rouet-Stora transformation[22]. If we introduce

$$\mathcal{A} = A + v \qquad (3.13)$$

and a corresponding field strength

$$\mathcal{F} = (d+\mathcal{J})\mathcal{A} + \mathcal{A}^2 \qquad (3.14)$$

we find easily that

$$\mathcal{F} = F . \qquad (3.15)$$

Therefore

$$\begin{aligned} (d+\mathcal{J})\,\omega_{2n-1}(A+v, F) &= P(F^n) \\ &= d\omega_{2n-1}(A,F) . \end{aligned} \qquad (3.16)$$

Let us expand in powers of v

$$\omega_{2n-1}(A+v, F) = \omega_{2n-1}{}^{0} + \omega_{2n-2}{}^{1} + \cdots + \omega_0{}^{2n-1} , \qquad (3.17)$$

where the superscript indicates the power of v and the subscript the degree of the form. Equation (3.16) implies a set of relations

$$\left.\begin{aligned} \mathcal{J}\omega_{2n-1}{}^{0} + d\,\omega_{2n-2}{}^{1} &= 0 \\ \mathcal{J}\omega_{2n-2}{}^{1} + d\,\omega_{2n-3}{}^{2} &= 0 \\ - - - - & \\ \mathcal{J}\omega_1{}^{2n-2} + d\,\omega_0{}^{2n-1} &= 0 \\ \mathcal{J}\omega_0{}^{2n-1} &= 0 \end{aligned}\right\} \qquad (3.18)$$

The consistent anomaly is given by the integral of $\omega_{2n-2}{}^{1}$. The consistency condition, which can be written as

$$\mathcal{J}\int \omega_{2n-2}{}^{1} = 0 , \qquad (3.19)$$

follows from the second of (2.19), the second terms integrates to zero. One can derive a convenient explicit formula for the anomaly [11]

$$\int \omega_{2n-2}{}^{1} = n(n-1)\int_0^1 dt(1-t)\int P(dv, A, F_t^{\,n-2}) . \qquad (3.20)$$

In the gauge transformation (3.11) the infinitesimal parameter is odd (anticommuting) and transforms like a Faddeyev-Popov ghost. If one prefers, one can rewrite the consistency condition in terms of gauge transformations T_Λ with even (commuting) infinitesimal parameter Λ

$$T_\Lambda A = D\Lambda = d\Lambda + [A,\Lambda]$$
$$T_\Lambda F = [F,\Lambda] \tag{3.21}$$

and T_Λ does not operate on the parameter itself. The anomaly, with Λ replaced for v, is a linear functional of Λ. Denoting it with $\Lambda\cdot G[A,F]$ (the dot indicates integration as well as summation over internal symmetry indices), it satisfies the consistency condition

$$T_\Lambda \Lambda'\cdot G - T_{\Lambda'} \Lambda\cdot G = [\Lambda,\Lambda']\cdot G \ , \tag{3.22}$$

which is equivalent to (3.19). This is the form used in Section 2; it follows from the definition

$$T_\Lambda W[A] = \Lambda\cdot G \tag{3.23}$$

and justifies the above construction. If we define the current (n-1)-form (which is dual to the usual current vector J^μ)

$$J = \frac{\delta W}{\delta A} \ , \tag{3.24}$$

(3.23) can be written as

$$\Lambda\cdot DJ \equiv \Lambda\cdot\left(dJ + \{A,J\}\right) = \Lambda\cdot G \tag{3.25}$$

(remember that J is odd).

How does J transform under gauge transformations? As explained in Section 2 we evaluate in two ways the commutator

$$\left(\delta\, T_\Lambda - T_\Lambda\, \delta\right) W[A] \tag{3.26}$$

where δ is defined by

$$\delta A = B \ , \ \delta F = DB = dB + \{A,B\} \ . \tag{3.27}$$

Here, the increment B is odd and the operation δ is even. Since

$$T_\Lambda = D\Lambda\cdot\frac{\delta}{\delta A} = \left(dA + [A,\Lambda]\right)\cdot\frac{\delta}{\delta A} \ , \tag{3.28}$$

the commutator equals

$$[B,\Lambda]\cdot\frac{\delta W}{\delta A} = [B,\Lambda]\cdot J \tag{3.29}$$

On the other hand, using (3.23), and again (3.24), the commutator equals

$$\delta\left(\Lambda\cdot G\right) - T_\Lambda\left(B\cdot J\right). \tag{3.30}$$

Equating (3.29) with (3.30) we obtain

$$\begin{aligned} T_\Lambda(B\cdot J) &= -[B,\Lambda]\cdot J + \delta(\Lambda\cdot G) \\ &= -B\cdot[\Lambda,J] + \delta(\Lambda\cdot G) \ . \end{aligned} \tag{3.31}$$

The first term in the right hand side would be the covariant transformation law appropriate to the adjoint representation. When there is an anomaly the second term shows that J does not transform covariantly. In (3.31) B is taken not to change under the gauge transformation generated by T_Λ. If instead we stipulate that B transforms according to the adjoint representation

$$T_\Lambda B = [B,\Lambda] \ , \tag{3.32}$$

(3.31) becomes simply

$$T_\Lambda (B \cdot J) = \delta(\Lambda \cdot G) . \qquad (3.33)$$

Together with (3.32) and

$$\delta \Lambda = 0 \quad , \qquad (3.34)$$

(3.33) completely specifies the transformation law of the current J in terms of the anomaly $\Lambda \cdot G[A,F]$.

Is it possible to find a <u>local</u> $(\nu -1)$-form X such that the new current

$$\tilde{J} = J + X \qquad (3.35)$$

transforms covariantly? This means

$$T_\Lambda (B \cdot \tilde{J}) = 0 \qquad (3.36)$$

and therefore we must require

$$T_\Lambda (B \cdot X) = - \delta(\Lambda \cdot G) \qquad (3.37)$$

This equation for X can be rewritten in terms of the anticommuting parameter v, instead of Λ

$$\mathcal{J}(B \cdot X) = \delta \int \omega_{2n-2}{}^{1} (v, A, F), \qquad (3.38)$$

with

$$\delta v = 0 \; . \qquad (3.39)$$

Here we have used the fact that the anomaly is given by

$$v \cdot G[A,F] = \int \omega_{2n-2}{}^{1} (v, A, F). \qquad (3.40)$$

Now, it is very easy to solve (3.38) in general. We use the relation, explained in Appendix A,

$$\delta = d\ell + \ell d \quad , \qquad (3.41)$$

where δ is defined by (3.27), (3.39), d is the exterior differentiation, and the odd operations ℓ is given by

$$\ell A = 0 \; , \quad \ell F = B \quad , \quad \ell v = 0 \, . \qquad (3.42)$$

Applying (3.41) to $\omega_{2n-2}{}^{1}$ we find

$$\begin{aligned} \delta \omega_{2n-2}{}^{1} &= d(\ell \omega_{2n-2}{}^{1}) + \ell \, d\omega_{2n-2}{}^{1} \\ &= d(\ell \omega_{2n-2}{}^{1}) - \ell \mathcal{J} \omega_{2n-1}{}^{0} \; , \end{aligned} \qquad (3.43)$$

where we have used (3.18). Now, the operators ℓ and $\mathcal{J}$ anticommute

$$\ell \mathcal{J} + \mathcal{J} \ell = 0 \; . \qquad (3.44)$$

Upon integration over (compactified) space-time, the first term in the right hand side of (3.43) vanishes and we obtain finally

$$\delta \int \omega_{2n-2}{}^{1} = \mathcal{J} \int \ell \, \omega_{2n-1}{}^{0} \; . \qquad (3.45)$$

