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Phase Structure of the O(N) Vector Model*

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ABSTRACT

In a large N analysis of the O(N) $\lambda\Phi^4$ theory in four dimensions, we find that the O(N) symmetric lowest lying state is a metastable state. The decay rate of this false vacuum into a broken symmetry state is however suppressed ($\sim$exp(-N)). Nevertheless, at finite temperature, we find that above a critical temperature T_c the only existing phase is a broken symmetry phase with N-1 goldstone bosons. This phase reflects the intrinsic instability of this theory, and the large Φ_c structure indicates that the renormalized $\lambda\Phi^4$ theory is inconsistent. The only acceptable version of this theory is its regularized form which becomes a free field theory as the regularization is removed.

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I. INTRODUCTION

Quantum field theories with N dynamical variables, $N\to\infty$, have recently attracted attention from several different perspectives.[1-4] Traditionally, these theories have served mostly as study grounds for extending one's intuition in handling the basic problems of quantum field theory. Aproximations used at large N possess many of the properties believed to be true in the exact solution. Thus, it is hoped that the theory's features revealed in the large N limit as well as the lessons learned in these analyses are worthwhile.

Indeed, the phase structure and the nature of the ground state of large N theories are of much interest. Recently, t'Hooft[3] has obtained results for the planar model when the perturbation expansion is performed in the phase where all particles are massive and the renormalized coupling constant magnitude is bounded from above. The stability of this phase at zero temperature as well as at finite temperature will be investigated for an analogous O(N) vector model in the present study. Quantum as well as thermal fluctuations will be calculated in the large N limit and the full phase structure of the theory will be elucidated.

Spontaneous symmetry breaking in the O(N)-symmetric vector model at large N was studied by Coleman, Jackiw and Politzer.[5,6] It was found that the ground state could have a broken or an unbroken O(N) symmetry. The theory was shown, however, to possess a tachyon and therefore be inconsistent.[7] Further analyses[8,9] revealed an O(N) symmetric phase free of tachyons and it was concluded that spontaneous symmetry breaking is impossible in the large N limit. The effective potential remained undefined at large values of the classical field in these analyses.[5-9] A detailed study of the phase structure of the O(N) vector model presented here shows some of these previous results to be in error.

Another problem which concerns us is the existence of the renormalized $\lambda\Phi^4$ theory as a non-trivial field theory in four dimensions. Momentum space or lattice regularization are usually regarded as intermediate steps on the way to obtain a finite physical result either in perturbation theory or in a nonperturbative analysis. It is then assumed that the limit $\Lambda^2\to\infty$ or $a\to 0$, result in a well-defined theory provided the correct vacuum has been chosen. In a lattice field theory, the ultraviolet fixed point reached must be the continuum theory originally meant to be analyzed. The existence of a renormalized $\lambda\Phi^4$ in four dimensions has been in doubt for some time.[10-12] There are no indications that the theory possesses a non-trivial ultraviolet fixed point[13] and recent rigorous results[14,15] indicate that the renormalized $\lambda\Phi^4$ may very well be a free field theory in $d\geq 4$. However we should emphasize that the regularized $\lambda\Phi^4$ theory may well describe the correct physics for a broad spectrum of processes below the cutoff scale.

In Sec. II we present a variational calculation combined with a large N approximation for the Hartree Fock ground state energy and the gap equation of the O(N) vector model. The resulting phase structure of the theory is described in Sec. III along with a comparison to previous results. We discuss, in some detail, the end point contribution in the variational calculation and also the appearance of tachyons and related instabilities for large values of ϕ_c. In Sec. IV we estimate the decay rate for the false vacuum ($m \neq 0$ phase) at zero temperature. We find it to be proportional to e^{-N}. Thermal fluctuations at finite temperature are calculated in Sec. V, where we find that at $T > T_c$ (finite) the only existing phase of the theory is a broken symmetry phase. A summary and discussion of our results is given in Sec. VI. In Appendix A we present a calculation where the oscillator frequency[16] is used as a variational parameter rather then the oscillator mass as in Sections II-V. The effective action that determines the dynamics involved in the decay of the false vacuum is calculated in Appendix B.

II. VARIATIONAL CALCULATION AND 1/N EXPANSION

The O(N) symmetric $\lambda\Phi^4$ theory is defined from the functional integral

$$Z(\vec{J}) = \int D\vec{\Phi}(x)\, \exp\left\{ \int d^4x \left[-\tfrac{1}{2}(\partial_\mu \vec{\Phi})^2 - \frac{\mu_0^2}{2}\vec{\Phi}^2 - \frac{\lambda_0}{4}(\vec{\Phi}^2)^2 + \vec{J}\cdot\vec{\Phi} \right] \right\} \qquad (2.1)$$

where $\vec{\Phi}(x)$ is an N component real scalar field and μ_0, λ_0 are the unrenormalized mass and coupling constant. The large N behavior of the theory can be studied[4-9] holding $\lambda_0 N$ fixed as $N\to\infty$. Here we will combine a variational calculation with the 1/N expansion in order to obtain an upper limit for the ground state energy of the theory. Writing $\vec{\Phi}$ as $\vec{\Phi} = \vec{\Phi}_c + \vec{\phi}$, where $\vec{\Phi}_c$ is a background classical field and $\vec{\phi}$ is the N component quantum dynamical variable of the theory, we first calculate the kinetic energy K built up from the quantum fluctuation in $\vec{\phi}$. The Hartree Fock variational ground state is defined as the state annihilated by the operators $a_\alpha(k)$ in the trial plane wave expansion of the quantum field $\vec{\phi}$.

$$\phi_\alpha(x,t) = \int \frac{d^3k}{2\omega_k} \left[a_\alpha^\dagger(k) f_k^*(x) + a_\alpha(k) f_k(x) \right] \qquad (2.2)$$

where

$$f_k(x) = (2\pi)^{-3/2}\exp(-ikx) \qquad \text{and} \qquad \omega_k = \sqrt{k^2+m^2} \quad .$$

We choose here the oscillator mass m^2 as our variational parameter. (Alternatively, one may vary ω_k instead of m^2; this possibility will be discussed in Appendix A.) The kinetic energy per unit volume given by

$$K(m^2) = \frac{1}{V}\langle 0|\int d^3x \; \frac{1}{2}\left[\vec{\Pi}^2(x,t) + (\vec{\nabla}\cdot\vec{\Phi}(x,t))^2\right]|0\rangle$$

$$= \frac{1}{4}\int \frac{d^3k}{(2\pi)^3}\left[\omega_k + \frac{\vec{k}^2}{\omega_k}\right] \qquad (2.3)$$

is defined with an ultraviolet cutoff Λ^2 or one may prefer to define the theory on a D dimensional Euclidean lattice with finite lattice spacing a. In what follows, whenever the regulated bare theory will be considered, it will be assumed that the theory has a nontrivial physical content at $k^2 \ll \Lambda^2$ (or $k^2 \ll (\pi/a)^2$) and not all of its physics is at the ultraviolet end. Thus, under certain circumstances the regulated theory may be regarded as an effective field theory at energies which are low on the scale of Λ^2 or a^{-2}.

The relation of $K(m^2)$ to the quantum fluctuation of $\vec{\Phi}(x,t)$ can be expressed also by

$$\frac{\partial K}{\partial m^2} = -\frac{1}{2}m^2\frac{\partial\langle\vec{\Phi}^2\rangle}{\partial m^2} = -\frac{m^2 N}{2}\frac{\partial}{\partial m^2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2} \qquad (2.4)$$

This can be also derived from the Euclidean path integral over $\vec{\phi}$.

$$\int D\vec{\phi}\; e^{-\int d^4x\, \mathcal{L}_E(\vec{\phi}(x))} \simeq e^{[-K(m^2) - \frac{1}{2} m^2 \langle \vec{\phi}^2 \rangle] VT} \tag{2.5}$$

where $\mathcal{L}_E(\vec{\phi})$ is the Euclidean Lagrangian of a free field $\vec{\phi}$ of mass m.

The ultraviolet cutoff will be defined from

$$\langle \vec{\phi}^2 \rangle = \frac{N}{(2\pi)^4} \int \frac{d^4k}{k^2 + m^2} = \frac{N}{16\pi^2} \left[e\Lambda^2 - m^2 \log\left(\frac{e\Lambda^2}{m^2}\right) \right] \tag{2.6}$$

and finally the kinetic energy is

$$K(m^2) = \frac{N}{64\pi^2} m^4 \log\left(\frac{e^{1/2}\Lambda^2}{m^2}\right) \tag{2.7}$$

Adding now the potential energy $\langle V(\vec{\phi}) \rangle_0$, we obtain the Hartree Fock energy in the large N limit

$$W(\vec{\phi}_c^{\,2}, m^2) = K(m^2) + \langle V(\vec{\phi}) \rangle_0 \simeq K(m^2) + V(\vec{\phi}_c^{\,2} + \langle \vec{\phi}^2 \rangle)$$

$$= \frac{N}{64\pi^2} m^4 \log\left(\frac{e^{\frac{1}{2}}\Lambda^2}{m^2}\right) + \frac{\mu_0^2}{2}(\vec{\phi}_c^{\,2} + \langle \vec{\phi}^2 \rangle) + \frac{\lambda_0}{4}(\vec{\phi}_c^{\,2} + \langle \vec{\phi}^2 \rangle)^2 \tag{2.8}$$

The variational ground state energy of the theory is now found by minimizing $W(\vec{\phi}_c^2, m^2)$ with respect to $\vec{\phi}_c$ and m^2. The physics of the model is thus determined from the extremum conditions in the $\{\vec{\phi}_c, m^2\}$ space.

