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Decoupling, Effective Lagrangian, and Gauge Hierarchy in Spontaneously Broken Non-Abelian Gauge Theories

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ABSTRACT

In spontaneously broken non-abelian gauge theories which admit gauge hierarchy at the tree level, we show, to all orders in perturbation theory, that (i) the super heavy particles decouple from the light sector at low energies, (ii) effective low energy renormalizable theory emerges together with appropriate counter terms, and (iii) gauge hierarchy can be consistently maintained in the presence of radiative corrections. These assertions are explicitly demonstrated for O(3) gauge theory with two triplets of Higgs particles in a manner easily applicable to more realistic grand unified theories. Furthermore, as a byproduct of our analysis, we obtain a systematic method of computing the parameters of the effective low energy theory via renormalization group equations to any desired accuracy.

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I. INTRODUCTION AND SUMMARY

A central idea behind the unification of forces of vastly different strength is that such an apparent hierarchy arises not from the difference of the fundamental coupling constants of the theory but rather from that of the masses of the exchanged particles. Although, ultimately, we hope to explain the mass hierarchy itself through some dynamical mechanism from a theory with a single coupling constant and a single mass scale, it is certainly worthwhile, at this stage, to try to achieve the unification along such a line of thought. This has come to be known as grand unification.[1]

In such unified theories, very heavy particles must inevitably occur. The behavior of the coupling constants of the now popular low energy $SU(3)_C \otimes SU(2)_L \otimes U(1)$ theory undeniably points to that direction[2]: If no new physics interferes with their evolution drastically, the mass scale of the heavy particles is as high as $10^{14} \sim 10^{15}$ GeV. It is perhaps not a coincidence that attempts of quark lepton unification (i.e. assigning them to the same irreducible representation of the fundamental gauge group) call for such a mass scale in order to secure the longevity of protons.[3]

Now in constructing a viable unified theory, these heavy particles must be incorporated into the structure with due caution. Among the most important requirements are (i) that super heavy particles must effectively decouple at low energies, (ii) that correct effective light particle theory must emerge at low energies, and (iii) that the mass hierarchy, arranged at the tree level, should be stable against radiative corrections. As we shall see, these requirements are deeply interrelated. None of them are trivial to satisfy. Indeed there have been numerous discussions of each of these problems in the literature,[4-6] with yet no clear-cut conclusion. In this paper we shall address ourselves to these questions and give solutions to all orders in perturbation theory. Specifically, we shall establish the following: In spontaneously broken non-abelian gauge theories which admit gauge hierarchy at the tree level, (i) superheavy particles decouple from the light sector at low energies, (ii) effective low energy renormalizable theory emerges together with appropriate counter terms, and (iii) gauge hierarchy can be consistently maintained in the presence of radiative corrections. These assertions will be demonstrated for O(3) gauge theory with two triplets of Higgs particles in a manner that does not depend on the details of the theory (hence readily generalizable to more realistic theories such as SU(5) theory).

Let us first explain the nature of each of the problems and point out the difficulties involved in resolving them.

1. Decoupling. Consider a theory with heavy (mass M) and light (mass m) particles. Heavy particles are said to decouple at low energies if their effects are physically undetectable in the limit $M \to \infty$. To be more precise, they are said to decouple if their effects are either of $O(1/M)$ (manifest decoupling) or, if non-vanishing (call it of $O(1)$), can be absorbed into finite renormalization of masses, coupling constants, and the wave functions of a low energy effective light particle renormalizable theory. In equation,

$$\Gamma^{(n)}_{full}(p_i, g, m, M, \mu) = Z^{-n/2}\Gamma^{(n)}_{light}(p_i, g_{eff}, m_{eff}, \mu) + O(1/M) \tag{1.1}$$

where μ is a renormalization scale and Z is a finite wave function renormalization constant. For theories without spontaneous symmetry breaking, Eq. (1.1) has already been establisehd by Symanzik, and Appelquist and Carazzone.[7] The essence of their demonstration is that while for graphs (or subgraphs) with dimension less than zero heavy particle effects are suppressed by the heavy propagators, for graphs (or subgraphs) with dimension greater than zero one can absorb all the apparently large heavy particle effects into the renormalization counter terms. Although the original demonstration was done in momentum subtraction scheme with small subtraction scale ($\mu << M$) so that the decoupling is manifest, the fact of decoupling is clearly independent of the renormalization scheme chosen. It is a matter of simple finite renormalization to go from one scheme to the other.

To discuss decoupling in theories with spontaneously broken symmetry, one must first specify the manner in which M becomes large. In such theories, the masses are proportional to gV, where g and V are, respectively, a generic coupling constant and a vacuum expectation value of a scalar field. Thus two cases should be clearly distinguished. Case (a) $g \to \infty$. In this case, the light particle sector becomes, in general, apparently nonrenormalizable, indicating its sensitivity to the physics at large mass scale. One therefore expects no decoupling. (This does not necessarily mean that heavy particle effects are easily observable in practice. Since these effects appear through radiative corrections, they are often suppressed by powers of remaining small coupling constants of the theory.) Indeed many examples of this phenomenon have been reported.[8] Besides, the theory in this limit will contain a strongly interacting sector. Although potentially very interesting, we shall not dwell on this case any further. Case (b) $V \to \infty$. This is the case relevant to the theories in the grand unification category, and is the one to be discussed in this paper. Contrary to case (a), light sector looks renormalizable and therefore one expects the decoupling to take place. Some explicit calculations have been performed to one loop order with results in support of this expectation.[4,5]

However, to go beyond explicit calculation is not an easy task. For spontaneously broken theories, the essential arguments of Symanzik, Appelquist and

Carazzone do not go through: Propagator suppression does not always work due to large three point vector-vector-scalar coupling and scalar self-coupling (and possibly large Yukawa coupling if very heavy fermions are present), and not every two, three, and four point counter term can be freely adjusted due to the gauge symmetry restriction. Added to these difficulties is the fact that, in general, light-heavy mixing occurs. I.e., even if we define light and heavy particles at the tree level, this identification is lost once one turns on the radiative corrections. Although conceptually one may re-identify light and heavy particles at each loop level, in practice it is certainly difficult, if not impossible, to implement such a procedure to all orders. Furthermore one must worry about the gauge hierarchy problem,[6] for without gauge hierarchy the very concept of decoupling does not make sense.

2. Effective Lagrangian: Now, complementary to the existence of decoupling is that of effective low energy light particle Lagrangian. As is already clear in the case of theories without spontaneously broken symmetry one cannot talk about decoupling without the existence of an effective light particle theory, because in its absence we cannot absorb the large mass effects by redefinition of the parameters of the light theory. These two concepts are, therefore, two sides of one and the same subject. Thus all the difficulties associated with decoupling are present in deriving an effective Lagrangian. Recently a method of obtaining an effective Lagrangian by "integrating out" "heavy fields" has been discusssed by several authors.[5] If implemented naively, this method produces an infinite series of seemingly nonrenormalizable-looking terms in the Lagrangian, which is due to the (unallowable) interchange of the limits Λ(cutoff)$\to\infty$ and $M\to\infty$. To avoid this, analysis of light-heavy overlapping graphs is called for and this has been done to two loop order.[5] Our method to be described in this

article deals exclusively with fully renormalized Green's functions and hence leads to systematic incorporation of the correct limiting procedure to all loop orders.

3. Gauge hierarchy: Finally the gauge hierarchy problem. There are two levels at which to discuss this problem. Of course the profound question is why such a hierarchy exists. Although some interesting ideas have been put forth (e.g. technicolor approach,[9] and Veltman's conjecture[10] of underlying supersymmetry) nobody has yet solved the problem. From this point of view, any nonsupersymmetric theory with elementary scalars is "unnatural" due to the existence of quadratic divergences. The less profound, yet important and practical question is whether we can maintain gauge hierarchy consistently in the presence of radiative corrections within the realm of perturbation theory.[6] The emphasis is on the "consistency," for, in a renormalizable theory, we have a certain number of free parameters and it is trivial to choose the renormalized value of small (v) and large (V) vacuum expectation values to be among our free parameters. The real question of consistency is whether, by so doing, all the masses (not all of them are free now) automatically come out to retain the desired hierarchical pattern. This is another question we address ourselves to in this paper and give the answer in the affirmative.

Now let us summarize the essence of how we solve all the problems mentioned previously, to all orders in perturbation theory. (This will at the same time serve to inform the organization of the paper.)

The model is the O(3) gauge theory with two triplets of Higgs particles. It is chosen so that we can easily arrange a gauge hierarchy at the tree level. (It is described in Sec. II.) The first step (the content of Sec. III) is to do away with the light-heavy mixing problem. We shall show that consideration of one light-particle-irreducible (hereafter denoted by LPI) Green's functions, which are the natural objects to study from the point of view of effective Lagrangian, takes care of the mixing automatically without the need of rediagonalization. Here "light particle" means the one defined at the tree level.

The next step (see Sec. IV), one of the two crucial steps in our program, is to decompose a given arbitrary renormalized LPI light graph into the part which survives in the $M \to \infty$ limit (call it O(1) part) and the remainder of O(1/M) in such a way that O(1) part is readily seen to be generated by an effective Lagrangian with effective parameters and appropriate counter terms. The basic idea here is that of factorization of the most "divergent" part, where "divergent" here is defined with respect to the limit $M \to \infty$. Factorization has been widely discussed lately[11] but there is, however, a novel feature in our case: the operators of interest are of dimension four and hence appear many times in a diagram. This is to be contrasted with, for instance, the usual operator product expansion, where one needs to consider only a single insertion. Therefore combinatorical and renormalization aspects of such multiple insertions present complications. Technically, this is handled by the construction of a new algebraic identity in the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) framework.[12] At the same time power counting theorems are stated (see the appendix for proof) which guarantee that what is claimed to be O(1/M) is indeed O(1/M). It is to be emphasized that we shall always deal with fully renormalized graphs so that nowhere do we interchange the limits $\Lambda \to \infty$ and $M \to \infty$. The fact that the identity is designed to closely follow the combinatorics of renormalization is responsible for (i) O(1) part can be generated from a Lagrangian and (ii) counter terms are automatically supplied.

Now that we have an effective Lagrangian, the final crucial step (Sec. V) is to (i) find what theory it describes, (ii) see if all the light particles remain light, and (iii) identify the effective coupling constants of the theory. It is at this point that the gauge symmetry of the theory plays the central role. (Power counting alone cannot exclude the possibility that light particles may become heavy.) From the original O(3) Becchi-Rouet-Stora (BRS) identity[13] (in fact what is relevant is an O(2) part of it around the direction of the large vacuum expectation value-the "residual" symmetry) we derive the BRS identity satisfied by the proper generating functional of the effective light particle theory. This puts such a severe constraint that the structure of the theory is completely determined: it is that of the O(2) gauge theory spontaneously broken by a small vacuum expectation value $\sim$ v. In particular, the masses of the gauge, Higgs and ghost particles (we shall use a renormalizable gauge[14]) are shown to be of order O(v). This demonstrates that the gauge hierarchy can be consistently maintained in the presence of radiative corrections.

Furthermore, as a byproduct of our analysis, we obtain a systematic and unambiguous method of computing the parameters of low energy effective theory to any desired accuracy via renormalization group equations (Sec. VI). The effective theory constructed in Sec. IV and V is provided with counterterms corresponding to zero momentum subtraction, and it is not particularly convenient from the point of view of renormalization group analysis. More advantageous in this respect is the effective theory made finite by minimal subtractions.[15] We shall describe an algorithm by which such a representation is obtained directly from the full theory without going through the intermediate stage of zero momentum subtractions.

In the minimal scheme, effective parameters are free of infrared (i.e. $m \to 0$) singularities and the decoupling takes place irrespective of the magnitude of the renormalization scale μ. By comparing renormalization group equations pertinent to the Green's functions of the full and the effective theory, exact equations satisfied by the effective parameters are obtained. Boundary conditions to be imposed at $\mu \sim M$ for the solutions of these equations are found in an unambiguous manner through the algorithm alluded to above. The merit of our method lies in that it is conceptually clear, and is systematic (backed by the all order analysis) so that the procedure can be carried through straightforwardly to any desired accuracy. Also included in this section is a comment on the "partially covariant" gauge fixing procedure recently proposed by Weinberg[16] in the context of effective Lagrangian. It will be pointed out that there exist certain difficulties associated with this procedure.

II. THE MODEL

Although our arguments and the techniques in studying the three problems stated in the introduction are quite general, it is certainly instructive to carry through the procedures explicitly for a definite model theory. We have chosen to work with the O(3) gauge theory with two triplets of Higgs fields, $\vec{H}$ and $\vec{h}$, so that a gauge hierarchy can be arranged at the tree level.

The starting Lagrangian is given by

$$\mathscr{L}_o = -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \tfrac{1}{2}(D_\mu \vec{H}\cdot D^\mu \vec{H}) + \tfrac{1}{2}(D_\mu \vec{h}\cdot D^\mu \vec{h}) - V(\vec{H}, \vec{h}) \quad , \tag{2.1}$$

$$V(\vec{H}, \vec{h}) = -\tfrac{1}{2}m_H{}^2H^2 - \tfrac{1}{2}m_h{}^2h^2 + \tfrac{1}{4}f_1(H^2)^2 + \tfrac{1}{4}f_2(h^2)^2$$

$$+ \tfrac{1}{2}f_3H^2h^2 + \tfrac{1}{4}f_4(\vec{H}\cdot\vec{h})^2 \quad , \tag{2.2}$$

where

$$D_\mu = \partial_\mu + gt^aA^a_\mu \qquad (a = 1,2,3) \tag{2.3}$$

$$(t^a)_{bc} = -\varepsilon_{abc} \tag{2.4}$$

is the covariant derivative in the triplet representation. We have assumed the symmetry under separate reflections, $\vec{H} \to -\vec{H}$ and $\vec{h} \to -\vec{h}$ and for simplicity no fermions have been added. The Lagrangian $\mathscr{L}_o$ is invariant under the O(3) gauge transformations,

$$\delta A^a_\mu(x) = \left(\varepsilon^{abc} A^c_\mu(x) - \frac{1}{g} \delta^{ab} \partial_\mu \right) \delta\theta^b(x) = -\frac{1}{g} (D_\mu)^{ab} \delta\theta^b(x) \tag{2.5a}$$

$$\delta H^a(x) = \varepsilon^{abc} H^c(x) \delta\theta^b(x) \tag{2.5b}$$

$$\delta h^a(x) = \varepsilon^{abc} h^c(x) \delta\theta^b(x) \quad . \tag{2.5c}$$

We shall, of course, add the gauge fixing and the corresponding ghost terms later.

As was discussed by Gildener,[6] one can easily arrange a gauge hierarchy at the tree level under the conditions

$$0 < f_i \quad (i = 1,2,4) \tag{2.6a}$$

$$-\sqrt{f_1 f_2} < f_3 < f_1 \frac{m_H^2}{m_h^2} \quad . \tag{2.6b}$$

As we shall work in perturbation theory, $f_i \sim f_j << 1$ will be assumed. The absolute minimum of the potential then occurs when

$$< 0 | \vec{H} | 0 > \cdot < 0 | \vec{h} | 0 > = 0 \quad . \tag{2.7}$$

We may therefore choose

$$<0|H_a|0> = \delta_{a1}V \tag{2.8a}$$

$$<0|h_a|0> = \delta_{a2}v \quad . \tag{2.8b}$$

V^2 and v^2 are given, in terms of the original parameters, by

$$V^2 = (f_2 m_H{}^2 - f_3 m_h{}^2)/(f_1 f_2 - f_3) \tag{2.9a}$$

$$v^2 = (f_1 m_h{}^2 - f_3 m_H{}^2)/(f_1 f_2 - f_3) \quad . \tag{2.9b}$$

Alternatively,

$$m_H{}^2 = f_1 V^2 + f_3 v^2 \tag{2.10a}$$

$$m_h{}^2 = f_3 V^2 + f_2 v^2 \quad . \tag{2.10b}$$

Gauge hierarchy is arranged by choosing

$$V^2 \gg v^2 \quad . \tag{2.11}$$

The original O(3) symmetry is broken by $\vec{V}$ to O(2) (i.e. U(1)), and then by $\vec{v}$ down to no symmetry.

