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THE POTENTIAL MODEL OF COLORED QUARKS: SUCCESS FOR SINGLE-HADRON STATES, FAILURE FOR HADRON-HADRON INTERACTIONS

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ABSTRACT

The success of the additive potential model of colored quarks for the masses, decay rates, and other properties of single mesons and baryons does not imply that this model can yield the observed meson-nucleon and nucleon-nucleon interactions. We give a comprehensive discussion of this issue. In agreement with previous authors, we conclude that, on the contrary, this model predicts inverse-power color-analog van der Waals potentials between separated hadrons which are in substantial contradiction with experimental data. We also discuss pathologies of non-abelian confining potentials, and show that the Hamiltonian is unbounded below for an arbitrary number of quarks and antiquarks in a definite color state for all color states, except the singlet, triplet, and antitriplet.

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I. INTRODUCTION

Very likely quantum chromodynamics (QCD), the local gauge theory of the interaction of colored quarks with colored gluons, should form the basis of a proper description of nucleons and mesons, as well as the basis for the derivation of the meson-nucleon and nucleon-nucleon interactions. It is attractive to attempt the simplest use of QCD by abstracting from it an additive two-body potential model in which the confining potential which binds quarks into nucleons and quarks and antiquarks into mesons is represented by a potential of the form

$$H_I = -\sum_{i<j} F_i \cdot F_j \, V(|\underset{\sim}{x}_i - \underset{\sim}{x}_j|) \,, \tag{1}$$

where the eight $F_i^{(\alpha)}$'s are the $SU(3)_{color}$ generators for the i^{th} quark or antiquark.[1] However, as emphasized in [1], it is important to note that, although the $F \cdot F$ color structure in (1) follows from one-gluon exchange in QCD, it is not clear that QCD implies that the confining potential, which does not come from one-gluon exchange alone, has this color structure.

Models in which hadrons are made of constituent quarks interacting via two-body forces were introduced in the early days of the quark model. These models provided a number of successful relations between hadron masses by using effective matrix elements as free parameters to describe the unknown quark forces [2-6]. At this level there was no understanding

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1. For quarks (antiquarks) $F^{(\alpha)}$ is $\frac{1}{2}\lambda^{(\alpha)}$ $(-\frac{1}{2}\lambda^{(\alpha)T})$. We abbreviate $\Sigma_\alpha F_i^{(\alpha)} F_j^{(\alpha)}$ by $F_i \cdot F_j$.

of the saturation of these forces at the quark-antiquark and three-quark systems, or of the relation between quark-antiquark forces in mesons and quark-quark forces in baryons. The introduction of the color degree of freedom solved the statistics problem [2,7,8] and partially explained saturation [7,8,9]. An interaction of the form (1) was introduced by Nambu who obtained a hadron mass formula which had the saturation property for color-singlet states and pushed non-singlet states to high mass. However, Nambu did not consider the spatial dependence of the potential, which introduces qualitatively different effects.

A model using an interaction of the form (1) with a general spatial dependence was shown to give the desired relation between strong quark-quark and quark-antiquark forces and to give very much weaker forces between color singlet hadrons [10]. Exotic bound multiquark states were shown to exist in models with sharply-cut-off short-range potentials, such as a square well, but not to exist for reasonably smooth potentials, such as Yukawa or power law potentials. Long range van der Waals forces were not considered, since the concept of confinement had not yet been introduced. It was natural at that time to assume a Yukawa potential with a long range cutoff, since no zero-mass gluons had been observed. However, the explicit results [10] for the two-quark-two-antiquark system hold for potentials with arbitrary spatial dependence, and are used below to calculate van der Waals forces for confining potentials.

Confining potentials which become infinite at large distances cause serious difficulty in multiquark systems, because the potential is not positive definite and can give negative infinite energies for systems of widely separated particles. In particular, Nambu's results do not hold for states which are not color singlets or triplets because the Hamiltonian is unbounded from below in such states.

The success of the additive potential model of colored quarks for the masses, decay rates, and other properties of single mesons and baryons does not imply that this model can yield the observed meson-nucleon and nucleon-nucleon interactions.[2] Indeed, the validity of this model for the description of the forces has been challenged, because this model leads to spurious long-range interactions between separated hadrons [1,10,13-16,21,22] which contradict experimental data. However, the literature contains conflicting results for the power behavior of the long-range interactions: some authors give an $a^{2\alpha-4}$ potential between hadrons separated by a distance a if the confining potential goes as r^{α} between quarks [15], and other authors give $a^{\alpha-4}$ [1,10,14,16-18]. We believe the latter result is correct. In addition, the negative conclusion about the validity of the model for the interhadron forces has itself been questioned [23]. Thus the situation remains somewhat confused.

The purpose of this article is to give a comprehensive discussion of the problem of unphysical long-range forces in potential models, and derive a number of rigorous results based on the use of the variational principle. In agreement with previous authors, we conclude that this

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2. Without trying to give an exhaustive set of references, we cite some articles which deal with color-analog van der Waals forces or with attempts to derive the nucleon-nucleon interaction from the quark model [1,10-22]. We emphasize that we use the phrase color van der Waals forces to refer to residual inverse-power interactions between separated hadrons, not the exponentially-decreasing strong interactions which are, presumably, also a residual effect of the color interactions among quarks and gluons.

model predicts inverse-power color-analog van der Waals potentials between separated hadrons which are in contradiction with present experimental data.