$$\frac{\partial W}{\partial m^2} = -\frac{1}{2}\frac{\partial\langle\vec{\phi}^2\rangle}{\partial m^2}\left[m^2 - \mu_0^2 - \lambda_0(\vec{\phi}_c^{\,2} + \langle\vec{\phi}^2\rangle)\right] = 0 \qquad (2.9)$$

$$\frac{\partial W}{\partial\vec{\phi}_c} = \vec{\phi}_c\left[\mu_0^2 + \lambda_0(\vec{\phi}_c^{\,2} + \langle\vec{\phi}^2\rangle)\right] = 0 \qquad (2.10)$$

One should note, however, that in order to find the lowest energy eigenstate, the end point values of $W(\vec{\phi}_c^2, m^2)$ must also be examined and compared to the extremal values found by solving Eqs. 2.9, 10. These extremum conditions can be expressed in terms of the renormalized parameters defined from

$$\lambda = \frac{\lambda_0}{1 + \frac{\lambda_0 N}{16\pi^2}\log\left(\frac{\Lambda^2}{M^2}\right)} \qquad (2.11)$$

$$\pm\frac{\mu^2}{\lambda} = \frac{\mu_0^2}{\lambda_0} + \frac{N}{16\pi^2}e\Lambda^2 \qquad (2.12)$$

where M^2 is a renormalization scale and $\mu^2/\lambda>0$. Eqs. 2.9 and 2.10 have now the form

$$\frac{\partial W}{\partial m^2} = -\frac{\lambda_0}{2}\frac{\partial\langle\vec{\phi}^2\rangle}{\partial m^2}\left[\frac{m^2}{\lambda_0} + \frac{N m^2}{16\pi^2}\log\left(\frac{e\Lambda^2}{m^2}\right) - \left(\vec{\phi}_c^{\,2} \pm \frac{\mu^2}{\lambda}\right)\right]$$

$$(2.13)$$

$$0 = -\frac{\lambda_0}{2}\frac{\partial\langle\vec{\phi}^2\rangle}{\partial m^2}\left[\frac{m^2}{\lambda} + \frac{N m^2}{16\pi^2}\log\left(\frac{eM^2}{m^2}\right) - \left(\vec{\phi}_c^{\,2} \pm \frac{\mu^2}{\lambda}\right)\right]$$

$$\frac{\partial W}{\partial \vec{\phi}_c} = \vec{\phi}_c \left[m^2 - \left(\frac{32\pi^2}{N}\right) \frac{\partial W}{\partial m^2} \left(\log\left(\frac{\Lambda^2}{m^2}\right) \right)^{-1} \right] = 0 \quad (2.14)$$

Since $\partial\langle\vec{\phi}^2\rangle/\partial m^2 = (-N/16\pi^2)\log(\Lambda^2/m^2)$ is non-zero in what is considered, as mentioned above, the physical region ($m^2 \ll \Lambda^2$) for the regulated theory, we see that the gap equation in Eq. 2.13 and $\partial W/\partial\vec{\phi}$ in Eq. 2.14 vanish simultaneously at $\vec{\phi}_c = 0$, $m^2 = \mu_0^2 + \lambda_0\langle\vec{\phi}^2\rangle \neq 0$ or $\vec{\phi}_c \neq 0$ $(\mu_0^2 + \lambda_0(\vec{\phi}_c^2 + \langle\vec{\phi}^2\rangle) = 0)$, $m^2 = 0$. We will consider these extremum points as well as the end points in m^2 (which do not satisfy Eqs. 2.14 and 2.15) by inserting these values ($m^2 = m^2(\vec{\phi}_c^2)$) in W and find the lowest energy from $W(\vec{\phi}_c^2, m^2(\vec{\phi}_c^2))$.

A convenient approach in discussing the energy eigenvalues determined by the extremum condition (gap equation Eq. 2.13) is to insert its solution into Eq. 2.8 . First, from Eqs. 2.6 and 2.8 we have

$$W(\vec{\phi}_c^2, m^2) = \frac{N}{64\pi^2} m^4 \log\left(\frac{e^{\frac{1}{2}}\Lambda^2}{m^2}\right) + \frac{\lambda_0}{4}\left[\vec{\phi}_c^2 \pm \frac{\mu^2}{\lambda} - \frac{N}{16\pi^2} m^2 \log\left(\frac{e\Lambda^2}{m^2}\right)\right]^2 \quad (2.15)$$

and finally, if the gap equation is satisfied one inserts the solution to Eq. 2.13 into Eq. 2.15 to obtain

$$W(\vec{\phi}_c^2(m^2), m^2) = \frac{m^4}{4}\left[\frac{1}{\lambda} + \frac{N}{16\pi^2}\log\left(\frac{e^{\frac{1}{2}}M^2}{m^2}\right)\right] \quad (2.16)$$

where $\vec{\phi}_c^2 = \vec{\phi}_c^2(m^2)$ is determined from Eq. 2.13. These steps are clearly traced in Fig. 1. The solid line is $m^2[1/\lambda + (N/16\pi^2)\log(eM^2/m^2)]$ which determines $\vec{\phi}_c^2(m^2) \pm \mu^2/\lambda$ (gap equation 2.13). The dashed line is $W(m^2)$ of Eq. 2.16. Thus, for a given ϕ_c^2 we find $m^2(\vec{\phi}_c^2)$ from the solid line and knowing $m^2(\vec{\phi}_c^2)$, we determine $W(\vec{\phi}_c^2) = W(\vec{\phi}_c^2, m^2(\vec{\phi}_c^2))$ from the dashed line.

As seen in Fig. 1, the gap equation has two branches to the right and left of

$$m_x^2 = M^2 \exp(16\pi^2/\lambda N) = \Lambda^2 \exp(16\pi^2/\lambda_0 N) \quad (2.17)$$

on which the m^2 dependence of $W(\vec{\phi}_c^2, m^2)$ is at the extremum. Note, however, that

$$\frac{\partial^2 W}{\partial^2 m^2} = \frac{\lambda_0 N}{32\pi^2}\log\left(\frac{\Lambda^2}{m^2}\right)\left[\frac{1}{\lambda_0} + \frac{N}{16\pi^2}\log\left(\frac{\Lambda^2}{m^2}\right)\right] - \frac{\partial W}{\partial m^2}\left[m^2\log\left(\frac{\Lambda^2}{m^2}\right)\right]^{-1} \quad (2.18)$$

is positive on the left hand side branch ($m^2<\Lambda^2$) if $\lambda_0>0$. In the case $\lambda_0<0$, $\partial^2W/\partial^2m^2$ is positive for $m^2>m_x^2$, namely, on the right hand branch.

The significance of $\lambda_0>0$ and $\lambda_0<0$ can be seen in Eq. 2.11. If λ is held fixed as $\Lambda^2\to\infty$, then clearly $\lambda_0\to 0^-$ (A necessary condition in a renormalized $\lambda\Phi^4$ is $\lambda_0<0$), whereas holding $\lambda_0>0$ implies $\lambda\to 0$ as $\Lambda^2\to\infty$. Otherwise, for a fixed λ and $\lambda_0>0$ the ultraviolet regulator must be kept fixed (finite Λ^2 or lattice spacing a). These different cases will be discussed in Section III.

III. PHASES OF THE $O(N)\lambda\phi^4$ VECTOR THEORY

3.1 Gap Equation and End Point Contribution - Case $\lambda_0>0$

The theory with a positive bare coupling constant exhibits the expected behavior. We will discuss the two cases (a) $\mu_0^2/\lambda_0 + (N/16\pi^2)e\Lambda^2 = \mu^2/\lambda$ and (b) $\mu_0^2/\lambda_0 + (N/16\pi^2)e\Lambda^2 = -\mu^2/\lambda$ (recall that we denote μ^2/λ always positive). As mentioned in Sec. II, since for $\lambda_0>0$ we have $m_x^2>\Lambda^2$ (eq. 2.17), the interesting physical region is the $m^2<<m_x^2$ branch in Fig. 1.

In case (a), as seen in Fig. 1, the classical field ϕ_c^2 reaches any value between $\phi_c^2=0$ at $m^2=m_1^2$ and $\phi_c^2 = (\phi_c^2)_{max}$ at $m^2=m_x^2$. Clearly, the ground state energy is $W(0,m_1^2)$ and the system is in the O(N) symmetric phase with pions of mass m_1^2. Note that m_1^2 is the solution of $\mu^2/\lambda = m_1^2[1/\lambda + (N/16\pi^2)\log(eM^2/m_1^2)]$, and since $M^2\exp(16\pi^2/\lambda N) = \Lambda^2\exp(16\pi^2/\lambda_0 N)$ we can keep $m_1^2<<\Lambda^2$ only if μ^2/λ is small enough. $W(\phi_c(m^2),m^2)$ (Eq. 2.16) in Fig. 1 gives the value of the Hartree Fock energy W when the gap equation $\partial W/\partial m^2 = 0$ in Eq. 2.13 is satisfied. The end point value of $W(\bar{\phi}_c^2,m^2)$ at $m^2=0$ can be read from Eq. 2.15 ($\partial W/\partial m^2 \neq 0$). We have $W(\bar{\phi}_c^2,0) = (\lambda_0/4)(\bar{\phi}_c^2 + \mu^2/\lambda)^2 > W(\bar{\phi}_c^2,m_1^2)$ and thus the end point value does not give a lower energy state. Case (a) is summarized in Fig. 2a.