Clearly we can drop the superscript zero in the right hand side. Comparing with the equation for X, (3.38), we see that it is solved by

$$B \cdot X = \int \ell \omega_{2n-1} \quad . \qquad (3.46)$$

The explicit formula (3.9) for ω_{2n-1} can be used to find an explicit formula for X, since the operator ℓ is easy to apply and, from (3.10),

$$\ell F_t = t B \; . \qquad (3.47)$$

One finds easily

$$\mathcal{B}\cdot X = n(n-1)\int_0^1 dt\, t \int P(\mathcal{B}, A, F_t^{\,n-2}) . \qquad (3.47)$$

For instance, for an internal symmetry such as SU(N) in four dimensions, we start from

$$\mathcal{P}(F^3) = c\, \mathrm{Tr}\, F^3 \qquad (3.49)$$

(where c is a known constant). Then

$$\omega_5 = c\, \mathrm{Tr}\left(F^2 A - \tfrac{1}{2} F A^3 + \tfrac{1}{10} A^5\right) \qquad (3.50)$$

and, applying ℓ directly,

$$\begin{aligned} \ell\omega_5 &= c\, \mathrm{Tr}\, B\left(FA + AF - \tfrac{1}{2} A^3\right) \\ &= c\, \mathrm{Tr}\, B\left(dA\,A + A\,dA + \tfrac{3}{2} A^3\right) , \end{aligned} \qquad (3.51)$$

which gives

$$X = c\left(dA\,A + A\,dA + \tfrac{3}{2} A^3\right) . \qquad (3.52)$$

Alternatively one can use (3.48) with exactly the same result

$$\mathrm{Tr}\, B X = 6c\ \mathrm{STr}\ BA\left(\tfrac{1}{3} F - \tfrac{1}{12} A^2\right) , \qquad (3.53)$$

where STr is the totally symmetrized trace (See Ref. [11]).

Since the current $\tilde{J} = J + X$ transforms covariantly, its covariant divergence must also be covariant. In order to compute it we need $DX = dX + \{A,X\}$. The simplest way to obtain this is to observe that, integrating by parts,

$$v . DX = Dv \cdot X \qquad (3.54)$$

(remember that v is odd), and it would seem that the right hand side can be obtained directly from (3.48) just by making the substitution

$$\mathcal{B} = Dv . \qquad (3.55)$$

Strictly speaking this is not allowed since both B and v are odd and so is the operation D. To be precise we must first rewrite (3.48) as

$$C\cdot X = n(n-1)\int_0^1 dt\, t \int P(C, A, F_t^{\,n-2}) , \qquad (3.56)$$

where C is even. Actually (3.56) follows from (3.48), and vice-versa. Now we can set correctly, in (3.56),

$$C = Dv , \qquad (3.57)$$

which gives

$$Dv . X = n(n-1)\int_0^1 dt\, t \int P\left(dv + \{A,v\}, A, F_t^{\,n-2}\right) . \qquad (3.58)$$

We also know that

$$v . DJ = Dv . J = n(n-1)\int_0^1 dt\,(1-t) \int P\left(dv, A, F_t^{\,n-2}\right) , \qquad (3.59)$$

where we have used the explicit form (3.20) for the anomaly. Adding (3.58) and (3.59) we obtain

$$\begin{aligned} v . D\tilde{J} &= n(n-1)\int_0^1 dt \int P\left(dv + t\{A,v\}, A, F_t^{\,n-2}\right) \\ &= n(n-1)\int_0^1 dt \int P\left(v, dA + t\{A,A\}, F_t^{\,n-2}\right) = \end{aligned}$$

$$= n\int_0^1 dt\,\frac{d}{dt}\int \mathcal{P}(v, F_t^{\,n-1})$$
$$= n\int \mathcal{P}(v, F^{\,n-1})\ . \qquad (3.60)$$

In going from the first to the second form of this expression we have used the invariance of the symmetric polynomial P. The last expression (3.60) shows the covariant form of the anomaly. This result should be compared with (3.59) or equivalently

$$v\cdot\mathcal{D}J = \int \omega_{2n-2}{}'(v, A, F)\ . \qquad (3.61)$$

Now, it is clearly

$$\omega_{2n-1}(A,F) = \mathcal{P}(A, F^{\,n-1}) + \cdots \qquad (3.62)$$

where the dots denote higher non linear terms. This implies that

$$\omega_{2n-2}{}'(v, A, F) = \mathcal{P}(v, F^{\,n-1}) + \cdots\ . \qquad (3.63)$$

Therefore the leading (least non linear) term in (3.60) is n times larger than the leading term in (3.61). The relation between n and the dimension ν of space time is

$$\nu + 2 = 2n\ . \qquad (3.64)$$

As mentioned in the introduction, this factor can be understood diagrammatically as a result of Bose symmetrization.

4. Purely Gravitational Anomalies

Infinitesimal Einstein transformations are specified in terms of infinitesimal parameters $\xi^\mu(x)$ and operate on tensors as Lie derivatives. For instance, on a scalar field A(x)

$$E_\xi A = \xi^\mu \partial_\mu A\ ,\quad \partial_\mu = \frac{\partial}{\partial x^\mu}\ , \qquad (4.1)$$

while on the metric tensor $g_{\mu\nu}(x)$

$$\begin{aligned} E_\xi g_{\mu\nu} &= \xi^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu \xi^\lambda g_{\lambda\nu} + \partial_\nu \xi^\lambda g_{\mu\lambda} \\ &= \mathcal{D}_\mu \xi_\nu + \mathcal{D}_\nu \xi_\mu\ . \end{aligned} \qquad (4.2)$$

They satisfy the commutation relations

$$[E_{\xi_1}, E_{\xi_2}] = E_{[\xi_1,\xi_2]}\ , \qquad (4.3)$$

where

$$([\xi_1, \xi_2])^\mu = \xi_2^\lambda \partial_\lambda \xi_1^\mu - \xi_1^\lambda \partial_\lambda \xi_2^\mu\ . \qquad (4.4)$$

If the connected vacuum functional in an external gravitational field $W[g_{\mu\nu}]$ is not Einstein invariant

$$E_\xi W = H_\xi\ , \qquad (4.5)$$

the anomaly H_ξ must satisfy the consistency condition

$$E_{\xi_1} H_{\xi_2} - E_{\xi_2} H_{\xi_1} = H_{[\xi_2,\xi_1]} \qquad (4.6)$$

which is the analogue of (2.4).