For the purpose of power counting, to be performed later, it is convenient to choose a renormalizable gauge for which the ghosts and the Goldstone bosons associated with the heavy gauge fields are also heavy. A suitable gauge[14] is specified by

$$\mathcal{L}_{gf} = -\frac{1}{2\alpha} F_a^{\ 2} = -\frac{1}{2\alpha}\left[\partial_\mu A^\mu_a - g_\alpha(\vec{h}, t^a\vec{v}) - g_\alpha(\vec{\chi}, t^a\vec{V})\right]^2 \tag{2.12}$$

where the shifted fields $\vec{\eta}$ and $\vec{\chi}$ are defined by

$$\vec{h} = \vec{\eta} + \vec{v} \tag{2.13a}$$

$$\vec{H} = \vec{\chi} + \vec{V} \tag{2.13b}$$

The corresponding ghost Lagrangian then takes the form

$$\mathcal{L}_{gh} = -\bar{c}_a \partial^\mu (D_\mu)^{ab} c_b - g^2\alpha\left(v^2\bar{c}_1 c_1 + V^2\bar{c}_2 c_2 + (V^2 + v^2)\bar{c}_3 c_3\right)$$

$$- g^2\alpha\left\{v(\eta_2\bar{c}_a c_a - \bar{c}_a\eta_a c_2) + V(\chi_1\bar{c}_a c_a - \bar{c}_a\chi_a c_1)\right\} \tag{2.14}$$

By shifting the fields as in Eq. (2.13) and diagonalizing the quadratic part of the resulting Lagrangian, we easily find the particle content of the theory at the tree level. The theory contains altogether 12 particles listed in Table 1. Note the following features: (i) For a gauge parameter $\alpha = O(1)$, the masses of a gauge boson and the associated ghost and the Goldstone boson are of the same order. (ii) Mixing occurred in (χ_1, η_2) and (η_1, χ_2) systems with small mixing angles, $\theta_1, \theta_2 \sim O(v/V)$. Later we shall see that the radiative corrections induce further mixings, of the same order, within these pairs, so that the heavy-light identifications made at the tree level will be upset. This annoying problem will be nicely resolved in Sec. III.

Finally, to complete the definition of the theory, we must specify the renormalization prescriptions. So far all the fields and the parameters in our Lagrangian are the bare quantities and they diverge in perturbation theory. From the general theory of renormalization of spontaneously broken gauge theories,[17] we know that these divergences are removed by the following renormalization transformations:

$$A_\mu^a = Z_3^{1/2} A_{\mu R}^a \quad , \quad H^a = Z_H^{1/2} H_R^a$$

$$h^a = Z_h^{1/2} h_R^a \quad , \quad c^a = \tilde{Z}^{1/2} c_R^a$$

$$g = (Z_1/Z_3^{3/2}) g_R \quad , \quad \alpha = Z_3 \alpha_R$$

$$f_1 = (Z_{f_1}/Z_H^2) f_{1R} \quad , \quad f_2 = (Z_{f_2}/Z_h^2) f_{2R}$$

$$f_3 = (Z_{f_3}/Z_H Z_h) f_{3R} \quad , \quad f_4 = (Z_{f_4}/Z_H Z_h) f_{4R}$$

$$m_H^2 = m_{HR}^2 - \delta m_H^2 \quad , \quad m_h^2 = m_{hR}^2 - \delta m_h^2 \quad . \tag{2.15}$$

As was first pointed out by Appelquist et al.[18] and later elaborated by Lee,[19] the use of renormalizable gauge with scalars requires an additional renormalization; namely $\vec{H}_R$ and $\vec{h}_R$ defined above still contain gauge-dependent divergent vacuum expectation values $\delta\vec{V}$ and $\delta\vec{v}$ (even in theories without spontaneous symmetry breaking) and these must be removed. The necessary counterterms are generated by writing

$$\vec{H}_R = \vec{X}_R + \vec{V}_R + \delta\vec{V} \tag{2.16}$$

$$\vec{h}_R = \vec{\eta}_R + \vec{v}_R + \delta\vec{v}$$

where now $\vec{X}_R$, $\vec{V}_R$, $\vec{\eta}_R$, $\vec{v}_R$ are completely finite. Furthermore, one can choose $\delta\vec{V}$ and $\delta\vec{v}$ such that the gauge fixing Lagrangian retains its form expressed in terms of renormalized quantities. To actually fix the counterterms we need to specify the subtraction procedure. Although any sensible gauge invariant scheme will do, for simplicity and definiteness we shall adopt the dimensional minimal subtraction[15] as the intermediate renormalization. After removing the infinities this way, we must still find the true vacuum and re-expand around it. This is automatically done by demanding that, at each loop order, appropriate sums of the tadpoles vanish. This amounts to making a finite renormalization on $\delta m_H{}^2$ and $\delta m_h{}^2$. As was emphasized in the introduction, we are free to choose V_R and v_R to be among our free parameters and insist that $V_R \gg v_R$. Then according to our prescription above, all the masses will be fixed in terms of V_R, v_R, and the coupling constants. We have now completed the definition of our model and are ready for its analysis.

III. RESOLUTION OF THE LIGHT-HEAVY MIXING PROBLEM

As has been pointed out, the identification of light and heavy particles made at the tree level may appear ephemeral; radiative corrections will introduce further mixings and it seems, at first sight, that rediagonalization must be performed at every loop level to re-establish correct identification. This vexing problem, however, has a neat natural solution to the order of accuracy we wish to achieve. This is the subject of discussion of this section.

The first question to be asked is whether the radiative corrections induce patterns of mixings different from those at the tree level. Fortunately this situation does not obtain. The key observation is that there exists a symmetry obeyed by every individual term (i.e. without even summing over the group indices) in the Lagrangian, which we shall call the index conservation. For the theory at hand, it can be described as follows. We shall assign the indices to various fields

$$\begin{aligned}
&0 \text{ for } \chi_1, \eta_2 \\
&1 \text{ for } A_\mu{}^1, \eta_3, c_1, \bar{c}_1 \\
&2 \text{ for } A_\mu{}^2, \chi_3, c_2, \bar{c}_2 \\
&3 \text{ for } A_\mu{}^3, \chi_2, \eta_1, c_3, \bar{c}_3
\end{aligned} \tag{3.1}$$

and impose the rules of composition

$$0 \cdot 0 = 0$$

$$i \cdot 0 = i$$

$$i \cdot i = 0$$

$$i \cdot j = k \qquad \text{where } i,j,k = \text{cyclic}$$
$$i,j,k = 1,2,3 \qquad . \tag{3.2}$$

Then one can easily verify that every term in the Lagrangian conserves indices. Some examples are given in Fig. 1. This assures that, by looking at two point functions, mixing can occur only among fields with the same index (see Fig. 2 for an illustration), demonstrating the assertion made above.

Having been assured that radiative corrections affect angles, but not the pattern, of mixing, let us describe how rediagonalization at each loop level is avoided. To do this we must first analyze what are the objects of interest that we want to compute. For the sake of clarity of discussion, let us concentrate on the mixing in the (χ_1, η_2) system and suppress all the other fields. Our starting point is the generating function

$$\exp iW(j_2, J_1) = \int \mathcal{D}\hat{\eta}_2 \, \mathcal{D}\hat{\chi}_1 \exp i(S + i j_2 \hat{\eta}_2 + iJ_1 \hat{\chi}_1) \tag{3.3}$$

which, upon tree level diagonalization becomes

$$\int \mathcal{D}\hat{\bar{\eta}}_2 \, \mathcal{D}\hat{\bar{\chi}}_1 \exp i(S + i\bar{j}_2 \hat{\bar{\eta}}_2 + i\bar{J}_1 \hat{\bar{\chi}}_1) \qquad . \tag{3.4}$$

This is the form from which we generate perturbation series. Now what we eventually wish to obtain is the effective Lagrangian for the light particles. Thus the object of interest is the effective action, defined, as usual, by the Legendre transformation

$$\bar{\Gamma}_{1PI}(\bar{\eta}_2, \bar{\chi}_1) = W - \bar{J}_2\bar{\eta}_2 - \bar{J}_1\bar{\chi}_1 \tag{3.5}$$

where

$$\bar{\eta}_2 \equiv \frac{\delta W}{\delta \bar{J}_2} \quad , \quad \bar{\chi}_1 \equiv \frac{\delta W}{\delta \bar{J}_1} \quad . \tag{3.6}$$

Because of the radiative mixing, we must re-express this in terms of the <u>true</u> light and heavy fields $\tilde{\eta}_2$ and $\tilde{\chi}_1$ which are orthogonal linear combinations of the tree level fields $\bar{\eta}_2$ and $\bar{\chi}_1$. (Actually we must do this for each Fourier component. I.e., the mixing angle is momentum dependent.) Thus we write

$$\bar{\Gamma}_{1PI}(\bar{\eta}_2, \bar{\chi}_1) = \tilde{\Gamma}_{1PI}(\tilde{\eta}_2, \tilde{\chi}_1) \quad . \tag{3.7}$$

We now wish to "integrate out" the heavy field $\tilde{\chi}_1$. This means computing $\tilde{\Gamma}_{1PI}$ in the absence of the source of $\tilde{\chi}_1$. It is expressed by

$$\tilde{J}_1 = -\frac{\delta\tilde{\Gamma}_{1PI}}{\delta\tilde{\chi}_1}(\tilde{\eta}_2, \tilde{\chi}_1) = 0 \quad . \tag{3.8}$$

Hence we can solve for $\tilde{\chi}_1$ in terms of $\tilde{\eta}_2$, $\tilde{\chi}_1 = \tilde{\chi}_1(\tilde{\eta}_2)$. We then obtain the desired object $\tilde{\Gamma}_{1PI}(\tilde{\eta}_2, \tilde{\chi}_1(\tilde{\eta}_2))$. This is nothing but $\tilde{\eta}_2$ (i.e. light particle)-irreducible generating function (since Legendre transformation for $\tilde{\chi}_1$ is not effective due to the condition $\tilde{J}_1 = 0$), which we denote by $\tilde{\Gamma}_{LPI}(\tilde{\eta}_2)$.

Now the question is: how do we compute this object? It turns out that, to O(1), this object is exactly the same as the LPI Green's functional $\overline{\Gamma}_{LPI}(\overline{\eta}_2) = \overline{\Gamma}_{1PI}(\overline{\eta}_2, \overline{\chi}_1)|_{\overline{J}_1=0}$ defined with respect to the tree level identification. In equation,

$$\tilde{\Gamma}_{LPI}(\tilde{\eta}_2) = \overline{\Gamma}_{LPI}(\overline{\eta}_2) + O(1/M) \quad . \tag{3.9}$$

To see this let us consider the actual process of diagonalization of fields. In $(\overline{\chi}_1, \overline{\eta}_2)$ basis, the inverse two point function matrix $\Delta^{-1}(p^2)$ is of the form

$$\Delta^{-1}(p^2) = \begin{pmatrix} p^2 - M^2 - \Sigma_{\chi_1\chi_1} & -\Sigma_{\chi_1\eta_2} \\ -\Sigma_{\chi_1\eta_2} & p^2 - m^2 - \Sigma_{\eta_2\eta_2} \end{pmatrix} , \tag{3.10}$$

where $\Sigma_{\chi_1\chi_1}$ etc. are the (renormalized) self energy operators. This is diagonalized by an orthogonal matrix

$$C(p^2) = \begin{pmatrix} \cos\theta(p^2) & \sin\theta(p^2) \\ -\sin\theta(p^2) & \cos\theta(p^2) \end{pmatrix} \tag{3.11}$$

so that we have

$$\tilde{\eta}_2 = \overline{\eta}_2 \cos\theta - \overline{\chi}_1 \sin\theta \quad , \tag{3.12a}$$

$$\tilde{\chi}_1 = \overline{\chi}_1 \cos\theta + \overline{\eta}_2 \sin\theta \quad . \tag{3.12b}$$

The mixing angle θ is easily computed to be

$$\tan 2\theta = \frac{2\,\Sigma_{\chi_1\eta_2}}{M^2 - m^2 + \Sigma_{\chi_1\chi_1} - \Sigma_{\eta_2\eta_2}} \quad . \tag{3.13}$$

At this juncture, we must borrow some results of the power counting theorem P1 described in Sec. IV. Applied to two point functions under consideration it tells us that

$$\Sigma_{\chi_1\chi_2} = O(M^2) \quad , \quad \Sigma_{\eta_2\eta_2} = O(m^2)$$

$$\Sigma_{\chi_1\eta_2} = O(Mm) \quad . \tag{3.14}$$

Substituting them into Eq. (3.13), we find that the mixing angle is small, i.e.,

$$\theta = O\left(\frac{m}{M}\right) \quad . \tag{3.15}$$

Therefore, we can write

$$-\sin\theta = \frac{\Sigma_{\chi_1\eta_2}}{p^2 - M^2 - \Sigma_{\chi_1\chi_1}}\left(1 + O\left(\frac{m^2}{M^2}\right)\right) \quad , \tag{3.16a}$$

$$\cos\theta = 1 + O\left(\frac{m^2}{M^2}\right) \quad . \tag{3.16b}$$

Equation (3.16a) has a nice diagrammatical interpretation depicted in Fig. 3. Let us now compute what the true light and heavy two point functions are. By elementary calculation one finds

$$\tilde{\Delta}^{-1}_{\chi_1\chi_1} = \left(p^2 - M^2 - \Sigma_{\chi_1\chi_1} \right) \left(1 + O\left(\frac{m^2}{M^2} \right) \right) \tag{3.17a}$$

$$\tilde{\Delta}^{-1}_{\eta_2\eta_2} = p^2 - m^2 - \Sigma_{\eta_2\eta_2} - \frac{\left(\Sigma_{\chi_1\eta_2} \right)^2}{p^2 - M^2 - \Sigma_{\chi_1\chi_1}} + O\left(\frac{m^2}{M^2} \right) \quad . \tag{3.17b}$$

Diagrammatically Eq. (3.17b) can be represented by Fig. 4. Equations (3.17a) and (3.17b) tell us that, within the accuracy desired, while "true" heavy propagator is equal to "tree" heavy propagator, the "true" light propagator is obtained by summing all the relevant LPI diagrams, where LPI is defined with respect to "tree" diagonal fields. As for the external lines, we learn from Eqs. (3.12a) and (3.16a,b) that proper projection of the true light field is done automatically by considering tree-LPI graphs. Figure 5 summarizes our findings in the case of a four point function. This clearly proves the statement Eq. (3.9).

Thus, to O(1), we have a very useful conclusion: Simply study the LPI Green's functions. The mixing problem is automatically taken care of.

IV. SEPARATION OF O(1) AND O(1/M) PARTS--A NEW ALGEBRAIC IDENTITY

We now come to the main part of the study. With the result of the previous section in mind, we can state our objective as follows: Given an arbitrary LPI light particle graph, which is made finite by the usual (in our case minimal) subtractions, we shall give a prescription to separate its contribution at low energies into the part that does not vanish as $M \to \infty$ (we shall call it O(1) part) and the rest which is of O(1/M) in such a fashion that O(1) part is manifestly obtainable from an effective light particle Lagrangian with effective coupling constants and effective masses.

To describe the basic idea, let us start with a simple example which well illustrates our approach. Consider a diagram shown in Fig. 6. Due to the large Higgs three-point couplings this diagram contains O(1) part in spite of a heavy Higgs exchange. Moreover, it originates from both low ($\ell^2 << M^2$) and high ($\ell^2 \sim M^2$) loop momentum regions. First we look at the low ℓ^2 region. In this region the dominant piece of the heavy propagator is almost a constant $\sim -1/M^2$. It is thus natural to make a decomposition

$$\frac{1}{\ell^2 - M^2} = \left(\frac{1}{\ell^2 - M^2} + \frac{1}{M^2} \right) + \left(- \frac{1}{M^2} \right)$$

$$\equiv \Delta a + \Delta b \qquad . \qquad (4.1)$$

Upon substituting this expression, the original graph splits into two graphs depicted in Figs. 7(a) and 7(b). Because of the improved ($\sim 1/M^4$) low energy behavior of Δ_a, the diagram 7(a) no longer produces O(1) contribution from low ℓ^2 region. To deal with high loop momentum region, however, this decomposition is not sufficient; although the sum is finite, 7(a) and 7(b) individually diverge as $\ell^2 \to \infty$. To remedy this we add and subtract the zero momentum values of the diagrams as shown in Figs. 8(a), (b), (c). Now each bracketed expression is finite. Moreover, recalling Fig. 7, 8(c) is nothing but the original graph evaluated at zero momentum. Notice that 8(a) is not only convergent but also is of $O(1/M)$ since potential $O(1)$ part is subtracted away together with the divergent contribution. (Reader can easily verify this explicitly.) At the same time the diagrams giving $O(1)$ contributions have exactly the diagrammatical structure pertinent to a light particle theory. Therefore for this example we have achieved our objective stated at the beginning of this section.

To see how this procedure may be generalized for an arbitrary graph, we must rephrase the above result from a different, more systematic point of view. Diagrammatically 8(b) and 8(c) can be obtained from the original graph by reducing non-trivial LPI light (sub)graphs of mass dimension ≥ 0 to a point. The resultant graphs contain light particle lines only and in this sense they are "fully-reduced." On the other hand, 8(a), which is not fully reduced (in fact not reduced at all in this case), is overly subtracted. I.e., all of its non-trivial LPI light subgraphs with dimemsion ≥ 0 were subtracted according to the nomimal naive dimension counting regardless of whether they actually diverged. What we have learned is that only the fully reduced graphs gave $O(1)$ contributions. These observations will be the key to our subsequent analysis--we shall make them more precise and express the above rule as an exact algebraic identity.

To handle the complicated combinatorics for arbitrary graphs, BPHZ[12] renormalization procedure with Zimmermann's forest formulation is well-tailored. Let us begin by making precise the various key concepts, some of which have already appeared in the example above.

(i) Renormalization operator t^γ: t^γ is defined for 1PI graphs only. It evaluates the divergent (pole) part of a 1PI graph γ.

