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Renormalization of U(1) Lattice Gauge Actions

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ABSTRACT

We investigate the space of variant mixed actions of compact U(1) lattice gauge theory through the Migdal renormalization recursion technique. We map and study the phase diagrams of actions specified through charge 1-charge 2, as well as charge 2-charge 5. Different phases of such diagrams are characterized by signals in their renormalized actions with the distinct patterns and periodicities which typify the collective behavior of their degrees of freedom.

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Lattice Euclidean gauge theories [1] are currently under vigorous investigation. They have revealed intricate phase structures in the space of mixed actions, i.e. actions with variable components of several characters of the gauge group elements defined on each plaquette. Such phase structures can be studied in a variety of ways, including renormalization techniques of the Migdal-Kadanoff type [2]. In a recent paper [3], we have illustrated how such techniques can yield information beyond the mere presence of a phase boundary: in a study of SU(2) gauge theory, we specified the role played by the phase boundaries to the universal approach towards infrared-stable lines of actions relevant to the continuum theory.

However, since the phase boundaries of SU(2) are observed or generally thought to have "windows," it is believed that the theory has only one phase, the confining phase. In contrast, the phase boundaries of U(1) gauge theory have not revealed (or hinted at the presence of) such windows into the confining phase [1,4]. In fact the phase boundaries of biperiodic compact U(1) gauge theory separate significantly different phases [4,5].

In this report we use the renormalization recursion technique to study the phase behavior of the charge 1-charge 2 space, whose phase boundary has already been determined by Monte Carlo [4]. We also study the charge 2-charge 5 space which is expected [6] to have an interesting two photon structure. The universal "Gaussian" stable line of actions [7] for each region of the phase diagrams investigated has a characteristic behavior, and allows us to identify the nature of the phase involved through its periodicity and the evolution of its effective coupling.

Let us outline the Migdal renormalization technique [2,3]. Details, conventions and assessment may be found in Ref. [3]. The (gauge invariant) compact U(1) actions under consideration are class functions, and, for this group, they depend on only one plaquette variable:

$$\theta = \frac{1}{r} \arccos \frac{U^r + (U^+)^r}{2} . \qquad (1)$$

The representation index r is the charge. The character expansion for U(1) even periodic class functions is simply the Fourier cosine expansion:

$$F(\theta) = F_0 + \sum_{r=1}^{\infty} F_r (2\cos r\theta)$$

$$F_r = \int_0^{2\pi} \frac{d\theta}{2\pi} \cos r\theta \; F(\theta)$$

$$\int_0^{2\pi} \frac{d\theta}{2\pi} (2\cos r\theta)(2\cos n(\theta-\theta')) = \delta_{rn} 2\cos r\theta' + 2\delta_{ro}\delta_{no} . \qquad (2)$$

We will be studying bare actions of the simplest alternative form:

$$-S(\theta) = \beta_r(2\cos r\theta) + \beta_n(2\cos n\theta) \equiv 2\beta(c_r\cos r\theta + c_n\cos n\theta), \quad (3)$$

where [r=1, n=0] denotes the Wilson action, and [r=1, n=2], [r=2, n=5] the actions analyzed below. We will be comparing renormalized actions to the periodic Gaussian actions of the Villain form [7,8], whose exponential (Gibbs factor) is:

$$e^{-S_V(\theta)} = \sum_{\ell=-\infty}^{\infty} e^{-\beta(\theta-2\pi\ell)^2} = \frac{1}{\sqrt{4\pi\beta}} \sum_{n=-\infty}^{\infty} e^{-n^2/4\beta} \cos n\theta \ . \quad (4)$$

This action approaches the Wilson action for very small and very large β [8]. For moderately large β it is quite close to the Manton action [7], which is a parabola in $[-\pi,\pi]$, and periodic beyond this interval:

$$-\beta\ \theta^2_{[-\pi,\pi]} = 4\beta\left(\frac{-\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos n\theta\right) . \quad (5)$$

Naturally, if the periodicity of the same function is multiplied by an integer factor k, all harmonics not divisible by k are missing:

$$-\beta\ \theta^2_{[-\pi/k,\pi/k]} = \frac{4\beta}{k^2}\left(\frac{-\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nk\theta\right). \quad (6)$$

Given a single plaquette action $S(\theta)$, Migdal's approximate renormalization recursion yields the effective single plaquette action describing the same theory in a lattice whose spacing has been increased by a factor λ:

$$e^{-S_{eff}(\theta,\lambda)} = \left[\sum_r F_r^{\lambda^2} (2\cos r\theta - \delta_{ro}) \right]^{\lambda^{d-2}}$$

where

$$F_r = \int_0^{2\pi} \frac{d\theta}{2\pi} \cos r\theta \; e^{-S(\theta)}, \tag{7}$$

d is the dimensionality of spacetime, taken to be 4 in this study, and λ is chosen [3] to be 1.05, which locates the phase transition of the Wilson action ($c_r=1$, $c_n=0$, in Eq. 3) at $\beta \simeq 0.49$, as dictated by Monte Carlo studies [9,4].