It is not difficult to find a solution of the consistency condition (4.6) in terms of the form $\omega_{2n-2}{}^1(v, A, F)$ which gives the anomaly in the case of gauge theories. In differential geometry the Levi-Civita connection $\Gamma_{\lambda\mu}{}^\rho$ plays the role of gauge potential and the Riemann tensor $R_{\nu\lambda\mu}{}^\rho$ the role of field strength. If we introduce the 1-forms

$$(\Gamma)_\mu{}^\rho = \Gamma_{\lambda\mu}{}^\rho \, dx^\lambda \,, \qquad (4.7)$$

the Riemann tensor is given by the 2-forms

$$(R)_\mu{}^\rho = (d\Gamma + \Gamma^2)_\mu{}^\rho = \frac{1}{2} R_{\nu\lambda\mu}{}^\rho \, dx^\nu dx^\lambda \,, \qquad (4.8)$$

with

$$R_{\nu\lambda\mu}{}^\rho = \partial_\nu \Gamma_{\lambda\mu}{}^\rho - \partial_\lambda \Gamma_{\nu\mu}{}^\rho + \Gamma_{\nu\mu}{}^\sigma \Gamma_{\lambda\sigma}{}^\rho - \Gamma_{\lambda\mu}{}^\sigma \Gamma_{\nu\sigma}{}^\rho \,. \qquad (4.9)$$

Under an infinitesimal Einstein transformation the connection transforms as

$$\begin{aligned} E_\xi \Gamma_{\lambda\mu}{}^\rho = {} & \xi^\sigma \partial_\sigma \Gamma_{\lambda\mu}{}^\rho + \partial_\lambda \xi^\sigma \Gamma_{\sigma\mu}{}^\rho \\ & + \partial_\mu \xi^\sigma \Gamma_{\lambda\sigma}{}^\rho - \Gamma_{\lambda\mu}{}^\sigma \partial_\sigma \xi^\rho - \partial_\lambda \partial_\mu \xi^\rho \,. \end{aligned} \qquad (4.10)$$

The last three terms have exactly the form of a gauge transformation with infinitesimal gauge parameter

$$\Lambda_\mu{}^\rho = - \partial_\mu \xi^\rho \,, \qquad (4.11)$$

while the first two terms have the form of a Lie derivative of $\Gamma_{\lambda\mu}{}^\rho$, treated as a vector with lower index λ and ignoring the other two indices μ and ρ. In terms of the 1-form (4.7) we can write

$$E_\xi \Gamma = \mathcal{L}_\xi \Gamma + T_\Lambda \Gamma \,, \qquad (4.12)$$

where

$$T_\Lambda \Gamma = D\Lambda = d\Lambda + [\Gamma, \Lambda] \qquad (4.13)$$

is a gauge transformation with infinitesimal parameter (4.11). In (4.12) the Lie derivative is defined as usual on forms, to operate only on those indices which are saturated with differentials, so that it corresponds to only the first two terms in (4.10). The well known formula applies

$$\mathcal{L}_\xi = d i_\xi + i_\xi d \,, \qquad (4.14)$$

where i_ξ is the (odd) inner product operator, for instance

$$i_\xi (\Gamma_{\lambda\mu}{}^\rho dx^\lambda) = \Gamma_{\lambda\mu}{}^\rho \xi^\lambda . \qquad (4.15)$$

In general, for a form of higher degree, i_ξ substitutes the vector ξ^μ for each differential (one after the other), for instance

$$\begin{aligned} i_\xi R_{\nu\lambda} dx^\nu dx^\lambda &= R_{\nu\lambda} \xi^\nu dx^\lambda - R_{\nu\lambda} dx^\nu \xi^\lambda = 2 R_{\nu\lambda} \xi^\nu dx^\lambda , \\ R_{\nu\lambda} &= - R_{\lambda\nu} \,. \end{aligned} \qquad (4.16)$$

The effect of an Einstein transformation on the Riemann curvature 2-form is given by

$$E_\xi R = \mathcal{L}_\xi R + T_\Lambda R \,, \qquad (4.17)$$

where now

$$T_\Lambda R = [R, \Lambda] . \qquad (4.18)$$

In a space-time of ν dimensions a ν-form has maximal degree and its differential vanishes. Therefore (4.14) becomes

$$\mathcal{L}_\xi \omega_\nu = d(i_\xi \omega_\nu) \qquad (4.19)$$

and the integral vanishes (with suitable boundary conditions)

$$\mathcal{L}_\xi \int \omega_\nu = \int d(i_\xi \omega_\nu) = 0 . \qquad (4.20)$$

In the dual description, more familiar to physicists, a ν-form corresponds to a density $\mathcal{D}$, (4.19) corresponds to

$$\mathcal{L}_\xi \mathcal{D} = \partial_\mu (\xi^\mu \mathcal{D}) , \qquad (4.21)$$

and (4.20) corresponds to

$$\mathcal{L}_\xi \int \mathcal{D}\, dx = \int \partial_\mu (\xi^\mu \mathcal{D})\, dx = 0 . \qquad (4.22)$$

The relation between Einstein and gauge transformations expressed by (4.12) and (4.17) shows that one can reduce the problem of finding consistent Einstein anomalies to that of finding consistent gauge anomalies. Indeed the gauge anomaly, in the form (3.40), immediately gives a consistent Einstein anomaly in the form

$$H_\xi = \Lambda\cdot G[\Gamma, R] = -\int \partial_\rho \xi^\mu G_\mu{}^\rho (\Gamma, R) , \qquad (4.23)$$

with the same function $G[\Gamma,R]$. Indeed

$$E_{\xi_1} H_{\xi_2} = (\mathcal{L}_{\xi_1} + T_{\Lambda_1}) \Lambda_2\cdot G = -\int \partial_\rho \xi_2^\nu \partial_\lambda (\xi_1^\lambda G_\nu^\rho) + T_{\Lambda_1} \Lambda_2 \cdot G , \qquad (4.24)$$

so that

$$E_{\xi_1} H_{\xi_2} - E_{\xi_2} H_{\xi_1} = \int (\xi_1^\lambda \partial_\lambda \partial_\rho \xi_2^\nu - \xi_2^\lambda \partial_\lambda \partial_\rho \xi_1^\nu) G_\nu{}^\rho + [\Lambda_1, \Lambda_2]\cdot G . \qquad (4.25)$$