(ii) Taylor operator τ^γ: This operator will be defined for 2, 3 and 4-point LPI light particle graphs (to be defined shortly as partition elements). Given such a graph γ, it extracts the superficially divergent part of its Taylor series around zero external momenta, where the superficial degree of divergence is determined according to the naive dimension counting, i.e., by the formula $\delta = 4 - B$ (B is the number of external lines). Since the actual superficial degree of divergence is given by $d = 4 - B - V_3$, where V_3 is the number of 3-point nonderivative couplings in the graph, τ^γ in general effects oversubtraction. Note that τ^γ also acts on graphs composed of light internal lines only. When it does, the difference $t^\gamma - \tau^\gamma$ is a finite renormalization. This is built into the algebraic identity so that at the end the light effective theory is consistently renormalized by zero momentum subtractions.

(iii) Partition element: A partition element π of a graph is a non-trivial (i.e. excluding tree light vertices) LPI light (sub)graph of mass dimension ≥ 0. The whole graph itself could be a partition element.

(iv) A reduction of a graph: A reduction of a graph is defined by a set of mutually disjoint partition elements $\{\pi_1, \pi_2, ..., \pi_n\}$. A reduced graph $\Gamma/\{\pi_1,...,\pi_n\}$ is then obtained by shrinking each π_i to a point. π_i's participating in a reduction will be called reduction elements. A reduction is said to be a full reduction if the reduced graph contains no heavy particle lines. Note that reductions are defined even for a graph that is composed of light particle lines only. In such a case every reduction is a full reduction.

(v) Concepts of type-t (or barred) and type-τ (or unbarred) elements: Later when we write down the relevant forest formulae we need to assign, for each element of the forest, the operator t^γ or τ^γ. An element will accordingly be called of type-t or of type-τ. For partition elements which are at the same time divergent we shall have occasions to include them twice in a forest, once as type-t elements and once as type-τ elements. This point will become clear later. When such a distinction is essential we shall put a bar on top to denote the type-t elements.

We are now in a position to state and prove the crucial algebraic identity. Let us first state it as a theorem and give a clear explanation of it. A proof will then follow.

Theorem 1. The following equation holds identically;

$$
\begin{aligned}
R_\Gamma &= \sum_{U_o \in \mathscr{F}_o(\Gamma)} \sum_{\gamma \in U_o} (-t^\gamma) I_\Gamma \\
&= \sum_{\{\pi_1,\dots,\pi_m\}} \left\{ \sum_{U \in \mathscr{F}(\Gamma/\{\pi_1,\dots,\pi_m\})} \prod_{\gamma \in U} (-\tau^\gamma) \right. \\
&\quad \left. \times \prod_{i=1}^{m} \left(\tau^{\pi_i} \sum_{U_i \in \mathscr{F}_o(\pi_i)} \prod_{\gamma' \in U_i} (-t^{\gamma'}) \right) \right\} I_\Gamma \quad . \qquad (4.2)
\end{aligned}
$$

Explanation of the theorem 1: Consider an arbitrary LPI light particle graph Γ, with its unrenormalized expression denoted by I_Γ. The renormalized expression R_Γ, obtained by the usual Zimmermann's forest formula, is recorded in the first line. Here $\mathscr{F}_o(\Gamma)$ is the collection of all the forests of Γ. U_o is a forest, γ is a renormalization part contained in the forest U. The subscript o means that the forests are defined in the ordinary way, i.e. according to the counting $d = 4 - B - V_3$. Now this R_Γ can be decomposed in the following way (the second line of Eq. (4.2)). First choose a particular reduction of Γ, defined by a collection of partition elements $\{\pi_1 ..., \pi_m\}$, which are to be reduced. Before reducing each π_i to a point, all the divergences within it must be subtracted. This is performed by the operation $\sum_{U_i \in \mathscr{F}_o(\pi_i)} \prod_{\gamma' \in U_i} (-t^{\gamma'})$. After this renormalization, τ^{π_i} acts on it to evaluate π_i at zero momentum and produces local vertices of dimension ≤ 4. (This is the precise meaning of the reduction.) Now we have obtained a reduced graph $\Gamma/\{\pi_1, ..., \pi_m\}$. The rest of the operation $\sum_{U \in \mathscr{F}(\Gamma/\{\pi_1,...,\pi_m\})} \prod_{\gamma \in U} (-T^{\gamma})$, where

$$T^{\gamma} \equiv \begin{cases} \tau^{\gamma} & \text{if } \gamma \text{ is a partition element of type-}\tau \\ \bar{t}^{\gamma} & \text{if } (\gamma =)\bar{\gamma} \text{ is a LPI renormalization part defined} \\ & \text{according to the naive counting } \delta = 4 - B \geq 0 \end{cases} \quad (4.3)$$

renormalizes the reduced graph with <u>consistent</u> oversubtraction for the partition elements of the reduced graph. A forest $U \in \mathscr{F}(\Gamma/\{\pi_1,...,\pi_m\})$ is composed of non-overlapping elements of the type listed in (4.3). In particular for fully reduced graphs all such elements are partition elements and effectively only τ^{γ}'s are operative. We then repeat the same procedure for all possible reduction patterns, including no-reduction case, and sum over all such contributions. The identity says that the sum reproduces precisely the original renormalized expression R_{Γ}. Note the complete parallel with the way the example was treated in the beginning of this section.

Proof of the Theorem 1: The theorem can be proved most transparently by focussing on the contribution, on the right hand side of Eq. (4.2), corresponding to the no-reduction case. So let us isolate it and rewrite the equation symbolically as

$$\sum_{\mathscr{F}_{NR}} = \sum_{\mathscr{F}_o} - \sum_a \sum_{\mathscr{F}_{R_a}} \tag{4.4}$$

where $\mathscr{F}_o$, $\mathscr{F}_{NR}$ and $\mathscr{F}_{R_a}$ are the set of forests pertinent to the original contribution, the no-reduction case, and one of the reduced cases respectively. Because of the extra subtractions performed for all of its partition elements, $\mathscr{F}_{NR}$ is the largest set of forests among them. Our aim is to enumerate the forests in $\mathscr{F}_{NR}$ and show that their contributions precisely match those of the right hand side of Eq. (4.4).

Let us take any forest U belonging to $\mathscr{F}_{NR}$. We shall define the set of minimal partition elements of U to be those partition elements of type-τ which do not contain any other partition elements of type-τ. This set is uniquely determined once U is chosen, including the possibility of being an empty set. Let us then write

$$U = \{\pi_1, \pi_2, ..., \pi_n; \{\gamma\}\} \tag{4.5}$$

where $\{\pi_1, ..., \pi_n\}$ is the set of minimal partition elements and $\{\gamma\}$ collectively denotes the rest of the elements of U. The operator associated with this U is

$$\prod_{i=1}^{n} (-\tau^{\pi_i}) \prod_{\gamma} (-T^{\gamma}) \tag{4.6}$$

where T^{γ} is either t^{γ} or τ^{γ} depending on the type of γ (see Eq. (4.3)). The crucial step now is to write the operator above as a sum of "factorized" forms, namely

$$\left\{(-\tau^{\pi_1})\right\}\left[(-\tau^{\pi_2})\dots(-\tau^{\pi_n})\prod_\gamma(-T^\gamma)\right]$$

$$+\left\{(-\tau^{\pi_2})\right\}\left[(-\tau^{\pi_1})(-\tau^{\pi_3})\dots(-\tau^{\pi_n})\prod_\gamma(-T^\gamma)\right]$$

$$+\dots$$

$$+\left\{(-\tau^{\pi_n})\right\}\left[(-\tau^{\pi_1})\dots(-\tau^{\pi_{n-1}})\prod_\gamma(-T^\gamma)\right]$$

$$+\left\{(-1)(-\tau^{\pi_1})(-\tau^{\pi_2})\right\}\left[(-\tau^{\pi_3})\dots(-\tau^{\pi_n})\prod_\gamma(-T^\gamma)\right]$$

$$+(-1)\ \text{(all pairs factorized)}$$

$$+(-1)^2\ \text{(all triples factorized)}$$

$$+\dots$$

$$+\left\{(-1)^{n-1}\prod_{i=1}^{n}(-\tau^{\pi_i})\right\}\left[\prod_\gamma(-T^\gamma)\right] \quad . \qquad (4.7)$$

Except for the sign, each term is identical. Alternating sign then assures that the sum is indeed equal to the original expression thanks to the simple combinatoric identity

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k} = 1 \quad . \qquad (4.8)$$

Note that the overall sign inside the curly bracket { } is always negative. This is the correct minus sign appearing in Eq. (4.4). Let us call what is inside { } a factorized part. We now sum over all $U \in \mathscr{F}_{NR}$ and collect terms having the same factorized part. It is clear that there exists a one to one correspondence between a factorization type and a reduction type. Focus on a particular factorization type. Apart from a common factorized part, say $\left\{ -\prod_{i=1}^{n} (\tau^{\pi_j}) \right\}$, the rest of the expression consists of contributions from all possible forests with the properties (i) that they appear together with the set $\{\pi_1, \ldots, \pi_m\}$ and (ii) that they do not contain any type-τ elements which are inside π_j's. (This is because these π_j's were among the set of minimal partition elements of a $U \in \mathscr{F}_{NR}$.) Recalling the form of Eq. (4.2), we recognize that they are nothing but the set of forests pertinent to the reduction defined by $\{\pi_1, \ldots, \pi_m\}$. What are left behind are those $U \in \mathscr{F}_{NR}$ which have no type-τ partition elements in them, but they precisely form $\mathscr{F}_o$. This completes the proof.

Some remarks are in order: (a) The spirit of the proof given above is similar to that for the original Zimmermann's algebraic identity,[20] which has been widely used in proving so-called factorization[11] of short and long distance physics. Our identity may be considered a generalization of it which allows for <u>multiple insertions of operators</u>. (b) Another useful interpretation of the identity is to regard it as performing a finite renormalization, although the operations are performed not only on 1PI functions but on a wider class of diagrams, namely the LPI functions. In terms of the symbols used in Eq. (4.4), $\sum_{\mathscr{F}_o}$ performs minimal subtraction whereas $\sum_{\mathscr{F}_{NR}}$ performs zero momentum subtractions for the light particle 2, 3, and 4 point functions. The difference then must give us diagrams where finite, light particle counter terms of renormalizable type are inserted. This is exactly what $\sum_a \sum_{\mathscr{F}_{R_a}}$ represents.

Let us now turn to the remaining portion of our analysis--the separation of O(1) and O(1/M) parts. (So far what we have obtained is an exact identity and hence involved no knowledge of the fact that $M \gg m$.) This naturally requires counting of the maximum powers of M of a graph. Again for clarity of presentation let us first state the result as a theorem and then indicate what sort of procedures are involved in proving it. The specific details will be given in the appendix.

Theorem 2. At low energies O(1) part of an arbitrary LPI light Green's function comes entirely from the fully-reduced portions, in the decomposition of theorem 1, of the relevant diagrams contributing to such a Green's function. Further, these fully-reduced diagrams can be generated by an effective light particle Lagrangian.

To prove the validity of this theorem, one must first understand the behavior of an arbitrary LPI graph as M becomes large, namely the maximum power of M that a diagram may generate. For tree diagrams this is rather trivial. However for diagrams with many loops one must systematically examine all possible regions of loop momentum space in order to obtain the correct power. This analysis is presented in the appendix. The result is nevertheless pleasantly simple and we shall quote it here as theorem P1.

Theorem P1. The maximum possible power n_{max} of M for an arbitrary 1PI and LPI Green's function with B_H heavy and B_L light external lines is given by

$$n_{max} = B_H \quad .$$

In particular for light particle Green's functions (i.e. $B_H = 0$) $n_{max} = 0$, meaning that dependence on M is at most logarithmic.

Remarks: (i) It should be emphasized that the theorem applies to Green's functions (i.e. relevant sum of diagrams) and is not necessarily correct for individual diagrams. This is due to the fact that individually two and three point light particle (sub)diagrams may contain $O(M^2)$ and $O(M)$ contributions respectively.

(This is the maximum information one can obtain by power counting procedure alone.) However in the sum which comprises the corresponding Green's functions to a certain loop order, these adverse powers cancel. The Ward identities ensuring this cancellation are discussed in the next section. (ii) n_{max} is the maximum possible power of M, and the actual power may be less. In fact in certain cases reflection symmetry of the theory does reduce the power. (See Sec. V for effective uses of this fact.) (iii) For purposes of dealing with the BRS-Ward identities, we shall have occasions to consider Green's functions which contain new vertices corresponding to the composite operators generated by the BRS transformations. (See Eq. (5.2).) Theorem P1 is valid for these Green's functions as well, provided that we regard the sources of these new vertices as effectively being composed of two light fields.

Now the central part of theorem 2 follows if we can prove the following:

Theorem P2. For any LPI light Green's function which contains at least one heavy internal line, extra zero momentum subtractions upon its partition elements (plus associated subtractions of the divergences generated for diagrams containing such partition elements) render it vanish as $M \to \infty$.

The proof of this second power counting theorem is again relegated to the appendix. It is quite similar to that of theorem P1 except that this time one takes into account the oversubtractions in counting the powers of M arising from various regions of loop momentum space.

Finally we must show that the remaining fully-reduced graphs can indeed be generated from an effective Lagrangian with effective masses and coupling constants. This is done as follows: We sum over the O(1) parts of all the diagrams making up a Green's function in a rearranged fashion. Namely, contributions represented by a common fully-reduced structure are bundled up first and then sum over the different structures is performed. This is schematically illustrated in Fig. 9 for a 4 point function. Note that the effective coupling λ^* is the 4 point LPI Green's function evaluated at zero momentum. This ensures the symmetry (under $p_i \leftrightarrow p_j$) property and leads to the correct combinatorical factor (½), for example, for the second graph. The fact that the combinatorics comes out precisely right is easily understood if we recall the remark (b) made immediately following the proof of theorem 1; from the point of view advocated there, O(1) contributions are made up of diagrams generated by the finite light particle counterterms only.

This then completes the construction of the effective light particle Lagrangian. It cannot be overly emphasized that throughout our procedure we dealt with finite renormalized quantities. It is this feature that allowed us to take the meaningful $M \to \infty$ limit and obtain the effective Lagrangian accompanied with appropriate counterterms. In the next section, we shall utilize the gauge symmetry of the theory to completely fix the structure of the effective Lagrangian.

V. BRS-WARD IDENTITIES

The final step of our analysis requires a detailed study of various BRS-Ward identities.[13] We shall see that the gauge symmetry constraints are sufficiently restrictive to dictate that all the desired structure of the theory be present. Specifically we shall establish (i) that the light-heavy mixing angle remains small (O(v/V)) to all orders, (ii) that the structure of the low energy effective Lagrangian is precisely that of spontaneously broken O(2) gauge theory (known as Abelian Higgs model), and (iii) that all the light particles remain light. Moreover all the parameters of the effective theory will be expressed explicitly in terms of the quantities in the full theory.