Since the effective action thus generated is also defined on the single plaquettes of the expanded lattice, the recursion can be iterated by computer a large number of times (typically 50), to allow direct evaluation of the functional integral by integration over plaquettes [2]. As described in Ref. [3], we use peaks of the heat capacity density $\mathscr{C}(\beta)$, for fixed c_r and c_n, to locate the phase boundaries of a given alternative action. Figures 1 and 2 indicate the phase boundaries thus specified in the β_r, $\beta_n \geqq 0$ quadrant of the [r=1, n=2] and [r=2, n=5] systems respectively. In Fig. 1, we are further superposing the known Monte Carlo phase boundary [4].

On each axis β_1, β_2 and β_5, the values of the critical couplings coincide ($\beta_c \simeq 0.49$), since the systems reduce to the Wilson action on these axes. In Fig. 1 and Fig. 2 the phase boundary for $\beta_2 \to \infty$ tends to the Z_2 phase transition value β_1, $\beta_5 \simeq 0.22$. In Fig. 2, for $\beta_5 \to \infty$, the two phase boundaries tend to the Z_5 phase transition [9.10] values $\beta_2 \simeq 0.5$ and $\beta_2 \simeq 0.6$.*

An exceptional feature of the recursion technique is that, by its very nature, it allows us to follow the evolution of the couplings and the functional forms of the renormalized actions with change of scale. Starting from a bare action (3), we directly monitor $S_{eff}(\theta)$ and its Fourier coefficients (or those of its Gibbs factor). We choose to do this every five iterations, which corresponds to an increase in scale of $\simeq$1.28. We can verify when the effective action approaches a periodic Gaussian by noting that the logarithum of the ratio of successive Fourier coefficients of the Gibbs factor (4) varies linearly with n, or, more easily, for large β, by noting that the Fourier coefficients of the action itself alternate in sign and decrease as $1/n^2$.

There is a known embarrassment of this approximate renormalization technique, concerning the lack of a sharp distinction between the confining and the free (sourceless QED) phases of the U(1) lattice theory, first observed in the study of the Wilson action: Migdal [2], and subsequently

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*The points marked by * in Fig. 2 do not actually represent peaks of $\mathscr{C}(\beta)$. Instead they locate the peaks of its crucial component $<(\cos 5\theta)^2>-<\cos 5\theta>^2$ (obtained by variations of β_5 instead of β) which responds to the transition between confined and unconfined photon-5 degrees of freedom, discussed below. The elevation of some of these peaks is so small that is is swamped by the tails of higher neighboring transition peaks in the full $\mathscr{C}(\beta)$ signal.

José et al. [8], have studied the two dimensional planar spin model--whose Migdal-Kadanoff recursions are identical to those of the four dimensional U(1) gauge theory [2]. These authors recognized that in the confining phase the inverse couplings β of the effective long distance actions iterate to zero, as they should. However, in the weak coupling phase (large β) the effective actions do not go to a completely free theory: after approaching some quasi-fixed point periodic Gaussian with inverse coupling β, that effective β drifts very slowly towards smaller values (extremely slowly for large β [8]: $d\beta/d(\text{iterations}) \propto -\beta^2 e^{-7\beta}$); eventually the trajectories accelerate and enter the confining phase. Thus, strictly speaking, there appears no phase transition separating the weak from the strong coupling regions.

Nevertheless, paying proper attention to this caveat, we can still distinguish the signal of the confining phase from those of the quasi-free phases. In Fig. 1 we observe that in the confining phase A' β_1 and β_2 iterate to zero, while higher harmonics develop smaller coefficients which then flow to zero as well. This behavior is quite similar to that of the confining region of the analogous SU(2) diagram [3].

In phases B' and C' of Fig. 1, the renormalized actions rapidly flow to quasi-fixed point periodic Gaussians (4)-(5) with periodicity 2 and 1 respectively, with couplings β continuously dependent on the bare couplings. This is represented, in C', by trajectories flowing to the dotted line of slope -1/4 which, on Fig. 1, represents the projection of Manton's action on the two parameter space of the two lowest harmonics. Once the effective actions reach these periodic Gaussians, the trajectories appear to stop (in contrast to the behavior of the corresponding SU(2) trajectories [3]), but they in fact do not. They flow to the origin, extremely slowly at first, and increasingly faster as they near the phase boundary, which they cross; from that point on, they appear like typical trajectories of the confining phase. Those trajectories which start in B' and C' very near to the interface with region A' do not slow down discontinuously after joining their respective periodic Gaussian--and in this regard they are not impressively different than those starting in region A'. However, the coefficient β of their dominant harmonic has been observed to always increase by some amount, before they start flowing to the origin. This is a general feature which the trajectories starting in the confining region A' uniformly lack (see Fig. 1): in this case the dominant coefficient β always decreases. This signal appears to us to constitute a practical clue for contrasting the confining from the nonconfining phases in the framework of the Migdal approximation.