Finally, using (4.11), the right hand side of (4.25) becomes equal to

$$-\int \partial_\rho (\xi_2^\lambda \partial_\lambda \xi_1^\nu - \xi_1^\lambda \partial_\lambda \xi_2^\nu) G_\nu{}^\rho = H_{\xi_2\cdot\partial\xi_1 - \xi_1\cdot\partial\xi_2} . \qquad (4.26)$$

Observe that the consistent gravitational anomaly given by (4.23) does not depend explicitly on the metric, but only on the connection (and through it on the metric) even though the connected vacuum functional $W[g_{\mu\nu}]$ cannot be expressed in terms of the connection

alone. For instance, in two dimensions, up to a known numerical factor, the consistent gravitational anomaly is

$$H_\xi \propto -\int \partial_\rho \xi^\mu \partial_\nu \Gamma_{\lambda\mu}{}^\rho \varepsilon^{\nu\lambda} d^2x , \qquad (4.27)$$

which corresponds to the non-abelian anomaly [see (1.1)]

$$\Lambda\cdot G \propto \int Tr (\Lambda\, \partial_\nu A_\lambda)\, \varepsilon^{\nu\lambda} d^2x . \qquad (4.28)$$

In higher dimensions the consistent gravitational anomaly is just as easily written, once the appropriate invariant polynomial (3.4) is known. In a Riemann space the Riemann tensor (4.8) is antisymmetric

$$R_{\mu\rho} = - R_{\rho\mu} . \qquad (4.29)$$

As a consequence the invariant symmetric polynomial (2.4) vanishes except for even n

$$n = 2m , \qquad (4.30)$$

which corresponds to a space-time dimension

$$\nu = 2n - 2 = 4m - 2 \qquad (4.31)$$

(see the analogous argument below, leading to (6.12) and (6.13).

In terms of the energy momentum "tensor"

$$\Theta^{\mu\nu} = 2 \frac{\delta W}{\delta g_{\mu\nu}} , \qquad (4.32)$$

(4.5) can be written

$$\int \xi_\nu D_\mu \Theta^{\mu\nu} dx = - H_\xi = - \Lambda\cdot G , \qquad (4.33)$$

or

$$\mathcal{D}_\mu \Theta^\mu{}_\nu = \partial_\mu G_\nu{}^\mu(\Gamma, R) . \qquad (4.34)$$

Here the covariant derivative is that appropriate to a symmetric tensor density

$$\mathcal{D}_\mu \Theta^{\mu\nu} = \partial_\mu \Theta^{\mu\nu} - \Theta^{\mu\rho} \Gamma_{\mu\rho}{}^\nu , \qquad (4.35)$$

but $\Theta^{\mu\nu}$ is not a tensor density, when the anomaly does not vanish. How does it transform under Einstein transformations? We follow an argument similar to that given in Section 2. Evaluate in two different ways the commutator

$$\left(E_\xi \delta_\varphi - \delta_\varphi E_\xi\right) W[g_{\mu\nu}] , \qquad (4.36)$$

where we define

$$\delta_\varphi = \int \varphi_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} dx , \qquad (4.37)$$

an operation which gives $g_{\mu\nu}$ an arbitrary symmetric increment $\varphi_{\mu\nu}$. Since

$$E_\xi = \int (E_\xi g_{\mu\nu}) \frac{\delta}{\delta g_{\mu\nu}} dx , \qquad (4.38)$$

the commutator equals

$$-\int (E_\xi \varphi_{\mu\nu}) \frac{\delta W}{\delta g_{\mu\nu}} dx . \qquad (4.39)$$

On the other hand, using (4.5), the commutator equals

$$\int \varphi_{\mu\nu} E_\xi \frac{\delta W}{\delta g_{\mu\nu}} dx - \delta_\varphi H_\xi . \qquad (4.40)$$

Equating (4.39) and (4.40), and using (4.32), we obtain

$$\int \left\{ \varphi_{\mu\nu} E_\xi \Theta^{\mu\nu} + (E_\xi \varphi_{\mu\nu}) \Theta^{\mu\nu} \right\} dx = 2 \delta_\varphi H_\xi \qquad (4.41)$$

or

$$E_\xi \int \varphi_{\mu\nu} \Theta^{\mu\nu} dx = 2 \delta_\varphi H_\xi . \qquad (4.42)$$

In this equation E_ξ transforms $\varphi_{\mu\nu}$ like a tensor, while $\Theta^{\mu\nu}[g_{\rho\sigma}]$ transforms as it follows from the transformation law of $g_{\rho\sigma}$ [Eq. (4.2)]. If the anomaly in the right hand side were zero, the left hand side would be Einstein invariant, i.e. $\Theta^{\mu\nu}$ would transform like a tensor density.

5. The Covariant Energy Momentum Tensor and its Covariant Anomaly

Is it possible to find a symmetric local $Y^{\mu\nu}$ such that the new energy momentum tensor

$$\tilde{\Theta}^{\mu\nu} = \Theta^{\mu\nu} + Y^{\mu\nu} \tag{5.1}$$

transforms like a tensor density? This means

$$E_\xi \int \varphi_{\mu\nu} \tilde{\Theta}^{\mu\nu} dx = 0 \tag{5.2}$$

and therefore we must require

$$E_\xi \int \varphi_{\mu\nu} Y^{\mu\nu} dx = -2\delta_\varphi H_\xi . \tag{5.3}$$

A solution $Y^{\mu\nu}$ of this equation can be easily found in terms of the solution of the analogous problem discussed in Section 3. There we found a $(\nu-1)$-form X which satisfied (3.3)

$$T_\Lambda B\cdot X = -\delta \Lambda\cdot G \quad , \tag{3,3}$$

where

$$\delta = \int B\cdot\frac{\delta}{\delta A} \quad . \tag{5.4}$$

In view of the relation (4.23) between G and H_ξ, it is clear that we can use (3.37) with the substitutions $A\to\Gamma$, $F\to R$ and taking also (4.11) and

$$B = B[\varphi] = \delta_\varphi\Gamma \quad , \tag{5.5}$$

since then $\delta\to\delta_\phi$ in the right hand side of (3.37). The result is that $Y^{\mu\nu}$ is given by

$$2B[\varphi]\cdot X = \int \varphi_{\mu\nu} Y^{\mu\nu} dx \quad . \tag{5.6}$$

In order to make this expression more explicit we observe that the standard expression for the Christoffel connection

$$\Gamma_{\lambda\mu}{}^{\rho} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\sigma g_{\lambda\mu} - \partial_\lambda g_{\mu\sigma} - \partial_\mu g_{\lambda\sigma}\right) \tag{5.7}$$

implies

$$\begin{gathered} B_{\lambda\mu}{}^{\rho}(\varphi) = \frac{1}{2} g^{\rho\sigma}\left(D_\sigma\varphi_{\lambda\mu} - D_\lambda\varphi_{\mu\sigma} - D_\mu\varphi_{\lambda\sigma}\right) , \\ (B[\varphi])_\mu{}^\rho = B_{\lambda\mu}{}^\rho(\varphi)\, dx^\lambda . \end{gathered} \tag{5.8}$$

Substituting into (5.6) and integrating by parts the covariant derivatives one finds easily the explicit form of $Y^{\mu\nu}$ in terms of that of X, but we shall not carry it out. We point out that the argument which leads to (5.6) is based on the identification

$$E_\xi = \mathcal{L}_\xi + T_\Lambda \tag{5.9}$$

and on the fact that $\mathcal{L}_\xi$ gives zero when applied to the quantitites we are interested in.

