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General Theory of Renormalization of Gauge Invariant Operators

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ABSTRACT

We study the question of renormalization of gauge invariant operators in the gauge theories. Our discussion applies to gauge invariant operators of arbitrary dimensions and tensor structure. We show that the gauge noninvariant (and ghost) operators that mix with a given set of gauge invariant operators form a complete set of local solutions of a functional differential equation. We show that this set of gauge noninvariant operators together with the gauge invariant operators close under renormalization to all orders. We obtain a complete set of local solutions of the differential equation. The form of these solutions has recently been conjectured by Kluberg-Stern and Zuber. With the help of our solutions, we show that there exists a basis of operators in which the gauge noninvariant operators "decouple" from the gauge invariant operators to all orders in the sense that eigenvalues corresponding to the eigenstates containing gauge invariant operators can be computed without having to compute the full renormalization metrix. We further discuss the substructure of the renormalization matrix.

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I. INTRODUCTION

The problem of operator product expansion in gauge theories[1] has been studied extensively following the initial work of Georgi and Politzer,[2] and Gross and Wilczek.[3] A salient feature of this problem, unique to gauge theories, is the possibility that the so-called Faddeev-Popov[4] ghost fields may participate in the operator product expansion of, say, two gauge invariant currents. Gross and Wilczek[3] dealt with this situation in the axial gauge, where the Faddeev-Popov ghost fields are absent (i.e., are free fields), and showed that the anomalous dimensions of gauge invariant operators are correctly given when possible couplings of the ghost fields are ignored in other gauges, at least in one-loop approximation. Subsequently, a number of authors, including Dixon and Taylor,[5] Kluberg-Stern and Zuber,[6,7] Sarkar and Strubbe,[8] have elaborated on and extended this result in some respects.

The purpose of this paper is to give a general discussion of the renormalization of gauge invariant operators of arbitrary dimension and twist, and valid to any order of perturbation theory, along the lines exploited previously by Dixon and Taylor,[5] and Kluberg-Stern and Zuber.[6,7] In the course of this discussion, we will extend their results and prove conjectures made by some of the previous workers.

We base our discussion on the proof of renormalizability of gauge theories in the form presented by one of us[9] and streamlined by

Zinn-Justin[10] by means of the Becchi-Rouet-Stora[11] (BRS) transformation. This is briefly summarized in Sec. II. The BRS transformation is a transformation of fields by an anticommuting c-number λ which leaves invariant the effective action $\mathcal{L}_{eff}$ defined in the gauge specified by f_α:

$$\delta A_i = c_\alpha D_i^\alpha \lambda ; \; D_i^\alpha = (\partial_i^\alpha + g t_{ij}^\alpha A_j) \quad ,$$

$$\delta c_\alpha = -\tfrac{1}{2} g f_{\alpha\beta\gamma} c_\beta c_\gamma \lambda \quad ,$$

$$\delta \bar{c}_\alpha = -\eta_0^{\frac{1}{2}} f_\alpha \lambda \quad , \qquad (1.1)$$

where A_i is the gauge field, c_α and $\bar{c}_\alpha$ are the Fadeev-Popov ghost fields, and g is the coupling constant.

In Sec. III, we show that only a subset of possible gauge noninvariant operators together with a set of gauge invariant operators of the same dimension and twist form a closed set under renormalization to all orders, and in Sec. IV, we give a complete characterization of the gauge noninvariant operators in this set. The problem reduces to finding a complete set of local functionals $H[\Phi, c, \partial\bar{c}, L]$ of a given dimension and ghost number which satisfy

$$\mathcal{G}' H = 0 \qquad (1.2)$$

where the differential operator $\mathcal{G}'$ is

$$\mathcal{G}' = c_\alpha D_i^\alpha \frac{\delta}{\delta A_i} + \tfrac{1}{2} f_{\alpha\beta\gamma} c_\alpha c_\beta \frac{\delta}{\delta c_\gamma}$$

(cont.)

$$- \frac{\delta \tilde{S}}{\delta A_i} \frac{\delta}{\delta (\partial \bar{c})_i} + \frac{\delta \tilde{S}}{\delta c_\alpha} \frac{\delta}{\delta L_\alpha} \quad , \tag{1.3}$$

L_α being the source of the operator $\frac{1}{2} f_{\alpha\beta\gamma} c_\beta c_\gamma$.

We solve Eq. (1.2) completely in Sec. IV, by a method suggested by Dixon and Taylor. Various mathematical lemmas necessary are proved in Appendices A and B. In all of these, the observation that

$$(\mathscr{G}')^2 = 0 \quad , \tag{1.4}$$

or more generally that the BRS transformation on A_i and c_α is nilpotent, plays a crucial role. We confirm the conjecture of Kluberg-Stern and Zuber[7] on solutions of Eq. (1.2), for arbitrary dimension and twist. Actually, the construction suggested by Dixon and Taylor does not make it clear the locality of solution,[12] but we have explicitly shown in Sec. IV and in Appendix C, the locality of solutions which is crucial to the arguments of Sec. V.

Section V is devoted to the study of the renormalization matrix of these operators. It is shown there that the renormalization matrix is in a block triangle form when the basis of operators closed under renormalization is appropriately chosen, and that, in this basis, eigenvalues of the matrix corresponding to the eigenvectors containing gauge invariant operators are computable by neglecting couplings to gauge noninvariant operators.

We do not discuss the operator product expansion of gauge invariant currents per se in this paper, leaving it to a future communication.

II. REVIEW

A. Preliminary

In this section we shall briefly review the definitions of generating functionals in a gauge theory, the BRS[11] (Becchi, Rouet, Stora) transformations and WT identities satisfied by the generating functionals. We shall use the condensed summation-integration convention as used for example in Ref. 9.

It is well-known that the Feynman rules for constructing Green's functions of a gauge theory can be deduced from the effective action $\mathcal{L}_{\text{eff}}$.

$$\mathcal{L}_{\text{eff}}[A, c, \bar{c}] = \mathcal{L}_0[A] - \tfrac{1}{2}\{f_\alpha[A]\}^2 + \bar{c}_\alpha M_{\alpha\beta} c_\beta \tag{2.1}$$

where $\mathcal{L}_0[A]$ is the Lagrangian for the Yang-Mills fields (possible interacting with matter fields, in which case A denotes collectively gauge fields and matter fields), and

$$\mathcal{L}_g[A] \equiv -\tfrac{1}{2}\{f_\alpha[A]\}^2 \tag{2.2}$$

are the gauge fixing terms and $\bar{c}_\alpha$, c_β are the Faddeev-Popov ghost fields.

The Lagrangian $\mathcal{L}_0[A]$ is invariant under the infinitesimal local gauge transformations of a compact Lie group G, which we shall choose to be a simple group solely for the sake of notational simplicity. Extention to the case of a direct product of simple compact groups is obvious.

Thus $\mathcal{L}_0[A]$ is invariant under the infinitesimal transformation,

$$A_i \to A_i = A_i + (\partial_i^\alpha + g_0 t_{ij}^\alpha A_j)\theta_\alpha$$
$$\equiv A_i + D_i^\alpha[A]\theta_\alpha \tag{2.3}$$

A_i's are hermitian fields so that the matrices $\{t^\alpha : (t^\alpha)_{ij} = t_{ij}^\alpha\}$ are real antisymmetric representation of the generators of G. g_0 is the (unrenormalized) coupling constant of the group G.

We shall work in linear gauges defined by,

$$f_\alpha[A] = \eta_0^{\frac{1}{2}} \partial_i^\alpha A_i \tag{2.4}$$

where η_0 is an arbitrary real positive number.

Then $M_{\alpha\beta}[A]$ of Eq. (2.1) is defined by

$$M_{\alpha\beta}[A] = \eta_0^{-\frac{1}{2}} \frac{\delta f_\alpha[A]}{\delta A_i} D_i^\beta[A] \quad . \tag{2.5}$$

The BRS supertransformations consist of

$$\delta A_i = c_\alpha D_i^\alpha \delta\lambda$$
$$\delta c_\alpha = -\tfrac{1}{2} g_0 f_{\alpha\beta\gamma} c_\beta c_\gamma \delta\lambda$$
$$\delta \bar{c}_\alpha = -\eta_0^{\frac{1}{2}} f_\alpha[A]\delta\lambda \tag{2.6}$$

where $\delta\lambda$ is an x-independent infinitesimal anticommuting c-number. We note that under the BRS transformations

$$\delta(\mathcal{L}_0) = 0 \;;\; \delta(c_\alpha D_i^\alpha) = 0 \;;\; \delta(-\tfrac{1}{2} f_{\alpha\beta\gamma} c_\beta c_\gamma) = 0 \;;$$

$$\delta(-\tfrac{1}{2} f_\alpha^2[A] + \bar{c}Mc) = 0 \;;$$

$$\delta(\mathcal{L}_{eff}[A, c, \bar{c}]) = 0 \quad . \tag{2.7}$$

We shall also find it useful to consider only the following transformations:

$$\delta A_i = c_\alpha D_i^\alpha \delta\lambda \quad ;$$

$$\delta \mathbf{c}_\alpha = -\tfrac{1}{2} g_0 f_{\alpha\beta\gamma} c_\beta c_\gamma \delta\lambda , \quad \delta \bar{c} = 0 \quad . \tag{2.8}$$