The signals observed in the five regions of the charge 2-charge 5 systems are more dramatic, and characterize the nature of the five phases involved. Renormalization trajectories in region A exhibit the conventional confinement behavior typified by all β's rapidly going to zero (Fig. 3a). Renormalized actions in region B display confining behavior for the components of the actions associated with even harmonics (their appropriate β's $\rightarrow 0$), but quasi-free behavior for those harmonics which are divisible by five (their β's go to quasi-fixed values). The quasi-fixed point actions are quite close to a Villain/Manton action of periodicity 5 (Fig. 3b). Symmetrically, the effective actions in region C are close to a Villain/Manton action of periodicity 2 (Fig. 3c). The effective actions in region D are in fact superpositions of Villain/Manton actions with periodicity 2 and 5 (Fig. 3d). Finally, the effective actions in region E approach quasi-fixed point Villain actions with periodicity 1 (Fig. 3e).

We are thus led to characterize each phase as the domain of attraction to a specific type of effective action with a distinct periodicity. The associated periodic Gaussian Gibbs factors correspond [11] to the limit distributions of statistics, where they describe the collective behavior of many weakly coupled degrees of freedom.

We recognize the phases of our diagram as of the types anticipated in the entropy-energy analysis of biperiodic compact QED of Horn et al. [5]. These authors predict that a system of the general form (3) will support two types of photon excitations and two kinds of monopoles distinguished by two different periodicities--in our particular example 2 and 5. We thus verify that:

a) In phase A (in which the monopoles condense), both photons are confined.

b) In phase B, photon-5 is free, and photon-2 is confined.

c) In phase C, photon-2 is free, and photon-5 is confined.

d) In phase D, both photon-2 and photon-5 are free. For large β_5 this phase is recognized as the intermediate phase of Z_5 [10].

e) Finally in phase E, both individual periodicities 2 and 5 have been lost, but the effective action retains the fundamental periodicty of the bare action (1: the largest common divisor of 2 and 5), which could not be violated through the iteration procedure. This phase displays features of free excitations since a quasi-fixed point is reached. It corresponds to magnetic confinement--condensation of electric current loops [6,10]. We can see that the induced current loops of Ref. [6] in fact couple to one linear combination of the two photon fields. Consequently this combination becomes massive in this phase; but the linearly independent combination, which couples only to monopoles, has periodicity 1 and is massless, thus providing the Gaussian signal observed.*

We have also repeated the same analysis for [r=1, n=6], and verified the specific results of Ref. [6]. In this case the fixed point actions in each region display the same pattern as above, with the corresponding periodicities 1 and 6. The boundary between regions C and E does not extend to infinity, and phase C blends smoothly into phase E--observe that the fundamental periodicty is 1 in both regions. This also occurs in the charge 1-charge 2 system.

In summary, we have illustrated that the numerical renormalization recursion technique can not only produce the phase diagrams of multiperiodic lattice actions, but it may actually identify the essential hallmarks of the different phases of these diagrams.

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*In the notation of Ref. [6], these linear combinations are

$$\bar{A}_\mu \equiv \bar{\beta}\left(\frac{N_2}{\beta_1} A^1_\mu - \frac{N_2}{\beta_2} A^2_\mu\right) \quad \text{, and}$$

$$A_\mu \equiv \sqrt{\frac{\bar{\beta}}{\beta_1\beta_2}}\,(N_1A_1 + N_2A_2) \quad \text{, where} \quad \frac{1}{\bar{\beta}} \equiv \frac{N_2^2}{\beta_1} + \frac{N_1^2}{\beta_2}.$$

In terms of these, the action (14) of that reference reads:

$$-\frac{F^2_{\alpha\beta}}{2} - \frac{1}{2\bar{\beta}}\left(\bar{F}_{\alpha\beta} + \frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta}W^{\mu\nu}\right)^2 - 2\pi i\left[\bar{A}_\mu(N_2m_\mu - N_1M_\mu) + \sqrt{\bar{\beta}\beta_1\beta_2}\,A_\mu\left(\frac{N_1}{\beta_2}\,m_\mu + \frac{N_2}{\beta_1}\,M_\mu\right)\right].$$

Since A_μ does not couple to the electric current, it does not develop a mass.