Let us first briefly review[21] the basic BRS-Ward identity relevant to our theory. Our Lagrangian including the gauge fixing and the ghost terms is invariant under the following BRS transformations:

$$\delta_\lambda A_\mu^a = D_\mu^{ab} c^b \delta\lambda$$

$$\delta_\lambda h^a = - g \varepsilon^{abc} h^c c^b \delta\lambda$$

$$\delta_\lambda H^a = - g \varepsilon^{abc} H^c c^b \delta\lambda$$

$$\delta_\lambda \bar{c}^a = \frac{1}{\alpha} F_a \delta\lambda$$

$$\delta_\lambda c^a = \frac{1}{2} g \varepsilon^{abc} c^b c^c \delta\lambda \tag{5.1}$$

where $\delta\lambda$ is an infinitesimal global anticommuting variable. It is convenient, as is customary, to introduce the sources for the composite operators appearing on the right hand side of Eq. (5.1). So we add to our Lagrangian,

$$\mathscr{L}_{cs} = K^a_\mu \frac{\delta A^a_\mu}{\delta\lambda} + k^a \frac{\delta h^a}{\delta\lambda} + K^a \frac{\delta H^a}{\delta\lambda} + \ell^a \frac{\delta c^a}{\delta\lambda}$$

$$= -K^a_\mu D^{\mu\, ab} c^b + g k^a \varepsilon^{abc} h_c c_b + g K^a \varepsilon^{abc} H_c c_b$$

$$- \tfrac{1}{2} \ell^a g \varepsilon^{abc} c_b c_c \quad . \tag{5.2}$$

Because of the nilpotency, $\delta_\lambda^{\,2} = 0$, of the BRS transformation, $\mathscr{L}_{cs}$ is by itself BRS invariant. Now by making a change of variables corresponding to the above BRS transformations in the functional integral representation of the generating functional, one obtains

$$\int d^4x < J^a_\mu \frac{\delta A^\mu_a}{\delta\lambda} + j^a \frac{\delta h_a}{\delta\lambda} + J^a \frac{\delta H^a}{\delta\lambda} + \frac{\delta \bar{c}_a}{\delta\lambda} \xi_a - \bar{\xi}_a \frac{\delta c_a}{\delta\lambda} >$$

$$= 0 \tag{5.3}$$

where the symbol < > denotes the vacuum to vacuum amplitude in the presence of the sources. This equation can be easily translated into the one in terms of the 1PI generating functional Γ. Namely we have

$$\int d^4x \left\{ \frac{\delta\Gamma}{\delta A^a_\mu(x)} \frac{\delta\Gamma}{\delta K^\mu_a(x)} + \frac{\delta\Gamma}{\delta h_a(x)} \frac{\delta\Gamma}{\delta k_a(x)} + \frac{\delta\Gamma}{\delta H_a(x)} \frac{\delta\Gamma}{\delta K_a(x)} \right.$$

$$\left. + \frac{\delta\Gamma}{\delta c_a(x)} \frac{\delta\Gamma}{\delta \ell_a(x)} + \frac{1}{\alpha} F_a(x) \frac{\delta\Gamma}{\delta \bar{c}_a(x)} \right\} = 0 \quad . \tag{5.4}$$

To make Eq. (5.4) finite we need to perform the renormalization transformation listed in Eq. (2.15) and in addition the following rescaling of the composite sources.

$$K^a_\mu = \tilde{Z}^{1/2} K^a_{\mu R}$$

$$k_a = (\tilde{Z} Z_3)^{1/2} Z_h^{-1/2} k_{aR}$$

$$K_a = (\tilde{Z} Z_3)^{1/2} Z_H^{-1/2} K_{aR}$$

$$\ell_a = Z_3^{1/2} \ell_{aR} \quad . \tag{5.5}$$

The Ward identity remains form-invariant. (We shall therefore omit the subscript R and consider Eq. (5.4) as already renormalized.) In particular one may choose $\delta\vec{v}$ and $\delta\vec{V}$ of Eq. (2.16) so that the gauge fixing term $1/\alpha\ F_a$ retains the same form

$$\frac{1}{\alpha}\left(\partial_\mu A^\mu_a - g\alpha(\vec{\eta}, t^a\vec{v}) - g\alpha(\vec{\chi}, t^a\vec{V}) \right) \tag{5.6}$$

when expressed in terms of the renormalized quantities. The Ward identity should be supplemented with another informative equation, i.e. the equation of motion for the ghost fields. It reads (in renormalized form),

$$\frac{\delta \Gamma}{\delta \bar{c}_a} = \partial_\mu \frac{\delta \Gamma}{\delta K^\mu_a} - g\alpha v \varepsilon_{ba2} \frac{\delta \Gamma}{\delta k_b} - g\alpha V \varepsilon_{ba1} \frac{\delta \Gamma}{\delta K_b} \quad . \tag{5.7}$$

Using this one can show that the quantity $\tilde{\Gamma}$ defined by

$$\tilde{\Gamma} \equiv \Gamma + \frac{1}{2\alpha}\int d^4x (F_a)^2 \tag{5.8}$$

satisfies a more compact BRS-Ward identity

$$\int d^4x \left\{ \frac{\delta \tilde{\Gamma}}{\delta A^A_\mu(x)} \frac{\delta \tilde{\Gamma}}{\delta K^\mu_a(x)} + \frac{\delta \tilde{\Gamma}}{\delta h_a(x)} \frac{\delta \tilde{\Gamma}}{\delta k_a(x)} + \frac{\delta \tilde{\Gamma}}{\delta H_a(x)} \frac{\delta \tilde{\Gamma}}{\delta K_a(x)} + \frac{\delta \tilde{\Gamma}}{\delta c_a(x)} \frac{\delta \tilde{\Gamma}}{\delta \ell_a(x)} \right\}$$

$$= 0 \quad . \tag{5.9}$$

Equations (5.7) and (5.9) are the fundamental equations that we shall utilize below.

Let us begin by showing that the light ghost field c_1 remains light. Consider the first component of the ghost equation of motion and apply $\int d^4y \frac{\delta}{\delta c_1(y)}$. After setting all the sources to zero, we obtain

$$-m_{gh}^2 \equiv \int d^4y \frac{\delta^2 \Gamma}{\delta c_1(y) \delta \bar{c}_1(x)} = -g\alpha v \int d^4y \frac{\delta^2 \Gamma}{\delta c_1(y) \delta k_3(x)} \quad . \tag{5.10}$$

The power counting theorem dictates that $\int d^4y \frac{\delta^2 \Gamma}{\delta c_1(y) \delta k_3(x)}$ can be at most of $O(v)$. This gives us

$$m_{gh}^2 = O(v^2) \quad . \tag{5.11}$$

Thus the light ghost field c_1 remains light.

Next we shall show that the same is true for the light Goldstone field η_3. For this and later purposes we must express the Ward identity (5.9) in terms of the tree-level diagonal fields. (For η_3 this is clearly enough for it does not mix with other fields--its mixing with A^μ_1 is irrelevant for its mass. Further, as was shown in Sec. III, it suffices for our subsequent analysis of LPI generating functional.) The result of the appropriate rotations is

$$\int d^4x \Bigg\{ \frac{\delta \tilde{\Gamma}}{\delta A^a_\mu(x)} \frac{\delta \tilde{\Gamma}}{\delta K^\mu_a(x)} + \frac{\delta \tilde{\Gamma}}{\delta c_a(x)} \frac{\delta \tilde{\Gamma}}{\delta \ell_a(x)} + \left(\cos\theta_1 \frac{\delta \tilde{\Gamma}}{\delta \bar{\eta}_1(x)} - \sin\theta_1 \frac{\delta \tilde{\Gamma}}{\delta \bar{\chi}_2(x)} \right) \frac{\delta \tilde{\Gamma}}{\delta k_1(x)}$$

$$+ \left(\cos\theta_2 \frac{\delta \tilde{\Gamma}}{\delta \bar{\eta}_2(x)} + \sin\theta_2 \frac{\delta \tilde{\Gamma}}{\delta \bar{\chi}_1(x)} \right) \frac{\delta \tilde{\Gamma}}{\delta k_2(x)} + \frac{\delta \tilde{\Gamma}}{\delta \eta_3(x)} \frac{\delta \tilde{\Gamma}}{\delta k_3(x)}$$

$$+ \left(\cos\theta_2 \frac{\delta \tilde{\Gamma}}{\delta \bar{\chi}_1(x)} - \sin\theta_2 \frac{\delta \tilde{\Gamma}}{\delta \bar{\eta}_2(x)} \right) \frac{\delta \tilde{\Gamma}}{\delta K_1(x)}$$

$$+ \left(\cos\theta_1 \frac{\delta \tilde{\Gamma}}{\delta \bar{\chi}_1(x)} + \sin\theta_1 \frac{\delta \tilde{\Gamma}}{\delta \bar{\eta}_1(x)} \right) \frac{\delta \tilde{\Gamma}}{\delta K_2(x)} + \frac{\delta \tilde{\Gamma}}{\delta \chi_3(x)} \frac{\delta \tilde{\Gamma}}{\delta K_3(x)} \Bigg\} = 0 \quad . \quad (5.12)$$

What we need to prove in fact is the Goldstone theorem[22] for the η_3 field. Let us apply $\int d^4y \frac{\delta}{\delta \eta_3(y)} \frac{\delta}{\delta c_1(z)}$ to Eq. (5.12) and set the sources to zero. Many terms seem to be produced but thanks to the index conservation discussed in Sec. III all except one term vanish. We thus obtain

$$\int d^4x d^4y \frac{\delta^2 \tilde{\Gamma}}{\delta \eta_3(y) \delta \eta_3(x)} \frac{\delta^2 \tilde{\Gamma}}{\delta c_1(z) \delta k_3(x)} = 0 \quad . \quad (5.13)$$

Using the translation invariance, this can be written as

$$\int d^4y \frac{\delta^2 \tilde{\Gamma}}{\delta \eta_3(y) \delta \eta_3(0)} \cdot \int d^4x \frac{\delta^2 \tilde{\Gamma}}{\delta c_1(x) \delta k_3(0)} = 0 \quad . \quad (5.14)$$

Since $\int d^4x \frac{\delta^2 \tilde{\Gamma}}{\delta c_1(x) \delta k_3(0)}$ has a nonzero tree level contribution it cannot vanish and we get the Goldstone theorem

$$\int d^4y \frac{\delta^2 \tilde{\Gamma}}{\delta \eta_3(y) \delta \eta_3(0)} = 0 \quad . \quad (5.15)$$

Recalling Eq. (5.8) this implies

$$m^2_{\eta_3} = -\int d^4y \frac{\delta^2\Gamma}{\delta\eta_3(y)\delta\eta_3(0)} = +\int d^4y \frac{\delta^2}{\delta\eta_3(y)\delta\eta_3(0)} \frac{1}{2\alpha}\int [F_a(x)]^2 d^4x \Big|_{\text{sources}=0}$$

$$= \alpha g^2 v^2 \quad . \tag{5.16}$$

i.e. the mass of the Goldstone boson η_3 is not renormalized and afortiori η_3 remains light.

Demonstration of the smallness (O(v/V)) of the light-heavy mixing angle requires slightly more complicated analysis. This time we apply to Eq. (5.12) the operation $\int d^4y d^4z \frac{\delta}{\delta\eta_3(y)} \frac{\delta}{\delta\bar{\eta}_2(z)} \frac{\delta}{\delta c_1(u)}$ and then set all the sources to zero. Again most of the terms vanish due to the index mismatch and to the Goldstone theorem just proved, and the surviving terms are

$$\int d^4y d^4z d^4x \Bigg\{ \left(\cos\theta_2 \frac{\delta^2\Gamma}{\delta\bar{\eta}_2(z)\delta\bar{\eta}_2(x)} - \sin\theta_2 \frac{\delta^2\Gamma}{\delta\bar{\eta}_2(z)\,\delta\bar{\chi}_1(x)} \right) \frac{\delta^3\Gamma}{\delta\eta_3(y)\delta c_1(u)\delta k_2(x)}$$

$$+ \frac{\delta^3\Gamma}{\delta\bar{\eta}_2(z)\delta\eta_3(y)\delta\eta_3(x)} \frac{\delta^2\Gamma}{\delta c_1(u)\delta k_3(x)}$$

$$+ \left(\cos\theta_2 \frac{\delta^2\Gamma}{\delta\bar{\eta}_2(z)\delta\bar{\chi}_1(x)} - \sin\theta_2 \frac{\delta^2\Gamma}{\delta\bar{\eta}_2(z)\delta\bar{\eta}_2(x)} \right) \frac{\delta^3\Gamma}{\delta\eta_3(y)\delta c_1(u)\delta K_1(x)} \Bigg\}$$

$$= 0 \quad . \tag{5.17}$$

In momentum space this is a relation between various Green's functions at zero momentum. We shall use obvious notations such as $\Gamma_{\eta_3 c_1 k_2}$ for $\int d^4y d^4x \frac{\delta^3\Gamma}{\delta\eta_3(y)\,\delta c_1(u)\delta k_2(x)}$, etc. From the power counting theorem we know that

$$\Gamma_{\eta_3 c_1 k_2} = O(1) \quad , \quad \Gamma_{\bar{\eta}_2 \eta_3 \eta_3} = O(V)$$

$$\Gamma_{c_1 k_3} = O(v) \quad , \quad \Gamma_{\bar{\eta}_2 \bar{\chi}_1} = O(vV)$$

$$\Gamma_{\eta_3 c_1 k_1} = O(1) \quad . \tag{5.18}$$

But the reflection symmetry under $h_a \to -h_a$, $v \to -v$, $k_a \to -k_a$, tells us that $\Gamma_{\bar{\eta}_2 \eta_3 \eta_3}$ cannot be of $O(V)$ but only of $O(v)$, and likewise the symmetry under $H_a \to -H_a$, $V \to -V$, $K_a \to -K_a$, dictates that $\Gamma_{\eta_3 c_1 k_1}$ can actually be at most of $O(v/V)$. Putting all the information into Eq. (5.17), we can deduce the order of magnitude of the two point function $\Gamma_{\bar{\eta}_2 \bar{\eta}_2}$. From

$$0 = \left[\Gamma_{\bar{\eta}_2 \bar{\eta}_2} \cos\theta_2 - O(Vv)\sin\theta_2\right] O(1)$$

$$+ O(v)O(v) + \left(O(Vv)\cos\theta_2 - \Gamma_{\bar{\eta}_2 \bar{\eta}_2} \sin\theta_2 \right) O\left(\frac{v}{V}\right) \tag{5.19}$$

we get

$$\Gamma_{\bar{\eta}_2 \bar{\eta}_2} = O(v^2) \quad . \tag{5.20}$$

This then is enough to secure the smallness of the full mixing angle and justifies the discussions of Sec. III.

What remains to be done is to derive the BRS-Ward identity satisfied by the $O(1)$ part of the LPI generating functional Γ^{LPI} which, as was shown in Sec. III, is nothing but the 1PI generating functional Γ^* for the effective light particle theory.

The Ward identity for Γ^{LPI} is obtained by setting all the heavy particle sources to zero in Eq. (5.12). We obtain

$$0 = \int d^4x \left\{ \frac{\delta \Gamma^{LPI}}{\delta A_\mu^1(x)} \frac{\delta \Gamma^{LPI}}{\delta K^1(x)} + \frac{\delta \Gamma^{LPI}}{\delta c_1(x)} \frac{\delta \Gamma^{LPI}}{\delta \ell_1(x)} \right.$$

$$+ \cos\theta_2 \frac{\delta \Gamma^{LPI}}{\delta \bar{\eta}_2(x)} \frac{\delta \Gamma^{LPI}}{\delta k_2(x)} + \frac{\delta \Gamma^{LPI}}{\delta \eta_3(x)} \frac{\delta \Gamma^{LPI}}{\delta k_3(x)}$$

$$\left. - \sin\theta_2 \frac{\delta \Gamma^{LPI}}{\delta \bar{\eta}_2(x)} \frac{\delta \Gamma^{LPI}}{\delta K_1(x)} + \frac{1}{\alpha} F_1(x) \frac{\delta \Gamma^{LPI}}{\delta \bar{c}_1(x)} \right\} \quad . \tag{5.21}$$

As a matter of fact $\delta \Gamma^{LPI}/\delta \ell_1(x)$ identically vanishes for the following reason: The ghost number conservation requires that the nonvanishing Green's function one can obtain from $\delta \Gamma^{LPI}/\delta \ell_1(x)$ is of the general form

$$\left. \frac{\delta^{2n+2} \Gamma^{LPI}}{\delta \ell_1 (\delta c_1)^2 (\delta \bar{c}_1 \delta c_1)^n} \right|_{\text{sources} = 0} \quad . \tag{5.22}$$

This, however, is incompatible with the index conservation. (Total index number for (5.22) is 1, not zero.) Furthermore the next to the last term in Eq. (5.21) can also be eliminated since it is down by v/V: Ghost number and the index conservations dictate that the allowed Green's function of lowest dimension involving $\delta \Gamma^{LPI}/\delta K_1$ is $(\delta^3 \Gamma^{LPI})/(\delta \eta_3 \delta c_1 \delta K_1)$, which by the power counting theorem, is of O(1). Green's functions with more legs are also at most of O(1). Therefore the presence of $\sin\theta_2 \sim O(v/V)$ allows us to drop this term within the accuracy of our approximation.