Under these transformations $(\mathcal{L}_0[A] + \bar{c}Mc)$ is invariant. This is expressed as

$$\mathcal{G}_0 (\mathcal{L}_0[A] + \bar{c}Mc) = 0 \tag{2.9}$$

where $\mathcal{G}_0$ is the differential operator defined by

$$\delta\{F[A,\bar{c},c]\} = \mathcal{G}_0 F[A,c,\bar{c}]\delta\lambda \tag{2.10a}$$

where δF refers to the change in the functional $F[A,c,\bar{c}]$ under the transformations of Eq. (2.8). If $F[A,c,\bar{c}]$ is a functional containing an equal number of $c, \bar{c}$ with all $\bar{c}$'s appearing before all c's , $\mathcal{G}_0$ may be expressed as

$$\mathcal{G}_0 \equiv c_\alpha D_i^\alpha \frac{\delta}{\delta A_i} + \tfrac{1}{2} g_0 f_{\alpha\beta\gamma} c_\beta c_\gamma \frac{\delta}{\delta c_\alpha} \quad . \tag{2.10b}$$

In terms of $\mathcal{G}_0$, the group condition on $D_i^\alpha[A]$ viz.,

$$\left[D_i^\alpha \frac{\delta}{\delta \Phi_i} , D_i^\beta \frac{\delta}{\delta \Phi_i} \right] = g_0 f^{\alpha\beta\gamma} D_i^\gamma \frac{\delta}{\delta \Phi_i} \tag{2.11a}$$

i.e.,

$$t_{ij}^\alpha D_j^\beta - t_{ij}^\beta D_j^\alpha = f^{\alpha\beta\gamma} D_i^\gamma \tag{2.11b}$$

can be expressed elegantly as the operator identity:

$$\mathscr{G}_0^2 \equiv 0 \quad . \tag{2.12}$$

B. Generating Functionals of a Gauge Theory; WT Identities

In the following, we shall be dealing with unrenormalized but dimensionally regularized quantities. Following Zinn-Justin, we introduce sources for the composite operators $c_\alpha D_i^\alpha$ and $\frac{1}{2} g_0 f_{\alpha\beta\gamma} c_\beta c_\gamma$ and define:

$$S[A, c, \bar{c}, \kappa, \ell] = \mathscr{L}_{eff}[A, c, \bar{c}] + \kappa_i D_i^\alpha c_\alpha - \tfrac{1}{2} g_0 f_{\alpha\beta\gamma} c_\beta c_\gamma \ell_\alpha \ . \tag{2.13}$$

We define the generating functional of Green's functions,

$$W[j, \xi, \bar{\xi}, \kappa, \ell] = \int [dA dc d\bar{c}] \ \mathrm{Exp}\ i\{ S[A, c, \bar{c}, \kappa, \ell] + j_i A_i + \bar{\xi}_\alpha c_\alpha + \bar{c}_\alpha \xi_\alpha \} \tag{2.14}$$

where ξ, $\bar{\xi}$ are anticommuting sources for the ghost fields. The generating functional of the connected Green's functions is defined by,

$$Z[j, \xi, \bar{\xi}, \kappa, \ell] = -i \ln W[j, \xi, \bar{\xi}, \kappa, \ell] \ . \tag{2.15}$$

We define expectation values of fields in presence of sources:

$$\phi_i[j, \xi, \bar{\xi}, \kappa, \ell] = \frac{\delta Z}{\delta j_i} \ , \quad \Omega_\alpha = \frac{\delta Z}{\delta \bar{\xi}_\alpha} \ ; \ \bar{\Omega}_\alpha = -\frac{\delta Z}{\delta \xi_\alpha} \tag{2.16}$$

where it is understood that the partial derivatives are taken with the rest of the sources fixed.

The generating functional of the proper vertices is given by

$$\Gamma[\phi, \Omega, \bar{\Omega}, \kappa, \ell] = Z[j, \xi, \bar{\xi}, \kappa, \ell] - j_i A_i - \bar{\xi}_\alpha c_\alpha - \bar{c}_\alpha \xi_\alpha \quad . \tag{2.17}$$

The sources j, ξ, $\bar{\xi}$ can be expressed as a functional of $\phi, \Omega, \bar{\Omega}, \kappa, \ell$ by the relations:

$$j_i = -\frac{\delta \Gamma}{\delta \phi_i} \qquad \xi_\alpha = -\frac{\delta \Gamma}{\delta \Omega_\alpha} \qquad \bar{\xi}_\alpha = \frac{\delta \Gamma}{\delta \Omega_\alpha} \tag{2.18}$$

where as a matter of convention partial derivatives of Γ such as $\frac{\delta \Gamma}{\delta \phi_i}$ stand for

$$\left.\frac{\delta \Gamma}{\delta \phi_i}\right|_{\phi', \Omega, \bar{\Omega}, \kappa, \ell} \tag{2.19}$$

With this convention, we also note,

$$\frac{\delta \Gamma}{\delta \kappa_i} = \frac{\delta Z}{\delta \kappa_i} \qquad \frac{\delta \Gamma}{\delta \ell_\alpha} = \frac{\delta Z}{\delta \ell_\alpha} \quad . \tag{2.20}$$

By consideration of an arbitrary change $\delta \bar{c}_\alpha$ in the $W[j, \xi, \bar{\xi}, \kappa, \ell]$ of Eq. (2.14), we can obtain the equation of motion for the antighost field. It is expressed in terms of Γ as

$$\frac{\delta \Gamma}{\delta \bar{\Omega}_\alpha} = \partial_i^\alpha \frac{\delta \Gamma}{\delta \kappa_i} \quad . \tag{2.21}$$

By considering the BRS transformations of Eq. (2.6) in the integration variables of W of Eq. (2.14), one can obtain the WT identity for W. It can be transformed into WT identity for Γ and simplified using Eq. (2.21). The final result is

$$\frac{\delta \Gamma_0}{\delta \phi_i} \frac{\delta \Gamma_0}{\delta \kappa_i} - \frac{\delta \Gamma_0}{\delta \Omega_\alpha} \frac{\delta \Gamma_0}{\delta \ell_\alpha} = 0 \tag{2.22}$$

with

$$\Gamma_0[\phi, \Omega, \bar{\Omega}, \kappa, \ell] = \Gamma[\phi, \Omega, \bar{\Omega}, \kappa, \ell] + \tfrac{1}{2}\{f_\alpha[\phi]\}^2 \tag{2.23}$$

The theory is made finite by wave-function, and coupling constant renormalizations:

$$A_i = A_i^{(r)} Z^{\frac{1}{2}} , \qquad c_\alpha = \tilde{Z}^{\frac{1}{2}} c_\alpha^{(r)} , \qquad \bar{c}_\alpha = \tilde{Z}^{\frac{1}{2}} \bar{c}_\alpha^{(r)} ,$$

$$g_0 = gX\tilde{Z}^{-1} Z^{-\frac{1}{2}} ,$$

$$\kappa_i = \tilde{Z}^{\frac{1}{2}} K_i , \qquad \ell_\alpha = Z^{\frac{1}{2}} L_\alpha \tag{2.24}$$

Alternatively, we can write the renormalized effective action as

$$S^{(r)}[A^{(r)}, c^{(r)}, \bar{c}^{(r)}, K, L] = S[A, c, \bar{c}, \kappa, \ell]$$

$$= S[A^{(r)}, c^{(r)}, \bar{c}^{(r)}, K, L] + \Delta S[A^{(r)}, c^{(r)}, \bar{c}^{(r)}, K, L] , \tag{2.25}$$

where ΔS represents local counterterms. It is a well-known result that Z, $\tilde{Z}$ and X in Eq. (2.24) can be chosen in successive loop approximations so that the resulting renormalized Γ is finite in terms of g and renormalized quantities.

III.

A. Generating Functionals With An Insertion of a Local Operator

In order to discuss renormalization of an operator one needs to relate two quantities: Green's functions, involving an insertion of the operator which are independent of the renormalization point -- the so-called unrenormalized Green's functions and the finite (i.e., renormalized) Green's functions which however depend on the renormalization point. In the following we shall give the definitions for the generating functionals for these two Green's functions.

We consider a local (though it is not necessary for these definitions) operator $O_i[A, c, \bar{c}]$, which may carry additional Lorentz indices. Let N_i denotes (in general, x-dependent) source for $O_i[A, c, \bar{c}]$. The generating functional of the unrenormalized Green's functions with an arbitrary number of insertions of O_i is given by

$$W[j, \xi, \bar{\xi}, \kappa, \ell, N] = \int [dAdcd\bar{c}] \exp i \{S[A, c, \bar{c}, \kappa, \ell] + j_i A_i + \bar{c}_\alpha \xi_\alpha + \bar{\xi}_\alpha c_\alpha + N_i O_i [A, c, \bar{c}]\} \tag{3.1}$$

whose derivatives with respect to various sources are independent of the renormalization point for obvious reasons. The corresponding generating functionals of connected Green's functions and of proper vertices are defined analogous to Eqs. (2.15) - (2.18) of the last section viz.

$$Z[j, \xi, \bar{\xi}, \kappa, \ell, N] = -i \ln W[j, \xi, \bar{\xi}, \kappa, \ell, N] \quad , \tag{3.2a}$$

$$\Gamma[\phi, \Omega, \bar{\Omega}, \kappa, \ell, N] = Z[j, \xi, \bar{\xi}, \kappa, \ell, N] - j_i\phi_i - \bar{\Omega}_\alpha \xi_\alpha - \bar{\xi}_\alpha \Omega_\alpha . \tag{3.2b}$$

We further note that as in Eq. (2.20)

$$\left.\frac{\delta\Gamma}{\delta N_i}\right|_{\phi, \Omega, \bar{\Omega}, \kappa, \ell} = \left.\frac{\delta Z}{\delta N_i}\right|_{j \xi, \bar{\xi}, \kappa, \ell} . \tag{3.3}$$

The (Fourier transform of the above quantity $\left\{\frac{\delta\Gamma}{\delta N_i}\right\}_{N=0=K=L}$ generates the unrenormalized proper vertices with a single insertion of O_i at an arbitrary momentum. Insertions at zero momentum can be obtained by considering $\int d^4x \left(\frac{\delta\Gamma}{\delta N(x)}\right)_{N=K=L=0}$.