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FIGURE CAPTIONS

Fig. 1: The phase diagram of the charge 1-charge 2 system and the projection of the associated renormalization trajectories on the β_1-β_2 plane. The x's indicate heat capacity peaks, and the dashed line provides the Monte-Carlo boundary of Ref. [4] for comparison. The dotted line represents the projection of Manton's action on this plane.

Fig. 2: The phase diagram of the charge 2-charge 5 system. The points o in each phase locate the representative bare actions whose evolution is given in Fig. 3.

Fig. 3: The effective actions of Fig. 2., after N(=1,11,...) iterations, where N is indicated next to each action. The inverse couplings (β_2, β_5) of each initial bare action are: a) (0.35,0.35); b) (0.35,1.0); c) (0.75,0.2); d) (0.65,0.65); e) (1.0,1.0).

Fig. 1

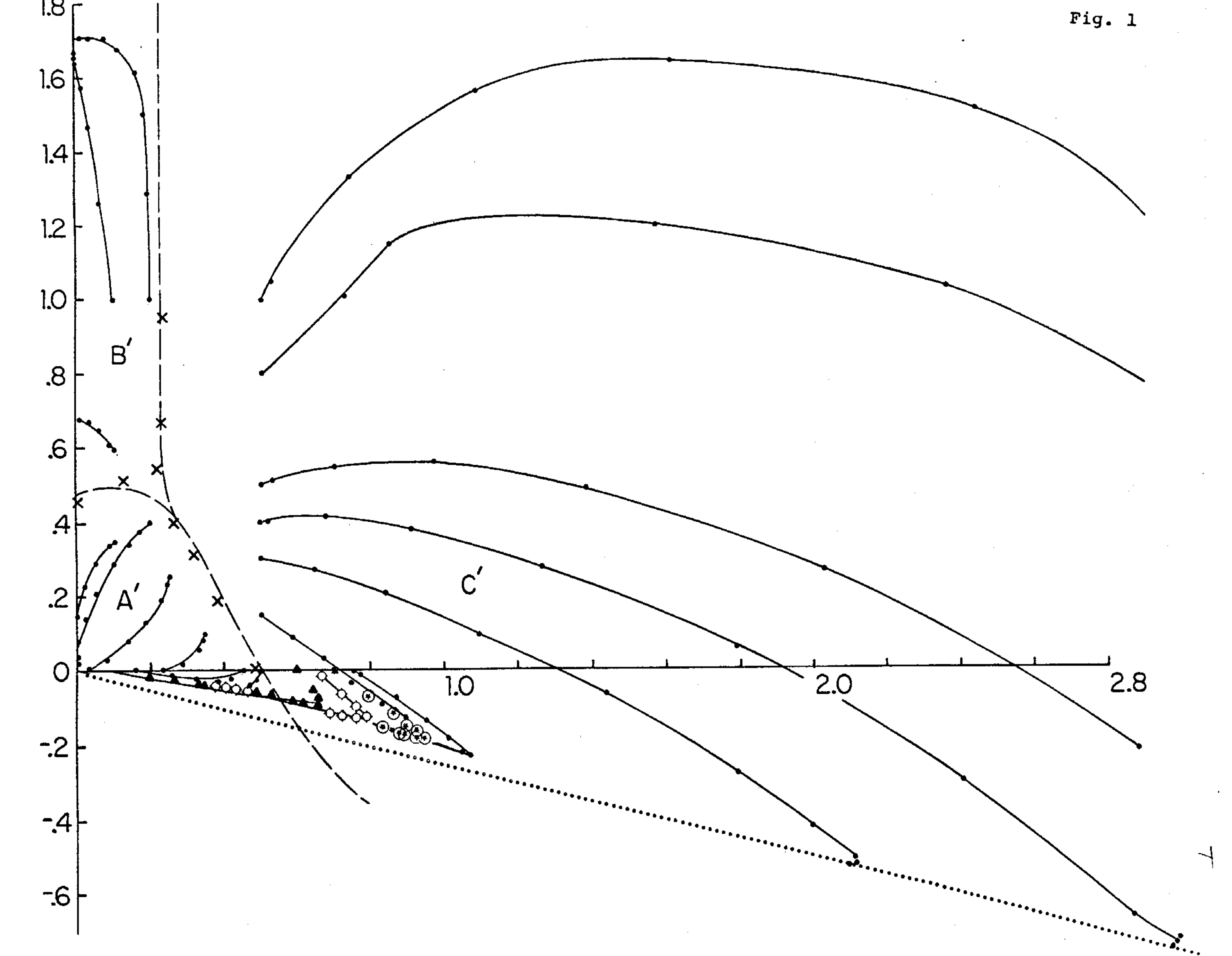


Fig. 2

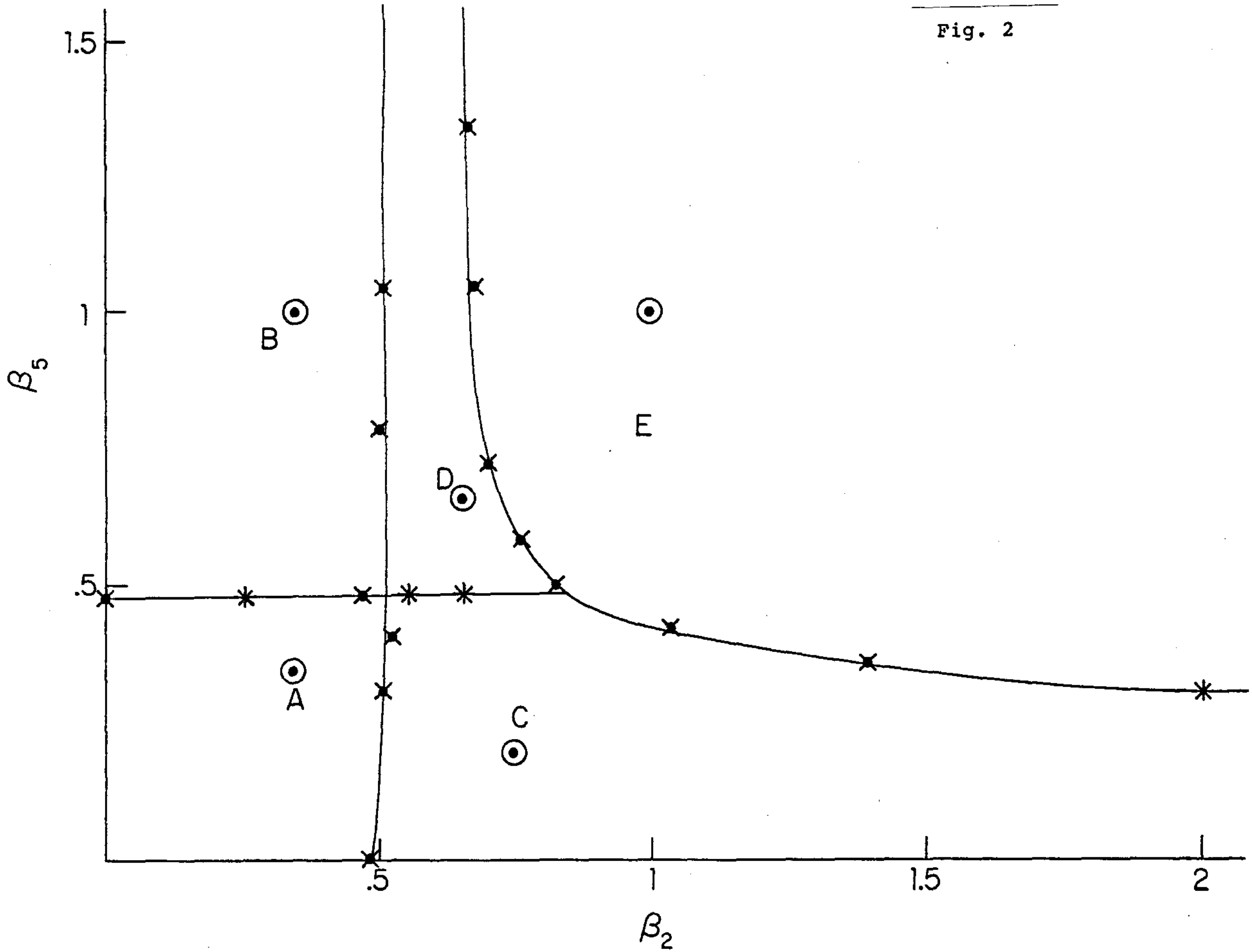


Fig. 3(a)