In case (b) ($\mu_0^2/\lambda_0 + (N/16\pi^2)e\Lambda^2 = -\mu^2/\lambda$, $\lambda_0>0$) we see in Fig. 1 that $\bar{\phi}_c^2$ has a minimum value $\bar{\phi}_c^2 = \mu^2/\lambda$ if the gap equation is satisfied. This value is obtained at $m^2=0$ and there $W(\mu^2/\lambda,0) = 0$. The end point value ($m^2=0$) of $W(\phi_c^2, m^2)$ gives $W(\bar{\phi}_c^2,0) = (\lambda_0/4)(\bar{\phi}_c^2 - \mu^2/\lambda)^2$ (Eq. 2.15); it coincides with the gap equation solution at $\bar{\phi}_c^2 = \mu^2/\lambda$ and gives a broken O(N) symmetry ground state. It also gives the energy in the range $\bar{\phi}_c^2<\mu^2/\lambda$ not attainable from the gap equation solution. Case (b) is summarized in Fig. 2b.

Thus, the O(N) vector theory with $\lambda_0>0$ is consistent with a lattice theory with a finite lattice spacing[11] $a\sim\Lambda^{-1}$. The physics at $k^2<\Lambda^2$ and in particular the phase structure is governed by the value of $\mu_0^2/\lambda_0 + (N/16\pi^2)e\Lambda^2$. We have an O(N) symmetric or asymmetric ground state if this quantity is positive or negative, respectively. (The zero mass N-1 pions in the broken symmetry phase are the goldstone bosons of the O(N)$\to$O(N-1) breaking). The theory can be viewed as an effective field theory in the regime $k^2<<\Lambda^2$ where Λ^2 must be kept fixed. If $\Lambda^2\to\infty$ ($a\to 0$) the renormalized coupling constant $\lambda\to 0$ (Eq. 2.11), and the theory is a free field theory.[12-15]

3.2 Gap Equation and End Point Contribution - Case $\lambda_0<0$

As mentioned in Sec. II, this case allows a finite renormalized λ as $\Lambda^2\to\infty$ and we are interested in determining whether a consistent $\lambda\Phi^4$ renormalized theory can be defined in four dimensions. Though the gap equation remains unchanged, both regimes $m^2<m_x^2$ and $m^2>m_x^2$ in Fig. 1 are now of physical interest. Moreover, it has been shown above (Eq. 2.18) that the gap equation solution for $m^2>m_x^2$ is a local minimum of W, whereas the solution of the gap equation for $m^2<m_x^2$ is a local maximum of W.

We would like now to relate our results for the case in which the gap equation is satisfied to those of refs. 5, 8, and 9. Eq. 2.15 can be rewritten in the form

$$W(\vec{\phi}_c^{\,2}, m^2) = \frac{m^4}{4}\left[\frac{1}{\lambda} + \frac{N}{16\pi^2}\log\left(\frac{e^{\frac{1}{2}}M^2}{m^2}\right)\right] \qquad (3.1)$$

$$+ \frac{m^2}{2}\left\{\vec{\phi}_c^{\,2} \pm \mu^2/\lambda - m^2\left[\frac{1}{\lambda} + \frac{N}{16\pi^2}\log\left(\frac{eM^2}{m^2}\right)\right]\right\}$$

$$+ \frac{\lambda_0}{4}\left\{\vec{\phi}_c^{\,2} \pm \mu^2/\lambda - m^2\left[\frac{1}{\lambda} + \frac{N}{16\pi^2}\log\left(\frac{eM^2}{m^2}\right)\right]\right\}^2$$

Note that the coefficients of $m^2/2$ and $\lambda_0/4$ in Eq. 3.1 are the gap equation (Eq. 2.13) and its square, respectively. Now, suppose we hold $\vec{\phi}_c^{\,2}$, m^2 fixed and let $\lambda_0\to 0^-$ ($\lambda_0\to 0^-$ in the renormalized theory with a fixed λ and $\Lambda^2\to\infty$). The last term in Eq. 3.1 now vanishes and we obtain the standard result for the effective potential, $V_{eff}(\vec{\phi}_c^{\,2}, m^2\equiv\chi)$, presented in refs. (5) and (8-9).

$$V_{eff}(\vec{\phi}_c^{\,2}, m^2 \equiv \chi) = W(\vec{\phi}_c^{\,2}, m^2)\Big|_{\lambda_0 \to 0^-} \tag{3.2}$$

$$= \frac{m^2}{2}\left(\vec{\phi}_c^{\,2} \pm \frac{\mu^2}{\lambda}\right) - \frac{m^4}{4}\left[\frac{1}{\lambda} + \frac{N}{16\pi^2}\log\left(\frac{e^{\frac{3}{2}}M^2}{m^2}\right)\right]$$

$V_{eff}(\phi_c^2, m^2)$ reproduces, of course the same gap equation as well as the same value of W on the gap equation branches. Indeed, Eq. 2.18 is replaced now by

$$\frac{\partial^2 V_{eff}}{\partial^2 m^2} = -\frac{1}{2}\left[\frac{1}{\lambda} + \frac{N}{16\pi^2}\log\left(\frac{M^2}{m^2}\right)\right]$$

$$= \frac{N}{32\pi^2}\log\left(\frac{m^2}{m_x^2}\right) \tag{3.3}$$

and thus V_{eff} has a local minimum as a function of m^2 in the range $m^2 > m_x^2$, namely, when satisfying the gap equation (Fig. 1 - solid line) on its right hand side branch. The value of the ground state energy can be read again from Fig. 1. In the case $\mu_0^2/\lambda_0 + (N/16\pi^2)e\Lambda^2 = \mu^2/\lambda$, V_{eff} is given by the dashed lines in Fig. 3a and Fig. 3b, and in the case $\mu_0^2/\lambda_0 + (N/16\pi^2)e\Lambda^2 = -\mu^2/\lambda$ by the dashed lines in Figs. 3c and 3d. This agrees with the results of refs. (8,9). We also plot on Fig. 3a-d the value of V_{eff} that can be read from the left hand side branch ($m^2 < m_x^2$) of the gap equation where V_{eff} has a maximum. This is the solid line in Fig. 3 a-d which agrees with the results of ref. (5).

The gap equation solutions have been considered above. These should be compared with the end point $m^2=0$ value of $W(\vec{\phi}_c^2, m^2)$ from Eq. 2.15. We have $W(\vec{\phi}^2{}_c, 0) = (\lambda_0/4)(\vec{\phi}_c^2 \pm \mu^2/\lambda)^2 < 0$ which goes to minus infinity as $\vec{\phi}_c^2 \to \infty$. From our point of view, this upper limit on the ground state energy, which is unbounded from below, renders the $\lambda_0<0$ case intrinsically unstable.[10-12] Restricting ourselves to V_{eff} of Eq. 3.2 ($\lambda_0 \phi_c^4 \to 0$ region) gives an end point contribution which is flat in this regime shown by the dotted curve in Fig. 3a-d.

Thus, in addition to the previously obtained results[5,8,9] in the 1/N expansion, our variational calculation shows the existence of a new broken symmetry phase. This phase comes from the contribution of the end point value in the variational parameter range $0<m^2<\infty$. Even for small values of ϕ_c the ground state is in this phase if μ_0^2 is negative and $|\mu_0^2|$ large enough. (Note: $\mu_0^2/\lambda_0 + (N/16\pi^2)e\Lambda^2 = \mu^2/\lambda$ in Figs. 3a and 3b). There exists a first order phase transition from the phase found in refs. 8-9 where $m^2 = e^{1/2}m_x^2 = e^{1/2}M^2\exp(16\pi^2/\lambda N) \equiv m_{W_0}^2$ to the $m^2 = 0$ new phase.[17] We note, however, as mentioned above, that the instability of this new phase at large ϕ_c^2 values implies a basic instability in the renormalized O(N) $\lambda\Phi^4$ vector theory.

Fig. 4 summarizes the different cases. We plot $V_{eff}(\vec{\phi}_c^2, m^2)$ (Eq. 3.2) as a function of m^2 for different ranges of $\vec{\phi}_c^2 \pm \mu^2/\lambda$. For the renormalized theory ($\lambda_0<0$) the maximum and minimum in these plots represent the O(N) symmetric phases found in refs. (5) and (8-9), respectively. The end point $m^2=0$ gives the broken symmetry phase. The decay rate of the O(N) symmetric metastable false vacuum into the broken symmetry ($m^2=0$) phase will be discussed in Sec. IV.