To renormalize

$$\Gamma_N[\phi, \Omega, \bar{\Omega}] = \left.\frac{\delta\Gamma}{\delta N_i}\right|_{N=K=L=0} \tag{3.4}$$

in one loop approximation, one needs to compute $\{\Gamma_N\}_1^{\text{div}}$ expressed in terms of <u>renormalized</u> field $\Phi, \omega, \bar{\omega}$ and <u>renormalized</u> parameters: [renormalized according to some convention] and define

$$\left\{\Gamma_N^R[\Phi, \omega, \bar{\omega}]\right\}_1 = \left\{\Gamma_N[\phi, \Omega, \bar{\Omega}]\right\}_1 - \left\{\Gamma_N[Z^{\frac{1}{2}}\Phi, \tilde{Z}^{\frac{1}{2}}\omega, \tilde{Z}^{\frac{1}{2}}\bar{\omega}]\right\}_1^{\text{div}}, \tag{3.4}$$

$$\Phi = Z^{-\frac{1}{2}}\phi, \quad \omega = \tilde{Z}^{-\frac{1}{2}}\Omega, \quad \bar{\omega} = \tilde{Z}^{-\frac{1}{2}}\Omega , \tag{3.4a}$$

so that Γ_N^R is finite in 4 space-time dimensions. Γ_N as a function of renormalized fields and parameters can be best obtained from another generating functional

$$W^{(r)}[J,\zeta,\bar{\zeta},K,L,N] = \int [dAdcd\bar{c}] \exp i\Big\{ S^{(r)}[A,c,\bar{c},K,L] + J_i A_i$$
$$+ \bar{\zeta}_\alpha c_\alpha + \bar{c}_\alpha \bar{\zeta}_\alpha + N_i O_i [Z^{\frac{1}{2}} A_i, \tilde{Z}^{\frac{1}{2}} c_\alpha, \tilde{Z}^{\frac{1}{2}} \bar{c}_\alpha, g\frac{X}{\tilde{Z}Z^{\frac{1}{2}}}, \eta Z^{-\frac{1}{2}}]\Big\} \quad . \tag{3.5}$$

Here, the sources $J, \zeta, \bar{\zeta}, K, L$ stand for the renormalized sources. This $W^{(r)}$ is in fact equal to $W^{(u\cdot r)}$ of Eq. (3.1). This can be seen by performing the transformations

$$j = Z^{-\frac{1}{2}} J, \qquad \xi = \tilde{Z}^{-\frac{1}{2}} \zeta, \qquad \bar{\xi} = \tilde{Z}^{-\frac{1}{2}} \zeta, \qquad \kappa = \tilde{Z}^{\frac{1}{2}} K, \qquad \ell = Z^{\frac{1}{2}} L$$

$$g_0 = gX/\tilde{Z}Z^{\frac{1}{2}}, \qquad \eta_0 = \eta Z^{-\frac{1}{2}}$$

with a simultaneous change of integration variables

$$A_i = Z^{\frac{1}{2}} A^{(r)}, \qquad c_\alpha = \tilde{Z}^{\frac{1}{2}} c_\alpha^{(r)}, \qquad \tilde{c}_\alpha = \tilde{Z}^{\frac{1}{2}} \tilde{c}_\alpha^{(r)}$$

and dropping the overall infinite constant and superscript (r).

[We shall find it convenient to switch back and forth between $W^{(u\cdot r.)}$ and $W^{(r)}$. $W^{(u\cdot r.)}$ is especially useful to see the symmetry properties of W while it is $W^{(r)}$ which can give us $\{\Gamma_N\}^{div}$ as a functional of renormalizaed fields and parameters .]

B. WT Identity For the Generating Functional of Proper Vertices With A Single Insertion of a Gauge Invariant Operator

Our object here is to write down the WT identity satisfied by $\Gamma_N[\Phi, \omega, \bar{\omega}]$ in one loop approximation, when N refers to the source of a gauge invariant operator and use it to find the properties of gauge noninvariant operators which enter in $\{\Gamma_N[\Phi, \omega, \bar{\omega}]\}_1^{div}$.

To do this, we consider the BRS transformations of Eq. (2.6) on the integration variables of $W[j,\xi,\bar{\xi},\kappa,\ell,N]$ of Eq. (3.1), with $O_i[A,c,\bar{c}] \equiv O^{GI}[A]$. We note that the extra term $NO^{GI}[A]$ is invariant under these transformations. Therefore the WT identity satisfied by $W[J,\zeta,\bar{\zeta},K,L,N]$ is [i.e., expressed in terms of renormalized sources] and by $\Gamma[\Phi,\omega,\bar{\omega},K,L,N]$ will be identical in form as the WT identities for these generating functionals at $N=0$. Thus,

$$\frac{\delta\Gamma_0}{\delta\omega_\alpha} = \partial_i^\alpha \frac{\delta\Gamma_0}{\delta K_i} \tag{3.6}$$

and

$$\frac{\delta\Gamma_0}{\delta\Phi_i}\frac{\delta\Gamma_0}{\delta K_i} - \frac{\delta\Gamma_0}{\delta\omega_\alpha}\frac{\delta\Gamma_0}{\delta L_\alpha} = 0 . \tag{3.7}$$

We differentiate Eqs. (3.8) and (3.7) with respect to N and set $N=0$. We thus obtain,

$$\frac{\delta\Gamma_N}{\delta\omega_\alpha} = \partial_i^\alpha \frac{\delta\Gamma_N}{\delta K_i} \tag{3.8}$$

$$\frac{\delta\Gamma_N}{\delta\Phi_i}\frac{\delta\Gamma_0}{\delta K_i} + \frac{\delta\Gamma_N}{\delta K_i}\frac{\delta\Gamma_0}{\delta\Phi_i} - \frac{\delta\Gamma_N}{\delta L_\alpha}\frac{\delta\Gamma_0}{\delta\omega_\alpha} - \frac{\delta\Gamma_N}{\delta\omega_\alpha}\frac{\delta\Gamma_0}{\delta L_\alpha} = 0, \tag{3.9}$$

with

$$\Gamma_N \equiv \Gamma_N[\Phi,\omega,\bar{\omega},K,L] = \left.\frac{\delta\Gamma_0[\Phi,\omega,\bar{\omega},K,L,N]}{\delta N}\right|_{N=0} . \tag{3.10}$$

Equating the one-loop divergence on both sides of the Eqs. (3.8) and (3.9), remembering that $\Gamma_0\big|_{N=0}$ is a finite functional, we learn,

$$\frac{\delta}{\delta \omega_\alpha} \{\Gamma_N\}_1^{div} = \partial_i^\alpha \frac{\delta}{\delta K_i} \{\Gamma_N\}_1^{div} \tag{3.11}$$

$$[\omega_\alpha D_i^\alpha \frac{\delta}{\delta \Phi_i} + \frac{1}{2} g f_{\alpha\beta\gamma} \omega_\beta \omega_\gamma \frac{\delta}{\delta \omega_\alpha}] \{\Gamma_N\}_1^{div} = - \frac{\delta \tilde{S}}{\delta \Phi_i} \frac{\delta}{\delta K_i} \{\Gamma_N\}_1^{div}$$

$$+ \frac{\delta \tilde{S}}{\delta \omega_\alpha} \frac{\delta}{\delta L_\alpha} \{\Gamma_N\}_1^{div} , \tag{3.12}$$

with

$$\tilde{S}[\Phi, \omega, \bar{\omega}, K, L] = \mathscr{L}_0[\Phi] + \bar{\omega}_\alpha M_{\alpha\beta}[\Phi] \omega_\beta + K_i D_i^\alpha \omega_\alpha - \tfrac{1}{2} L_\alpha f_{\alpha\beta\gamma} \omega_\beta \omega_\gamma \tag{3.13}$$

$\{\Gamma_N[\Phi, \omega, \bar{\omega}, K, L]\}_1^{div}$ is a local polynomial of Φ, ω, $\bar{\omega}$, K, L of dimensions equal to that of $O^{GI}[\Phi]$. Thus $\{\Gamma_N[\Phi, \omega, \bar{\omega}, K, L]\}_1^{div}$ can be expressed in terms of a complete set of local polynomial solutions of the equations:

$$\frac{\delta}{\delta \bar{\omega}_\alpha} O[\Phi, \omega, \bar{\omega}, K, L] = \partial_i^\alpha \frac{\delta}{\delta K_i} O[\Phi, \omega, \bar{\omega}, K, L] \tag{3.14}$$

$$\left[\mathscr{G}_0 - \frac{\delta \tilde{S}}{\delta \omega_\alpha} \frac{\delta}{\delta L_\alpha} + \frac{\delta \tilde{S}}{\delta \Phi_i} \frac{\delta}{\delta K_i}\right] O[\Phi, \omega, \bar{\omega}, K, L] = 0. \tag{3.15}$$