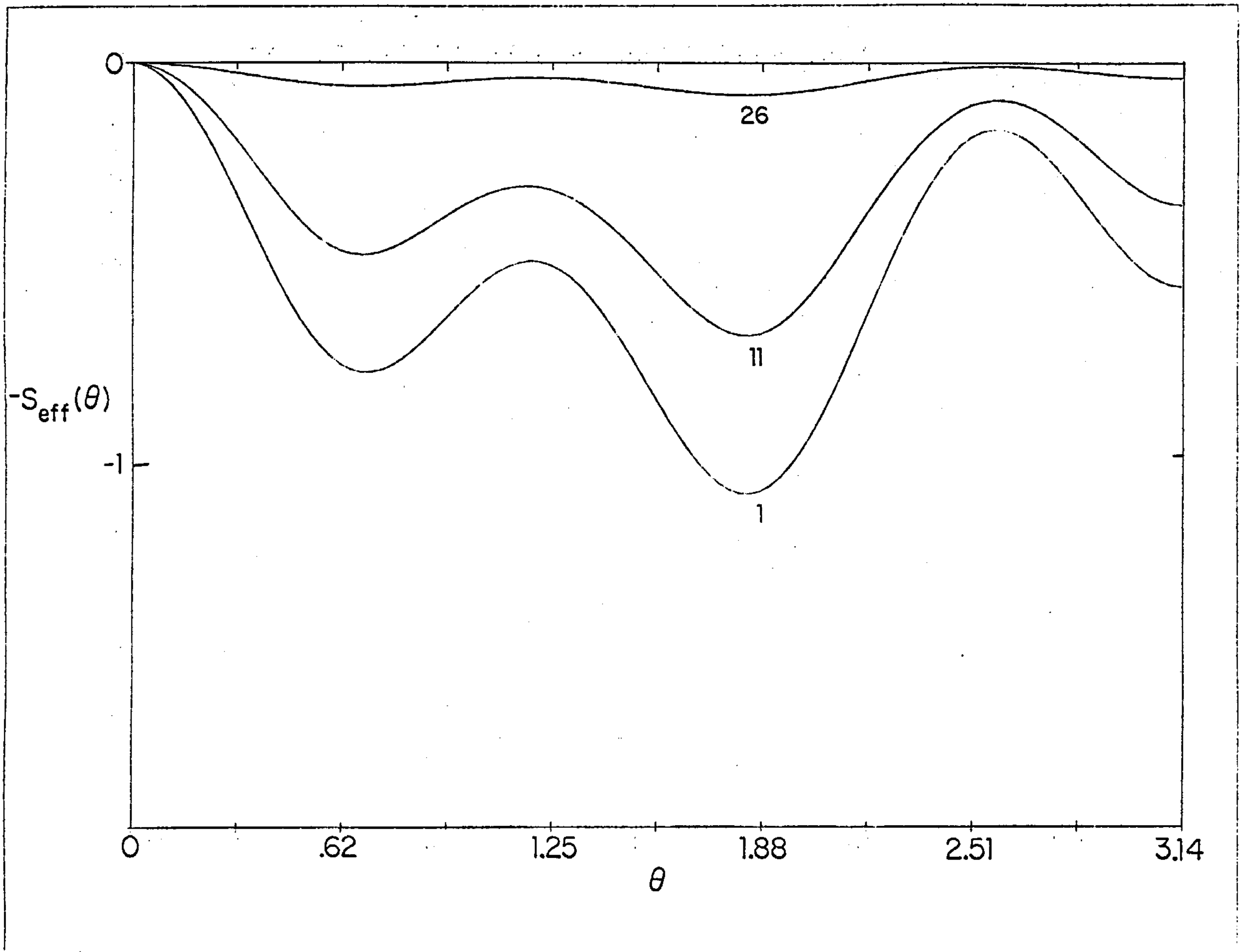
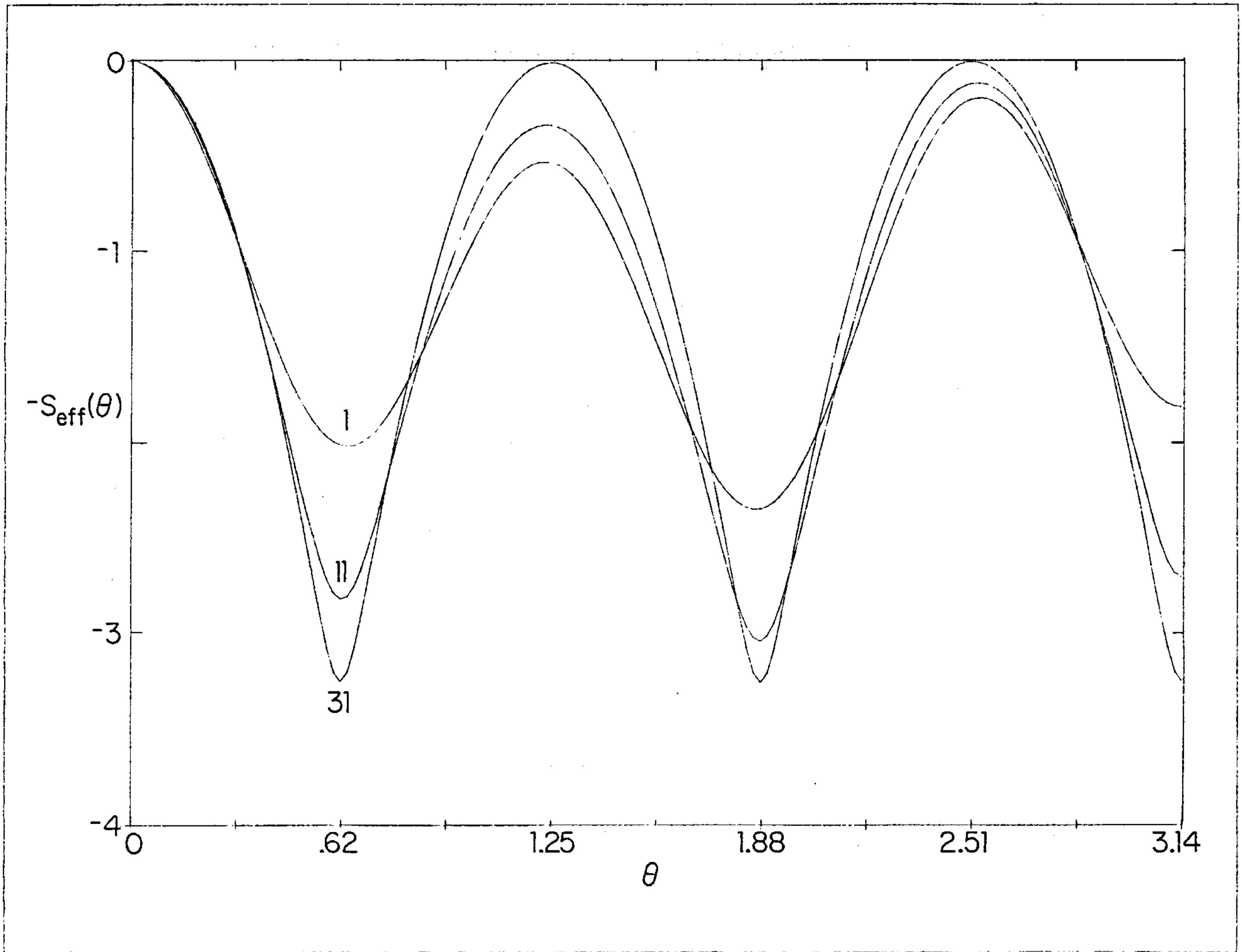