3.3 Tachyons - Case $\lambda_0 > 0$

We wish to examine first the scattering amplitude in the broken symmetry phase where the ϕ_σ component of the field $\vec{\Phi}$ has vacuum expectation value $\langle\phi\rangle$ and the fields $\vec{\Phi}_\pi$ describe the N-1 goldstone boson degrees of freedom. The appearance of poles at positive Euclidean momenta will be our main concern. To lowest order in 1/N, the $\pi\pi$ scattering amplitude is calculated by summing the "bubble diagrams" giving (in the Euclidean region) the amplitude

$$A_{\pi\pi} \sim \frac{p^2 + m^2}{(p^2 + m^2)\left[\frac{1}{2\lambda_0} + B(p^2, m^2, \Lambda^2)\right] + \langle\vec{\Phi}\rangle^2} \tag{3.4}$$

where $B(p^2,m^2,\Lambda^2)$ is the π loop integral[18]

$$\begin{aligned} B(p^2, m^2, \Lambda^2) &= \frac{N-1}{2}\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+m^2)((k+p)^2+m^2)} \\ &= \frac{N-1}{32\pi^2}\int_0^1 d\alpha \, \log\left[\frac{\Lambda^2}{\alpha(1-\alpha)p^2 + m^2}\right] \end{aligned} \tag{3.5}$$

The σ propagator is given by

$$iD_{\sigma\sigma} = \frac{\frac{1}{2\lambda_0} + B(p^2, m^2, \Lambda^2)}{(p^2+m^2)\left[\frac{1}{2\lambda_0} + B(p^2, m^2, \Lambda^2)\right] + \langle\phi\rangle^2} \tag{3.6}$$

In the broken symmetry phase ($\langle\phi\rangle \neq 0$, $m^2=0$) the denominator in Eq. 3.6 ($d(p^2) \equiv p^2[1/2\lambda_0 + B(p^2,0,\Lambda^2)] + \langle\phi\rangle^2$) is positive at $p^2=0$ and $d(p^2) \to -\infty$ as $p^2 \to \infty$. Since $B(p^2,0,\Lambda^2) = (N/32\pi^2)\log(e^2\Lambda^2/p^2)$, we find a tachyon pole $(p^2>0)$, at $p^2=m_T^2 > \Lambda^2 \exp(16\pi^2/N\lambda_0)$. In the case $\lambda_0>0$ this pole is outside the physical region. This phase has the ground state energy determined from Fig. 2b.

In the O(N) symmetric phase $(\langle\phi\rangle=0,\ m^2\neq 0)$ the denominator in the $\pi\pi$ scattering amplitude in Eq. 3.4 is $d(p^2) = 1/2\lambda_0 + B(p^2, m^2, \Lambda^2)$. At $p^2=0$ we have $d(0) = 1/2\lambda_0 + (N/32\pi^2)\log(\Lambda^2/m^2)$, which is positive on the left hand branch of the gap equation in Fig. 1. Since $B(p^2, m^2, \Lambda^2)$ decreases as $p^2\to\infty$ (Eq. 3.5) a tachyon will appear if $d(p^2=0)>0$ Indeed, this is the case in the $(\lambda_0>0)$ O(N) symmetric phase presented in Fig. 2a which has been derived from the gap equation solution on the left hand branch of Fig. 1. But again, the tachyon pole $p^2=m_T^2$ will appear at $m_T^2>\Lambda^2$ since $p^2=m_T^2$ is now the solution of

$$\frac{1}{2\lambda_0} + \frac{N}{32\pi^2}\log\left(\frac{e^2\Lambda^2}{p^2}\right) + O\left(\frac{m^2}{p^2}\right) = 0 \qquad (3.7)$$

Thus, we see that the symmetric as well as the broken symmetry phase in the $\lambda_0>0$ case do not have tachyons in the physical region $p^2<\Lambda^2$. This is not the situation in the $\lambda_0<0$ case as we will see next.

3.4 Tachyons - Case $\lambda_0<0$

Following the discussion in Sec. II , a transparent way for studying the $\lambda_0<0$ case is to use the renormalized theory. Indeed, only for $\lambda_0<0$ we may keep λ fixed and finite as $\Lambda^2\to\infty$. The denominator in the $\pi\pi$ scattering amplitude in Eq. 3.4 can be now written (using Eq. 2.11) as:

$$d(p^2) = \frac{1}{2}(p^2+m^2)\left[\frac{1}{\lambda_0}+\frac{N}{16\pi^2}\int_0^1 d\alpha \log\left(\frac{\Lambda^2}{\alpha(1-\alpha)p^2+m^2}\right)\right] + \langle\phi\rangle^2$$

$$= \frac{1}{2}(p^2+m^2)\left[\frac{1}{\lambda}+\frac{N}{16\pi^2}\int_0^1 d\alpha \log\left(\frac{M^2}{\alpha(1-\alpha)+m^2}\right)\right] + \langle\phi\rangle^2 \tag{3.8}$$

In the broken symmetry phase ($\langle\phi\rangle\neq 0$, $m^2=0$) a tachyon pole will appear at a value of p^2 which is a solution of

$$\frac{1}{\lambda} + \frac{2\langle\phi\rangle^2}{p^2} = \frac{N}{16\pi^2}\log\left(\frac{p^2}{e^2M^2}\right) \tag{3.9}$$

This pole at positive Euclidean p^2 further demonstrates the intrinsic instability discussed in Sec. II of the ground state of the renormalized O(N) vector theory.

The system in the O(N) symmetric phase ($\langle\phi\rangle=0$, $m^2\neq 0$) as shown in Sec. 3.2, is in a metastable state. Searching the spectrum in this phase for tachyons does not reveal its instability and indeed no tachyons are found here.[8-9] The decay of this false vacuum will be studied in Sec. IV, where we will also discuss its apparent stability at large N.

The absence of a tachyon pole in the $\pi\pi$ scattering amplitude in the O(N) symmetric phase is clearly viewed in Eq. 3.4. One can define[8] (using Eq. 3.4 with $\langle\phi\rangle=0$) an effective four pion coupling constant

$$\lambda_{eff}(p^2) = \frac{1}{1/\lambda_0 + B(p^2, m^2, \Lambda^2)}$$
$$= \frac{\lambda}{1 + \frac{\lambda N}{16\pi^2}\int_0^1 d\alpha \log\left[\frac{M^2}{\alpha(1-\alpha)p^2 + m^2}\right]} \tag{3.10}$$

which is negative for all values of p^2 provided the gap equation is satisfied on the right hand branch in Fig. 1. Indeed, we have seen in Sec. II that the extremum of the Hartree Fock energy as a function of m^2 gives a local minimum (see also Fig. 4) on the right hand side branch and thus $m^2 > M^2 \exp(16\pi^2/\lambda N)$, which assures that

$$\lambda_{eff}(p^2) = \frac{\lambda}{1 + \frac{\lambda N}{16\pi^2}\log\left(\frac{M^2}{m^2}\right)} < 0 \tag{3.11}$$

Since $\lambda_{eff}(p^2) \to 0^-$ as $p^2 \to \infty$, no tachyon poles are found.

Note that the tachyon found in ref. 5 (CJP) are due to using the solution of the gap equation on the left hand branch of Fig. 1, which is not a local minimum (but in fact, a local maximum in Fig. 4). Indeed, in this case, $\lambda_{eff}(p^2=0)>0$ and changes sign as $p^2\to\infty$.

IV. THE DECAY OF THE FALSE VACUUM

There are two types of instabilities detectable in the phase structure of the O(N) vector theory in four dimensions. The first one, an intrinsic instability extensively discussed in Secs.II and III, originates from the fact that the lowest lying state has an energy unbounded from below as the classical background field $\phi_c \to \infty$. If one limits the magnitude of ϕ_c below some finite ϕ_c^{max} the ground state is then determined by the value of $\phi_c^2 \pm \mu^2/\lambda$ as seen in Fig. 4a-f. The system can then be in one of two distinct phases: an O(N) symmetric phase with massive pions (discussed previously also in refs. 8-9) or in a spontaneously broken symmetry phase with N-1 massless goldstone bosons found in Sec. II. The different phases are also depicted in Fig. 3. The lowest energy state in the O(N) symmetric phase ($\phi_c = \langle\phi\rangle = 0$, $m^2 \neq 0$) is not the lowest lying state of the system if μ^2/λ is large (e.g. Fig. 3b and Fig. 4e), and thus this false vacuum will eventually decay into the lower lying broken symmetry phase even if ϕ_c is kept small. This is the second type of instability which will be discussed now. We recall also that the first instability revealed itself also by the presence of tachyons in the $m^2 = 0$ broken symmetry phase (in Sec. 3.4). On the other hand, no tachyons were found in the metastable O(N) symmetric phase. This renders the stability of this phase very interesting.

The ground state is in the O(N) symmetric phase as long as (Fig. 4c)

$$\vec{\Phi}_c^2 \pm \frac{\mu^2}{\lambda} < \frac{N}{32\pi^2} e^{\frac{1}{2}} m_x^2 \qquad \left[= \frac{N}{32\pi^2} M^2 \exp\left(\frac{16\pi^2}{\lambda N} + \frac{1}{2}\right) \right] \tag{4.1}$$

We will discuss here the $\phi_c^2 \simeq 0$ region and thus as long as μ^2/λ is limited from above, as seen in Eq. 4.1, the system is in the massive O(N) symmetric ground state. Since $\mu^2/\lambda = \mu_0^2/\lambda_0 + (N/16\pi^2)e\Lambda^2$ (Eq. 2.12) and $\lambda_0 < 0$, we see that for a fixed ultraviolet regulator, μ^2/λ increases as μ_0^2 decreases. Indeed, as expected, for μ_0^2 negative and large, μ^2/λ reaches $(N/32\pi^2)$ $M^2 \exp(16\pi^2/\lambda N + 1/2)$ and the ground state is now degenerate with the broken symmetry state with $m^2=0$ (Fig. 4d and going from Fig. 3a to 3b). For larger values of μ^2/λ, the system, whose V_{eff} is shown now in Fig. 4e, should decay into its true vacuum, which is in the $m^2=0$ phase. In order to calculate the rate of decay we will have to know the effective action that governs the dynamics of the system. Following ref. 5 we multiply the original Euclidean action functional integral

$$Z(\vec{J}) = \int D\vec{\Phi}(x) \exp\left\{ \int d^4x \left[-\frac{1}{2}(\partial_\mu \vec{\Phi})^2 - \frac{\mu_0^2}{2}\vec{\Phi}^2 - \frac{\lambda_0}{4}(\vec{\Phi}^2)^2 + \vec{J}\cdot\vec{\Phi} \right] \right\} \tag{4.2}$$

by an integral on the field $\chi(x)$

$$\int D\chi_{(x)}\ \exp\left\{\int d^4x\left[\frac{1}{4\lambda_0}\left(\chi-\lambda_0\vec{\Phi}^2-\mu_0^2\right)^2\right]\right. \tag{4.3}$$

which is well defined for $\lambda_0<0$ and a contour rotation is needed for $\lambda_0>0$.