Thus $\{\Gamma_N[\Phi, \omega, \bar{\omega}]\}_1^{div} = \{\Gamma_N[\Phi, \omega, \bar{\omega}, K, L]\Big|_{K=L=0}\}_1^{div}$ can be expressed in terms of the complete set of local solutions of the equation

$$\frac{\delta O[\Phi, \omega, \bar{\omega}]}{\delta \bar{\omega}_\alpha} = \partial_i^\alpha Q_i[\Phi, \omega, \bar{\omega}] \tag{3.16}$$

and

$$\mathscr{G}_0 O[\Phi, \omega, \bar{\omega}] = -Q_i[\Phi, \omega, \bar{\omega}] \frac{\delta \tilde{S}}{\delta \Phi_i} + R_\alpha[\Phi, \omega, \bar{\omega}] \frac{\delta \tilde{S}}{\delta \omega_\alpha} \tag{3.17}$$

where $Q_i[\Phi, \omega, \omega]$ and $R_\alpha[\Phi, \omega, \omega]$ are local functionals of Φ, ω, ω possessing appropriate transformation properties under the global transformations of group G and Lorentz transformations. [As we shall see later the integrability of Eqs. (3.16), (3.17) will restrict the set $\{ Q_i \}$ and $\{ R_\alpha \}$.]

We shall obtain the complete set of local solutions of the Eqs. (3.14), (3.15) in the next section. In this section, we wish to show that the set of operators $\{ O^{GI}[\Phi] \} \oplus \{ O[\Phi, \omega, \bar{\omega}] \}$ [the latter being the set of all gauge non-invariant solutions of Eqs. (3.16), (3.17)] ; closes under renormalization to all orders. For this purpose, it is not necessary to know the set $\{ O[\Phi, \omega, \bar{\omega}] \}$ explicitly.

C. Closure Property in One Loop Approximation:

We consider the generating functional of Green's functions W of Eq. (3.5) with $O_i[A, c, \bar{c}]$ being any one of the gauge noninvariant operators of the set. Our objective is to show that $\{ \Gamma_N[\Phi, \omega, \bar{\omega}] \}_1^{div}$ thus computed for any of the gauge noninvariant operators can be expanded in terms of the same set $\{ O^{GI}[\Phi] \} \oplus \{ O[\Phi, \omega, \bar{\omega}] \}$.

We shall find it convenient to use $W^{(u.r.)}$ [in terms of unrenormalized sources] to exhibit its symmetry properties:

$$W^{(u.r.)} = \int [dAdcd\bar{c}] \exp i \Big\{ \mathscr{L}_0[A] + \bar{c}Mc - \frac{1}{2} f_\rho^2 [A] + \kappa_i' D_i^{\alpha'} c_\alpha$$

$$- \frac{1}{2} \ell_\alpha' f_{\alpha\beta\gamma} c_\beta c_\gamma + N' O[A, c, \bar{c}] + j_i A_i + \bar{c}_\alpha \xi_\alpha + \bar{\xi}_\alpha c_\alpha \Big\} . \qquad (3.18)$$

We note that $\{\mathscr{L}_0 + \bar{c}Mc - \frac{1}{2} f_\rho^2 [A] + N'O[A,c,\bar{c}]\}$ is invariant under the transformations:

$$\delta A_i = (D_i^\alpha c_\alpha + N'Q_i)\delta\lambda \equiv c_\alpha D_i'^\alpha \delta\lambda$$

$$\delta c_\alpha = \left(-\frac{1}{2} g_0 f_{\alpha\beta\gamma} c_\beta c_\gamma + N' R_\alpha\right) \delta\lambda \equiv -g_0 \frac{1}{2} f'_{\alpha\beta\gamma} c_\beta c_\gamma$$

$$\delta\bar{c}_\alpha = -\eta_0^{1/2} f_\alpha [A] \delta\lambda \quad , \tag{3.19}$$

to the 1st order in N'. Here Q_i and R_α are the functions corresponding to $O[A,c,\bar{c}]$ that enter Eqs. (3.16), (3.17). To see this, we note

$$\delta\{\mathscr{L}_{eff}[A,c,\bar{c}] + N'O[A,c,\bar{c}]\} = 0 + O(N') + O(N'^2) + \ldots$$

and $O(N')$ terms are given by,

$$N'\left\{+\frac{\delta\mathscr{L}_{eff}}{\delta A_i} Q_i - \frac{\delta\mathscr{L}_{eff}}{\delta c_\alpha} R_\alpha\right\}\delta\lambda$$

$$+ N'\left\{\mathscr{G}_0 O + \frac{\delta O}{\delta\bar{c}_\alpha} \eta_0^{1/2} f_\alpha\right\}\delta\lambda \quad , \tag{3.20}$$

$$= N'\left\{+\frac{\delta\tilde{S}}{\delta A_i} Q_i - \frac{\delta\tilde{S}}{\delta c_\alpha} R_\alpha + \mathscr{G}_0 O\right\}\delta\lambda + N'\left\{-\eta_0^{1/2} f_\alpha \partial_i^\alpha Q_i\right.$$

$$\left. + \eta_0^{1/2} f_\alpha \frac{\delta O}{\delta\bar{c}_\alpha}\right\}$$

$$= 0 \tag{3.21}$$

on account of Eqs. (3.16) and (3.17). We further note that

$$\delta\left(D_i^{\prime\alpha} c_\alpha\right) = O(N') \neq 0$$

$$\delta\left(\frac{1}{2} f'_{\alpha\beta\gamma} c_\beta c_\gamma\right) = O(N') \neq 0 \ . \tag{3.22}$$

We use the invariance of Eq. (3.21) under transformations of Eq. (3.19) to write down a WT identity of $W^{(u.r)}$ of Eq. (3.18). We note that the change in the Jacobian for the transformations of Eq. (3.19) is

$$\propto (Q_{i,i} - R_{\alpha,\alpha}) \propto \delta^4(x) \text{ or its dirivatives at } x \equiv 0 ,$$

since Q_i and R_α are local functionals. We are using dimensional regularization, so that the Jacobian can be taken to be unity. We thus write the WT identity for $W^{u.r.}$ [See Eq. (3.18)].

$$0 = \int [dA dc d\bar{c}] \left\{ j_i D_i^{\prime\alpha} c_\alpha - \frac{1}{2} g_0 \bar{\xi}_\alpha f_{\alpha\beta\gamma} c_\beta c_\gamma + \eta_0^{1/2} f_\alpha [A] \xi_\alpha \right.$$
$$\left. + \text{terms of } O(\kappa' N', \ell' N', N'^2 \ldots) \right\}$$
$$\times \operatorname{Exp} i \ \{\ldots\ldots\} \ . \tag{3.23}$$

We note that the WT identity of Eq. (3.23) is the same as the WT identity for the insertion of a gauge invariant operator, except for terms of $O(\kappa' N', \ell' N', N'^2, \ldots)$. Further by transformation of the antighost field $\bar{c}_\alpha \to \bar{c}_\alpha + \delta \bar{c}_\alpha$ [where $\delta \bar{c}_\alpha$ are independent of fields], we obtain the equation of motion for the antighost field.

$$0 = \int [dA\,dc\,d\bar{c}] \left\{ \partial_i^{\alpha} D_i^{\beta} c_{\beta} + N' \frac{\delta O}{\delta \bar{c}_{\alpha}} + \xi_{\alpha} \right\} \exp i \; \{\ldots.\}. \tag{3.24}$$

Using Eq. (3.16) and the definition of $D_i'^{\alpha}[A]$, we may write Eq. (3.24) as

$$0 = \int [dA\,dc\,d\bar{c}] \left\{ \partial_i^{\alpha} D_i'^{\beta} c_{\beta} + \xi_{\alpha} \right\} \exp i \; \{\ldots.\}. \tag{3.25}$$

We can transcribe Eqs. (3.23) and (3.25) in terms of the renormalized sources:

$$0 = \left\{ \left[J_i \frac{\delta}{\delta K_i'} + \bar{\zeta}_{\alpha} \frac{\delta}{\delta L_{\alpha}'} + \eta^{1/2} f_{\alpha}\left[-i \frac{\delta}{\delta J}\right] \zeta_{\alpha} \right] \right.$$

$$\left. W^{(r)} [J, \zeta, \bar{\zeta}, K', L', N'] \right\}_{K' = L' = N' = 0} \tag{3.26}$$

and

$$0 = \left[\partial_i^{\alpha} \frac{\delta}{\delta K_i'} + \zeta_{\alpha} \right] W^{(r)} [J, \zeta, \bar{\zeta}, K', L', N']. \tag{3.27}$$

These can be translated in terms of the WT identity for proper vertices in the usual manner [and Eq. (3.26) simplified with the help of Eq. (3.27)] yielding,

$$\mathscr{G}_0 \left\{ \Gamma_{N'} \; [\Phi, \omega, \omega] \right\}_1^{\text{div}} = - \frac{\delta \tilde{S}}{\delta \Phi_i} \left[\frac{\delta \Gamma_{N'}}{\delta K_i} \bigg| \; K' = L' = 0 \right]_1^{\text{div}}$$

$$+ \frac{\delta \tilde{S}}{\delta \omega_{\alpha}} \left\{ \frac{\delta \Gamma_{N'}}{\delta L_{\alpha}} \bigg| \; K = L = 0 \right\}_1^{\text{div}} \tag{3.28}$$

and

$$\frac{\delta}{\delta \omega_\alpha} \left\{ \Gamma'_N [\Phi, \omega, \omega] \right\}_1^{\mathrm{div}} = \partial_i^\alpha \left\{ \frac{\delta \Gamma_{N'}}{\delta K_i} \Bigg|_{K=L=0} \right\}_1^{\mathrm{div}} . \tag{3.29}$$

It is therefore clear that $\{\Gamma_{N'} [\Phi, \omega, \omega]\}_1^{\mathrm{div}}$ is also expressible in terms of the complete set of solutions of Eqs. (3.16) and (3.17). i.e. $\{ O^{GI}[\Phi]\} \oplus \{O[\Phi, \omega, \bar{\omega}]\}$. This completes the proof of the closure property in one loop approximation.