Fig. 3(b)



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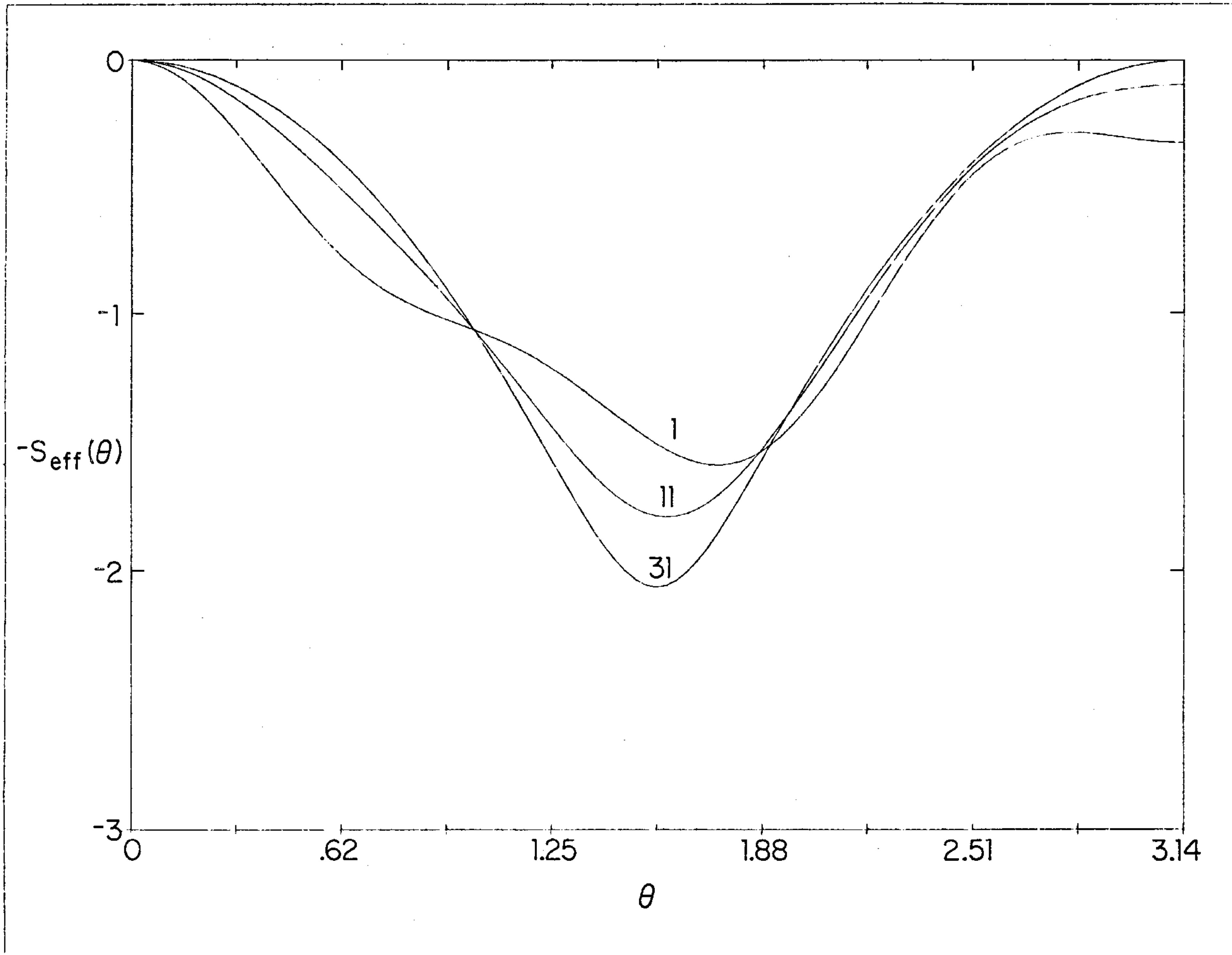