In Appendix B we present the calculation of the effective action in terms of χ by integrating out the degrees of freedom $\vec{\Phi} = \vec{\phi} + \vec{\phi}_c$. This gives

$$Z(\vec{J}) = \int D\chi_{(x)}\ \exp\left\{-\Gamma\{\chi\}\right. \tag{4.4}$$

$$\left.+\int d^4z\left[\frac{1}{2}\vec{\phi}_c\Box\vec{\phi}_c + \vec{J}\cdot\vec{\phi}_c\right]\right\}$$

where

$$\Gamma\{\chi\} = \int d^4x\left[\frac{1}{2}\left(\vec{\phi}_c^{\,2}+\frac{\mu_0^2}{\lambda_0}\right)\chi - \frac{\chi^2}{4\lambda_0}\right] \tag{4.5}$$

$$+\frac{N}{2}\,\mathrm{Tr}\,\log(-\Box+\chi)$$

In the region of small $(\partial\chi)^2/\chi^3$ we calculated the effective action in terms of ψ where

$$\psi^2_{(x)} = \frac{N}{192\pi^2}\ \chi_{(x)} \tag{4.6}$$

We obtained

$$\Gamma\{\psi\} \simeq \int d^4z \left(-\frac{1}{2}\psi\Box\psi + V(\psi) \right) \qquad (4.7)$$

where

$$V(\psi) = \frac{1}{2}\mathcal{M}^2\psi^2 - \frac{K}{4}\psi^4 \log\left(\frac{\tilde{M}^2 e^{\frac{3}{2}}}{\psi^2}\right) \qquad (4.8)$$

The constants $\mathcal{M}^2$, K, $\tilde{M}^2$ in Eq. 4.8 are given in Eq. B.17 in terms of μ^2, λ, M^2 and N. The shape of the potential $V(\psi)$ is similar to the potential shown in Fig. 4 (when m^2 is replaced by ψ^2).

We would like now to estimate the transition rate from the O(N) symmetric metastable ground state with $m^2>0$ ($\langle\psi\rangle_0>0$) to the symmetry broken lower lying state ($m^2=0$, $\langle\psi\rangle_0=0$). Following ref. 19 we have to estimate the "bounce" which is the total Euclidean action $B = \int_{-\infty}^{\infty} d\tau \mathcal{L}_E$, since the width of the unstable symmetric state is proportional[20] to $\exp(-B/\hbar)$. The calculation of B involves the solution of the imaginary-time equation of motion in the potential $-V(\psi)$, with the initial and final condition shown in Fig. 5. The initial and final points are just the solution of the gap equation

$$\frac{\partial V(\psi)}{\partial\psi} = \psi\left[\mathcal{M}^2 - K\psi^2 \log\left(\frac{\tilde{M}^2 e}{\psi^2}\right)\right] \qquad (4.9)$$

The bounce $\int_{-\infty}^{\infty} d\tau \mathcal{L}_E$ is clearly proportional to N since $\chi \sim O(1)$ on the gap equation and thus $\psi^2 \sim O(N)$. The decay rate of the metastable O(N) symmetric state is thus proportional to exp(-N) and asymptotically small for large N.

The absence of tachyons in the metastable O(N) symmetric state at large N noted in Sec. 3.4 is due to the apparent stability of this state in the $N \to \infty$ limit. It seems proper to conclude that a perturbation expansion[3] around this false vacuum is not obviously inconsistent. Though the system is only metastable it can survive time scales arbitrarily large as $N \to \infty$. At finite N, the finite life time must be large compared to any time scale in the problem (e.g. universe life time, etc.) in order that an expansion around this false vacuum has any practical physical application. The situation is however very different at finite temperature as we find in Sec. V. At $T > T_c$ there is no metastable O(N) symmetric ground state for the renormalized $\lambda\Phi^4$ theory ($\lambda_0 < 0$) and no consistent perturbation expansion can be defined as the ultraviolet regulator is removed ($\Lambda^2 \to \infty$).

V. THE O(N) VECTOR MODEL AT FINITE TEMPERATURE AND THE FATE OF THE FALSE VACUUM

The discussion in the previous section dealt with the theory at zero temperature. The effect of a finite temperature on the phase structure and transition rate from the false vacuum will be discussed here.

The finite temperature effects in quantum field theories have been discussed in refs. 21 and 22. The large N analysis combined with a variational calculation (used in Section II to compute the quantum fluctuations to first order in 1/N), will be used now to calculate the thermal fluctuations at finite temperature $T=\beta^{-1}$.

The $\vec{\phi}^2$ vacuum expectation value at finite β is given by using the finite temperature Feynman rules

$$\langle\vec{\phi}^2\rangle_\beta = \frac{N}{(2\pi)^4}\int d^4k\, D_\beta(k) = \frac{N}{\beta}\sum_{n=0,\pm1,\pm2\ldots}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{k^2+m^2} \tag{5.1}$$

where $k_0 = \omega_n = (2\pi/\beta)n$. In Eq. 5.1

$$\sum_{n=0,\pm1,\pm2\ldots}\left(\frac{4\pi^2n^2}{\beta^2}+\vec{k}^2+m^2\right)^{-1} = \frac{1}{E_k^2}\left\{1+\frac{\beta E_k}{2}\sum_{n=0,1,2}\left[\frac{\beta E_k/2\pi}{n^2+(\beta E_k/2\pi)^2}\right]\right\} \tag{5.2}$$

where $E_k = (\vec{k}^2+m^2)^{1/2}$. Using

$$\sum_{n=1,2\ldots}^{\infty} \frac{x}{n^2+x^2} = -\frac{1}{2x} + \frac{\pi}{2}\coth(\pi x)$$

one finds

$$\begin{aligned} \langle \vec{\phi}^2 \rangle_\beta &= N \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{2E_k} + \frac{1}{E_k} \left(\frac{1}{e^{\beta E_k} - 1} \right) \right] \\ &= N \int \frac{d^4k}{(2\pi)^4} \left[\frac{1}{k^2+m^2} + \frac{2\pi\delta(k^2+m^2)}{e^{\beta E} - 1} \right] \end{aligned} \tag{5.3}$$

The second term originates from the Bose-Einstein distribution of the free boson gas. This expression clearly separates the quantum and thermal fluctuations of $\langle \vec{\phi}^2 \rangle_\beta$. Following Eq. 2.4 and 2.8, the Hartree Fock free energy at finite temperature can be written as

$$W_\beta(\vec{\phi}_c^{\,2}, m^2) = K_\beta(m^2) + \langle V(\vec{\Phi}) \rangle_\beta \tag{5.4}$$

where

$$\frac{\partial K_\beta(m^2)}{\partial m^2} = -\frac{m^2}{2} \frac{\partial \langle \vec{\phi}^2 \rangle_\beta}{\partial m^2} \tag{5.5}$$

and at large N

$$\langle V(\vec{\Phi})\rangle_\beta \simeq V(\vec{\Phi}_c^{\,2} + \langle\vec{\Phi}^2\rangle_\beta)$$

From Eq. 5.3

$$\langle\vec{\Phi}^2\rangle_\beta = \frac{N}{16\pi^2}\Lambda^2 - \frac{N}{16\pi^2}m^2\log\left(\frac{e\Lambda^2}{m^2}\right) + \frac{NT^2}{24}F\left(\frac{m^2}{T^2}\right) \quad (5.6)$$

and one finds

$$K_\beta(m^2) = \frac{N}{64\pi^2}m^4\log\left(\frac{e^{\frac{1}{2}}\Lambda^2}{m^2}\right) - \frac{N}{48}T^4\int_0^{m^2/T^2} dy\; y\left(\frac{d}{dy}F(y)\right) \quad (5.7)$$

where the first term is the T=0 variational kinetic energy term in Eq. 2.8. The rest of the expression in Eq. 5.7 contains the $T\neq 0$ contributions. F(x) (shown in Fig. 6) is given by

$$F(x) = \frac{6}{\pi^2}\int_0^\infty \frac{\zeta^2\,d\zeta}{\sqrt{\zeta^2+x}}\left[\frac{1}{\exp(\sqrt{\zeta^2+x}) - 1}\right] \quad (5.8)$$

at $x\to 0$ ($m^2/T\to 0$ limit) $F(0) = 1$ and $F(x)\to e^{-x}$ as $x\to\infty$. From Eq. 5.4 one finds:

$$W_\beta(\vec{\phi}_c^{\,2}, m^2) = \frac{N}{64\pi^2} m^4 \log(e^{\frac{1}{2}}\Lambda^2/m^2) - \frac{N}{48} T^4 \int_0^{m^2/T^2} dy\; y\left(\frac{d}{dy}F(y)\right)$$
$$+ \frac{\lambda_0}{4}\Big[\vec{\phi}_c^{\,2} + \mu_0^2/\lambda_0 + \frac{N}{16\pi^2} e\Lambda^2 - \frac{N}{16\pi^2} m^2 \log\left(\frac{e\Lambda^2}{m^2}\right)$$
$$+ \frac{N}{24} T^2 F\left(\frac{m^2}{T^2}\right)\Big]^2 \tag{5.9}$$

which can be compared with Eq. 2.15. After some algebra $W_\beta(\vec{\phi}_c^{\,2}, m^2)$ can be written in the form (compare to Eq. 3.1)

$$W_\beta(\vec{\phi}_c^{\,2}, m^2) = \frac{m^4}{4}\left[\frac{1}{\lambda} + \frac{N}{16\pi^2}\log\left(\frac{e^{\frac{1}{2}}M^2}{m^2}\right)\right] - \frac{N}{48} T^4 \int_0^{m^2/T^2} dy\; y\left(\frac{d}{dy}F(y)\right)$$
$$+ \frac{m^2}{2}(\text{G. Eq.}) + \frac{\lambda_0}{4}(\text{G. Eq.})^2 \tag{5.10}$$

where

$$\text{G. Eq.} \equiv \vec{\phi}_c^{\,2} \pm \mu^2/\lambda - \frac{m^2}{\lambda} - \frac{N}{16\pi^2} m^2 \log\left(\frac{eM^2}{m^2}\right) + \frac{N}{24} T^2 F\left(\frac{m^2}{T^2}\right) \tag{5.11}$$