II. PECULIAR PROPERTIES OF CONFINING POTENTIALS

We first note that the interaction (1) has very peculiar properties. If it is attractive and confining, i.e. $V(r) \geqslant 0$ and $V(r) \to \infty$, $r \to \infty$, in the color-singlet quark-antiquark state and in the color-antitriplet quark-quark state, then it is repulsive and anticonfining, i.e. $\langle H_I \rangle$ is unbounded below, in the color-octet quark-antiquark and color-sextet quark-quark states.

For a quark and antiquark in the state ψ, which is either a color-singlet or a color-octet state, we use the Casimir operator[3] and

$$C(\psi) = (F_1+F_2)^2 = F_1^2 + 2F_1\cdot F_2 + F_2^2 = C(3) + 2F_1\cdot F_2 + C(3^*) \qquad (2)$$

to conclude that

$$2F_1\cdot F_2 = C(\psi) - 2C(3), \qquad (3)$$

so that (for localized static quarks -- we comment later that introducing

3. The second-order Casimir operator of SU(3) is given by

$$C = F\cdot F = \tfrac{1}{3}(f_1^2 - f_1 f_2 + f_2^2) + f_1$$

for the irreducible representation whose Young tableau has f_1 boxes in the first row and f_2 in the second. C is the same for a representation and its complex conjugate. We give C for representations of small dimension; $C(1) = 0$, $C(3) = 4/3$, $C(6) = 10/3$, $C(8) = 3$, and $C(10) = 6$.

wave packets and kinetic energy does not change our conclusions)

$$\langle 1(q\bar{q})|H_I|1(q\bar{q})\rangle = \frac{4}{3}V(r_{q\bar{q}}) \geq 0 \tag{4}$$

and

$$\langle 8(q\bar{q})|H_I|8(q\bar{q})\rangle = -\frac{1}{6}V(r_{q\bar{q}}) \to -\infty, \qquad r_{q\bar{q}} \to \infty . \tag{5}$$

For three quarks in the color-singlet (baryon) state, a similar argument using the fact that each qq state is in a color antitriplet shows that

$$\langle 1(qqq)|H_I|1(qqq)\rangle = \frac{2}{3}[V(r_{12}) + V(r_{23}) + V(r_{31})] \geq 0. \tag{6}$$

This interaction is unbounded both above and below[4] in a color-octet three-quark (baryon) state, since

$$\langle 8(qqq)|H_I|S(qqq)\rangle = \frac{1}{2}V(r) \to \infty,\ r \to \infty, \tag{7}$$

for a state with $r_{12} = r_{23} = r_{31} = r$; while

$$\langle 8(qqq)|H_I|8(qqq)\rangle \simeq -\frac{1}{6}V(a) \to -\infty,\ a \to \infty, \tag{8}$$

for a state with the quarks at $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ in a 3* state, where r_{12} is finite and fixed, and $a = |\underset{\sim}{x}_3 - \frac{1}{2}(\underset{\sim}{x}_1+\underset{\sim}{x}_2)|$. For three quarks in a color-decuplet state,

$$\langle 10(qqq)|H_I|10(qqq)\rangle = -\frac{1}{3}[V(r_{12}) + V(r_{23}) + V(r_{31})], \tag{9}$$

since each qq pair is in a color sextet, and thus $\langle H_I\rangle$ is negative semi-definite and unbounded below in this case.

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4. The instability of the color-octet three-quark states was discussed in Ref. [24,25].

Now we consider the general case of an arbitrary number of quarks and antiquarks in a given color representation r, labeled by a Young tableau with f_1 boxes in the first row and f_2 boxes in the second row. We will show that for each r except r = 1, 3, and 3* we can choose a sequence of wave functions for which the energy is unbounded below. If $f_1 \geqslant f_2+1$, separate a subsystem I (for example, a quark) in a color-triplet state by a large distance $\underline{a}$ from the remaining particles in subsystem II. This latter subsystem can be chosen to be in the representation with Young tableau (f_1-1,f_2). Then

$$\langle r|H_I|r\rangle \approx -\frac{1}{6}(2f_1-2-f_2)V(a) \to -\infty,\ a \to \infty, \tag{10}$$

unless $(f_1,f_2) = (1,0)$; i.e., unless r is a color triplet. If $f_1 = f_2$, separate a subsystem I' (for example, an antiquark) in a color-antitriplet state by a large distance $\underline{a}$ from the remaining particles in subsystem II'. This latter subsystem can be chosen to be in the representation with Young tableau (f_1-1,f_2-1). Then

$$\langle r|H_I|r\rangle \approx -\frac{1}{6}(f_1+f_2-2)V(a) \to -\infty,\ a \to \infty, \tag{11}$$

unless $(f_1,f_2) = (0,0)$ or $(1,1)$; i.e., unless r is a color singlet or a color antitriplet.

Inclusion of kinetic energy and the constraints of the Pauli principle do not change the above conclusions. Both effects can be taken into account by using properly antisymmetrized wave functions with the subsystem which is separated by the large distance $\underline{a}$ in, for example, a Gaussian wave packet. The kinetic energy remains constant, and the exchange terms decrease with

Gaussian rapidity as $\underset{\sim}{a} \to \infty$. Thus H_I and H are unbounded below in all color sectors except the singlet, triplet, and antitriplet.