Fig. 3(d)

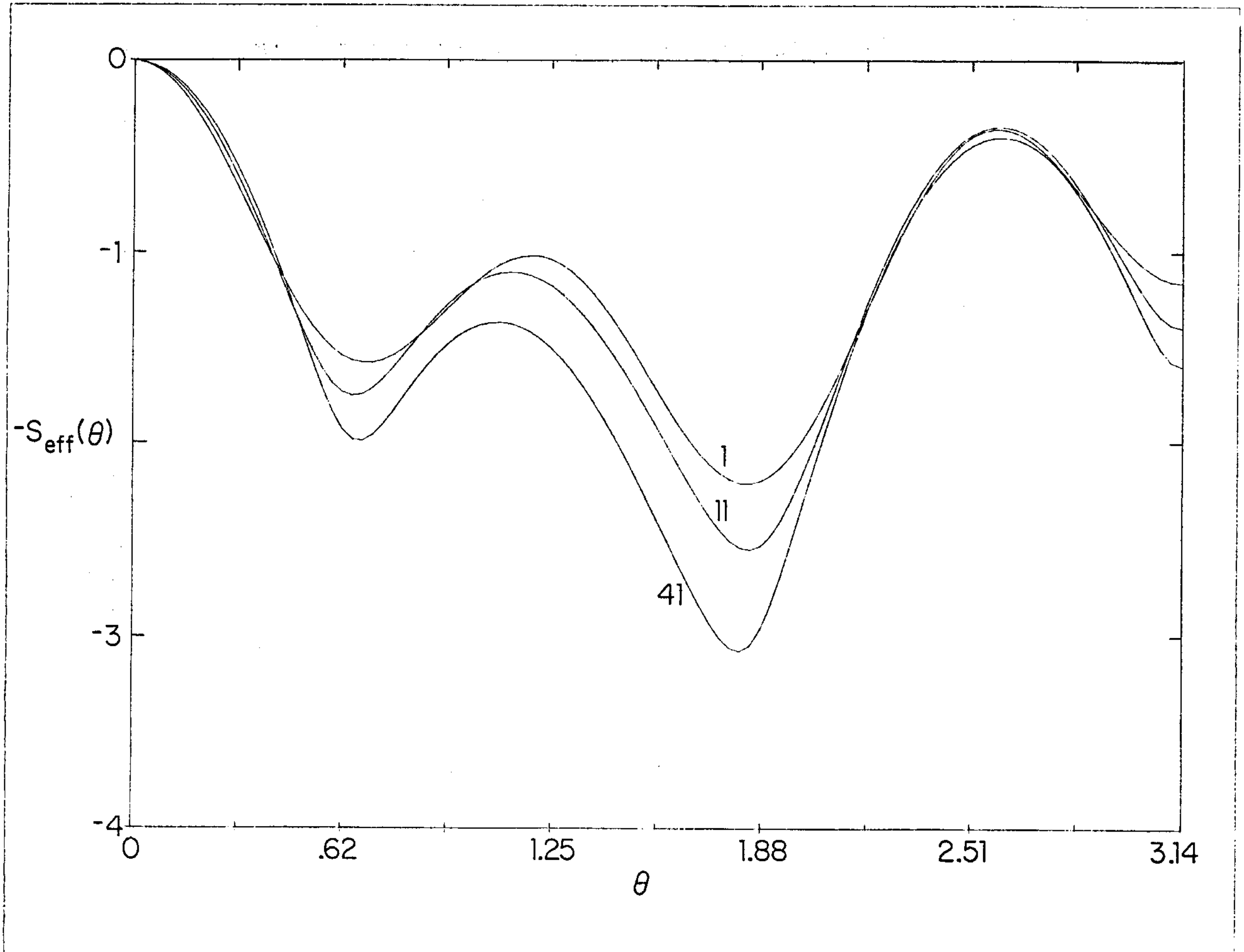

Fig. 3(e)