This is in fact the gap equation in the case $T \neq 0$ since

$$\frac{\partial W_\beta}{\partial m^2} = \frac{\lambda_0}{2}\left[\frac{N}{24}\, \frac{\partial}{\partial x}F\Big|_{x=m^2/T^2} - \frac{N}{16\pi^2}\log\left(\frac{\Lambda^2}{m^2}\right)\right] \times (\text{G. Eq.}) \tag{5.12}$$

The coefficient in front of the G.Eq. is always negative in the physical region. $(m^2 < \Lambda^2)$ and thus the only solution to $\partial W/\partial m^2 = 0$ is G.Eq. $= 0$. Analogous to Eq. 3.2, we have here

$$V_{eff}^{\beta}(\vec{\phi}_c^{\,2}, m^2) = \frac{m^2}{2}\left(\vec{\phi}_c^{\,2} \pm \frac{\mu^2}{\lambda}\right) \tag{5.13}$$
$$- \frac{m^4}{4}\left[\frac{1}{\lambda} + \frac{N}{16\pi^2}\log\left(\frac{e^{\frac{3}{2}}M^2}{m^2}\right) - \frac{N}{12}\left(\frac{T}{m}\right)^4 \int_0^{m^2/T^2} F(y)\,dy\right]$$

that reproduces, of course, the same gap equation.

We will now discuss the renormalized theory (λ finite, $\Lambda^2 \to \infty$ and $\lambda_0 \to 0^-$). Fig. 7 represents the gap equation solution in case of $T \neq 0$. Clearly, as seen from Eq. 5.11, the only change is that $\phi_c^2 \pm \mu^2/\lambda$ in Fig. 1 is now replaced by $\phi_c^2 \pm \mu^2/\lambda + (N/24)T^2F(m^2/T^2)$. The effect, however, is very interesting; as T grows, the gap equation solutions m_L^2 and m_R^2 approach m_x^2 and finally, at large enough finite $T=T_c$, there is no solution. Viewed in Fig. 8, which is the $T \neq 0$ version of Fig. 4, we see the change in the effective potential due to the thermal fluctuations.

Thus, at high temperature ($T>T_c$) the large N renormalized $\lambda\Phi^4$ theory can be only in one phase in which the O(N) symmetry is broken. We see here the less common effect of a system that becomes asymmetric as the temperature increases.[23,24] A perturbation expansion in the symmetric vacuum ($m^2 \neq 0$) is bound to break down as the temperature increases. The implication of this result on the work of ref. 3 and in cosmological models may be of much interest. Indeed if we start at a very high temperature in the early universe, the metastable O(N) symmetric state will never be obtained as the universe cools down. The only existing (broken symmetry) phase suffers however from the intrinsic instability of the renormalized theory, namely the ground state energy is unbounded from below as seen in Sec. III. We conclude that the renormalized O(N) $\lambda\Phi^4$ vector theory is futile for $T\to\infty$ studies and the only consistent form of the theory in this limit is its regularized version with $\lambda_0>0$ and a finite cutoff, Λ^2.

The theory with a finite ultraviolet regulator and $\lambda_0>0$ has the usual behavior. Namely, as T increases, the ground state becomes symmetric even if we start at T=0 in an asymetric phase. This is clearly seen in Fig. 7. In the $\lambda_0>0$ and finite Λ^2 case, only the left hand branch of the gap equation is of interest (since $m_x^2>\Lambda^2$ as discussed in Sec. III). The O(N) symmetric phase at T=0 in Fig. 2a will remain symmetric at $T\neq 0$. At $\phi_c = \langle\phi\rangle = 0$ we see from Fig. 7 that the effective mass m_{LT}^2 grows with T and the O(N) symmetric lower energy state in Fig. 2a continues to be the ground state of the system.

The broken symmetry phase shown in Fig. 2b (gap equation solution) will have $\phi_c = \langle\phi\rangle \to 0$ as T increases and finally as seen in Fig. 7 at $-\mu^2/\lambda + (N/24)T^2F(0) > 0$, namely $T>\sqrt{24\mu^2/\lambda N}$ the system will be in an O(N) symmetric phase (with $m_\pi^2 \neq 0$).

As discussed in Sec. III we have to restrict the range of possible μ^2/λ values so that $m^2 \ll m_x^2$ and thus stay within what we called the physical region of the regularized theory namely, the low energy regime on the Λ^2 scale. Here we also must restrict the temperature range to $T^2 < \Lambda^2$ in order to stay in the physical region. This also allows a properly behaved solution to the gap equation for all values of T, as seen in Fig. 7, contrary to the case $\lambda_0 < 0$ of the renormalized theory where there is no solution to the gap equation as $T \to \infty$.

VI. SUMMARY AND DISCUSSION

Combining a variational calculation and a detailed large N analysis of the O(N) $\lambda\Phi^4$ vector model, the following results have been obtained:

(1). We have proved the existence of a broken symmetry phase with N-1 massless goldstone bosons in the large N limit of the renormalized $\lambda\Phi^4$ vector theory. This phase has been found here when the end point contribution of the variational parameters were studied. Previous analyses[8,9] concluded that there is no spontaneous symmetry breaking at large N. The symmetric and broken symmetry phases correspond to the minimum at $m^2 \neq 0$ and the end point $m^2=0$ in Fig. 4, respectively.

(2) We have pointed out, however, the existence of an inherent instability in the renormalized version of the theory. The upper limit of the ground state energy is unbounded from below at large values of the classical field (see Section 3.1 e.g. Eq. 3.1). If the analysis is limited to the small ϕ_c region, one finds a broken or an unbroken symmetry ground state (depending on the value of μ^2/λ - Figs. 3,4). The instability reflects itself, however, through either limiting[5,8,9] the possible ϕ_c values below a certain ϕ_c^{max} or through the existence of tachyons in the theory. We believe that this inherent instability persists in higher order calculations and is not a peculiarity of the 1/N expansion but rather a property of

the exact solution. Its origin can be traced back to the negative sign of λ_0 in the renormalized theory.

(3) From (1) and (2) we have concluded that the renormalized version of the large N $\lambda\Phi^4$ O(N) vector model in four dimensions is inconsistent. The broken and unbroken phases mentioned above are metastable. We have not calculated their decay rate at large ϕ_c values. If, however, the lifetimes are large, the theory may still have practical use, though ultimately unstable.

(4) Taking into account the last remark in (3) we have concentrated on the small ϕ_c region of the renormalized theory. The new broken symmetry phase we have found contains, for a certain region of the parameters, the true vacuum of the theory. We have shown that the decay rate of the O(N) symmetric ($m_\pi \neq 0$) false vacuum is small (proportional to e^{-N}) in the large N limit.

(5) At finite temperature and above some finite value of T_c we have found that the symmetric phase disappears and only the broken symmetry phase of the theory can exist at high temperature. This phase transition structure is of a less common type (although, not unknown); the system becomes asymmetric as the temperature increases.[23,24]

(6) The theory with positive λ_0 does not suffer from the instabilities mentioned in (2) and it may very well be the only meaningful theory. If $\lambda_0>0$, then the theory is defined as an effective field theory at momenta scale small compared to a fixed cutoff (Λ^2 or inverse lattice spacing

a^{-1}). At energies above this cutoff the physics must be modified from $\lambda\Phi^4$. If we take $\Lambda^2\to\infty$ ($a\to 0$) then the renormalized coupling $\lambda\to 0$ and thus the O(N) $\lambda\Phi^4$ vector theory. becomes a free theory.[14,15]

(7) The results (4) and (5) imply that indeed the O(N) symmetric ($m_\pi^2 \neq 0$) false vacuum may be used as a ground state for performing perturbation calculations (e.g., Ref. 3) since its decay rate is $\sim e^{-N}$. However, the disappearance of this false vacuum at high temperature (Sec. V) implies that little use for practical physical applications can be made of the theory at $T>T_c$. Thus, any early universe model incorporating this theory will never find the system in the O(N) symmetric false vacuum phase, as the universe cools down (seen in Fig. 8). The system will remain in the $m^2=0$ phase, where $W\to-\infty$ as $\phi_c^2\to\infty$, thus demonstrating the inconsistency in the renormalized $\lambda\Phi^4$.