Even in the color-singlet sector of physical interest, unphysical states of large negative energy can occur for potentials which increase too rapidly at large distances. Consider a color-singlet system with 2n quarks and 2n antiquarks, for the general color group SU(N). Assume that the antiquarks are all in the neighborhood of the origin, and in the totally symmetric color state. Assume that n of the quarks are in the neighborhood of -a on the x-axis, and that the other n quarks are in the neighborhhod of +a on the x-axis, with both sets of n quarks separately in the n-particle symmetric color state. The potential energy of this configuration is

$$V = F_n(-a)\cdot F_{2\bar{n}}(0) + F_{2\bar{n}}(0)\cdot F_n(a) + F_n(-a)\cdot F_n(a) \ . \tag{12}$$

Standard formulas [26] give $C_2^{(N)}(\bar{n}) = n(N-1)(n+N)/N$. Using $(F_n+F_{2\bar{n}})^2 = C_2^{(N)}(\bar{n})$, and $(F_n+F_n)^2 = C_2^{(N)}(2\bar{n})$, we find

$$V = kn(N-1)[2^\alpha n-2(2n+N)]a^\alpha/N \tag{13}$$

for a confining potential kr^α. This potential is unbounded below for

$$\alpha > 2 + \ln[1+(N/2n)]/\ln 2 \to 2,\ n \to \infty,\ N \text{ fixed.} \tag{14}$$

(As usual in such arguments, the particles at the three locations can be placed in fixed wavefunctions with finite interparticle separations to avoid getting large kinetic energy.)

For smaller values of α, states with unbounded negative energy probably do not occur. This can be shown rigorously for the harmonic oscillator potential, $\alpha = 2$, where V is positive semi-definite for color-singlet states with any number of quarks and antiquarks, since for this case [11],

$$H_I = k\{\Sigma_\alpha[\underset{\sim}{D}^{(\alpha)}]^2 - \Sigma_\alpha[\Sigma_i F_i^{(\alpha)} \underset{\sim}{x}_i^2, F^{(\alpha)}]_+\} , \tag{15}$$

where $F^{(\alpha)} = \Sigma_i F_i^{(\alpha)}$, $[A,B]_+ = AB + BA$, and $\underset{\sim}{D}^{(\alpha)} = \Sigma_i \underset{\sim}{x}_i F_i^{(\alpha)}$. The stated result follows from the fact that $F^{(\alpha)}$ annihilates color-singlet states.

Similar results hold for the abelian case [11,21,22]. For a neutral system ($n_q = n_{\bar{q}}$), H is unbounded below for $\alpha > 2$, as is shown by the example of two quarks and two antiquarks, with the quarks across one diagonal of a square of side a and the antiquarks across the other diagonal. Then

$$V = k[4a^\alpha - 2(\sqrt{2}a)^\alpha] \to -\infty, \; a \to \infty, \text{ for } \alpha > 2 . \tag{16}$$

The boundary case of the harmonic oscillator is also pathological, since the sum of all the two-body potential terms satisfies an algebraic identity, so that

$$\begin{aligned} V &= k[\sum_{i,j} |\underset{\sim}{x}_i - \underset{\sim}{y}_j|^2 - \sum_{i<j} (|\underset{\sim}{x}_i - \underset{\sim}{x}_j|^2 + |\underset{\sim}{y}_i - \underset{\sim}{y}_j|^2)] \\ &= k[\Sigma_i \underset{\sim}{x}_i - \Sigma_j \underset{\sim}{y}_j]^2, \end{aligned} \tag{17}$$

where the $\underset{\sim}{x}_i$ ($\underset{\sim}{y}_i$) are quark (antiquark) coordinates for N quarks and N antiquarks.

The Schrödinger equation with the potential (17), which is exactly solvable, does not have a spectrum corresponding to N mesons [11]. For N quark-antiquark mesons, the 6N degrees of freedom consist of N 3-dimensional degrees of freedom with discrete spectra corresponding to internal energies (one for each meson), N-1 3-dimensional translational degrees of freedom of relative motion of the mesons, and one 3-dimensional degree of freedom of translation of the whole system. Neglecting the kinetic energies, the ground state energy of the set of N mesons grows as N. For the system with the potential (17), there is only one 3-dimensional degree of freedom whose spectrum is a discrete, internal energy, 6N-6 continuous internal degrees of freedom, and one 3-dimensional degree of freedom of translation of the whole system. Neglecting the energy associated with the continuous degrees of freedom, the ground state energy of the discrete internal degree of freedom grows as $N^{1/2}$. Further pathologies in both abelian and non-abelian models are given in [21,22].

Although potentials with $\alpha = 4$ have recently been suggested in connection with QCD sum rules [27], confining potentials used in particle physics usually do not increase faster than the harmonic oscillator, and unphysical states with large negative energy do not occur. However this discussion shows the delicate nature of the cancellations between confining and anticonfining forces which are necessary to avoid this difficulty. Results for hadron interactions which are very sensitive to the details of this cancellation can be questioned. In particular, extreme care must be taken to avoid unphysical effects in approximation methods which do not rigorously restrict all allowed wave functions to the color-singlet sector. Variational methods, for example, can obtain a spurious lowering of the energy by adding tiny amounts of non-singlet wave functions with infinite negative energy.

One example of this difficulty is in the Hartree-Fock approximation for a many-quark system, in which each quark is assumed to move in the average field of the others. The Hartree-Fock wave function is not a pure color singlet; the color of the odd quark must be correlated in a complicated manner with the colors of the other quarks in order to produce a color-singlet state. Similarly in the treatment of large systems as infinite quark matter, by analogy with infinite nuclear matter, it is not sufficient for the local color density to be a singlet only on the average. Any fluctuations in color density at different points must be correlated rigorously in order to give a pure singlet state for the infinite system. Otherwise tiny admixtures of nonsinglet states with high negative energies can introduce spurious effects.