D. Renormilazation of Insertion of Operators and the Closure Property in Higher Orders.

Next we shall consider the renormalization of Green's functions (and proper vertices) with one insertion of operators in higher orders; and show that the closure property holds to all orders. To show the closure property it is sufficient to know how to renormalize

$$\Gamma_N \Big|_{K=0=L}, \quad \frac{\delta \Gamma_N}{\delta K} \Bigg|_{K=L=0}, \quad \frac{\delta \Gamma_N}{\delta L} \Bigg|_{K=L=0} .$$

We consider the complete set of independent triplets of functionals $\{[O^{(p)}[\Phi, \omega, \bar{\omega}],\ Q_i^{(p)}[\Phi, \omega, \bar{\omega}],\ R_\alpha^{(p)}[\Phi, \omega, \bar{\omega}]\}$ (and these include gauge invariant operators) such that each member (triplet) of the set satisfies Eqs.(3.16) and (3.17). [Some entries in the triplets may be zero. e.g. There may be more nonvanishing $Q_i^{(p)}$ than there are gauge noninvariant operators etc.] Then

$$\left\{ \Gamma_{N_p} [\Phi, \omega, \bar{\omega}] \right\}_1^{\mathrm{div}} = \sum_q z_{pq}^{(1)} (\epsilon) \, O^{(q)} [\Phi, \omega, \bar{\omega}]$$

$$\left\{ \frac{\delta \Gamma_{N_p}}{\delta K_i} \bigg|_{K = L = 0} \right\}_1^{\mathrm{div}} = \sum_q z_{pq}^{(1)} (\epsilon) \, Q_i^{(q)} [\Phi, \omega, \bar{\omega}]$$

$$\left\{ \frac{\delta \Gamma_{N_p}}{\delta L_\alpha} \bigg|_{K = L = 0} \right\}_1^{\mathrm{div}} = \sum_q z_{pq}^{(1)} (\epsilon) \, R_\alpha^{(q)} [\Phi, \omega, \bar{\omega}] \tag{3.30}$$

(q runs over the complete set.)

And thus

$$\left\{ \Gamma_{N_p}^{\mathrm{u.r.}} [\Phi, \omega, \bar{\omega}] \right\}_1 = \left\{ Z_{pq} \, \Gamma_{N_q}^{R} [\Phi, \omega, \bar{\omega}] \right\}_1$$

$$\left\{ \frac{\delta \Gamma_{N_p}^{\mathrm{u.r.}}}{\delta K_i} \bigg|_{K = L = 0} \right\}_1 = \left\{ Z_{pq} \frac{\delta \Gamma^R}{\delta K_i} \quad K = L = 0 \right\}_1$$

$$\left\{ \frac{\delta \Gamma_{N_p}^{\mathrm{u.r.}}}{\delta L_\alpha} \bigg|_{K = L = 0} \right\}_1 = \left\{ Z_{pq} \frac{\delta \Gamma^R}{\delta L_\alpha} \quad K = L = 0 \right\}_1 \tag{3.31}$$

where
$$Z_{pq} = \delta_{pq} + a \cdot z_{pq}^{(1)} \,, \tag{3.32}$$

where a is the loop expansion parameter.

The generating functional of Green's functions with a single operator insertion, in which all the internal subtractions up to one loop approximation [for the three types of Green's functions in Eq. (3.28)] have been performed, is expressed as

$$W[j, \xi, \bar{\xi}, \kappa', \ell', N_p] = \int [dAdcd\bar{c}] \exp i \Big\{ \mathscr{L}_{eff} + j_i A_i + \bar{\xi}_\alpha \bar{c}_\alpha + c_\alpha \xi_\alpha$$
$$+ N_p Z_{pq} O^{(q)}[A, c, \bar{c}]$$
$$+ \kappa_i' (D_i^\alpha c_\alpha + N_p Z_{pq} Q_i^{(q)}$$
$$+ \ell_\alpha' \Big[-\frac{1}{2} f_{\alpha\beta\gamma} c_\beta c_\gamma + N_p Z_{pq} R_\alpha^{(q)} \Big] \Big\} \quad (3.33)$$

We have expressed W in terms of the unrenormalized sources only because the expression is simpler. The overall two-loop divergence is to be computed by expressing the corresponding generating functional for proper vertices in terms of renormalized quantities.

It is obvious that the definition of Eq. (3.31) can be extended to all orders simply by determining the renormalization matrix Z_{pq} in successive orders:

$$Z_{pq} = \delta_{pq} + a\, z_{pq}^{(1)} + a^2\, z_{pq}^{(2)} + \ldots \quad (3.34)$$

where a is the loop expansion parameter, provided the counter terms needed are restricted to the set $\{[O^{(q)}, Q_i^{(q)}, R_\alpha^{(q)}]\}$; in other words, if the closure property holds to all orders. In the following, we shall show that this is true.

The proof proceeds by induction; and is very similar to the proof for the case of the one loop approximation. Let us assume that the closure property holds up to (n-1)-loop approximation. Then Z_{pq} is determined up to (n-1)-loop approximation in Eq. (3.33). We note that under the transformation of integration variables,

$$\delta A_i = (D_i^\alpha c_\alpha + N_p Z_{pq} Q_i^{(p)})\,\delta\lambda$$

$$\delta c_\alpha = \left(-\frac{1}{2} g_0 f_{\alpha\beta\gamma} c_\beta c_\gamma + N_p Z_{pq} R_\alpha^{(p)}\right)\delta\lambda$$

$$\delta \bar{c}_\alpha = -\eta_0^{1/2} f_\alpha [A]\,\delta\lambda \tag{3.35}$$

$$\delta\left\{\mathscr{L}_{\text{eff}}[A, c, \bar{c}] + N_p Z_{pq} O^{(p)}[A, c, \bar{c}]\right\} = O[N^2] \tag{3.36}$$

which is again a consequence of the fact that $\{O^{(p)}, Q_i^{(p)}, R_\alpha^{(p)}\}$ satisfy the Eqs. (3.14) and (3.15). One can derive the WT identity satisfied by $\Gamma[\Phi, \omega, \bar{\omega}, K, L, N]$. As before the WT identity satisfied by single insertions of any of the operators $\{O^{GI}\} \oplus \{O[A, c, \bar{c}]\}$ is the same, viz.

$$\left\{\frac{\delta\Gamma_N}{\delta\Phi_i}\frac{\delta\Gamma_0}{\delta K_i'} + \frac{\delta\Gamma_0}{\delta\Phi_i}\frac{\delta\Gamma_N}{\delta K_i'} - \frac{\delta\Gamma_0}{\delta\omega_\alpha}\frac{\delta\Gamma_N}{\delta L_\alpha} - \frac{\delta\Gamma_N}{\delta\omega_\alpha}\frac{\delta\Gamma_0}{\delta L_\alpha'}\right\}_{K'=L'=0} = 0 \tag{3.37}$$

We equate the n-loop divergence on both sides noting

$$\left\{\frac{\delta^2\Gamma_N}{\delta K_i'}\right\}_{\substack{r\text{-loop}\\ K'=L'=N=0}} = \text{finite} \qquad \left\{\frac{\delta\Gamma_N}{\delta L_\alpha}\right\}_{\substack{r\text{-loop}\\ K'=L'=N=0}} = \text{finite} \qquad r \le n-1 \tag{3.38}$$

we get

$$0 = \Bigg[\left\{ \frac{\delta \Gamma_0}{\delta K_i'} \right\}_0 \left\{ \frac{\delta \Gamma_N}{\delta \Phi_i} \right\}_n^{\text{div}} + \left\{ \frac{\delta \Gamma_0}{\delta \Phi_i} \right\}_0 \left\{ \frac{\delta \Gamma_N}{\delta K_i} \right\}_n^{\text{div}}$$

$$- \left\{ \frac{\delta \Gamma_0}{\delta \omega_\alpha} \right\}_0 \left\{ \frac{\delta \Gamma_N}{\delta L_\alpha'} \right\}_n^{\text{div}} - \left\{ \frac{\delta \Gamma_0}{\delta L_\alpha'} \right\}_0 \left\{ \frac{\delta \Gamma_N}{\delta \omega_\alpha} \right\}_n^{\text{div}} \Bigg]_{K'=L'=0} \tag{3.39}$$

and thus

$$\mathscr{G}_0 \left\{ \Gamma_N \right\}_n^{\text{div}} = - \frac{\delta \tilde{S}}{\delta \Phi_i} \left\{ \frac{\delta \Gamma_N}{\delta K_i'} \Bigg|_{K'=L'=0} \right\}_n^{\text{div}}$$

$$- \frac{\delta \tilde{S}}{\delta \omega_\alpha} \left\{ \frac{\delta \Gamma_N}{\delta L_\alpha} \Bigg|_{K'=L'=0} \right\}_n^{\text{div}} \tag{3.40}$$

Similarly the equation of motion for the $\bar{c}$ field has the same form.

$$\frac{\delta}{\delta \omega_\alpha} \left[\Gamma_N \right]_n^{\text{div}} = \partial_i^\alpha \left\{ \frac{\delta \Gamma_N}{\delta K_i'} \Bigg|_{K'=L'=0} \right\}_n^{\text{div}} \tag{3.41}$$

Thus, it is clear that $\{\Gamma_N[\Phi, \omega, \bar{\omega}]\}_n^{\text{div}}$ can be expressed in terms of the same set of operators $\{O^{GI}[\Phi]\} \oplus \{O[\Phi, \omega, \bar{\omega}]\}$. Thus the proof by induction is complete.