Since the new energy momentum tensor is really a tensor density, its covariant divergence will also be covariant. We can work it out without unnecessary computations if we observe that

$$\int (D_\mu\xi_\nu + D_\nu\xi_\mu) Y^{\mu\nu} dx = -2\int \xi_\nu D_\mu Y^{\mu\nu} dx . \tag{5.10}$$

Therefore we can use (5.6) with the substitution

$$\varphi_{\mu\nu} = D_\mu\xi_\nu + D_\nu\xi_\mu = E_\xi\, g_{\mu\nu} \quad , \tag{5.11}$$

which gives

$$\delta_\varphi = E_\xi \tag{5.12}$$

and, from (5.5), (4.12) and (4.13),

$$\mathcal{B}[\varphi_{\mu\nu}] = E_\xi \Gamma = \mathcal{L}_\xi \Gamma + \mathcal{D}\Lambda \quad . \qquad (5.13)$$

We obtain in this way

$$\int \xi_\nu \mathcal{D}_\mu Y^{\mu\nu} dx = -\mathcal{L}_\xi \Gamma \cdot X + \Lambda \cdot \mathcal{D} X \quad . \qquad (5.14)$$

We shall now use the result (3.60) which represents the solution of the analogous problem for the non-abelian current. Equation (3.60) can be written as (use Λ instead of v)

$$\Lambda \cdot \mathcal{D} X - \Lambda \cdot G = n \int \mathcal{P}(\Lambda, F^{n-1}) \quad . \qquad (5.15)$$

Combining (4.33), (5.14) and (5.15) with the definition (5.1) we obtain

$$\int \xi_\nu \mathcal{D}_\mu \tilde{\Theta}^{\mu\nu} dx = n \int \mathcal{P}(\Lambda, R^{n-1}) - \mathcal{L}_\xi \Gamma \cdot X \quad . \qquad (5.16)$$

The right hand side is still not obviously covariant but the two terms can be combined because, as we shall show below,

$$\mathcal{L}_\xi \Gamma \cdot X = -n \, \mathcal{P}(i_\xi \Gamma, R^{n-1}) \, . \qquad (5.17)$$

Since

$$\begin{aligned} \Lambda_\mu{}^\nu + (i_\xi \Gamma)_\mu{}^\nu &= -\partial_\mu \xi^\nu + \xi^\lambda \Gamma_{\lambda\mu}{}^\nu \\ &= -\mathcal{D}_\mu \xi^\nu \equiv M_\mu{}^\nu \quad , \end{aligned} \qquad (5.18)$$

we finally obtain the fully covariant result

$$\int \xi_\nu \mathcal{D}_\mu \tilde{\Theta}^{\mu\nu} dx = n \int \mathcal{P}(M, R^{n-1}) \, . \qquad (5.19)$$

Note that, again, the leading (least non linear) term in (5.19) is

$n = \frac{\nu + 2}{2}$ times larger than the corresponding term in the right hand side of (4.33). The covariant form of the anomaly given by the right hand side of (5.19) is also expressed in terms of the connection alone, the metric does not occur explicitly, just as it does not in the consistent form.

It remains for us to prove (5.17). In ν space-time dimensions, $(\nu + 1)$ forms vanish, therefore

$$\Gamma^2 \cdot X = d\Gamma \cdot X = 0 \quad . \qquad (5.20)$$

If we apply to these forms the operator i_ξ, where the vector ξ^μ is tangent to the ν-dimensional space-time manifold, we still get zero

$$\begin{aligned} 0 = i_\xi(\Gamma^2 \cdot X) &= (i_\xi \Gamma \Gamma) \cdot X - (\Gamma i_\xi \Gamma) \cdot X + \Gamma^2 \cdot i_\xi X \\ &= i_\xi \Gamma \cdot (\Gamma X + X \Gamma) + \Gamma^2 \cdot i_\xi X \end{aligned} \qquad (5.21)$$

and

$$0 = i_\xi(d\Gamma \cdot X) = (i_\xi d\Gamma) \cdot X + d\Gamma \cdot i_\xi X \quad . \qquad (5.22)$$

Now

$$\begin{aligned} \mathcal{L}_\xi \Gamma \cdot X &= i_\xi d\Gamma \cdot X + d \, i_\xi \Gamma \cdot X \\ &= -d\Gamma \cdot i_\xi X - i_\xi \Gamma \cdot d X \quad , \end{aligned} \qquad (5.23)$$

Subtracting (5.21) from (5.23) we obtain

$$\mathcal{L}_\xi \Gamma \cdot X = -R \cdot i_\xi X - i_\xi \Gamma \cdot \mathcal{D} X \, . \qquad (5.24)$$

Now, again in $\nu = 2n - 2$ dimensions, the $(2n-1)$-form

$$\omega_{2n-1}(\Gamma, R) = 0 \tag{5.25}$$

vanishes. Apply again i_ξ

$$0 = i_\xi \int \omega_{2n-1} = i_\xi \Gamma \cdot \frac{\delta}{\delta \Gamma} \int \omega_{2n-1} + i_\xi R \cdot \frac{\delta}{\delta R} \int \omega_{2n-1}$$

$$= i_\xi \Gamma \cdot G + i_\xi R \cdot X \quad , \tag{5.26}$$

where we have used equations analogous to (3.17) and (3.46). This gives

$$R \cdot i_\xi X = i_\xi \Gamma \cdot G \quad . \tag{5.27}$$

Combining (5.27) with (5.24) we obtain

$$\alpha_\xi \Gamma \cdot X = - i_\xi \Gamma \cdot (\mathcal{D} X + G) \tag{5.28}$$

and finally, using (3.60), we prove (5.17). Observe that occasionally, in our derivations, we use results proven earlier for odd quantities or operations and apply them to even quantities or operations and vice-versa. This is permissible if proper care is exercised and we leave it to the reader to be properly careful so a not to make sign mistakes.