III. COLOR VAN DER WAALS FORCES

Another manifestation of these unphysical long-range interactions occurs in the generation of long-range van der Waals interactions between hadrons. The interaction (1) has a vanishing expectation value between two separated color-singlet hadrons; however, this comes about because the attractive and repulsive interactions between pairs in the two separated hadrons exactly cancel for singlet states. Because of this cancellation, small errors relative to the confining potential can lead to large errors relative to the van der Waals potentials which we discuss below. The state of two separated color-singlet hadrons is not an eigenfunction of the interaction (1), which has matrix elements connecting this state with states in which both hadrons are color octets coupled to an overall color singlet. It immediately follows from the variational principle that the energy of a state of two separated color-singlet hadrons can be lowered by admixing a small amount of the separated color-octet state, since the only contribution to the energy which is linear in the admixed amplitude comes from the off-diagonal matrix elements of H_I and the phase can be adjusted to be negative. When H_I is treated in second-order perturbation theory, this gives the standard type of van der Waals interaction which results from mutual polarization of the two separated objects and is attractive and decreases like a power of the distance. Because the polarization exists in color space as well as configuration space and the potential is confining, the power law is different from the conventional non-confining abelian case. Modified second-order perturbation theory for the energy of two hadrons separated by a large distance $\underset{\sim}{a}$ gives

$$E_2(a) \propto [\nabla^2 V(a) \langle r^2 \rangle]^2 / V(a), \tag{18}$$

where $\langle r^2 \rangle$ is the mean square radius of the individual hadron and $a^2 \gg \langle r^2 \rangle$.

The expression (18) is just the ratio of the square of the interaction matrix element to the energy denominator. The interaction is expanded in powers of r/a and the first nonvanishing term comes from the quadratic term. An additional power of V appears in the energy denominator because the attractive and repulsive confining potentials no longer cancel when the individual hadrons are not color singlets, and this term dominates the energy denominator at large distances. For the case of a power law potential, $V(r) \propto r^{\alpha}$.

$$E_2(a) \propto \langle r^2 \rangle^2 V(a)/a^4 \propto a^{\alpha-4} . \tag{19}$$

For the particular case of the Quigg-Rosner logarithmic potential, $V \propto \log(r/r_o)$

$$E_2(a) \propto \langle r^2 \rangle^2 a^{-4}/\log(a/r_o) \propto 1/a^4 \log(a/r_o) . \tag{20}$$

We now demonstrate the validity of the result (18) in a number of specific cases. We should like to go beyond perturbation theory and use the variational principle just to verify that the peculiarities of confinement do not destroy the validity of the perturbation expansion. Because the realistic case of two three-quark baryons with three colors and the full antisymmetrization required by the Pauli principle is complicated, we consider two simplified cases:

1. Two quark-antiquark mesons with three colors but with different flavors for the quark and antiquark pairs so that antisymmetrization is not necessary.

2. Two two-quark baryons in a model with two colors, spin and flavor suppressed, and the full permutation symmetry required by the Pauli principle.

In both cases the qualitative results of (18) are obtained, thus confirming the conclusion that the additive two-body confining model possesses long-range interactions which are in contradiction with experiment and can be used neither to calculate nucleon-nucleon forces nor the bulk properties of quark matter. The physical reason for this difficulty is clearly indicated by noting that the long-range interactions between hadrons must come from multigluon exchange, since single-gluon exchange between quarks does not give confinement. There is no obvious reason why multigluon exchange over large distances should be parameterized as a two-body interaction between quarks, even though this parameterization agrees with experiment for the short-distance interactions responsible for the static properties of hadrons. There are retardation effects, known both in the abelian and non-abelian cases, which reduce the effective van der Waals interaction [1,28,29]. In the non-abelian case there are, in addition, color oscillations which must be correlated between two non-singlet hadrons which are in an overall color-singlet state. For two color-octet hadrons in a color-singlet state, each individual hadron has color quantum numbers which oscillate about the eight states of the octet at a frequency given by the energy denominator of eq. (18), and these oscillations must be correlated between the two hadrons in order to have an overall color singlet.

This model clearly must break down when the period of the color oscillations becomes short in comparison with the time required for a gluon to traverse the distance between the two hadrons. This occurs when

$$\omega t \approx a\, V(a)/\hbar c >> 1, \tag{21}$$

where $\omega \simeq V(a)/\hbar$ is the frequency of color oscillations and $t = a/c$ is the time required for a gluon to travel between the two hadrons. Note that the color correlations introduced by this model are not Einstein-Podolsky-Rosen correlations initially introduced into a system which is later separated. Here we begin with two separated color-singlet states which are completely uncorrelated at very large distances. The interaction

(1) creates color polarizations, correlations, and oscillations at intermediate distances satisfying the condition (21) when the two hadrons are brought together. These effects are clearly unphysical and result from the use of an instantaneous "action at a distance" potential which introduces unphysical correlations between events separated by space-like intervals.

A correct description of long-range forces between quarks must also take multiparticle channels into account. Even in the simplest case of charmonium, the single-channel potential picture breaks down above charm threshold and the coupling to open decay channels seriously affects the spectroscopy [30]. The infinitely rising confining potential has neither real theoretical nor phenomenological justification. The separation of non-singlet quark clusters does not require infinite energy for large distances. The act of separation polarizes the vacuum, creates quark-antiquark pairs, and eventually produces free hadrons. If it is possible at all to treat interhadron forces with a fixed number of quarks and antiquarks, the interquark potential should develop an imaginary part at large distances, rather than remaining real and becoming infinite.