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APPENDIX A

OSCILLATOR FREQUENCY AS A VARIATIONAL PARAMETER AND END POINT CONTRIBUTION

In Sec. II we used the mass m^2 defined from the oscillator frequencies $\omega_k = (k^2+m^2)^{1/2}$ as a variational parameter. One may instead let the frequencies ω_k vary, and then choose the best oscillator frequency that will determine the lowest lying state. Eq. 2.8 can be written instead as a functional of ω_k

$$W\{\omega_k\} = K\{\omega_k\} + \langle V(\vec{\Phi})\rangle_0$$

$$= \frac{N}{2}\int \frac{d^3k}{(2\pi)^3}\left(\frac{\omega_k}{2} + \frac{\vec{k}^2}{2\omega_k}\right) + \frac{\mu_0^2}{2}\left(N\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\right)$$

$$+ \frac{\lambda_0}{4}\left(N\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\right)^2 \qquad (A.1)$$

where

$$\langle\vec{\Phi}^2\rangle = \frac{N}{(2\pi)^3}\int\frac{d^3k}{2\omega_k}$$

The variation of ω_k gives

$$\frac{\delta W}{\delta \omega_k} = \frac{N}{32\pi^3}\left[1 - \frac{\vec{k}^2}{\omega_k^2} - \frac{\mu_0^2}{\omega_k^2} - \frac{\lambda_0}{\omega_k^2} N \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_k}\right] \quad (A.2)$$

and thus $\delta W/\delta\omega_k = 0$ implies

$$\omega_k^2 = \vec{k}^2 + \mu_0^2 + \lambda_0 N \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_k} \quad (A.3)$$

which gives the gap equation (compare with Eq. 2.9):

$$\omega_k^2 = \vec{k}^2 + \mu_0^2 + \lambda_0 \langle \vec{\phi}^2 \rangle \quad (A.4)$$

(or[16]

$$\langle \vec{\phi}^2 \rangle = \frac{N}{2} \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{\sqrt{\vec{k}^2 + \mu_0^2 + \lambda_0 \langle \vec{\phi}^2 \rangle}} \quad)$$

The physics in these calculations is clearly seen in Eq. A.3. in the case $\lambda_0 > 0$ and $\mu_0^2 > 0$, there will be no condensation of the zero mode in the $\vec{k}$ summation in $\langle \vec{\phi}^2 \rangle$ and the mass m^2 will be determined by the gap equation. After renormalization, Eq. A.4 gives (using Eqs. 2.6 and 2.11-12)

$$\omega_k^2 = \vec{k}^2 + m^2 + \lambda_0\left[\pm \frac{\mu^2}{\lambda} - \frac{m^2}{\lambda} - \frac{N}{16\pi^2}\, m^2 \log\left(\frac{M^2 e}{m^2}\right)\right] \quad (A.5)$$

When a background classical field ϕ_c is added, ϕ_c will vary and acquire the value $\phi_c = \langle\phi\rangle = 0$ in the ground state. The coefficient of λ_0 in Eq. A.5 gives the gap equation as seen in Eq. 2.13.

If $\mu_0^2<0$ (still $\lambda_0>0$) a condensation of the zero mode will give $\langle\phi\rangle\neq 0$ and thus Eq. A.3 becomes

$$\omega_k^2 = \vec{k}^2 + \mu_0^2 + \lambda_0\left(\langle\phi\rangle^2 + N\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_k}\right) \qquad (A.6)$$

With $\langle\phi\rangle^2 = -\mu_0^2/\lambda_0$ (or $\langle\phi\rangle^2 = -\mu^2/\lambda$ if renormalization is taken into account) a massless phase is created (Fig. 2b). No further condensations of the higher modes will occur in this case ($\lambda_0>0$) since $\langle\phi\rangle^2 = -\mu_0^2/\lambda_0$ stabilizes the system. If a classical background field ϕ_c is introduced it will acquire the value $\phi_c^2 = \langle\phi\rangle^2 = -\mu_0^2/\lambda_0$ in the ground state. In the region $\phi_c^2<-\mu_0^2/\lambda_0$ the effective mass is negative and the system is unstable.[5,25] We have seen that the solution to the gap equation (e.g., Fig. 2b) does not define the effective potential below $\phi_c^2 = -\mu_0^2/\lambda_0$. When the end points of $W(\phi_c^2, m^2)$ had been checked in the $\lambda_0>0$ case in Sec. III, we did not find any lower energy state below the one found from the gap equation.

The situation is different in the $\lambda_0<0$ case. Here the condensation of the $\omega_k=0$ mode does not suffice to stabilize the system and the gap equation above does not reveal all the physics. In Eq. A.3 after the $\omega_k=0$ mode condenses and therefore $<\phi>\neq 0$, there is a preference for higher modes to condense as well when $\lambda_0<0$. The gap equation now is however not much different from the $\lambda_0>0$ case and gives (Eq. A.5 including $\vec{\phi}_c^{\,2}$)

$$\vec{\phi}_c^{\,2} \pm \mu^2/\lambda - \frac{m^2}{\lambda} - \frac{N}{16\pi^2} m^2 \log\left(\frac{eM^2}{m^2}\right) = 0 \qquad (A.7)$$

It allows a $m^2\neq 0$, $\phi_c=<\phi>=0$ solution which gives an O(N) symmetric phase but the instability mentioned above reveals itself in the end point contribution of $W(\phi_c^2,m^2)$ Indeed the energy of the $m^2=0$, $\phi_c=<\phi>\neq 0$ broken symmetry phase has been shown in Sec. III to be lower than the energy of the lowest lying state in the $m^2\neq 0$ phase. In fact, as we have seen in Sec. III, the lowest lying state in the $m^2=0$ phase has an energy which is unbounded from below at large ϕ_c.

APPENDIX B

The effective action $\Gamma\{\chi\}$ in Eqs. 4.5 and 4.6 which governs the dynamics of the large N theory will be calculated here. $\Gamma\{\chi\}$ is defined from the Euclidean functional integral

$$Z(\vec{J}) = \int D\chi\, D\vec{\Phi}\, \exp\Big\{-\int d^4z\Big[\tfrac{1}{2}(\partial_\mu\vec{\Phi})^2 + \tfrac{\mu_0^2}{2}\vec{\Phi}^2 + \tfrac{\lambda_0}{4}(\vec{\Phi}^2)^2 - \vec{J}\cdot\vec{\Phi} - \tfrac{1}{4\lambda_0}(\chi - \lambda_0\vec{\Phi}^2 - \mu_0^2)^2\Big]\Big\}$$

$$= \int D\chi\, D\vec{\Phi}\, \exp\Big\{-\int d^4z\Big[\tfrac{1}{2}(\partial_\mu\vec{\Phi})^2 + \tfrac{1}{2}\chi\vec{\Phi}^2 - \tfrac{1}{4\lambda_0}\chi^2 + \tfrac{1}{2}\tfrac{\mu_0^2}{\lambda_0}\chi - \vec{J}\cdot\vec{\Phi}\Big]\Big\} \qquad (B.1)$$

After integrating out the $\vec{\Phi} = \vec{\phi} + \vec{\phi}_c$ degrees of freedom, we have

$$Z(\vec{J}) = \int D\chi\, \exp\Big\{\int d^4z\Big[\tfrac{1}{2}\vec{\phi}_c\Box\vec{\phi}_c - \tfrac{1}{2}\chi\vec{\phi}_c^2 + \tfrac{1}{4\lambda_0}\chi^2 - \tfrac{1}{2}\Big(\tfrac{\mu_0^2}{\lambda_0}\Big)\chi + \vec{J}\cdot\vec{\phi}_c\Big] - \tfrac{N}{2}\mathrm{Tr}\log(-\Box+\chi)\Big\} \qquad (B.2)$$

$$= \int D\chi\, \exp\Big\{-\Gamma\{\chi\} + \int d^4z\big(\tfrac{1}{2}\vec{\phi}_c\Box\vec{\phi}_c + \vec{J}\cdot\vec{\phi}_c\big)\Big\}$$

where we denote

$$\Gamma\{\chi\} = \bar{\Gamma}\{\chi\} - \int d^4z \left[\frac{1}{4\lambda_0}\chi^2 - \frac{1}{2}\left(\vec{\Phi}_c^{\,2} + \frac{\mu_0^2}{\lambda_0}\right)\chi \right] \qquad (B.3)$$

and

$$e^{-\bar{\Gamma}\{\chi\}} = \int D\vec{\phi}\ \exp\left\{ -\int d^4z \left[\frac{1}{2}\vec{\phi}(-\Box+\chi)\vec{\phi} \right] \right\}$$

$$= \exp\left[-\frac{N}{2}\,\mathrm{Tr}\log(-\Box+\chi) \right] \qquad (B.4)$$

In what follows we will consider the subspace of slowly varying functions $\chi(z)$ for which $(\partial\chi)^2/\chi^3 \ll 1$. $\bar{\Gamma}(\chi)$ can then be written as a local expansion[26]

$$\bar{\Gamma}\{\chi\} = \int d^4z \left[F_0(\chi) + F_1(\chi)(\partial_\mu\chi)^2 + \dots \right] \qquad (B.5)$$

where $F_0(\chi)$ and $F_1(\chi)$ are local functions of $\chi(z)$. $F_0(\chi)$ is found by calculating the $\mathrm{Tr}\log(-\Box+\chi)$ in Eq. B.4 with a constant χ. Up to an (infinite) constant

$$\frac{N}{2}\mathrm{Tr}\log(-\Box+\chi) = \frac{N}{2}\int d^4z \int \frac{d^4p}{(2\pi)^4} \log\left(1+\frac{\chi}{p^2}\right)$$

$$= \frac{N}{32\pi^2}\int d^4z \int_0^{e\Lambda^2} dy\ y \log\left(1+\frac{\chi}{y}\right) \qquad (B.6)$$

$$= \frac{N}{32\pi^2}\int d^4z \left[e\Lambda^2\chi - \frac{\chi^2}{2}\log\left(\frac{e^{\frac{3}{2}}\Lambda^2}{\chi}\right) \right]$$

or alternatively use Eq. 2.6 and a constant χ

$$\frac{\delta \bar{\Gamma}\{\chi\}}{\delta \chi} = \frac{1}{2}\langle \vec{\phi}^2 \rangle = \frac{N}{32\pi^2}\left(e\Lambda^2 - \chi \log\left(\frac{e\Lambda^2}{\chi}\right)\right) \equiv \frac{\partial F_0(\chi)}{\partial \chi} \tag{B.7}$$

and thus

$$F_0(\chi) = \frac{N}{32\pi^2}\left[e\Lambda^2\chi - \frac{\chi^2}{2}\log\left(\frac{e^{\frac{3}{2}}\Lambda^2}{\chi}\right)\right] \tag{B.8}$$

as is found also in Eq. B.6.