6. Einstein Anomalies and Lorentz Anomalies are Equivalent

As explained in the introduction, local Lorentz invariance could also be spoiled by anomalies. In this case the connected vacuum functional must be considered as a functional of the vielbein field $e_{\mu a}$ and cannot be assumed to depend on the metric tensor. Let us work in the Euclidean. Under local rotations of infinitesimal parameter $\theta_{ab} = -\theta_{ba}$ the vielbein field transofrms as

$$L_\theta e_{\mu a} = e_{\mu b} \theta_{ba} \; , \tag{6.1}$$

while under Einstein transformations we have

$$E_\xi e_{\mu a} = \xi^\lambda \partial_\lambda e_{\mu a} + \partial_\mu \xi^\lambda e_{\lambda a} \quad . \tag{6.2}$$

It is easy to see that the full Lie algebra consists of (4.3), (4.4) together with

$$[L_{\theta_1}, L_{\theta_2}] = L_{[\theta_1, \theta_2]} \tag{6.3}$$

and

$$[L_\theta, L_\xi] = L_{\xi \cdot \partial \theta} \quad . \tag{6.4}$$

If there are Lorentz anomalies

$$L_\theta W = K_\theta \quad , \tag{6.5}$$

they must satisfy the consistency conditions

$$L_{\theta_1} K_{\theta_2} - L_{\theta_2} K_{\theta_1} = K_{[\theta_1, \theta_2]} \tag{6.6}$$

and

$$L_\theta H_\xi - E_\xi K_\theta = K_{\xi \cdot \partial \theta} \quad . \tag{6.7}$$

It is consistent to assume that there are only Einstein anomalies ($K_\theta = 0$) and we have discussed this case in section 4. It is also consistent to assume that there are only rotational anomalies ($H_\xi = 0$). A consistent form for the rotational anomaly is easy to find, the orthogonal rotation group can be treated like an internal symmetry. The gauge potential is the Cartan-Weyl connection

$$\alpha_{ab} = -\alpha_{ba} = \alpha_{\mu ab}\, dx^\mu \tag{6.8}$$

and the field strength is the Riemann tensor referred to local orthonormal frames

$$R_{ab} = -R_{ba} = (d\alpha + \alpha^2)_{ab} = \frac{1}{2} R_{\mu\nu ab}\, dx^\mu dx^\nu \tag{6.9}$$

(customarily the connection (6.8) is denoted by the letter ω; here we depart from the usual notation in order to avoid confusion with the forms ω of section 3.). The solution of the consistency condition (6.6) can be written immediately in terms of (3.9) and (3.17)

$$K_\theta = \int \omega_{2n-2}{}^1(\theta, \alpha, R) = \theta \cdot G[\alpha, R]\,. \tag{6.10}$$

We note again that, because of the antisymmetry of the matrix R_{ab},

$$P(R^n) = (-1)^n P(R^n)\,. \tag{6.11}$$

Therefore, in the case of the orthogonal group, the symmetric polynomial P will vanish unless n is even

$$n = 2m \tag{6.12}$$

which corresponds to a space-time dimension

$$\nu = 2n - 2 = 4m - 2\,. \tag{6.13}$$

Only in these dimensions can there be rotational (or Lorentz) anomalies.

Just as the Einstein anomaly (4.23) does not depend explicitly on the metric, but only through the connection Γ, so the rotational anomaly (6.10) does not depend explicitly on the vielbein, but only through the connection α. Indeed, the functional forms of the two anomalies are directly related. However, there is a vielbein field, and in this the theory of gravitation is not like other gauge theories, a fact which cannot be sufficiently stressed. Let us use matrix notation and denote by E the vielbein matrix $e_{\mu a}$. The field H defined by

$$E = e^H \tag{6.14}$$

behaves, in a certain sense, like a Goldstone field for both Einstein transformations and local rotations. Under an infinitesimal Einstein transformation

$$E_\xi H = \xi^\lambda \partial_\lambda H + T_\Lambda H\,, \tag{6.15}$$

where $T_\Lambda H$ is defined by

$$T_\Lambda e^H = -\Lambda e^H \tag{6.16}$$

and Λ is given by (4.11). The finite version of (6.16) is

$$e^{T_\Lambda} e^H = e^{-\Lambda} e^H\,. \tag{6.17}$$

Similarly, under a finite rotation

$$e^{L_\theta} e^H = e^H e^\theta \tag{6.18}$$

This suggests that, using the vielbein field, one should be able to construct a local functional whose Einstein variation gives the Einstein anomaly and whose Lorentz variation gives the rotational anomaly. This is indeed possible by mimicking the solution of the anomalous chiral Ward identities obtained using a Goldstone field [6]. Define the functional

$$S[E,\Gamma] = \int_0^1 dt \int_x \mathrm{Tr}\left(H\, G[\Gamma_t]\right) , \qquad (6.19)$$

where

$$\Gamma_t = e^{-tH}\, \Gamma\, e^{tH} + e^{-tH}\, d\, e^{tH} . \qquad (6.20)$$

In Appendix B we verify that

$$E_\xi S = \int_x \partial_\mu \xi^\nu G_\nu{}^\mu[\Gamma] = - H_\xi \quad . \qquad (6.21)$$

On the other hand, one can express S in terms of E and α , instead of E and Γ . We recall the relation between the Levi-Civita and the Cartan connection

$$\Gamma = E\alpha E^{-1} + E\, d\, E^{-1} . \qquad (6.22)$$

This implies that

$$\Gamma_t = e^{(1-t)H}\, \alpha\, e^{-(1-t)H} + e^{(1-t)H}\, d\, e^{-(1-t)H} . \qquad (6.23)$$

Changing the integration variable from t to

$$\tau = 1 - t \quad , \qquad (6.24)$$

we see that

$$S[E,\Gamma] = \int_0^1 d\tau \int_x \mathrm{Tr}\, H\, G[\alpha_\tau] \equiv S'[E,\alpha] , \qquad (6.25)$$

where

$$\alpha_\tau = e^{\tau H}\, \alpha\, e^{-\tau H} + e^{\tau H}\, d\, e^{-\tau H} . \qquad (6.26)$$

Using perfectly analogous arguments as for (6.21) one shows that

$$L_\theta S = L_\theta S' = K_\theta \quad . \qquad (6.27)$$

The functional S (or S')) is local, in the sense that it is the integral of an expression constructed with derivatives of the vielbein and of the connection up to a finite order. It is highly non linear and uniquely defined only for relatively weak fields. Nevertheless, it can be used to redefine the connected vacuum functional so as to eliminate either the Einstein anomalies (by changing W into W + S) or the Lorentz anomalies (by changing W into W - S'). In this sense Einstein and Lorentz anomalies are different aspects of the same thing. It seems convenient to choose the pure Lorentz anomaly (vanishing Einstein anomaly) as the canonical form of the gravitational anomaly: the formalism is then more directly related to the case of internal gauge symmetries and the absence of Einstein anomalies gives a more satisfactory geometrical picture.[F2,F3]

Finally, we remark that formulas (6.19) and (6.25) for the functional S' can be written in a more intrisic form (see Ref. [12]). We have preferred to use here the special choice of local coordinates of (6.19) and (6.25) in order to render manifest the locality in x of the functional.