With the above-mentioned two terms eliminated thus, Γ^*, the O(1) part of Γ^{LPI}, satisfies the equation

$$\int d^4x \left\{ \frac{\delta\Gamma^*}{\delta A^1_\mu(x)} \frac{\delta\Gamma^*}{\delta K^\mu_1(x)} + \frac{\delta\Gamma^*}{\delta\bar{\eta}_2(x)} \frac{\delta\Gamma^*}{\delta k_2(x)} + \frac{\delta\Gamma^*}{\delta\eta_3(x)} \frac{\delta\Gamma^*}{\delta k_3(x)} + \frac{1}{\alpha} F_1(x) \frac{\delta\Gamma^*}{\delta\bar{c}_1(x)} \right\} = 0 \quad . \tag{5.23}$$

For completeness let us record the relevant light ghost equation of motion below (obtained from the first component of Eq. (5.7)).

$$\frac{\delta\Gamma^*}{\delta\bar{c}_1(x)} = \frac{\delta\Gamma^*}{\delta K^1_\mu(x)} - g\alpha v \frac{\delta\Gamma^*}{\delta k_3(x)} \quad . \tag{5.24}$$

As we may have anticipated, these are identical in form to the BRS-Ward identity and the ghost equation of motion for O(2) gauge theory known as Abelian Higgs model. These equations must be satisfied, in particular, by the effective light particle Lagrangian (or more accurately the tree action). By writing down a most general light particle action of dimension zero and substituting it into the above two equations (we leave this as an exercise to the reader), we can easily fix the structure of the action of the effective theory. The result can be conveniently cast into the following form:

$$\mathscr{L}^* = -\frac{1}{4} z_3 F^1_{\mu\nu} F^{\mu\nu 1} + \frac{1}{2}\left[(\partial_\mu + g^* t^1 A^1_\mu) \begin{pmatrix} 0 \\ z_{\eta_2}^{1/2}(\bar{\eta}_2 + v^*) \\ z_{\eta_3}^{1/2}\eta_3 \end{pmatrix} \right]^2$$

$$- \frac{\lambda^*}{4!} \left(z_{\eta_2}(\bar{\eta}_2 + v^*)^2 + z_{\eta_3}\eta_3^2 \right)^2$$

$$- \frac{1}{2\alpha} \left(\partial_\mu A^\mu_1 - g\alpha v \eta_3 \right)^2 - \tilde{z}\bar{c}_1 \partial^2 c_1 - m^2_{gh} \bar{c}_1 c_1$$

$$- \alpha g g^* v \tilde{z} \left(z_{\eta_2} / z_{\eta_3} \right)^{\frac{1}{2}} \bar{c}_1 c_1 \bar{\eta}_2$$

$$- \tilde{z} K^1_\mu \partial^\mu c_1 - \tilde{z} g^* \left(z_{\eta_3} / z_{\eta_2} \right)^{\frac{1}{2}} k_2 \eta_3 c_1$$

$$+ \tilde{z} g^* \left(z_{\eta_2} / z_{\eta_3} \right)^{\frac{1}{2}} k_3 (\bar{\eta}_2 + v^*) c_1$$

$$+ \text{counter terms defined at zero momentum} \quad . \tag{5.25}$$

In the above equation various parameters and finite wave function renormalization constants can be <u>precisely identified in the full theory</u> in terms of LPI Green's functions and their derivatives at zero momentum:

$$z_{\eta_2} = \frac{\partial}{\partial p^2} \Gamma^{LPI}_{\eta_2 \eta_2} \Big|_{p=0} = O(1) \tag{5.26}$$

$$z_{\eta_3} = \frac{\partial}{\partial p^2} \Gamma^{LPI}_{\eta_3 \eta_3} \Big|_{p=0} = O(1) \tag{5.27}$$

$$\lambda^* z^2_{\eta_2} = \Gamma^{LPI}_{\eta_2 \eta_2 \eta_2 \eta_2} \Big|_{p_i=0} = O(1) \tag{5.28}$$

$$\Gamma^{LPI}_{A^1_\mu A^1_\nu} = \left(g_{\mu\nu} p^2 - p_\mu p_\nu \right) \pi_1 + g_{\mu\nu} \pi_2$$

$$z_3 = \pi_1 \Big|_{p=0} = O(1) \tag{5.29}$$

$$4 z_{\eta_2} g^{*2} g_{\mu\nu} = \Gamma^{LPI}_{A^1_\mu A^1_\nu \eta_2 \eta_2} \Big|_{p_i=0} = O(1) \tag{5.30}$$

$$\tilde{z} = \frac{\partial}{\partial p^2} \Gamma^{LPI}_{c_1 \bar{c}_1} \Big|_{p=0} = O(1) \tag{5.31}$$

$$m_{gh}^2 = -\Gamma^{LPI}_{c_1\bar{c}_1}\Big|_{p=0} = O(v^2) \tag{5.32}$$

$$v^* = \frac{m_{gh}^2 z_{\eta_2}^{1/2}}{\alpha g g^* v \tilde{z} z_{\eta_3}^{1/2}} = O(v) \quad . \tag{5.33}$$

The masses of A_μ^i, and $\bar{\eta}_2$ (those of η_3 and c_1 are already given in Eq. (5.16) and Eq. (5.32)) can be read off from Eq. (5.25) as

$$m_{A_1}^2 = g^{*2} z_{\eta_2} v^{*2} \tag{5.34}$$

$$m_{\eta_2}^2 = \frac{1}{2}\lambda^* z_{\eta_2}^2 v^{*2} \quad . \tag{5.35}$$

Thus we see that all the light particles remained light. One can of course bring $\mathscr{L}^*$ into the standard form (i.e. without the finite z factors) by making appropriate finite rescalings. Note that counter terms of the effective theory are the ones corresponding to zero momentum subtractions, in spite of the fact that the full theory is renormalized by minimal subtraction. This is due to our procedure of separating O(1) and O(1/M) parts at zero momentum. If one wishes to obtain the light theory which is renormalized also by minimal subtraction, all one needs to do is to simply make a finite renormalization within the light effective theory.

This completes the demonstration of all the three results announced in the introduction.

VI. DISCUSSIONS AND COMMENTS

In this final section we shall discuss an important application of our result, namely the renormalization group equations governing the heavy mass dependence of the effective parameters. There have been many discussions on this subject[23] but our emphasis will be on the all order aspect, especially a systematic method of computing the appropriate boundary conditions, which results from our analysis of previous sections. Also included in this section is a comment on the gauge fixing procedure recently proposed by Weinberg[16] in the context of effective Lagrangian.

As was described in the previous section, our general analysis has enabled us to make a precise identification of the parameters of the effective theory in terms of the zero momentum values of appropriate LPI Green's functions of the full theory. Among other things this will in turn allow us to write down exact renormalization group equations satisfied by these effective parameters. Although conceptually clean and unambiguous, direct execution of this procedure is not quite practical due to the fact that the effective theory, as was derived in the last section, is the one renormalized at zero momentum. Specifically, in such a scheme the effective parameters contain two different types of logarithms, namely $(\ln M/m)^n$ and $(\ln M/\mu)^n$, and this makes the integration of the equations cumbersome. For the purpose of actual computation, it is more advantageous to work with the effective theory made finite by minimal subtraction (especially the so-called $\overline{\mathrm{MS}}$ scheme[15]), which can be obtained through finite renormalization within the effective theory.

Now a point we wish to make here is that there exists an algorithm to go directly from the full theory to the minimally subtracted effective theory without going through the intermediate stage of zero momentum subtraction. Let us briefly describe this algorithm. Suppose one wants to compute an effective coupling constant, call it g_I^*, in terms of the coupling constants g_i of the full theory. With g_I^* are associated appropriate LPI diagrams of the full theory. (For example, if g_I^* is the coupling constant λ^* of our model O(3) theory, the relevant diagrams are LPI four point graphs with four $\overline{\eta}_2$ particles in the external legs.) At the tree level g_I^* is simply the sum of the relevant tree graphs evaluated at zero momentum. (Of course we may need to extract appropriate tensor structure before setting the momenta to zero depending on the type of the vertex. This procedure will always be tacitly understood when we say "evaluate at zero momentum.") At the one loop level, we shall do the following: Take the sum of all the one loop LPI diagrams relevant to g_I^*, renormalize (via minimal subtraction), and evaluate it at zero momentum. At the same time we use the effective Lagrangian previously obtained at the tree level to compute the relevant 1PI Green's function, at zero momentum, to one loop order with renormalization performed also via minimal subtraction. The difference of the two then gives g_I^* at one loop level, which we shall denote by $g_I^{*(1)}$. To obtain $g_I^{*(n)}$, in general, repeat the same procedure as above except (i) that we must use the effective Lagrangian obtained up to $n-1$ loop level (with $g_I^* = g_I^{*(0)} + g_U^{*(1)} + \ldots + g_I^{*(n-1)}$, etc.) and (ii) that "n loop" contribution to be subtracted is defined by the compounded loop number, i.e., including the loop number associated with the coupling constants, masses and the wave function renormalization factors, in addition to the actual number of loops of the diagrams. One can easily convince oneself that the above procedure precisely effects, order by order, finite renormalization relative to the effective theory constructed previously by zero momentum subtraction.

Now an important feature of minimal subtraction algorithm is that the infrared structure of the full theory is precisely inherited by the effective theory due to the purely ultraviolet nature of subtractions. This manifests itself in the fact that the contribution to the effective parameters come solely from the ultraviolet region of the overall loop momentum space and hence they are free of singularity as $m \to 0$. In other words they do not contain logarithms of the type $(\ln M/m)^n$; they are functions only of dimensionless coupling constants of the full theory and $(\ln M/\mu)^n$'s.

Another point of immense significance is that in the minimal scheme decoupling takes place irrespective of the magnitude of μ --μ may be taken as large as one likes. (The only requirement is $M^2 \gg p_i^2, m^2$.) This is in sharp contrast to the momentum subtraction scheme, where μ^2 must also be much smaller than M^2 to have decoupling.

With these understandings in mind we can formulate how to compute g_I^* by renormalization group equations to any desired accuracy. Let us start from the statement of decoupling at low energy for an arbitrary Green's function. We have

$$\Gamma_{LPI}\Big(\{p_i\}, \{g_i\}, M, \{m_i\}, \mu\Big) = z_\Gamma \Gamma^*_{1PI}\Big(\{p_i\}, \{g_I^*\}, \{m_I^*\}, \mu\Big) + O(1/M) \,. \quad (6.1)$$

where z_Γ is a finite wave function renormalization factor, and the starred quantities are those for the effective theory. (We have suppressed the dependence on the gauge fixing parameter, which is inessential for subsequent discussions.) By the standard procedure we can derive the renormalization group equations satisfied by Γ_{LPI} and Γ^*_{1PI}, viz.,

$$\left(\mu\frac{\partial}{\partial\mu} + \mu\frac{\partial g_i}{\partial\mu}\frac{\partial}{\partial g_i} + \mu\frac{\partial M}{\partial\mu}\frac{\partial}{\partial M} + \mu\frac{\partial m_i}{\partial\mu}\frac{\partial}{\partial m_i} + \gamma_\Gamma\right)\Gamma_{LPI} = 0 \quad (6.2)$$

$$\left(\mu\frac{\delta}{\delta\mu} + \mu\frac{\delta g_I^*}{\delta\mu}\frac{\partial}{\partial g_I^*} + \mu\frac{\delta m_I^*}{\delta\mu}\frac{\partial}{\partial m_I^*} + \gamma_\Gamma^*\right)\Gamma^*_{1PI} = 0 \quad (6.3)$$

where γ_Γ and γ_Γ^* are the usual anomalous dimensions associated with Γ_{LPI} and Γ^*_{1PI} respectively, and we have used the symbol $\delta/\delta\mu$ to indicate that the derivative is taken in the effective theory. Now by substituting Eq. (6.1) into Eq.

(6.2), we obtain another equation for Γ^*_{1PI} valid at low energy. This equation can be compared with Eq. (6.3) above, after using simple chain rules such as

$$\mu \frac{\partial}{\partial \mu} = \mu \frac{\delta}{\delta \mu} + \mu \frac{\partial g_I^*}{\partial \mu} \frac{\partial}{\partial g_I^*} + \mu \frac{\partial m_I^*}{\partial \mu} \frac{\partial}{\partial m_I^*} \tag{6.4}$$

and this gives us renormalization group equations satisfied by the effective parameters. The one satisfied by g_I^* reads

$$\left(\mu \frac{\partial}{\partial \mu} + \mu \frac{\partial g_i}{\partial \mu} \frac{\partial}{\partial g_i} + \mu \frac{\partial M}{\partial \mu} \frac{\partial}{\partial M} \right) g_I^* = \mu \frac{\delta g_I^*}{\delta \mu} \quad . \tag{6.5}$$

This, however, is nothing but

$$\mu \frac{d}{d\mu} g_I^* = \beta_I^* \left(\{ g_J^* \} \right) \tag{6.6}$$

where $d/d\mu$ is the total derivative and β_I^* is the β function computed in the effective theory. As emphasized before, in minimal subtraction scheme, decoupling is independent of μ and Eq. (6.6) is <u>exact for any μ</u>. Thus it is a matter of supplying the correct boundary conditions to integrate Eq. (6.6) and the previously described algorithm precisely allows us to do so to any desired accuracy. To be a little more specific, we shall choose to make contact with the full theory at a scale $\mu = M'$, which is of the order of M. (It does not matter whether one chooses $M' = M$ or $M' = 2M$ as long as $\ln M/M'$ and $g_i(M')$ are small so that the perturbation theory is reliable.) We then apply the minimal subtraction algorithm to compute $g_I^*(M')$'s in powers of $g_i(M)$'s and use them as the boundary conditions in solving Eq. (6.6). To our mind this is the most systematic and unambiguous method of computation.

Finally in connection with the effective light particle Lagrangian, we shall make a comment on the gauge fixing procedure recently proposed by Weinberg.[16]

Consider a scenario in which a simple gauge group G is spontaneously broken down to a smaller group $\tilde{G}$ by a large vacuum expectation value V along a certain direction in G. Now it would certainly be nice if the effective action $\tilde{I}$ for the light particles, obtained after "integrating out" heavy fields, is gauge invariant under $\tilde{G}$. Weinberg contends that it can be done by choosing the gauge fixing term for the heavy gauge bosons to be invariant under $\tilde{G}$. Namely he proposes to choose

$$\mathcal{L}^{\text{heavy}}_{\text{gauge fixing}} = -\frac{1}{2\xi} \sum_A F_A{}^2 \qquad (6.7)$$

where

$$F_A = \left(\partial_\mu \delta_{AB} + g f_{ABa} A^a_\mu \right) A^\mu_B + ig\xi(V, t_{AS} S) \quad . \qquad (6.8)$$

Here capital (small) Latin indices refer to broken (unbroken) generators, ξ is the gauge fixing parameter, f_{ABa} is the structure constant, S is a scalar field, and t_{AS} is a generator in the representation of the scalar S. It was then argued that the Fadeev-Popov determinant $|\delta F_\alpha / \delta\theta_\beta| \equiv |M_{\alpha\beta}|$ (Greek indices refer to both broken and unbroken generators) associated with this gauge fixing effectively factorizes into a product of determinants $|M_{ab}|\,|M_{AB}|$ where $|M_{AB}|$ is invariant under $\tilde{G}$, and that this ensures invariance of $\tilde{I}$ under $\tilde{G}$.

This apparently nice procedure, however, is correct only for $\xi = 0$. To see the inapplicability of Weinberg's prescription for nonzero ξ, it is better to start from the very beginning, i.e., the Fadeev-Popov procedure. For clarity of argument let us suppress all the fields except the gauge field and study the generating functional

$$Z = \int \mathcal{D} A^\alpha_\mu e^{iS_0} \qquad (6.9)$$

Since Eq. (6.9) is ill-defined, we appeal to the well-known trick of inserting the expression

$$1 = \Delta(A_\mu)\int \prod dg \prod_a \delta\left(F_\alpha(A_\mu^g) - C_\alpha \right) \tag{6.10}$$

in order to extract out the field independent infinities. Here A_μ^g denotes a gauge transform of A_μ and C_α is an arbitrary function. (Later we shall operate $\int \mathcal{D} C_\alpha \exp\left(-\frac{i}{2\xi}\right)(C_\alpha)^2 d^4x$ to get the usual covariant gauges.) Due to the gauge invariance of the measure $\prod dg$, $\Delta(A_\mu)$ is gauge invariant. To compute $\Delta(A_\mu)$ we perform infinitesimal transformations around the configuration defined by $F_\alpha(A_\mu) = C_\alpha$. Then we have

$$1 = \Delta(A_\mu)\int \prod_\beta d\theta^\beta \prod_\alpha \delta\left(F_\alpha(A_\mu) + \frac{\delta F_\alpha(A_\mu)}{\delta\theta_\beta}\theta_\beta - C_\alpha \right) \tag{6.11}$$

$$= \Delta(A_\mu)\int \prod_\beta d\theta^\beta \prod_\alpha \delta\left(\frac{\delta F_\alpha(A_\mu)}{\delta\theta_\beta}\theta_\beta \right) \quad . \tag{6.12}$$

If we perform the integration all at once we of course get the usual result

$$\Delta(A_\mu) = \det\left(\frac{\delta F_\alpha(A_\mu)}{\delta\theta_\beta} \right) \equiv |M_{\alpha\beta}| \quad .$$

Instead, the idea of Weinberg is to separate out the indices corresponding to broken and unbroken generators and to factorize the above determinant. So let us write

$$1 = \Delta(A_\mu)\int \prod_b d\theta^b \prod_B d\theta^B \prod_a \delta(M_{ab}\theta_b + M_{aB}\theta_B) \quad . \tag{6.13}$$

$$\times \prod_A \delta(M_{Ab}\theta_b + M_{AB}\theta_B)$$

Using the second set of δ-functions first, we get

$$1 = \Delta(A_\mu)\int \prod_b d\theta^b \prod_B d\theta^B \prod_a \delta(M_{ab}\theta_b + M_{aB}\theta_B)$$

$$\times \frac{1}{|M_{AB}|} \prod_A \delta(\theta_B + M^{-1}_{BD}M_{Db}\theta_b)$$

$$= \frac{\Delta(A_\mu)}{|M_{AB}|}\int \prod_b d\theta^b \prod_a \delta(M_{ab}\theta_b - M_{aB}M^{-1}_{BD}M_{Db}\theta_b)$$

$$= \Delta(A_\mu)/|M_{AB}|\,|M_{ab} - M_{aB}M^{-1}_{BD}M_{Db}| \quad . \qquad (6.14)$$

Thus we obtain a factorized form

$$\Delta(A_\mu) = |M_{AB}|\,|M_{ab} - M_{aB}M^{-1}_{BD}M_{Db}| \quad . \qquad (6.15)$$

This is a completely general result. Now if we choose the Weinberg's gauge, F_A transforms covariantly under the unbroken group, i.e.,

$$M_{Db} = \frac{\delta F_D}{\delta\theta_b} = -f_{DBb}F_B \quad . \qquad (6.16)$$

If the gauge fixing condition were $F_B = 0$, which corresponds to the Landau type gauge ($\xi = 0$), this vanishes and hence we obtain $\Delta(A_\mu) = |M_{AB}|\,|M_{ab}|$ as Weinberg advocates. But for $\xi \neq 0$, we must take $F_B = C_B$ and integrate over C_B with a Gaussian weight. Then Eq. (6.15) does not simplify and one can only achieve block diagonalization of ghost sector at the expense of introducing an unpleasant, if not disastrous, non-local object like M^{-1}_{Db}.