We now consider the two cases in detail

1. Two quark-antiquark mesons in $SU(3)_{color}$.

For two quarks and two antiquarks interacting with a non-abelian color-exchange interaction, the van der Waals interaction has been calculated using the same approach of modified second order perturbation theory as for the abelian case. However, the intermediate state in the perturbation treatment for the force between two separated clusters involves a polarization of each cluster in color space as well as in configuration space. This arises because an interaction having a color dependence $F_i \cdot F_j$ acting on quarks i and j in two separated color-singlet clusters changes each cluster to an octet in $SU(3)_{color}$ (or to the adjoint representation of SU(n) for the general case of $SU(n)_{color}$). The energy for producing this color-excited state thus appears in the energy denominator. This additional energy term comes from a term in the interaction which vanishes for color-singlet clusters but does not vanish for non-singlet clusters and gives a contribution equal to the two-body potential. Thus for a power-law potential which goes as r^α, the abelian result that the van der Waals force goes as $a^{2\alpha-4}$ is changed to $a^{\alpha-4}$ in modified second order perturbation theory, because the energy denominator introduces an additional factor of a^α.

The same result is obtained in a static treatment which neglects kinetic energies and diagonalizes the potential exactly. For the case of color-singlet states of two quarks and two antiquarks the potential is a 2x2 matrix in color space which has been diagonalized for the case of $SU(3)_{color}$ to give the eigenvalues [10,17]

$$U' = (7/16)(u_\alpha + u_\beta) + (1/8)u_q \pm (3/16)\ [8(u_\alpha - u_\beta)^2 + (u_\alpha + u_\beta - 2u_q)^2]^{1/2}, \quad (22a)$$

where

$$u_\alpha = u_{13} + u_{24}; \quad u_\beta = u_{14} + u_{23}; \quad u_q = u_{12} + u_{34}, \quad (22b)$$

and u_{ij} is the interaction between particles i and j; particles 1 and 2 are quarks and particles 3 and 4 are antiquarks.

To study the van der Waals problem, we consider the potential in the configuration where quark-antiquark pairs (13) and (24) are separated by a distance <u>a</u> which is large compared to the size of each cluster. We expand the potentials u_{12}, u_{23}, u_{14} and u_{34} around $\underset{\sim}{r}_{12}+\underset{\sim}{r}_{34} \simeq 2\underset{\sim}{a}$. The zero order values of u_β and u_q for this case are $u_\beta = u_q = 2u(a)$. Thus $u_\beta - u_q$ is higher order, and the potential can be expanded in powers of $u_\beta - u_q$ to give

$$U' = u_\alpha - \frac{(u_\beta - u_q)^2}{9(u_\beta - u_\alpha)} + O(u_\beta - u_q)^3 . \tag{23}$$

The first term is just the binding energy of the two clusters. The second term is the van der Waals interaction between the two clusters. Taking the expectation value of this term with the wave function chosen to give the color-spatial correlations to minimize the potential energy gives the result

$$\langle U' \rangle = \langle u_\alpha \rangle - \frac{[2(\frac{1}{r}\frac{du}{dr})^2_{r=a} + (\frac{d^2u}{dr^2})^2_{r=a}]\langle r_{13}{}^2\rangle^2}{162[u(a)-u(r_{13})]} , \tag{24}$$

where we have kept terms up to order $r_{13}{}^2$, and evaluated the expectation value by assuming $\langle r_{13}\rangle^2 = \langle r_{24}\rangle^2$ and assuming isotropic distributions in the wave functions for r_{13} and r_{24}. $u(r)$ denotes the spatial dependence of u_{ij}. Thus we can write

$$U_{vdW} = - \frac{[2(\frac{1}{r}\frac{du}{dr})^2_{r=a} + (\frac{d^2u}{dr^2})^2_{r=a}]\langle r_{13}{}^2\rangle^2}{162[u(a)-u(r_{13})]} , \tag{25}$$

where U_{vdW} denotes the van der Waals force given by the second term of (24). For a power law potential $u = kr^\alpha$, $\alpha > 0$,

$$U_{vdW} = -[k\alpha^2(\alpha^2-2\alpha+3)a^{\alpha-4}\cdot\langle r_{13}{}^2\rangle^2]/162 . \tag{26}$$

For a logarithmic potential $u = V_o \log(r/r_o)$,

$$U_{vdW} = V_o a^{-4} \langle r_{13}{}^2\rangle^2/54 \log(a/r_{13}) \quad . \tag{27}$$

We can include the contribution from the kinetic energy by using the virial theorem. We assume that the kinetic energy is due entirely to the relative motion within the two clusters, and neglect the kinetic energy of the motion of the cluster as a whole. This corresponds to the degrees of freedom denoted by $\mathcal{L}_{12}$ and $\mathcal{L}_{34}$. The virial theorem then gives the expectation value of the kinetic energy as

$$\langle T\rangle = \quad \langle r_{13} \frac{dU'}{dr_{13}} + r_{24} \frac{dU'}{dr_{24}} \rangle = 2\langle r_{13} \frac{dU'}{dr_{13}} \rangle \quad , \tag{28}$$

where we have used the symmetry between the two clusters.

Since the virial theorem is derived from the variational principle by minimizing the energy with respect to scale changes, the expression (28) holds for any approximate wave function where the cluster size is adjusted to minimize the energy by scale transformations. Note that there is a small change in cluster size produced by the van der Waals force (25). The contribution of this force gives a lower energy if the size of the cluster, represented by $\langle r_{13}{}^2\rangle$ is increased. If the cluster size is set by minimizing the energy of the clusters themselves, it is shifted slightly by the action of the van der Waals force. This then also gives a gain in kinetic energy. It is just this gain in kinetic energy which is automatically included by the use of the virial theorem. Substitution of (25) into (28) gives

$$\langle T\rangle = (1/2) \langle r\ du/dr\rangle - 2\ U_{vdW} \ . \tag{29}$$

We thus find that the total energy of the system is given by

$$E = \langle H\rangle = E_o - 3\ U_{vdW} \ , \tag{30}$$

where E_0 is the total energy, kinetic plus potential, of the two clusters when the van der Waals force is neglected. Thus the effect of the van der Waals force is enhanced by the kinetic energy contribution, but has the same power behavior.