7. Conclusion

An anomaly is a local expression which satisfies the consistency condition, but which is not the gauge variation of a local functional of the gauge potential, or the metric tensor, in the gravitational case. Here a local functional of certain fields means an integral over x of an expression constructed with the fields and their x derivatives up to some finite order. The consistent anomalies discussed in this paper satisfy both the above conditions. In order to show that they cannot be obtained from a local functional one has to enumerate all possible candidate expressions of the correct dimension and with the correct power of the fields and check that there is no combination which reproduces the anomaly when one performs a gauge variation. In general, the proof is rather cumbersome, but it can be considerably simplified by going over to the covariant form of the anomalies. Since the covariant current $\tilde{J}_\mu$ is obtained from the original current J_μ by adding to it a local expression, one can reduce the problem to that of finding a covariant current which is a local expression in the gauge potential and whose covariant divergence gives the covariant anomaly. Similarly, in the gravitational case, one can ask whether there exists a covariant energy momentum tensor (which means that it is really a tensor) constructed locally from the metric tensor and satisfying the anomalous equation. The number of possible candidates is greatly restricted by the condition that these quantities be tensors. A further restriction comes from the fact that the covariant anomaly has a known form possessing "abnormal parity", i.e. it is constructed with epsilon tensors (corresponding to it being an exterior form). This would require the current and the energy momentum tensor also to have abnormal parity, since no epsilon tensors can be generated by taking the

covariant divergence. With these restrictions, it is not difficult to show that no such local quantities exist [17] (for the energy momentum tensor one must also use the fact that it is symmetric).

Acknowledgements

The authors wish to thank O. Alvarez and I. Singer for numerous illuminating discussions and L. Alvarez-Gaumé, E. Witten and J. Thierry-Mieg for useful correspondence. Discussions and correspondence with R. Stora have been very helpful and stimulating.

Some additional recent papers on anomalies are listed as Refs. [24].

Appendix A. Algebraic Structure

In this appendix we collect and rederive, in part, the results and techniques described in Refs. [11-13] and used in the main text of this paper. The reader will notice that the structure described below is completely algebraic. In the text we use the resulting formulas for the gauge potential 1-form A and the field strength 2-form F, and for forms which are functions of them. However, the arguments given in this appendix apply to any expression which is a polynomial in two free variables A and F (free means not restricted by algebraic relations), say with complex coifficients. In particular we do not assume that the polynomials are symmetric or invariant, nor do we assume that A and F commute or that they satisfy specific commutation relations. In addition to A and F we shall also use two more variables v and B. A, v and B are odd (anticommuting), F is even. On these variables we define the (odd) antiderivatives d, $\mathfrak{s}$ and ℓ and the (even) derivative δ with the properties

$$dA = F - A^2 \quad , \quad dF = FA - AF \quad , \tag{A.1}$$

$$\mathfrak{s}F = Fv - vF \quad , \quad \mathfrak{s}v = -v^2 \, , \quad \mathfrak{s}B = -vB - Bv \, , \tag{A.2}$$

$$\ell A = 0 \quad , \quad \ell F = B \quad , \quad \ell v = 0 \quad , \quad \ell B = 0 \tag{A.3}$$

$$\delta A = B \quad , \quad \delta v = 0 \quad , \quad \delta B = 0 \tag{A.4}$$

and

$$\mathfrak{s}A + dv = -vA - Av \quad , \tag{A.5}$$

$$\delta F - dB = AB + BA \quad . \tag{A.6}$$

The differentiation operators satisfy

$$d^2 = \ell^2 = \mathcal{J}^2 = 0\,, \tag{A.7}$$

$$d\mathcal{J} + \mathcal{J}d = \ell\mathcal{J} + \mathcal{J}\ell = 0 \tag{A.8}$$

and

$$\ell d + d\ell = \delta \quad . \tag{A.9}$$

The algebraic consistency of all these relations (A.1) to (A.9) is not hard to verify. For instance, to see that $d^2 = 0$, apply d^2 on A

$$\begin{aligned} d(dA) &= d(F - A^2) = dF - dA\,A + A\,dA \\ &= FA - AF - (F - A^2)A + A(F - A^2) = 0\,. \end{aligned} \tag{A.10}$$

Similarly, on F

$$\begin{aligned} d(dF) &= d(FA - AF) = dF\,A + F\,dA - dA\,F + A\,dF \\ &= (FA - AF)A + F(F - A^2) - (F - A^2)F + A(FA - AF) \\ &= 0 \end{aligned} \tag{A.11}$$

Now apply d on (A.5). After a little algebra we find, using (A.1) and (A.5),

$$d\mathcal{J}A + d^2 v = -\mathcal{J}dA\,. \tag{A.12}$$

This shows that $d^2 = 0$ on v as well, provided d and $\mathcal{J}$ anticommute.

Let us also verify, in few cases, the important relation (A.9). On A

$$\ell(dA) = \ell(F - A^2) = \ell F = B \tag{A.13}$$

and

$$d(\ell A) = 0\,. \tag{A.14}$$

Comparing the sum of (A.13) and (A.14) with (A.4), we see that (A.9) is valid on A. Let us check it on F:

$$\ell(dF) = \ell(FA - AF) = BA + AB \tag{A.15}$$

and

$$d(\ell F) = dB\,. \tag{A.16}$$

Therefore

$$(\ell d + d\ell)F = dB + BA + AB = \delta F\,, \tag{A.17}$$

using (A.6).

Because of the properties of derivatives and antiderivatives all this extends immediately to polynomials in the variables A, F, v and B.