What if we use the Landau type gauge? There are still some complications worth mentioning. First the non-linearity of the gauge makes renormalization program more tricky: As was shown by a recent study,[23] in quadratic gauges it is not possible to renormalize the theory without breaking BRS invariance. Secondly, at least in the context of our formalism, it is not useful, since the ghost and the Goldstone fields, in particular the ones associated with heavy gauge fields, are no longer massive. This makes proper diagrammatic separation of heavy and light sectors difficult to perform. Besides, one would have to take due caution for infrared divergences with such massless particles present in the theory.

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APPENDIX. POWER COUNTING THEOREMS.

In the following we shall establish the power counting theorems P1 and P2 quoted in Sec. IV.

First let us classify the type of vertices that occur in a renormalizable theory with bosonic fields, such as our model. (Inclusion of fermions is straightforward but we shall not bother to do it here.) From the point of view of power counting, Lorentz and group indices are irrelevant and we only need to distinguish heavy (H) and light (L) fields. Then different types of vertices and the number of such vertices occurring in a given diagram can be denoted by

	type	No. of vertices	
non-derivative	$H^{n-m}L^m$	V_{nm}	$(0 \leq m \leq n \leq 4)$
(single) derivative	$(H^{n-m}L^m)_d$	D_{nm}	$(0 \leq m \leq n \leq 3)$. (A.1)

In addition to these ordinary types of vertices, we have special vertices corresponding to the composite operators that appear in connection with the BRS-Ward identities discussed in Sec. V. (See Eq. (5.2).) For the purpose of power counting, however, we can treat them as ordinary vertices provided we count the dimension 2 sources K^a_μ, k^a, K^a, ℓ^a, as L^2 and allow them to appear only as external legs. We shall also employ the following notations:

$$P_H(P_L) = \text{number of heavy (light) propagators}$$

$$B_H(B_L) = \text{number of heavy (light) external legs} \quad . \qquad (A.2)$$

Now by counting the number of heavy and light fields respectively, we obtain two topological relations

$$2P_H + B_H = \sum_{m=0}^{4} (4 - m)V_{4m} + \sum_{m=0}^{3} (3 - m)(V_{3m} + D_{3m}) \tag{A.3}$$

$$2P_L + B_L = \sum_{m=0}^{4} mV_{4m} + \sum_{m=0}^{3} m(V_{3m} + D_{3m}) \quad . \tag{A.4}$$

The maximum power of M coming from the overall high loop momentum region is intimately related to the superficial degree of divergence d of a graph. Noting that

$$\left\{ \begin{array}{l} \text{No. of } \delta^{(4)} \text{ functions } = \sum_{m=0}^{4} V_{4m} + \sum_{m=0}^{3} (V_{3m} + D_{3m}) \\ \text{No. of derivatives } = \sum_{m=0}^{3} D_{3m} \\ \text{No. of integrations } = P_H + P_L \end{array} \right. \tag{A.5}$$

d is easily computed to be

$$d = 4 - (B_H + B_L) - \sum_{m=0}^{3} V_{3m} \quad . \tag{A.6}$$

(This of course is the maximum possible superficial degree of divergence. The actual degree may be lower due to gauge invariance and/or explicit appearance of powers of external momenta.)

To count the maximum power of M, we must examine every possible subintegration--both high momentum ($\sim M$) and low momentum ($<< M$) regions. Let us start with the tree graphs.

Tree level: Let the maximum power of M be denoted by n_ℓ, where ℓ signifies low momentum. We must attach M to every 3 point non-derivative vertex, except for the completely light particle vertex L^3, for which the coefficient can only be of order m. (One can check this either explicitly or by recalling the reflection symmetry of the theory under $h_a \to -h_a$, $v \to -v$, $k_a \to -k_a$.) Clearly n_ℓ is given by

$$n_\ell = \sum_{m=0}^{2} V_{3m} - 2P_H \quad . \tag{A.7}$$

Using Eq. (A.3), we may rewrite (A.7) as

$$n_\ell = B_H - \left\{ \sum_{m=0}^{4} (4 - m)V_{4m} + \sum_{m=0}^{2} (2 - m)V_{3m} + \sum_{m=0}^{3} (3 - m)D_{3m} \right\} \leq B_H \quad . \tag{A.8}$$

The equality holds only when the diagram consists of vertices of type L^4, L^3, $(L^3)_d$, HL^2 alone. In particular for light particle graphs (i.e. $B_H = 0$), $n_\ell \leq 0$ holds.

One loop level. For low loop momentum region, counting is identical to the tree case and (A.8) holds without modification. For high loop momentum, except for L^3 vertices, counting should go by naive dimension counting. (In other words, each derivative should contribute a power of M.) In this way we get

$$n_h = 4 - (B_H + B_L) - V_{33} \quad , \tag{A.9}$$

where the subscript h stands for high loop momentum. Let us examine the special case where $B_H = 0$. Then $n_h > 0$ only for two and three point functions. For a two point function, the Ward identity tells us (see Sec. V) that it is actually of $O(m^2)$

when all the relevant diagrams are added. On the other hand a light three point function cannot have a power of M due to the reflection symmetry which has been mentioned many times already. Thus for light Green's functions, $n_h \leq 0$. For the general case, similar consideration shows that $n_h \leq B_H$. Combining this with (A.8) we obtain

$$n = \max\{ n_\ell, n_h \} \leq B_H \quad . \tag{A.10}$$

<u>Two loop level</u>. Two loop level presents a new situation, the understanding of which will then lead immediately to all order formula. Let the loop momenta be ℓ_1 and ℓ_2. There are three regions of momentum space: (i) $\ell_1, \ell_2 << M$. Here we may apply overall low energy counting, which is the same as the tree level counting. (ii) $\ell_1, \ell_2 \sim M$. This is the overall high momentum region and the counting is the same (including the use of Ward identity and the reflection symmetry) as for n_h in the one loop case. (iii) $\ell_1 << \ell_2 \sim M$ (or $\ell_2 << \ell_1 \sim M$). This is the new situation alluded to above. Here the counting should go as follows. First do the high momentum counting for the subdiagram through which $\ell_2 \sim M$ flows. Shrink this to a point with the power $M^{n_h(\ell_2)}$ attached and then perform the low momentum counting for this reduced graph. Suppose the effective vertex is of the type $H^a L^b$. Then $n_h(\ell_2) = a$. The topological relation of type (A.3) for the reduced graph gives

$$2P_H + B_H = \sum_{m=0}^{3} (4 - m)V_{4m} + \sum_{m=0}^{2} (3 - m)(V_{3m} + D_{3m})$$

$$+ aV_{a+b,b} \tag{A.11}$$

where $V_{a+b,b}$ is the number of vertices of type H^aL^b (here it is one). Now the maximum power of M for the reduced graph is

$$a + \sum_{m=0}^{2} V_{3m} - 2P_H = B_H - \left\{ \sum_{m=0}^{4} (4 - m)V_{4m} + \sum_{m=0}^{3} (3 - m)D_{3m} + \sum_{m=0}^{2} (2 - m)V_{3m} \right\}$$

$$\leq B_H \tag{A.12}$$

which is identical to Eq. (A.8). Putting all three cases together, we easily obtain

$$n \leq B_H \quad . \tag{A.13}$$

Higher loop level. It should now be clear how to proceed. A particular hierarchy of loop momenta corresponds to a particular way of reducing the diagram. At every stage of reduction, counting is always $n \leq B_H$ for each subdiagram reduced. This persists throughout the entire procedure and hence we have $n \leq B_H$ for any 1PI Green's function.

This result can be readily extended to LPI Green's functions. A typical LPI graph is depicted in Fig. 10. It consists of LPI blobs and heavy propagators connecting them as in a tree graph. Let each 1PI blob be of the type $H^{a_i}L^{b_i}$. The number of heavy propagators appearing in the diagram is clearly $\frac{1}{2}(\sum_i a_i - B_H)$. Since each blob can carry, according to the result obtained above, maximum power a_i, the maximum power of the whole graph is given by

$$\sum_i a_i - 2 \times \tfrac{1}{2} \left(\sum_i a_i - B_H \right) = B_H \quad . \tag{A.14}$$

This completes the proof of theorem P1.

Remarks: (i) The power counting should best be done by Wick-rotating the integrals into the corresponding Euclidean version. (ii) Renormalization subtractions do not interfere with our power counting. This is because we have been counting the maximum possible powers of M without any resort to "improvement" which might arise from such subtractions. (Besides in minimal subtractions, such improvement does not occur.)

Proof of theorem P2. The spirit of the proof is quite analogous to that for Theorem P1 just described. We must examine all the regions of loop momentum space, this time with oversubtractions taken into account. Let us organize the argument in the form of a mathematical induction in the number of loops.

Tree level. Let the maximum power of M of a graph be n. A LPI tree graph with at least one heavy line does not contain any light propagators. Therefore we may apply simple dimension counting to get

$$n = 4 - B_L \quad . \tag{A.15}$$

This shows that for $B_L \geq 5$ the graph is already of $O(1/M)$ without any subtraction. Now there exists no 2 and 3 point relevant graphs at the tree level. The only 4 point function is the one shown in Fig. 11. For this, obviously an extra subtraction at zero momentum renders it to be $O(1/M^2)$. Thus the assertion is proved for tree graphs.

Inductive proof to all orders. Suppose the theorem is correct for up to n loop graphs and let us study an arbitrary n + 1 loop graph. If the graph consists entirely of heavy propagators, there is only one relevant scale, namely M, and the naive dimension counting Eq. (A.15) applies. For $B_L = 2$, 3, and 4 (the actual n is zero because of previous arguments), an oversubtraction performed to the whole graph renders it to vanish as $M \to \infty$. If the graph contains (at least one) light propagator, they must occur in loops. (Remember we are dealing with a LPI graph.) A useful classification of the loop momentum space is as follows: (a) the momenta flowing through these light propagators are all very large, $\backsim M$. In this region, from the power counting point of view, they may as well be regarded as heavy and the argument reduces to that for the case already discussed. (b) We are left with the case in which there is at least one light propagator through which a small momentum flows. In this case, however, as far as the power counting goes, we may split open such a graph into two pieces illustrated as in Fig. 12. The number of loops is now reduced to n and we may apply the induction hypothesis. One may wonder if oversubtractions applied to those partition elements which contain the particular loop split open might upset the result. This does not happen because what one is subtracting is a piece which is already of $O(1/M)$. This is illustrated in Fig. 13. We have thus proved the validity of the theorem to n + 1 loop level and hence to all orders by induction.

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Table 1. Spectrum of the theory at the tree level.

Particles		Squared Mass
Gauge bosons	A_μ^1	g^2v^2
	A_μ^2	g^2V^2
	A_μ^3	$g^2(V^2 + v^2)$
Ghosts	c_1	αg^2v^2
	c_2	αg^2V^2
	c_3	$\alpha g^2(V^2 + v^2)$
Goldstone Bosons	η_3	αg^2v^2
	χ_3	αg^2V^2
	$\overline{\chi}_2$	$\alpha g^2(V^2 + v^2)$
Physical Higgs Scalars	$\overline{\eta}_1$	$\frac{1}{2} f_4(V^2 + v^2)$
	$\overline{\chi}_1$	$f_1V^2 + f_2v^2 + \sqrt{(f_1V^2 + f_2v^2)^2 - 4(f_1f_2 - f_3)V^2v^2}$ $\simeq 2f_1V^2(1 + O(v^2/V^2))$
	$\overline{\eta}_2$	$f_1V^2 + f_2v^2 - \sqrt{(f_1V^2 + f_2v^2)^2 - 4(f_1f_2 - f_3)V^2v^2}$ $\simeq \frac{2v^2}{f_1}(f_1f_2 - f_3)(1 + O(v^2/V^2))$

where $\overline{\chi}_2 = \chi_2\cos\theta_1 - \eta_1\sin\theta_1$, $\overline{\eta}_1 = \eta_1\cos\theta_1 + \chi_2\sin\theta_1$

$\overline{\chi}_1 = \chi_1\cos\theta_2 + \eta_2\sin\theta_2$, $\overline{\eta}_2 = \eta_2\cos\theta_2 - \chi_1\sin\theta_2$

$$\tan\theta_1 = v/V$$

$$\tan\theta_2 = \frac{2f_3Vv}{f_1V^2 - f_2v^2}$$

$$\simeq \frac{f_3v}{f_1V}(1 + O(v^2/V^2))$$

FIGURE CAPTIONS

Fig. 1: Examples of composition of indices.

Fig. 2: An illustration of "index conservation" for $\eta_2 - \chi_1$ two point function.

Fig. 3: Graphical interpretation of the mixing angle $\sin\theta$. Shaded round blobs represent 1PI two point functions.

Fig. 4: Graphical depiction of how LPI two point light function is formed. Shaded blobs and square carry the same meaning as in Fig. 3.

Fig. 5: Diagrams illustrating the equality of four point LPI light functions defined with respect to true fields (double lines) and tree fields (single lines) to O(1) accuracy.

Fig. 6: A simple example, analyzed in Sec. IV, in which suppression due to a heavy propagator (solid line) is offset by large ($\sim M$) Higgs self couplings.

Fig. 7: A decomposition of the diagram in Fig. 6 as explained in the text.

Fig. 8: Complete decomposition of the diagram in Fig. 6 in which each bracketed expression ((a), (b) and (c)) is finite.

Fig. 9: An example of how the O(1) part of the full LPI light function is reproduced from diagrams of effective light particle theory according to the rules of the algebraic identity. Effective coupling constant and light particle (inverse) propagator are explicitly identified.

Fig. 10: A typical LPI light function. Shaded blobs represent 1PI functions.

Fig. 11: The only non-trivial four point LPI light function at the tree level.

Fig. 12: Illustration of how power counting is done for the case in which low (<< M) momentum flows through a light propagator.

Fig. 13: Illustration of why an "oversubtraction" does not upset the power counting procedure depicted in Fig. 12. The key point is that the subtracted piece is of order O(1/M).

Fig. 1.

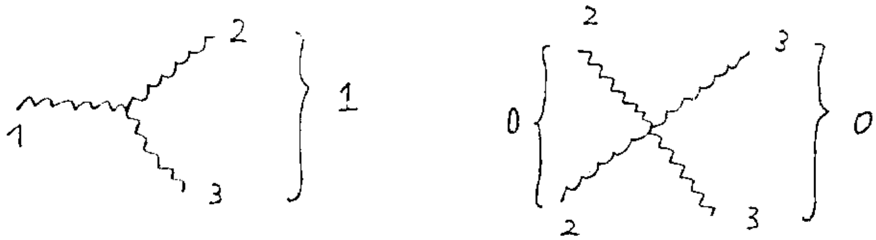
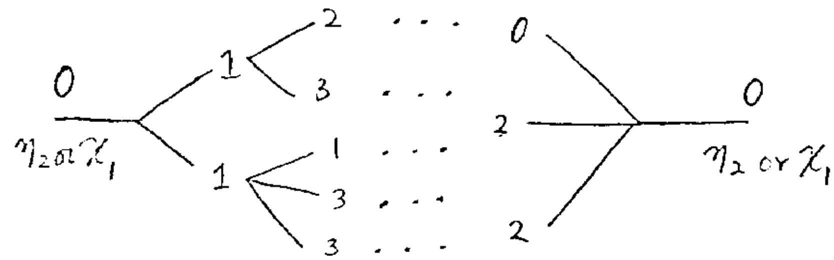


Fig. 2



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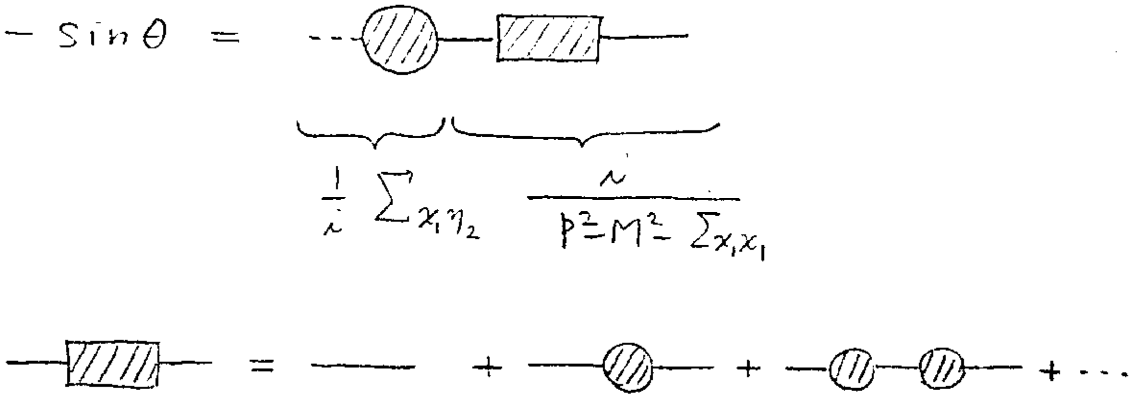