IV. FORM OF THE GAUGE NON-INVARIANT OPERATORS

In this section we shall show that the complete set of gauge noninvariant local solutions[14] of the Eqs. (3.16) and (3.17) can be written in the form

$$O[\Phi,\omega,\bar{\omega}] = \frac{\delta S[\Phi,\omega,\omega]}{\delta \Phi_i} \frac{\delta F[\Phi,\omega,\bar{\omega}]}{\delta (\partial\bar{\omega})_i} + \frac{\delta S[\Phi,\omega,\bar{\omega}]}{\delta \omega_\alpha} X_\alpha[\Phi,\omega,\bar{\omega}]$$

$$+ \mathscr{G}_0[\Phi,\omega,\bar{\omega}] \; F[\Phi,\omega,\bar{\omega}] \tag{4.1}$$

where $F[\Phi,\omega,\bar{\omega}]$ and $X_\alpha[\Phi,\omega,\bar{\omega}]$ are arbitrary local polynomials of $\Phi,\omega,\bar{\omega}$ possessing the appropriate ghost number and appropriate transformation properties under global transformations of G and under Lorentz transformations. This form of the operators $\{O[\Phi,\omega,\bar{\omega}]\}$ will be the basis of our proof of "decoupling" in the next section.

Terms in $Q_i[\Phi,\omega,\bar{\omega},K,L]$ which vanish when multiplied by ∂_i^α do not have a counter part in $O[\Phi,\omega,\bar{\omega},K,L]$ as seen from Eq. (3.14). We may therefore choose to write $O[\Phi,\omega,\bar{\omega},K,L]$ such that

$$\frac{\delta O[\Phi,\omega,\bar{\omega},K,L]}{\delta (\partial\bar{\omega})_i} = Q_i[\Phi,\omega,\bar{\omega},K,L] \tag{4.2}$$

We shall find it convenient to consider Eq. (3.14), (3.15) at K = 0 instead of Eqs. (3.16), (3.17). Replacing Q_i in Eq. (3.15) by the expression (4.2) and setting $K = 0$, we obtain

$$\left\{\mathscr{G}_0 - \frac{\delta \tilde{S}}{\delta \omega_\alpha}\frac{\delta}{\delta L_\alpha} + \frac{\delta \tilde{S}}{\delta \Phi_i}\frac{\delta}{\delta (\partial\bar{\omega})_i}\right\} O[\Phi,\omega,\bar{\omega},K,L] = 0 \tag{4.3}$$

We define a new differential operator

$$\mathscr{G}' = \mathscr{G}_0 - \frac{\delta \tilde{S}}{\delta \omega_\alpha} \frac{\delta}{\delta L_\alpha} + \frac{\delta \tilde{S}}{\delta \Phi_i} \frac{\delta}{\delta (\partial \bar{\omega})_i} \tag{4.4}$$

which can be easily shown to satisfy

$$\mathscr{G}'^2 = 0 \quad . \tag{4.5}$$

It is easy to see that all functionals of the form $\mathscr{G}' F[\Phi, \omega, \bar{\omega}]$ are solutions of Eq. (4.3). It is also easy to see that all gauge invariant operators are solutions of Eq. (4.3). Our objective is to show that the two types of solutions are the <u>only</u> <u>local</u> solutions of Eq. (4.3). Thus we state our theorem:

<u>Main Theorem</u>: The complete set of independent local solutions of the Eq. (4.3) with ghost number zero can be expressed as

$$\{ \mathscr{G}' F[\Phi, \omega, \bar{\omega}] \} \oplus \{ O^{GI}[\Phi] \}'$$

where $F[\Phi, \omega, \bar{\omega}]$ are arbitrary local polynomials with ghost number (-1). Here, the ghost number is defined as the difference between the powers of ω and $\bar{\omega}$ in the expression. $\{ O^{GI}[\Phi] \}'$ refers to the set of gauge invariant operators not expressible as $\mathscr{G}' F$, i.e., as $\frac{\delta \mathscr{L}}{\delta \Phi_i} F_i'$.[14]

We shall prove first a lemma which we shall find very useful in the proof. The method is essentially similar to that employed by Dixon and Taylor.

<u>Lemma 1 (Dixon-Taylor)</u>: Let $\mathscr{G}$ be a local differential operator in $\Phi, \omega, \bar{\omega}, L$. Let it be possible to expand $\mathscr{G}$ in powers of some parameter β as

$$\mathscr{G}(\beta) = A + \beta B + \beta^2 C \quad , \tag{4.6}$$

and let $\mathscr{G}(\beta)$ satisfy,

$$\mathscr{G}^2(\beta) = 0 \quad , \qquad \text{for arbitrary } \beta \quad . \tag{4.7}$$

Then all polynomial solutions of the equation

$$\mathscr{G}(\beta)H(\beta) = 0 \quad , \tag{4.8}$$

where $H(\beta)$ can be expanded as a power series in β :

$$H(\beta) = \sum_{n=0} H^{(n)}\beta^n \quad , \tag{4.9}$$

can be expressed as

$$H(\beta) = \mathscr{G}(\beta)J(\beta) \quad , \qquad J(\beta) = \sum_{n=0} \beta^n J^{(n)} \quad , \tag{4.10}$$

provided that the equation $AY = 0$ implies that there exists a X such that

$$Y = AX \quad , \tag{4.11}$$

where Y has the same quantum numbers as H .

<u>Proof:</u> Equation (4.7) can be written as

$$A^2 = 0 \quad , \tag{4.12a}$$

$$AB + BA = 0 \quad , \tag{4.12b}$$

$$CA + AC + B^2 = 0 \quad , \tag{4.12c}$$

$$BC + CB = 0 \quad , \tag{4.12d}$$

$$C^2 = 0 \quad . \tag{4.12c}$$

Equation (4.9) gives,

$$AH^{(n)} + BH^{(n-1)} + CH^{(n-2)} = 0 \quad . \tag{4.13}$$

We have to show that $J^{(n)}$ exists such that

$$H^{(n)} = AJ^{(n)} + BJ^{(n-1)} + CJ^{(n-2)} \quad . \tag{4.14}$$

Let us assume that we have determined $J^{(n)}$ for $n < r$. We thus know that

$$H^{(r-1)} = AJ^{(r-1)} + BJ^{(r-2)} + CJ^{(r-3)} \quad ,$$

$$H^{(r-2)} = AJ^{(r-2)} + BJ^{(r-3)} + CJ^{(r-4)} \quad . \tag{4.15}$$

Then Eq. (4.13) yields,

$$AH^{(r)} = -B[\, AJ^{(r-1)}+BJ^{(r-2)}+CJ^{(r-3)}] - C[\, AJ^{(r-2)}+BJ^{(r-3)}+CJ^{(r-4)}] \quad . \tag{4.16}$$

Using Eqs. (4.12) in Eq. (4.16),

$$AH^{(r)} = ABJ^{(r-1)} + ACJ^{(r-2)} \quad ,$$

$$\text{i.e.,} \quad A[\, H^{(r)} - BJ^{(r-1)} - CJ^{(r-2)}] = 0 \quad . \tag{4.17}$$

We can then determine the desired $J^{(r)}$ if the Eq. (4.17) implies that there exists $J^{(r)}$ such that

$$H^{(r)} - BJ^{(r-1)} - CJ^{(r-2)} = AJ^{(r)} \quad . \tag{4.18}$$

Applying the same argument for $r = 0$, we learn that we can determine $J^{(0)}$ if the equation

$$AH^{(0)} = 0 \tag{4.19}$$

implies that there exists a $J^{(0)}$ such that

$$H^{(0)} = AJ^{(0)} \quad . \tag{4.20}$$

Thus Eqs. (4.19), (4.20) and Eqs. (4.17), (4.18) imply that $J^{(0)}$ and $J^{(r)}$ $r \geq 1$ can be determined successively if the equation $AY = 0$ implies that there exists an X such that $Y = AX$, proving the lemma.

Comments: (i) If Eq. (4.11) holds, $J(\beta)$ can be found. However this $J(\beta)$ is not unique, for we may add to $J(\beta)$ a polynomial $K(\beta)$ such that

$$\mathscr{G}(\beta)K(\beta) = 0 \quad . \tag{4.21}$$

In particular $K(\beta)$ need not start as β^0, it may start as some positive or some negative integral power of β. If $J(\beta)$ exists,

$$J'(\beta) = J(\beta) + K(\beta)$$

is the most general expression satisfying $\mathscr{G}J' = H$. Therefore, there is no loss of generality in assuming that $J(\beta)$ starts as β^0.