Appendix B. Solution of the Anomaly Equation

In this appendix we verify that there exists a local functional whose Einstein variation gives the Einstein anomaly and whose Lorentz variation gives the Lorentz anomaly. The functional is given by (6.19)

$$S[E,\Gamma] = \int_0^1 dt \int_X \mathrm{Tr}\, H\, G[\Gamma_t] \ , \qquad (B.1)$$

where we use matrix notation and denote by E the vielbein matrix $e_{\mu a}$. Then

$$E = e^{H} \qquad (B.2)$$

and

$$\Gamma_t = e^{-tH}\,\Gamma\, e^{tH} + e^{-tH}\, d\, e^{tH} . \qquad (B.3)$$

An Einstein transformation is given by (5.9)

$$E_\xi = \mathcal{L}_\xi + T_\Lambda \ , \qquad (B.4)$$

where

$$\Lambda_\mu{}^\nu = -\partial_\mu \xi^\nu \qquad (B.5)$$

and the effect of T_Λ on Γ is given by (4.13)

$$T_\Lambda \Gamma = d\Lambda + [\Gamma, \Lambda] . \qquad (B.6)$$

We see that (B.4) is valid also on E, if we take

$$T_\Lambda E = -\Lambda E \qquad (B.7)$$

and

$$\mathcal{L}_\xi E = \xi^\lambda \partial_\lambda E \ , \qquad (B.8)$$

which agrees with (4.14)

$$\mathcal{L}_\xi = i_\xi d + d i_\xi \qquad (B.9)$$

if we treat E as a zero-form.

Now, since S is the integral of a form of maximum degree, $\mathcal{L}_\xi$ applied to S gives zero. Therefore, we need only evaluate the effect of T_Λ. It is easy to verify that

$$T_\Lambda \Gamma_t = d\Lambda_t + [\Gamma_t, \Lambda_t] \ , \qquad (B.10)$$

where

$$\Lambda_t = e^{-tH}\Lambda e^{tH} + e^{-tH}\, T_\Lambda e^{tH} . \qquad (B.11)$$

Observe that, from (B.2) and (B.7),

$$T_\Lambda e^{H} = -\Lambda e^{H} . \qquad (B.12)$$

Therefore

$$\Lambda_0 = \Lambda \ , \quad \Lambda_1 = 0 \qquad (B.13)$$

and

$$\frac{\partial \Lambda_t}{\partial t} = -H e^{-tH}\Lambda e^{tH} + e^{-tH}\Lambda e^{tH} H$$
$$- H e^{-tH}\, T_\Lambda e^{tH} + e^{-tH}\, T_\Lambda (e^{tH} H)$$

$$= \left[e^{-tH} \Lambda e^{tH} + e^{-tH} T_\Lambda e^{tH} , H \right] + T_\Lambda H$$

$$= \left[\Lambda_t , H \right] + T_\Lambda H \ . \qquad (B.14)$$

It is not necessary to know explicitly $T_\Lambda H$ and $T_\Lambda e^{tH}$. Let us compute

$$T_\Lambda S = \int_0^1 dt \int_X \mathrm{Tr} \left(T_\Lambda H \, G[\Gamma_t] + H \, T_\Lambda G[\Gamma_t] \right). \qquad (B.15)$$

Now, according to (B.10),

$$T_\Lambda G[\Gamma_t] = T_{\Lambda_t} G[\Gamma_t] \qquad (B.16)$$

if we define

$$T_{\Lambda_t} \Gamma_t = d\Lambda_t + [\Gamma_t , \Lambda_t] \ . \qquad (B.17)$$

The consistency condition gives

$$\int_X \mathrm{Tr}\, H \, T_{\Lambda_t} G[\Gamma_t] = \int_X \mathrm{Tr}\, \Lambda_t \, T_H G[\Gamma_t] + \int_X \mathrm{Tr} \left([\Lambda_t , H] \, G[\Gamma_t] \right), \qquad (B.18)$$

where we have defined

$$T_H \Gamma_t = dH + [\Gamma_t , H] = \frac{\partial \Gamma_t}{\partial t} \ , \qquad (B.19)$$

So, (B.15) gives

$$T_\Lambda S = \int_0^1 dt \int_X \mathrm{Tr} \left\{ \left(T_\Lambda H + [\Lambda_t , H] \right) G[\Gamma_t] + \Lambda_t \frac{\partial G[\Gamma_t]}{\partial t} \right\}$$

$$= \int_0^1 dt \int_X \mathrm{Tr} \left\{ \frac{\partial \Lambda_t}{\partial t} G[\Gamma_t] + \Lambda_t \frac{\partial G[\Gamma_t]}{\partial t} \right\}$$

$$= \int_0^1 dt \frac{\partial}{\partial t} \int_X \mathrm{Tr}\, \Lambda_t G[\Gamma_t] = - \int_X \mathrm{Tr}\, \Lambda G[\Gamma] \ . \qquad (B.20)$$

In conclusion

$$E_\xi S = \mathcal{L}_\xi S + T_\Lambda S = T_\Lambda S = - H_\xi \ . \qquad (B.21)$$

The effect of local rotations can be evaluated in an analogous manner, using the expression (6.25) in terms of E and α . Here there is not even the Lie derivative term, since local rotations are exactly like ordinary gauge transformations. The result is equation (6.27).

Figures

F1 Actually, in two-dimensional Minkowski space-time, the anomaly can be written in one of the forms

$$c\, \partial_\mu A_\lambda \left(\varepsilon^{\lambda\mu} \pm \eta^{\lambda\mu} \right)$$

which differ respectively from (1.1) by the gauge variation of the local functionals

$$\pm \frac{c}{2} \int A_\mu A_\lambda \eta^{\mu\lambda} = \pm \frac{c}{2} \int A_\mu A^\mu \, .$$

Roman Jackiw has emphasized that these forms are more natural than (1.1) since for chiral (antichiral) spinors in two dimensions the the Dirac Lagrangian depends only on $A_o + A_1$, $(A_o - A_1)$ and therefore the anomaly should also depend only on those combinations. We prefer to ignore this peculiarity of the two-dimensional Minkowski case and illustrate our point using the form (1.1) which is perfectly analogous to the abnormal parity expressions valid in four and higher dimensions.

F2 Tom Banks has observed that it is possible (for instance in six space-time dimensions) to regularize the Lagrangian for chiral spinors with a mass term which is Einstein covariant. Such a regularization, however, cannot be Lorentz covariant. Evaluated in this way the anomaly would naturally appear as a pure Lorentz anomaly.

F3 In the case of an internal gauge symmetry based on the compact Lie group $\mathcal{G}$, one knows [12] that the existence of anomalies requires that the homotopy group $\Pi_{\nu+1}[\mathcal{G}]$ contain the group Z of the integers (here ν is the dimensions of space-time). For the orthogonal group of local rotations $O(\nu)$, one is then led to consider $\Pi_{\nu+1}[O(\nu)]$. Now, it is known that this homotopy group contains Z only for $\nu = 4m - 2$ (m an integer ≥ 1) and otherwise is finite (see e.g. Ref. [23]). So, one expects that only for these space-time dimensions there can be a topological Lorentz anomaly. This condition is the same as given in (6.13). For $\nu = 2$ there is no topological anomaly, but there still is a Lorentz anomaly, in the local sense discussed in this paper. The connection between the local and the topological meaning of the anomalies will be discussed in Ref. [17].

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