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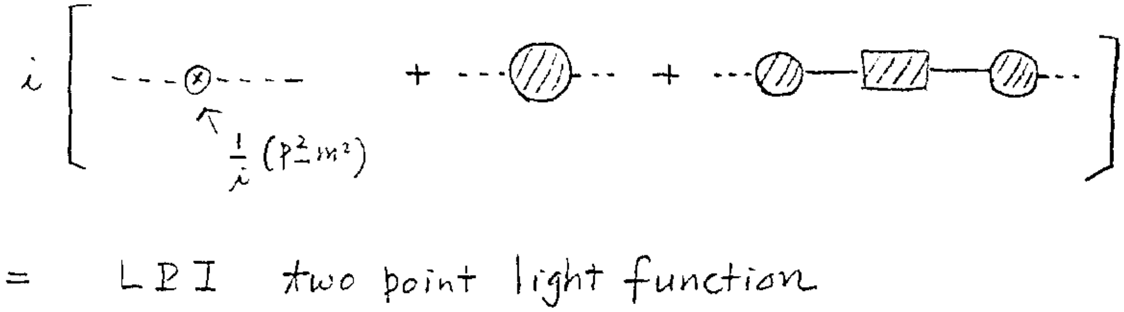

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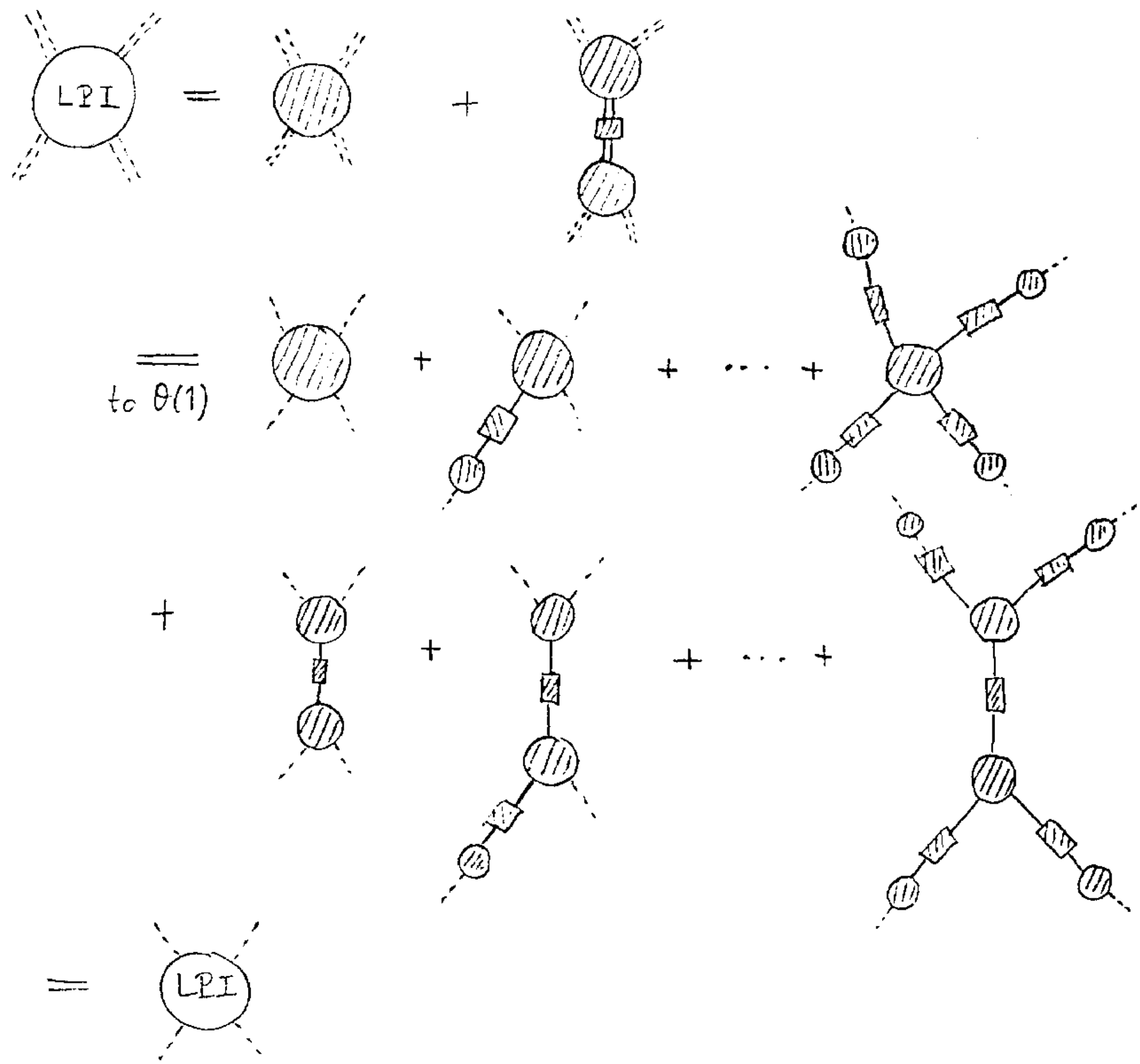


Fig. 6

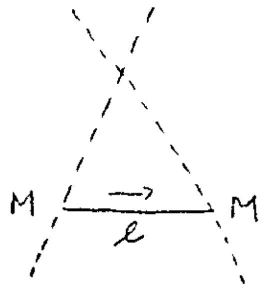


Fig. 7

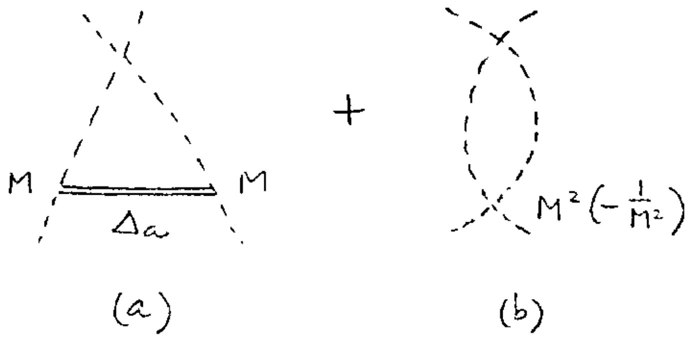


Fig. 8.

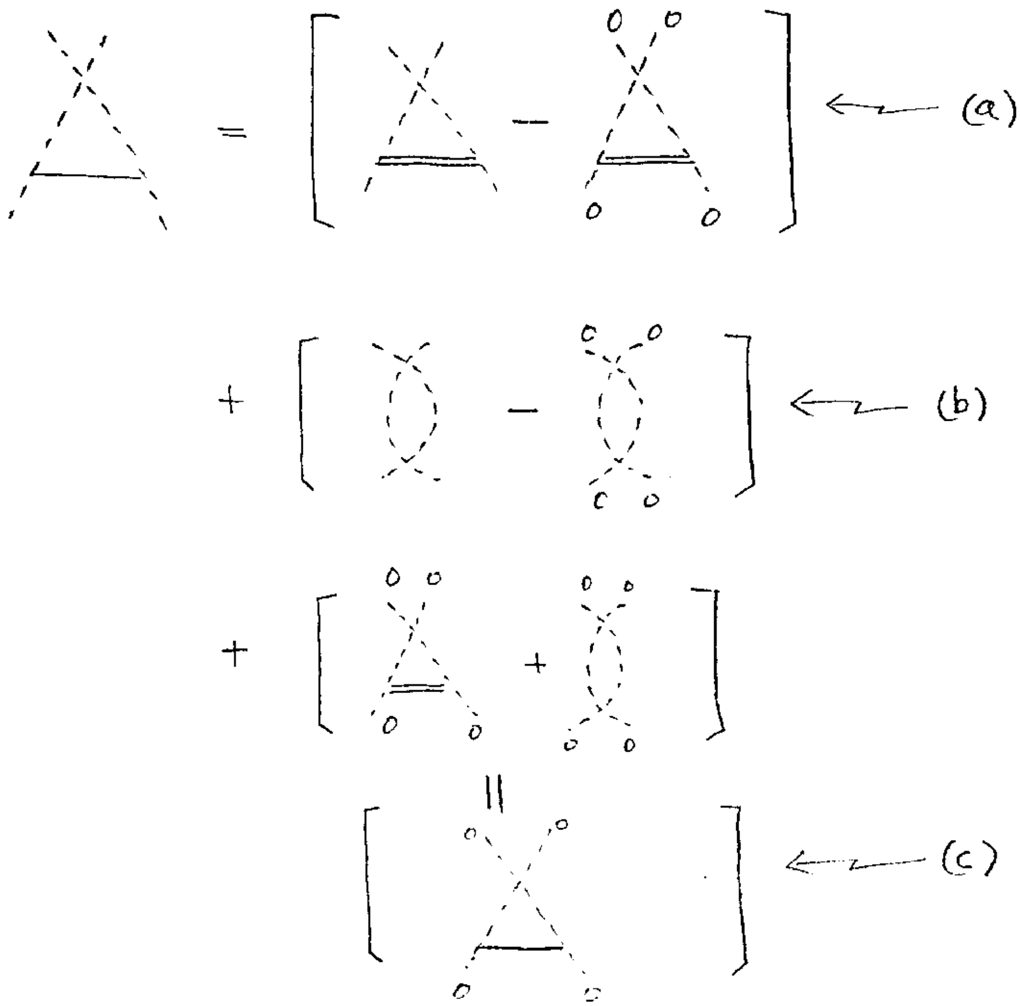


Fig. 9.

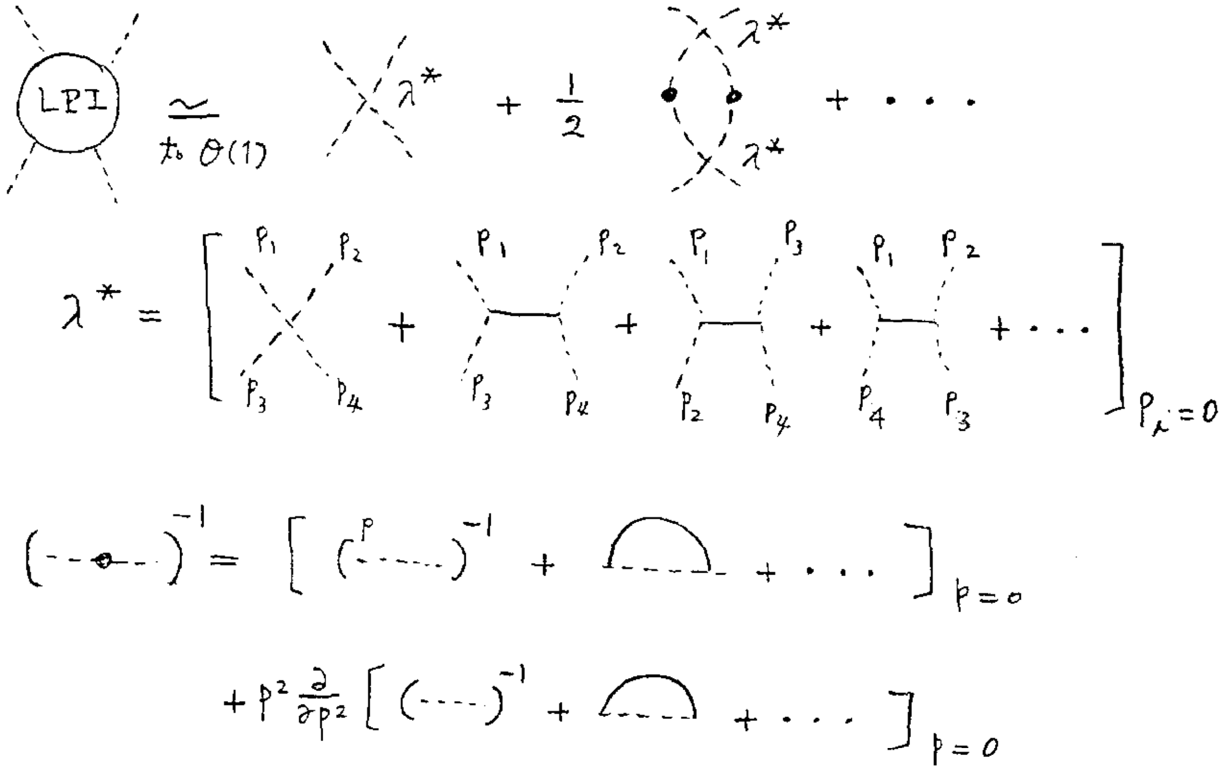


Fig. 10

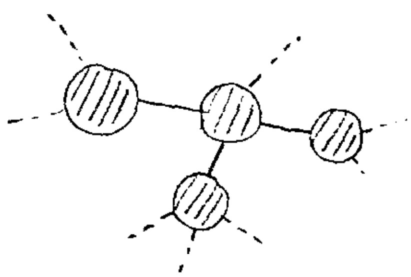


Fig. 11

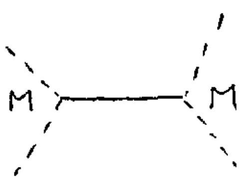
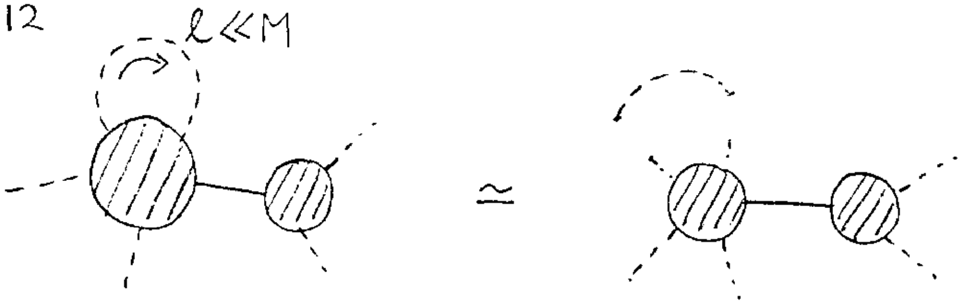
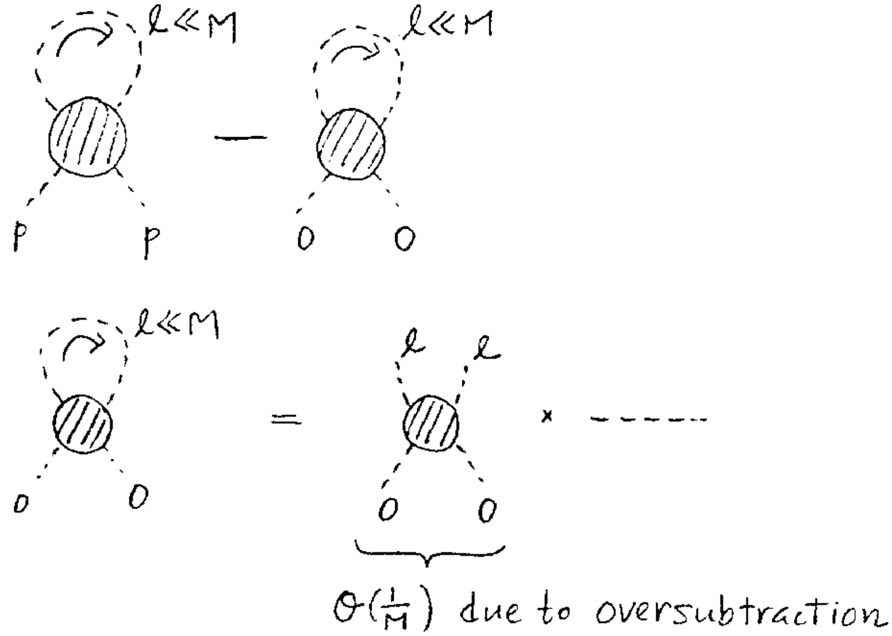


Fig. 12



$l \ll M$
$\simeq$

Fig. 13



$l \ll M$
$l \ll M$
p p
0 0
$l \ll M$
0 0
=
l l
0 0
×
$\mathcal{O}(\frac{1}{M})$ due to oversubtraction