2. Two baryons in $SU(2)_{color}$.

To make our discussion as simple as possible, we consider a model in which the color group is SU(2), so that two quarks in an antisymmetric color state correspond to a baryon. For $SU(2)_{color}$, the three Pauli matrices replace the eight λ's in (1). We also suppress the spin and flavor degrees of freedom of the quarks, and assume the quarks are nonrelativistic. Since we are interested in the residual interaction between two color-singlet baryons in our SU(2) model, we consider a four-quark state whose color representation belongs to the four-box square Young tableau (2,2). We construct two orthogonal color isospin tensors which correspond to four quarks in a color singlet:

$$A^{\mu_1\mu_2\mu_3\mu_4} = 2^{-1}\cdot 3^{-1/2}\,(\varepsilon^{\mu_3\mu_1}\,\varepsilon^{\mu_4\mu_2} - \varepsilon^{\mu_2\mu_3}\,\varepsilon^{\mu_4\mu_1}), \tag{31a}$$

and

$$B^{\mu_1\mu_2\mu_3\mu_4} = 6^{-1}\,(\varepsilon^{\mu_3\mu_1}\,\varepsilon^{\mu_4\mu_2} + \varepsilon^{\mu_2\mu_3}\,\varepsilon^{\mu_4\mu_1} - 2\varepsilon^{\mu_1\mu_2}\,\varepsilon^{\mu_4\mu_3}). \tag{31b}$$

These tensors transform under the symmetric group S_4 according to the square Young tableau; this representation is two dimensional. We satisfy the Pauli principle by requiring the wavefunctions F_1 and F_2 in the representation of the four-particle system to have the appropriate permutation symmetry under permutations of the coordinates $\underset{\sim}{x}_1$ through $\underset{\sim}{x}_4$,

$$F^{\mu_1\mu_2\mu_3\mu_4}(\underset{\sim}{x}_1,\underset{\sim}{x}_2,\underset{\sim}{x}_3,\underset{\sim}{x}_4) = A^{\mu_1\mu_2\mu_3\mu_4}\, F_1(\underset{\sim}{x}_1,\underset{\sim}{x}_2,\underset{\sim}{x}_3,\underset{\sim}{x}_4) + B^{\mu_1\mu_2\mu_3\mu_4}\, F_2(\underset{\sim}{x}_1,\underset{\sim}{x}_2,\underset{\sim}{x}_3,\underset{\sim}{x}_4) \ . \tag{31c}$$

The wavefunction for two quarks in a color singlet (i.e. an SU(2) "baryon") has the form

$$F^{(1)\mu_1\mu_2}(\underset{\sim}{x}_1,\underset{\sim}{x}_2) = \varepsilon^{\mu_1\mu_2}\, F^{(1)}(\underset{\sim}{x}_1,\underset{\sim}{x}_2),\ F^{(1)} \text{ symmetric;} \tag{32}$$

where $F^{(1)}$ satisfies the Schrödinger equation

$$\left[-\frac{1}{2m}\left(\nabla^2_{\underset{\sim}{x}_1} + \nabla^2_{\underset{\sim}{x}_2}\right) + \frac{k}{2}\quad V_{12}\right] \quad F^{(1)}(\underset{\sim}{x}_1,\underset{\sim}{x}_2) = EF^{(1)}(\underset{\sim}{x}_1,\underset{\sim}{x}_2) \ . \tag{33}$$

where $V_{ij} = |x_i - x_j|^{\alpha}$. Two quarks in a color triplet have the wavefunction

$$F^{(3)\mu_1\mu_2}(\underset{\sim}{x}_1,\underset{\sim}{x}_2) = \eta^{\mu_1\mu_2}F^{(3)}(\underset{\sim}{x}_1,\underset{\sim}{x}_2),\ \eta \text{ symmetric and } F^{(3)} \text{ antisymmetric;} \tag{34}$$

where $F^{(3)}$ satisfies the Schrödinger equation

$$\left[-\frac{1}{2m}\left(\nabla^2_{\underset{\sim}{x}_1} + \nabla^2_{\underset{\sim}{x}_2}\right) - \frac{k}{6}\quad V_{12}\right] \quad F^{(3)}(\underset{\sim}{x}_1,\underset{\sim}{x}_2) = EF^{(3)}(\underset{\sim}{x}_1,\underset{\sim}{x}_2) \ . \tag{35}$$

The spectrum of Eq. (35) extends to minus infinity; thus two quarks in a color-triplet state form an unstable system and the quarks repel each other and go into infinity. The Schrödinger equation for the four-quark color-singlet system expressed as a matrix equation for the wavefunctions F_1 and F_2 is

$$\left[-\frac{1}{2m}\sum_1^4 \nabla^2_{\underset{\sim}{x}_i}\,\mathbb{1} + k\begin{pmatrix} \frac{J}{3} - \frac{K}{6} & \frac{L}{2\sqrt{3}} \\ \frac{L}{2\sqrt{3}} & \frac{K}{2} \end{pmatrix}\right]\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = E\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} , \tag{36}$$

where

$$J = V_{13} + V_{23} + V_{14} + V_{24} , \tag{37a}$$

$$K = V_{12} + V_{34} , \tag{37b}$$

$$L = V_{13} + V_{24} - V_{14} - V_{23} , \tag{37c}$$

and $V_{ij} = |x_i - x_j|^\alpha$.