In order to calculate $F_1(\chi)$ one notes that the connected part $\langle \vec{\phi}^2(x)\vec{\phi}^2(y)\rangle_c$ can be written as

$$-\frac{\delta^2 \bar{\Gamma}\{\chi\}}{\delta \chi(x)\,\delta \chi(y)} = \frac{1}{4}\left[\langle \phi^2(x)\,\phi^2(y)\rangle - \langle \phi^2(x)\rangle\langle \phi^2(y)\rangle\right] \tag{B.9}$$

It can be calculated from Eq. B.4 with a constant χ

$$\begin{aligned}-\frac{\delta^2 \bar{\Gamma}\{\chi\}}{\delta \chi(x)\,\delta \chi(y)} &= \frac{N}{2}\int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)} \int \frac{d^4p}{(2\pi)^4}\frac{1}{((p-k)^2+\chi)(p^2+\chi)} \\ &= \frac{N}{32\pi^2}\int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\int_0^1 d\alpha \log\left(\frac{\Lambda^2}{\alpha(1-\alpha)k^2+\chi}\right)\end{aligned} \tag{B.10}$$

This can be expanded in the form

$$\begin{aligned}-\frac{\delta^2 \bar{\Gamma}\{\chi\}}{\delta \chi(x)\,\delta \chi(y)} &= \frac{N}{32\pi^2}\log\left(\frac{\Lambda^2}{\chi}\right)\delta(x-y) + \frac{N}{192\pi^2}\frac{\Box\,\delta(x-y)}{\chi} \\ &\quad + O(\Box^2\delta(x-y))\end{aligned} \tag{B.11}$$

Using Eq. B.5 we can now identify $F_0(\chi)$ and $F_1(\chi)$ with the constant χ expansion of Eq. B.11

$$\frac{\partial^2 F_0(\chi)}{\partial^2 \chi} = -\frac{N}{32\pi^2} \log\left(\frac{\Lambda^2}{\chi}\right) \qquad (B.12\,a)$$

$$2\, F_1(\chi) = \frac{N}{192\,\pi^2}\; \frac{1}{\chi} \qquad (B.12\,b)$$

Eq. B.12a reproduces the result of Eq. B.6 and Eq. B.7 (given in Eq. B.8). We finally have in Eq. B.3

$$\Gamma\{\chi\} = \int d^4 z \left[\frac{N}{24(4\pi)^2} \frac{(\partial_\mu \chi)^2}{\chi} + \frac{1}{2}\left(\vec{\phi}_c^{\,2} + \frac{\mu_0^2}{\lambda_0} + \frac{N}{(4\pi)^2} e\Lambda^2\right)\chi - \frac{N}{64\pi^2}\chi^2 \log\left(\frac{e^{\frac{3}{2}}\Lambda^2}{\chi}\right) - \frac{1}{4\lambda_0}\chi^2 \right] \qquad (B.13)$$

Using Eqs. 2.11 and 2.12 $\Gamma\{\chi\}$ can be written in terms of the renormalized parameters

$$\Gamma\{\chi\} = \int d^4 z \left[\frac{N}{24(4\pi)^2} \frac{(\partial_\mu \chi)^2}{\chi} + \frac{1}{2}\left(\vec{\phi}_c^{\,2} \pm \frac{\mu^2}{\lambda}\right)\chi - \frac{N}{4(4\pi)^2}\chi^2 \log\left(\frac{e^{\frac{3}{2}} M^2}{\chi}\right) - \frac{1}{4\lambda}\chi \right] \qquad (B.14)$$

If we now define

$$\psi^2_{(z)} = \frac{N}{192\,\pi^2}\,\chi_{(z)} \qquad (B.15)$$

then

$$\Gamma\{\psi\} = \int d^4 z \left[\frac{1}{2} (\partial_\mu \psi)^2 + \frac{\mathfrak{M}}{2} \psi^2 - \frac{K}{4} \psi^4 \log\left(\frac{e^{\frac{3}{2}} \tilde{M}^2}{\psi^2}\right) \right] \tag{B.16}$$

where

$$\mathfrak{M}^2 = \frac{192\pi^2}{N} \left(\vec{\phi}_c^{\,2} \pm \mu^2/\lambda \right) \quad , \qquad K = \frac{144(4\pi)^2}{N}$$

$$\tilde{M}^2 = M^2 \left(\frac{N}{192\pi^2} \right) \exp\left(\frac{(4\pi)^2}{N} \right) \tag{B.17}$$

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FIGURE CAPTIONS

Fig. 1 Solid line is the gap equation (Eq. 2.13) $m^2(1/\lambda + (N/16\pi^2)\log(eM^2/m^2)) = \phi_c^2 \pm \mu^2/\lambda$. Dashed line is the Hartree Fock energy W at m^2 values $(m^2 = m^2(\phi_c^2))$ for which the gap equation is satisfied (Eq. 2.16) $W(m^2) = (m^4/4)(1/\lambda + (N/16\pi^2)\log(\sqrt{e}\Lambda^2/m^2))$. $m_{1,2}^2$ solve the gap equation at $\phi_c^2 = 0$ (in the case $+\mu^2/\lambda$), $m_{G_0}^2$, $m_{W_0}^2$ are the zeroes of the solid and dashed lines whose maximum is at m_x^2 $(m_{G_0}^2 = em_x^2 = \sqrt{e}\, m_{W_0}^2$, $m_x^2 = M^2\exp(16\pi^2/\lambda N) = \Lambda^2\exp(16\pi^2/\lambda_0 N))$.

Fig.2 The Hartree Fock ground state energy W in the case $\lambda_0 > 0$ and finite Λ^2. The dashed line is the value of W as obtained from its extremum points when the gap equation $\partial W/\partial m^2 = 0$ is satisfied (Eq. 2.13). The dotted line is the end point values of W at $m^2 = 0$. The positive and negative $\mu_0^2/\lambda_0 + (N/16\pi^2)e\Lambda^2 = \pm\mu^2/\lambda$ cases are shown in Figs. 2a and 2b, respectively. The ground state is in an O(N) symmetric phase (2a) or in a broken symmetry (2b) in the case of $+\mu^2/\lambda$ or $-\mu^2/\lambda$, respectively.

Fig. 3 Case $\lambda_0 < 0$: As in Fig. 2, we plot the Hartree Fock ground state energy as obtained from the gap equation (Eq. 2.13) solution (dashed line - case $m^2 > m_x^2$, solid line - case $m^2 < m_x^2$ in Fig. 1) and the end point value ($m^2 = 0$ - the dotted line). Here we plot W in the limit $\lambda_0 \to 0^-$ and finite ϕ_c^2; this reproduces the results of refs. 5 and 8-9, and we denote W in this limit as V_{eff}. In the broken symmetry phase ($m^2=0$). V_{eff} is independent of $\vec{\phi}_c^2$ but $W\to-\infty$ as $\vec{\phi}_c^2\to\infty$ (not shown in these figures, see Eq. 2.15 with $m^2=0$).

Fig. 4 $V_{eff}(\vec{\phi}_c^2, m^2)$ in different $\vec{\phi}_c^2\pm\mu^2/\lambda$ regions. The gap equation solutions in Fig. 3 are the result of the minimum and maximum contributions shown here (dashed and solid lines of Fig. 3, respectively). The end point contribution is the $m^2=0$ point. The decay of the false vacuum in case (e) is discussed in Sec.IV.

Fig. 5 The potential $-V(\psi)$ in Eq. 4.8 (compare also to Fig. 4e), which governs the decay of the $m^2>0$ false vacuum. The "bounce" $B = \int d\tau \mathcal{L}_E$ is of order N and thus the decay rate $O(e^{-N})$.

Fig. 6 The shape of F(x) which appears in Eqs. 5.6 - 5.13.

Fig. 7 The effect of the finite temperature T on the gap equation solution of Fig. 1. For a given $\vec{\phi}_c^2$, the m^2 solutions change from m^2_{RO}, m^2_{LO} to m^2_{RT}, m^2_{LT}. Clearly for high enough temperature there is no solution to the gap equation in the case of ($\lambda_0<0$) the renormalized $\lambda\Phi^4$ (see also Fig. 8). In the case $\lambda_0>0$ and finite Λ^2 there will always be a solution in the physical region (p^2, $T^2 << \Lambda^2$).

Fig. 8 The disappearance of the false vacuum (gap equation solution with $m^2>0$) as the temperature increases beyond T_c. (See also Fig. 7). Here we see the behavior for the renormalized $\lambda\Phi^4 (\lambda_0<0)$. Recall that in the case $\lambda_0>0$, and finite Λ^2, the normal expected behavior occurs in the physical region as T^2 increases (always below Λ^2).