(ii) We may use special cases of the lemma such as putting $C = 0$.

(iii) Given that H is local, the lemma does not make any statement as to whether J can be chosen to be local.

Now, let us return to the Eq. (4.3) which we have to solve. Let us expand a local solution $O^{(p)}[\Phi, \omega, \bar{\omega}]$

$$O^{(p)}[\Phi, \omega, \bar{\omega}, L] = F^{(p)}[\Phi] + (\partial\bar{\omega})_i Q_i^{(p)}[\Phi, \omega, \bar{\omega}] + \text{terms proportional to } L \,. \tag{4.22}$$

[We have used Eq. (3.14).] We note that the terms proportional to L involve at least two factors of ω. Substituting Eq. (4.22) in Eq. (4.3) and comparing the coefficients of the lowest power in $\omega, \bar{\omega}$ [i.e., $(\bar{\omega})^0 \omega^1$] , we get

$$D_i^\alpha \frac{\delta F[\Phi]}{\delta \Phi_i} = Q_i^\alpha[\Phi] \mathscr{L}_{0,i}[\Phi] \tag{4.23}$$

where,

$$Q_i^\alpha[\Phi] = \frac{\delta Q_i}{\delta \omega_\alpha} [\Phi, \omega, \bar{\omega}] \Bigg|_{\omega = \bar{\omega} = 0} \tag{4.24}$$

Our method will be

(i) To show that all particular solutions of Eq. (4.23) can be expressed as

$$F[\Phi] = \mathscr{L}_{0,i} S_i[\Phi] \quad . \tag{4.25}$$

$S_i[\Phi]$ being an arbitrary local functional. (Theorem I).

(ii) To isolate all the solutions of Eq. (4.3) which must have a nonzero $F[\Phi]$ in Eq. (4.22). The rest of the independent solutions may be assumed to contain at least one ghost (ω) .

(iii) To solve for such solutions (containing at least one ghost) by using the Dixon-Taylor Lemma in two stages:

Step I: We write Eq. (4.23) as

$$\omega_\alpha D_i^\alpha \frac{\delta F[\Phi]}{\delta \Phi_i} = \mathscr{G}_0 F[\Phi] = \omega_\alpha Q_i^\alpha \mathscr{L}_{0,i} \tag{4.26}$$

where $\mathscr{G}_0$ is defined in Eq. (2.10) and satisfies $\mathscr{G}_0^2 = 0$. We thus have

$$\mathscr{G}_0(\omega_\alpha Q_i^\alpha \mathscr{L}_{0,i}) = 0 \quad . \tag{4.27}$$

We define an operator

$$\mathscr{G} = \mathscr{G}_0 \text{ - covariance terms ,} \tag{4.28}$$

where covariant terms are obtained by 'varying' the free indices of the functional on which it acts according to the following law:

$$\begin{aligned}(\text{covariance terms})A_i[\Phi, \omega, \bar{\omega}] &= -g\, t^{\alpha}_{ij}\omega_{\alpha} A_j[\Phi, \omega, \bar{\omega}] \\ (\text{covariance terms})A_{\alpha}[\Phi, \omega, \omega] &= -g\, f_{\alpha\beta\gamma}\omega_{\beta} A_{\gamma}[\Phi, \omega, \bar{\omega}]\end{aligned} \tag{4.29}$$

It can be shown that, in general,

$$\mathscr{G}^2 = 0 \,. \tag{4.30}$$

We note that

$$\mathscr{G}[\mathscr{L}_{0,i}] \equiv \mathscr{G}_0 \mathscr{L}_{0,i} - g\, t^{\alpha}_{ij}\omega_{\alpha}\mathscr{L}_{0,j} = 0 \,. \tag{4.31}$$

Using Eq. (4.31), Eq. (4.27) can be written as

$$\mathscr{G}(\omega_{\alpha} Q^{\alpha}_i[\Phi])\mathscr{L}_{0,i} = 0 \,. \tag{4.32}$$

This is the integrability condition for a solution of Eq. (4.23) to exist given Q^{α}_i.

Using Lemma (AII) in Appendix A, we infer from Eq. (4.32) that

$$\mathscr{G}(\omega_{\alpha} Q^{\alpha}_i[\Phi]) = \omega_{\alpha}\omega_{\beta} X_{\alpha\beta\gamma}[\Phi] D^{\gamma}_i + \omega_{\alpha}\omega_{\beta}\, Y^{\alpha\beta}_{ij}[\Phi]\mathscr{L}_{0,j} \,, \tag{4.33}$$

where

$$Y^{\alpha\beta}_{i_1 i_2}[\Phi] = \sum_r Y^{\alpha\beta(\gamma)}_{i_1 i_2 \cdots i_{r+2}}[\Phi]\, \mathscr{L}_{0,i_3} \cdots \mathscr{L}_{0,i_{r+2}} \,, \tag{4.34}$$

with

$$\sum_{\{\eta\}} Y^{\alpha\beta(r)}_{i_{\eta_1} \cdots i_{\eta_{\gamma+2}}} = 0. \tag{4.35}$$

It is shown in the Appendix B that Eq. (4.33) implies that there exist local functions T and U such that

$$\omega_\alpha Q^\alpha_i = \omega_\alpha T_{\alpha\beta} D^\beta_i + \omega_\alpha U^\alpha_{ij} \mathscr{L}_{0,j}$$

$$+ \text{ solutions of } \mathscr{G}(\omega_\alpha Q^\alpha_i [\Phi]) = 0 \tag{4.36}$$

where

$$U^\alpha_{i_1 i_2} [\Phi] = \sum_{r=0} U^{\alpha(r)}_{i_1 i_2 \cdots i_{r+2}} \mathscr{L}_{0,i_3} \cdots \mathscr{L}_{0,i_{r+2}} \tag{4.37}$$

$$\sum_{\{\eta\}} U^{\alpha(r)}_{i_{\eta_1} \cdots i_{\eta_{r+2}}} = 0. \tag{4.38}$$

Therefore the set of $Q^\alpha_i[\Phi]$'s for which a solution to Eq. (4.23) exist and hence satisfy Eq. (4.32) can be divided into two classes

$$\text{(i)} \quad \left\{ Q^\alpha_i \right\}_{\text{class 1}} = \left\{ T_{\alpha\beta} D^\beta_i \right\} \oplus \left\{ U^\alpha_{ij} \mathscr{L}_{0,j} \right\} . \tag{4.39}$$

They satisfy

$$Q^\alpha_i \text{ (class 1) } \mathscr{L}_{0,i} = 0 \tag{4.40}$$

and therefore a particular solution $F[\Phi]$ for Q^α_i in class 1 can be taken to be zero.

(ii) $Q_i^\alpha[\Phi]$ which satisfy

$$\mathscr{G}(\omega_\alpha Q_i^\alpha) = 0 . \tag{4.41}$$

As shown by Dixon and Taylor, for each such $Q_i^\alpha[\Phi]$ satisfying Eq. (4.41), there exists a polynomial $S_i[\Phi]$ such that

$$\omega_\alpha Q_i^\alpha = \mathscr{G} S_i[\Phi] . \tag{4.42}$$

[These follow essentially by the use of the Dixon Taylor Lemma for the operator $\mathscr{G}$ of Eq. (4.28).] It is shown in Appendix C that $S_i[\Phi]$ can be chosen to be local. Therefore a particular solution of Eq. (4.23) may be chosen to be

$$F[\Phi] = \mathscr{L}_{0,i} S_i[\Phi] . \tag{4.43}$$

We also note that the particular solution of Eq. (4.23) is unique modulo gauge invariant functionals.

Thus we have proved theorem I.

Step II:

We note that because of the property $\mathscr{G}'^2 = 0$, $\mathscr{G}' S_i^{(p)}[\Phi](\partial\bar{\omega})_i$ is a solution of Eq. (4.3). Therefore, we separate independent solutions of Eq. (4.3) in to two classes:

$$\{O^{(p)}[\Phi,\omega,\bar{\omega},L]\} = \{\mathscr{G}' S_i^{(p)}[\Phi](\partial\bar{\omega})_i\} \oplus \{O^{(p)}[\Phi,\omega,\bar{\omega},L] - \mathscr{G}' S_i^{(p)}[\Phi](\partial\bar{\omega})_i\} , \tag{4.44}$$

where $S_i^{(p)}$ is related to $F^{(p)}$ of Eq. (4.22) by Eq. (4.43).

We note that

$$\mathscr{G}' S_i^{(p)}[\Phi](\partial\bar{\omega})_i = \mathscr{L}_{0,i}[\Phi]\, S_i^{(p)}[\Phi] + g\,(\partial\bar{\omega})_j\, t^{\alpha}_{ji}\,\omega_\alpha\, S_i^{(p)}$$

$$+ \ \mathscr{G}_0\,(\partial\bar{\omega})_i\, S_i^{(p)}$$

$$= F^{(p)}[\Phi] - (\partial\bar{\omega})_i\,[\mathscr{G} S_i^{(p)}] \quad .$$

Thus, we see that

(i) $O^{(p)}[\Phi,\omega,\bar{\omega},L] - \mathscr{G}' S_i^{(p)}[\Phi]$ does not contain terms independent of $\omega,\bar{\omega}$

(ii) The terms linear in $\bar{\omega},\omega$ are

$$(\partial\bar{\omega})_i \left[Q_i^{\alpha(p)}[\Phi]\,\omega_\alpha - \mathscr{G} S_i^{(p)}[\Phi] \right],$$

which are expressible as $(\partial\bar{\omega})_i \left(T^{(p)}_{\alpha\beta}\, D_i^{\beta} + U_{ij}^{\alpha(p)}\, \mathscr{L}_{0,j} \right) \omega_\alpha$ by virtue of Eqs. (4.39) and (4.42).