We find the dependence of the energy of two baryons on the distance of separation $\underline{a}$ using a variational calculation of an approximate Born-Oppenheimer Hamiltonian in which we neglect the translational kinetic energy of the baryons.[5] To make this explicit, we introduce co-ordinates $\underset{\sim}{R} = 4^{-1}(\Sigma_1^4 \underset{\sim}{x}_i)$, $\underset{\sim}{a} = 2^{-1}(\underset{\sim}{x}_3+\underset{\sim}{x}_4-\underset{\sim}{x}_1-\underset{\sim}{x}_2)$, $\underset{\sim}{r}_{12} = \underset{\sim}{x}_1-\underset{\sim}{x}_2$, and $\underset{\sim}{r}_{34} = \underset{\sim}{x}_3-\underset{\sim}{x}_4$. The Born-Oppenheimer Hamiltonian is

$$\hat{H} = -\frac{1}{m}(\nabla^2_{\underset{\sim}{r}_{12}} + \nabla^2_{\underset{\sim}{r}_{34}})\,\mathbb{I} + k\begin{pmatrix} \frac{J}{3}-\frac{K}{6} & \frac{L}{2\sqrt{3}} \\ \frac{L}{2\sqrt{3}} & \frac{K}{2}\end{pmatrix}, \tag{38}$$

where $\underset{\sim}{r}_{13} = -\underset{\sim}{a} + 2^{-1}(\underset{\sim}{r}_{12}-\underset{\sim}{r}_{34})$, $\underset{\sim}{r}_{24} = -\underset{\sim}{a} + 2^{-1}(\underset{\sim}{r}_{34}-\underset{\sim}{r}_{12})$, $\underset{\sim}{r}_{14} = -\underset{\sim}{a} + 2^{-1}(\underset{\sim}{r}_{12}+\underset{\sim}{r}_{34})$, and $\underset{\sim}{r}_{23} = -\underset{\sim}{a} - 2^{-1}(\underset{\sim}{r}_{12}+\underset{\sim}{r}_{34})$.

It is convenient to write the functions F_1 and F_2 in terms of sums of products of wavefunctions for two particle systems. The most general form of the functions F_1 and F_2 in terms of such products of two-particle clusters in which the clusters are either both in the isocolor-singlet state with amplitude $\cos\theta$ or are both in the isocolor-triplet state with amplitude $\sin\theta$ (in either case the overall four-quark system is in a singlet state) is

5. We are indebted to J. Sucher for suggesting this approach.

$$F_1 = 2^{-3/2}[F_{14}^{(1)}G_{23}^{(1)} + F_{23}^{(1)}G_{14}^{(1)} - F_{13}^{(1)}G_{24}^{(1)} - F_{24}^{(1)}G_{13}^{(1)}]\cos\theta$$
$$+ 2^{-3/2}\,3^{-1/2}[2F_{12}^{(3)}G_{34}^{(3)} + 2F_{34}^{(3)}G_{12}^{(3)} + F_{13}^{(3)}G_{24}^{(3)} +$$
$$+ F_{24}^{(3)}G_{13}^{(3)} - F_{14}^{(3)}G_{24}^{(3)} - F_{23}^{(3)}G_{14}^{(3)}]\sin\theta \tag{39a}$$
$$F_2 = 2^{-3/2}\,3^{-1/2}[2F_{12}^{(1)}G_{34}^{(1)} + 2F_{34}^{(1)}G_{12}^{(1)} - F_{13}^{(1)}G_{24}^{(1)} - F_{24}^{(1)}G_{13}^{(1)} -$$
$$- F_{14}^{(1)}G_{23}^{(1)} - F_{23}^{(1)}G_{14}^{(1)}]\cos\theta$$
$$- 2^{-3/2}[F_{13}^{(3)}G_{24}^{(3)} + F_{24}^{(3)}G_{13}^{(3)} + F_{14}^{(3)}G_{23}^{(3)} + F_{23}^{(3)}G_{14}^{(3)}]\sin\theta\ , \tag{39b}$$

where one cluster is represented by the function F and the other cluster separated by a distance $\underline{a}$ is represented by the function G and where $F(\underset{\sim}{x}_1-\underset{\sim}{x}_2)$ is abbreviated by F_{12}, etc., and, finally, $F^{(1)}$ or $G^{(1)}$ stands for a singlet state and $F^{(3)}$ or $G^{(3)}$ stands for a triplet state. The Pauli principle requires that the singlet functions are symmetric and the triplet functions are antisymmetric under interchange of their arguments. We take the singlet functions to be ground-state eigenfunctions of the two-body singlet potential, (33) and the triplet functions to be the first antisymmetric excited eigenfunction of the same equation. We can then use (33) to eliminate the kinetic energy in favor of the appropriate two-body potential and the energy eigenvalue. When the clusters are well separated, exchange terms are exponentially small, and we drop them. In addition, as stated above, we have dropped the center of mass energy of each cluster. The expectation value of the Hamiltonian in this variational wavefunction is

$$\langle H\rangle = E(a) = \int d^3r_{12}\, d^3r_{34}\ [A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta]\ , \tag{40}$$

where

$$A = 2\varepsilon_o (F_{12}^{(1)} G_{34}^{(1)})^2 \ , \tag{41a}$$

$$B = 3^{-1/2} k (V_{13}+V_{24}-V_{14}-V_{23}) F_{12}^{(3)} G_{34}^{(3)} F_{12}^{(1)} G_{34}^{(1)} \ , \tag{41b}$$

and

$$C = [2\varepsilon_1 - (2k/3)(V_{12}+V_{34}) + (k/3)(V_{13}+V_{23}+V_{14}+V_{24})](F_{12}^{(3)} G_{34}^{(3)})^2 \ . \tag{41c}$$