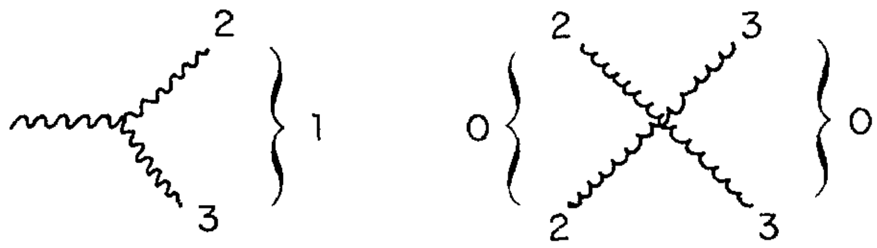


Fig. 1

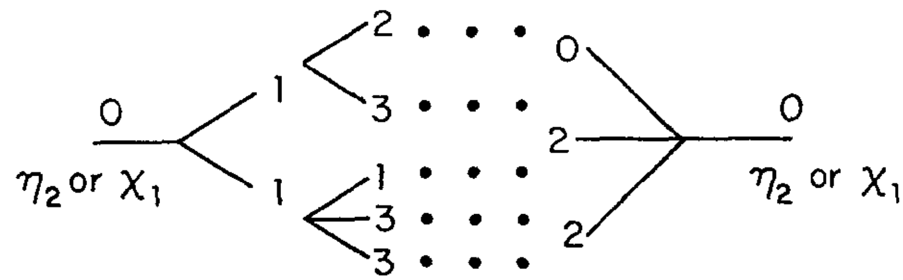
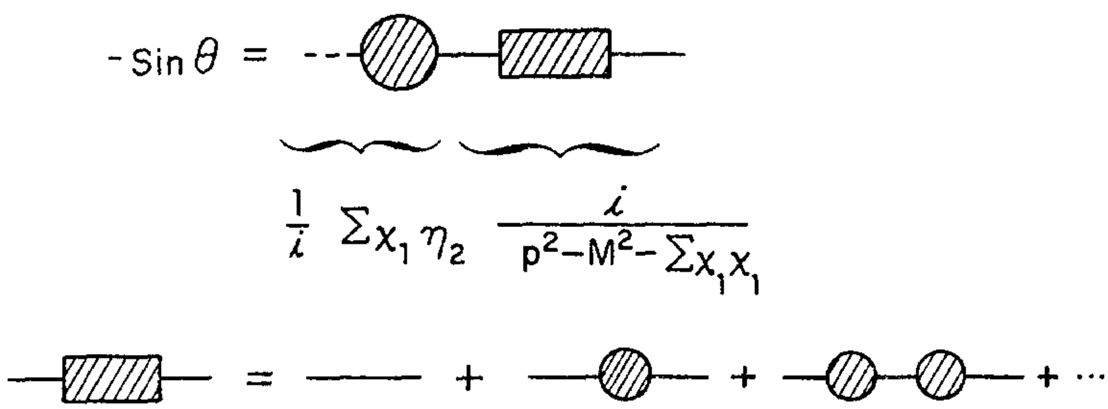


Fig. 2



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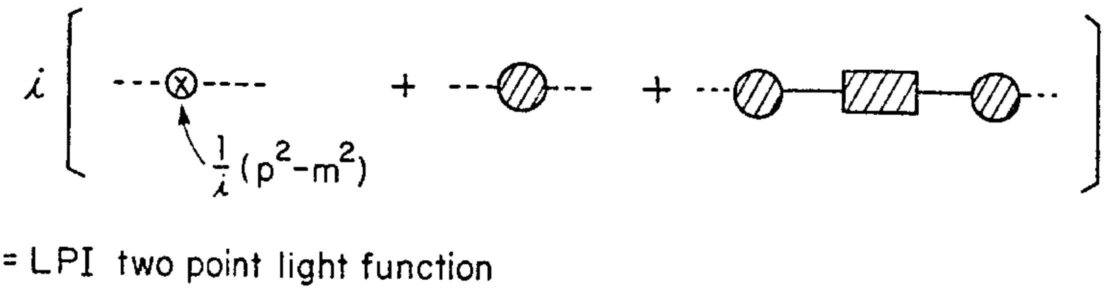
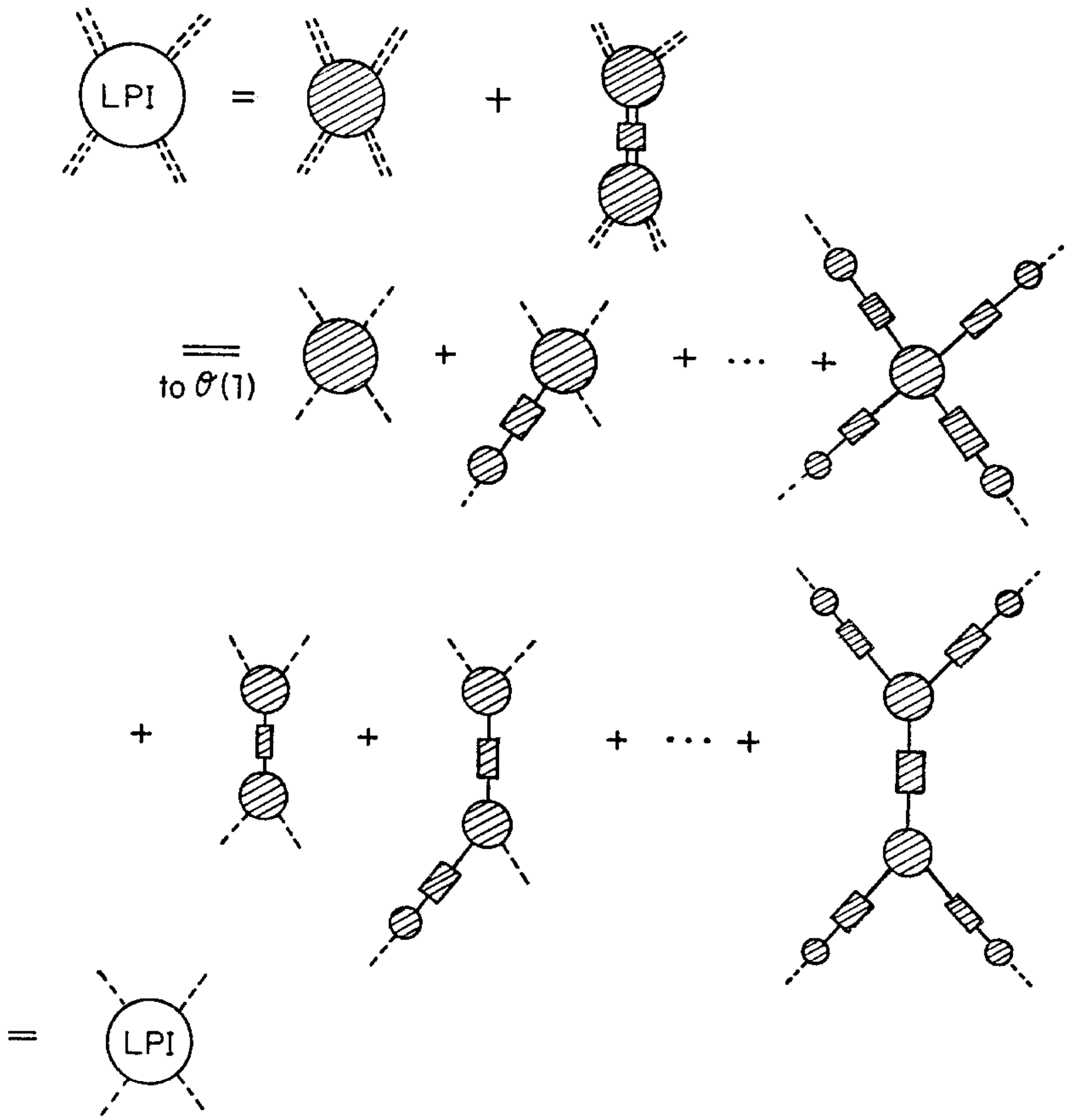


i
$\frac{1}{i}(p^2-m^2)$
= LPI two point light function

Fig. 4



LPI
to $\mathcal{O}(1)$
LPI

Fig. 5

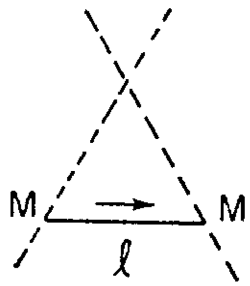


Fig. 6

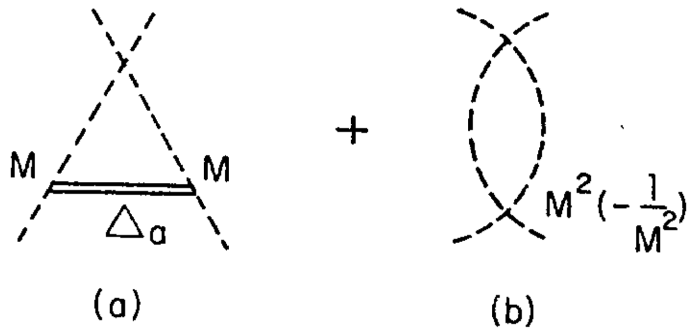


Fig. 7

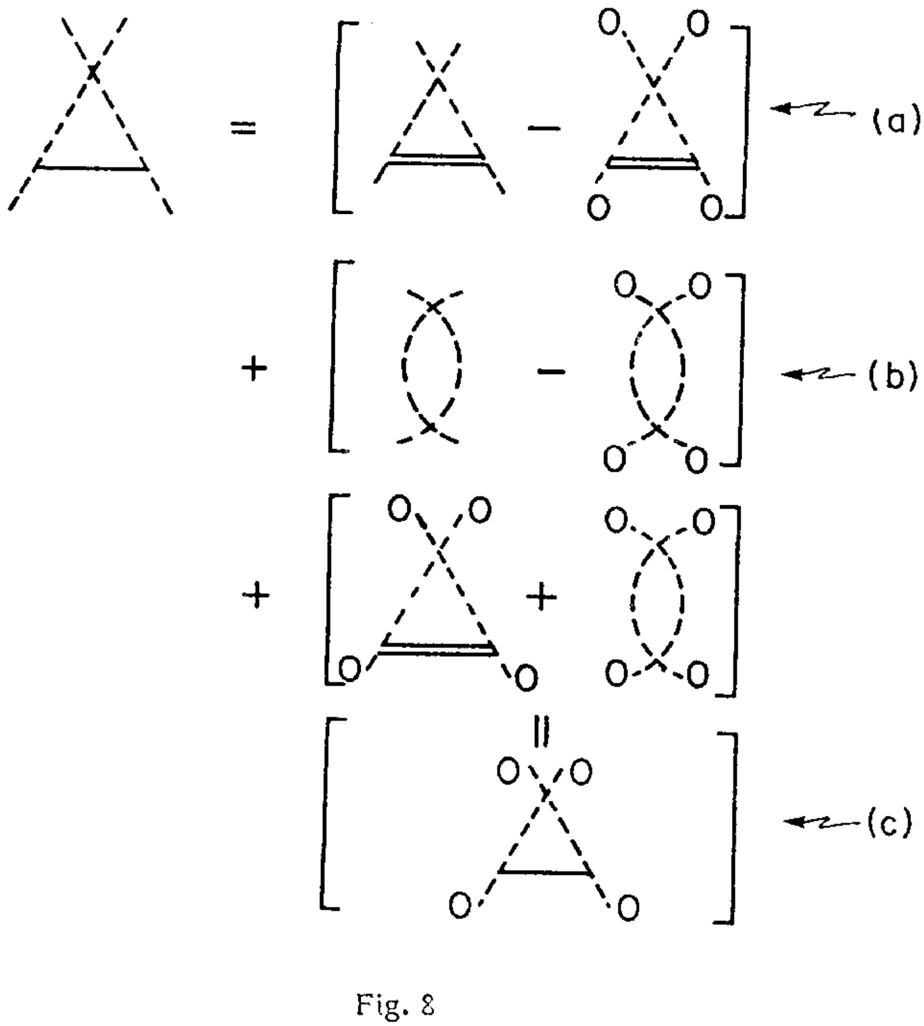


Fig. 8

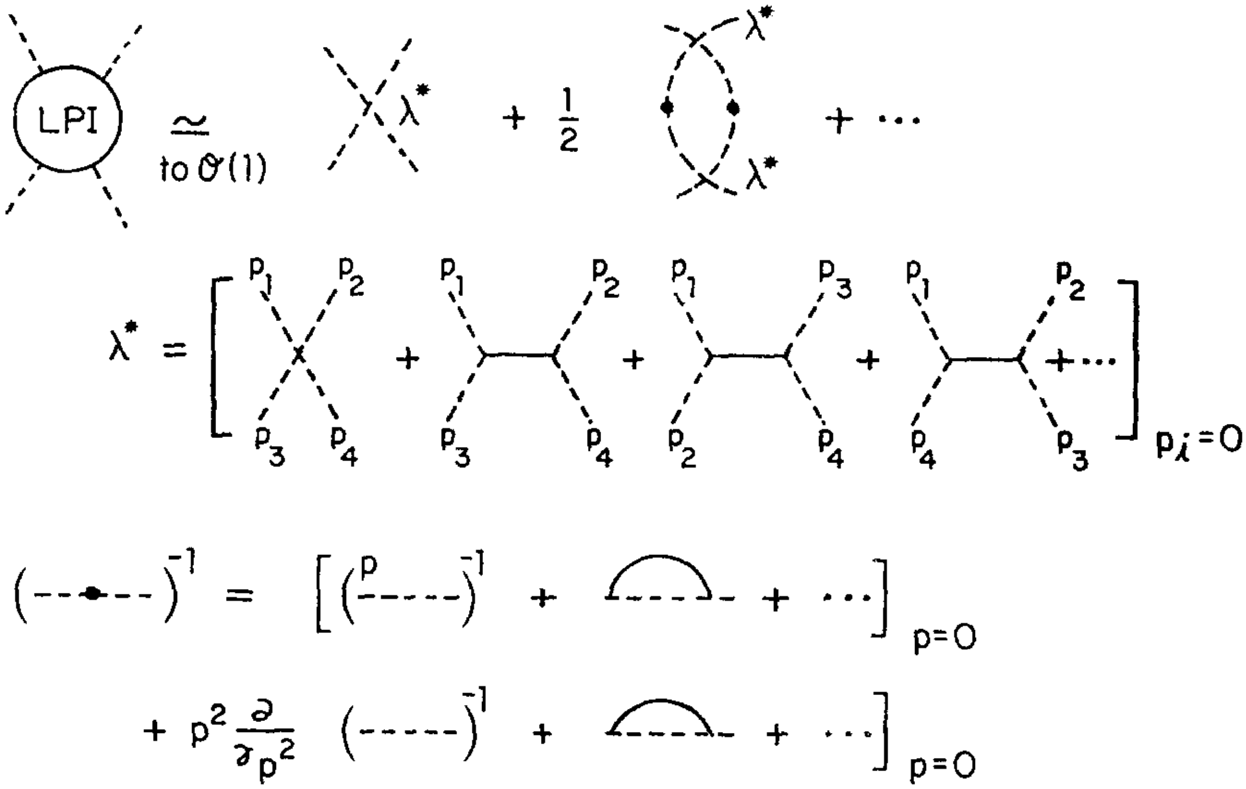


Fig. 9

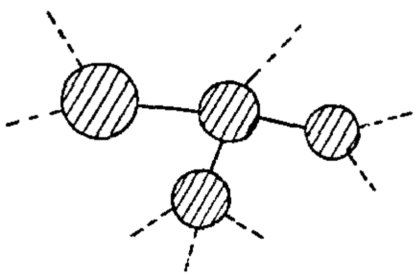


Fig. 10

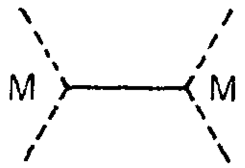


Fig. 11

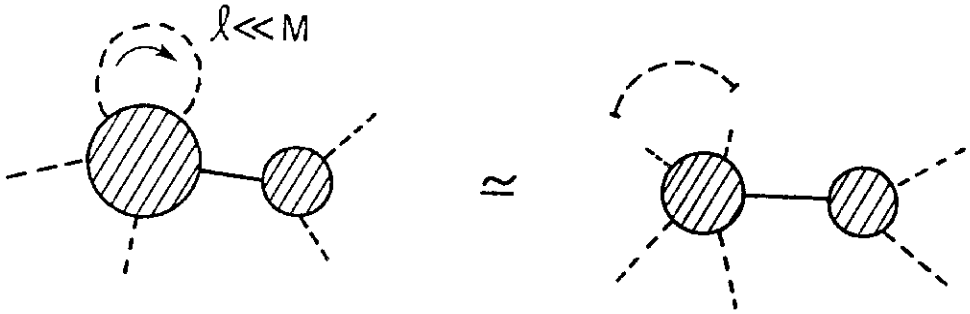


Fig. 12

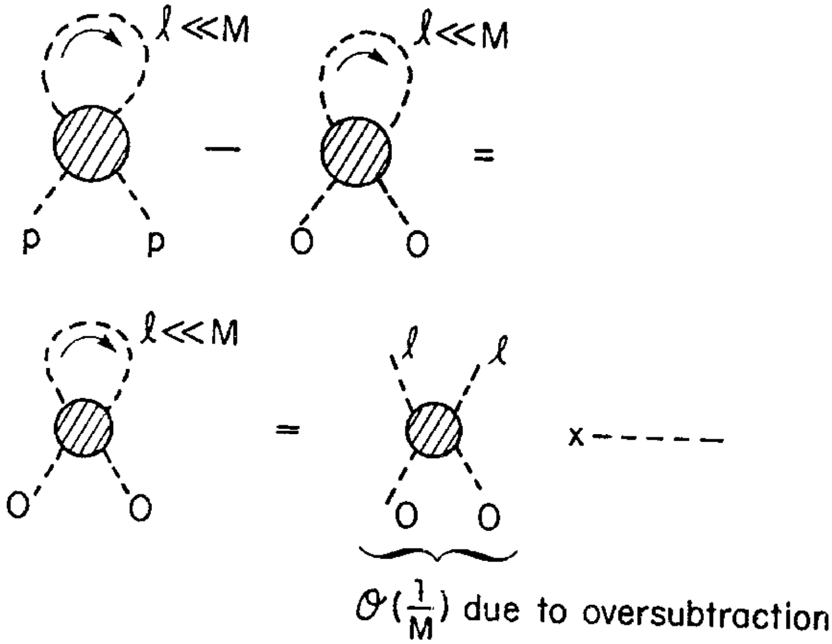


Fig. 13