Using the normalization properties of the two-body states, introducing the color-polarization dipole matrix elements, and keeping terms relevant for $0<\alpha<5$, we can write the energy as a function of separation.

$$E(a) = A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta \ , \tag{42}$$

where

$$A = 2\varepsilon_o \ , \text{ where } \varepsilon_o \text{ is the ground state "baryon" energy,} \tag{43a}$$

$$B = -3^{-1/2} \, k\alpha p \ , \tag{43b}$$

$$C = (4k/3)a^{\alpha} \ , \tag{43c}$$

$$p = <\underset{\sim}{r}_F^{(31)}>\cdot<\underset{\sim}{r}_G^{(31)}>+(\alpha-2)(\hat{\underset{\sim}{a}}\cdot<\underset{\sim}{r}_F^{(31)}>)(\hat{\underset{\sim}{a}}\cdot<\underset{\sim}{r}_G^{(31)}>)]a^{\alpha-2} \ , \tag{43d}$$

and

$$<\underset{\sim}{r}_F^{(31)}> = \int F^{(3)*}(\underset{\sim}{r}) \, \underset{\sim}{r} \, F^{(1)}(\underset{\sim}{r}) d^3 r, \text{ etc.} \tag{43e}$$

An alternative form for the energy is

$$E(a) = \frac{1}{2}(A+C) + \frac{1}{2}[(A-C)^2+B^2]^{1/2} \sin(2\theta+\phi) \ , \tag{44a}$$

where

$$\sin\phi = (A-C)[(A-C)^2+B^2]^{-1/2} \ . \tag{44b}$$

The minimum of the energy is

$$E_{min} = \frac{1}{2}(A+C) - \frac{1}{2}[(A-C)^2+B^2]^{1/2} , \tag{45a}$$

where the angle at the minimum is

$$\theta_{min} = -\frac{1}{2}(\phi + \frac{\pi}{2}) . \tag{45b}$$

Our final results are that the minimum energy is[6]

$$E_{min} \simeq 2\varepsilon_o - (1/16)\ k\alpha^2\ p^2\ a^{\alpha-4} \tag{46a}$$

and the mixing angle is

$$\theta_{min} \simeq -(\sqrt{3}/4)\alpha p . \tag{46b}$$

Since the variational calculation gives an upper bound to the energy, we are assured that the van der Waals term decreases no faster than $a^{\alpha-4}$. For the most likely case of the linear potential ($\alpha=1$), Feinberg and Sucher [1] find the constraint

$$|U_{vdW}(a)| = \lambda_3\ (a_o/a)^3\ 200\ \text{MeV},\ \lambda_3 \leq 10^{-12} , \tag{47}$$

$a_o = 1$fm, from data on Cavendish-type experiments. Comparison with the result of (46a),

$$U_{vdW}(a) \simeq -(1/16)\ k\ p^2 a^{-3} , \tag{48}$$

6. For completeness, we note that for $\alpha > 2$ the minimum value of p is $p_{min} = (1-\alpha)\langle r_F\rangle\cdot\langle r_G\rangle$, which occurs when the color-dipole transition moments are collinear with the separation between the clusters and have opposite direction, and that for $\alpha < 2$ $p_{min} = -\langle \tilde{r}_F\rangle\cdot\langle \tilde{r}_G\rangle$, which occurs when the transition moments also have opposite direction, but are orthogonal to the separation between the clusters.

for $\alpha = 1$, using the estimate $p \sim 1\ fm^2$, gives

$$k < 1.6 \times 10^{-11}\ fm^{-2}\ ; \tag{49a}$$

to be compared with

$$k = 5\ fm^{-2} \tag{49b}$$

found from charmonium data by Eichten, et al. [31]. Thus the van der Waals interaction which follows from a linear confining interaction in the potential model is very strongly ruled out [1,14,18] The potential model does not include retardation effects. Inclusion of retardation seems likely to change the van der Waals interaction from $a^{\alpha-4}$ to $a^{\alpha-5}$[1]. For the linear potential this changes a^{-3} to a^{-4}, for which the bounds of [1] are weaker, but still significant. Using the estimate

$$U^{ret}_{vdW} \simeq -(1/16)k\ p^{5/2}\ a^{-4} \tag{50}$$

for α=1 with retardation, and, again, taking $p \sim 1\ fm^2$, the bound

$$|U_{vdW}(a)| = \lambda_4(a_o/a)^4\ 200\ \text{MeV},\ \lambda_4 \lesssim 3 \times 10^{-3} \tag{51}$$

gives

$$k < 4.8 \times 10^{-2}\ fm^{-2}\ , \tag{52}$$

which is still two orders of magnitude smaller than (49b). Thus, even with retardation, the additive two-body potential model for the confinement of

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7. We comment briefly on simplifying assumptions made in this calculation. The use of non-relativistic kinematics is not important, since confinement is a low-energy or long-distance phenomenon in which relativisitc corrections should not be important. The use of $SU(2)_{color}$ instead of $SU(3)_{color}$ should not be qualitatively significant.

quarks leads to van der Waals interactions which are in strong contradiction with experiment, and this model cannot be used to derive the nucleon-nucleon interaction.

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