The problem is thus, reduced to finding solutions of Eq. (4.3) which contain at least one factor of ω_α.

Step III:

We note that the operator $\mathscr{G}'$ can be expressed as

$$\mathscr{G}' = A + g\ B + g^2\, C \tag{4.45}$$

where

$$A = \omega_\eta\, \partial_k^{\eta}\, \frac{\delta}{\delta\Phi_k} - D_{ij}\,\Phi_j\, \frac{\delta}{\delta(\partial\bar{\omega})_i} + (\partial\bar{\omega})_i\, \partial_i^{\alpha}\, \frac{\delta}{\delta L_\alpha} \tag{4.46}$$

$$D_{ij}\,\Phi_j = \mathscr{L}_{0,i}\Big|_{g=0}\ , \tag{4.47}$$

and $\mathscr{G}'$ satisfies $\mathscr{G}'^2 = 0$.

Solutions of our interest can be expanded in powers of g. [We may assume that the expansion begins as g^o]

$$O[\Phi,\omega,\bar{\omega},L] = \sum g^n O^{(n)}[\Phi,\omega,\bar{\omega},L] . \qquad (4.48)$$

The Dixon-Taylor Lemma tells us that there exists a functional $F[\Phi,\omega,\bar{\omega},L]$ such that,

$$O[\Phi,\omega,\bar{\omega},L] = \mathscr{G}' F[\Phi,\omega,\bar{\omega},L] , \qquad (.4.49)$$

if the equation ($Y[\Phi,\omega,\bar{\omega},L]$ has the same quantum numbers as O)

$$A\ Y[\Phi,\omega,\bar{\omega},L] = 0 , \qquad (4.50)$$

implies that there exists a functional $X[\Phi,\omega,\bar{\omega},L]$ such that

$$Y[\Phi,\omega,\bar{\omega},L] = A\,X[\Phi,\omega,\bar{\omega},L] . \qquad (4.51)$$

We thus have to prove Equation (4.51) given Eq. (4.50) to do this, we consider a scaling transformation on ω and L:

$$\bar{\omega} = \alpha\bar{\omega}', \ L = \alpha L', \ \Phi' = \Phi, \ \omega' = \omega \ . \qquad (4.52)$$

Then, in terms of the new variables Eq. (4.50) becomes (dropping primes)

$$(A_0 + \frac{1}{\alpha} A_1)\ Y[\Phi,\omega,\alpha\bar{\omega}, \alpha^2 L] = 0 , \qquad (4.53)$$

where

$$A_0 = \omega_\eta \partial^\eta_k \frac{\delta}{\delta\Phi_k} , \qquad (4.54)$$

$$A_1 = - D_{ij} \Phi_j \frac{\delta}{\delta(\partial\overline{\omega})_i} + (\partial\overline{\omega})_i \partial_i^\alpha \frac{\delta}{\delta L_\alpha} \ . \tag{4.55}$$

We note that in Eq. (4.50), we need to consider $X[\Phi, \omega, \overline{\omega}, L]$ which are local polynomials in $\overline{\omega}$ and L and therefore $Y[\Phi, \omega, \alpha\overline{\omega}, \alpha^2 L]$ contains a finite maximum power of α which may be determined given the dimension of the operators we are interested in. Multiplying Eq. (4.53) by α, we get

$$(A_1 + \alpha A_0) \{Y[\Phi, \omega, \alpha\overline{\omega}, \alpha^2 L]\} = 0 \ . \tag{4.56}$$

We further note that,

$$(A_1 + \alpha A_0)^2 = 0 \text{ for any } \alpha \ . \tag{4.57}$$

Thus, we may apply the Dixon Taylor Lemma to Eq. (4.56) and deduce that there exists a <u>local</u> polynomial $Y[\Phi, \omega, \overline{\omega}, L, \alpha]$ such that

$$Y[\Phi, \omega, \alpha\overline{\omega}, \alpha^2 L] = (A_1 + \alpha A_0) X[\Phi, \omega, \overline{\omega}, L, \alpha] \ , \tag{4.58}$$

provided for any <u>local</u> functional $Z[\Phi, \omega, \overline{\omega}, L]$ with the same quantum numbers as $Y\{$and therefore $O[\Phi, \omega, \overline{\omega}, L]\}$ satisfying

$$A_1 Z[\Phi, \omega, \partial\overline{\omega}, L] = 0 \ , \tag{4.59}$$

implies that there exists a <u>local</u> functional $V[\Phi, \omega, \partial\overline{\omega}, L]$ such that

$$Z[\Phi, \omega, \partial\overline{\omega}, L] = A_1 V[\Phi, \omega, \partial\overline{\omega}, L] \ . \tag{4.60}$$

Since $O[\Phi, \omega, \partial\overline{\omega}, L]$ is proportional to at least one ω, so must be Z of Eq. (4.59).

[At this point we note that we could have written Eq. (4.53) as

$$\gamma^N (A_0 + \gamma A_1) \quad Y[\Phi, \omega, \overline{\omega}/\gamma, L/\gamma^2] = 0 \tag{4.61}$$

with $\gamma = 1/\alpha$ and N is some finite integer; and attempted proving statement of Eq. (4.60) for A_0 instead of A_1. However since,

$$\begin{aligned} A_0 &= \omega_\eta \; \partial_i^\eta \frac{\delta}{\delta \Phi_\eta} \\ &= \omega_\eta \frac{\delta}{\delta \Phi_\eta^L} , \end{aligned} \tag{4.62}$$

where Φ_η^L is the longitudinal gauge field defined by,

$$\Phi_i = \Phi_i^T + \partial_i^\eta \Phi_\eta^L \quad ; \quad \partial_i^\alpha \Phi_i^T = 0 \tag{4.63}$$

and thus involves a derivative with respect to Φ^L which is not a local functional of Φ. Thus even though statement analogous to Eq. (4.60) may hold, it is difficult to decide whether the corresponding V can be chosen to be local. In fact, there are instances when a V exists but can never be chosen to be local.]

To prove Eq. (4.60) given Eq. (4.59), we expand Z in powers of L:

$$Z[\Phi, \omega, \partial\overline{\omega}, L] = \sum_{n=0}^{m} Z^{(n)} ,$$

$$Z^{(n)} = \frac{1}{n!} Z^{(n)\alpha_1 \ldots \alpha_n} L_{\alpha_1} \ldots L_{\alpha_n} . \tag{4.64}$$

Compare the coefficients $(L)^n$ on both sides of Eq. (4.59):

$$(D\Phi)_i \frac{\delta}{\delta(\partial\bar{\omega})_i} Z^{(n)} + (\partial\bar{\omega})_i \, \partial_i^\alpha \frac{\delta Z^{(n+1)}}{\delta L_\alpha} = 0 ,$$

$$0 \le n \le m-1 , \tag{4.65}$$

and

$$(D\Phi)_i \; \delta(\partial\bar{\omega})_i \; Z^{(m)\alpha_1 \ldots \alpha_m} = 0 . \tag{4.66}$$

In Eq. (4.64) we can always assume $Z^{(m)\alpha_1 \ldots \alpha_n}$ to be completely symmetric in $\alpha_1 \ldots \alpha_n$:

$$S[\alpha_1 \ldots \alpha_n] \; Z^{(n)\alpha_1 \ldots \alpha_n} = Z^{(n)\alpha_1 \ldots \alpha_n} . \tag{4.67}$$

We will begin with Eq. (4.66). We expand $Z^{(m)}$ further in $\partial\bar{\omega}$:

$$Z^{(m)\alpha_1 \ldots \alpha_m} = \sum_r (\partial\bar{\omega})_{i_1} \ldots (\partial\bar{\omega})_{i_r} Z^{(m,r)\alpha_1 \ldots \alpha_m}_{i_1 \ldots i_r} , \tag{4.68}$$

where we may choose

$$A[i_1 \ldots i_r] \; Z^{(m,r)\alpha_1 \ldots \alpha_m}_{i_1 \ldots i_r} = Z^{(m,r)\alpha_1 \ldots \alpha_m} . \tag{4.69}$$

Here A is the total antisymmetrizer of its arguments:

$$A = \frac{1}{r!} \sum_P \delta_P \; P ,$$

$\delta_P = \pm 1$, according as P is an even or odd permutation of $(1, 2, \ldots, r)$.

An application of the reasoning of Lemma AI to Eq. (4.66), taking due account of the antisymmetry in $i_1, \ldots, i_r$, leads to the decomposition

$$Z^{(m,r)\alpha_1\ldots\alpha_m}_{i_1\ldots i_r} = Z'^{\alpha_1\ldots\alpha_m}_{i_1\ldots i_r}$$

$$+\left(A[\beta_1\ldots\beta_r]\, J^{\alpha_1\ldots\alpha_m}_{\beta_1\ldots\beta_r}\right) \partial^{\beta_1}_{i_1}\, \partial^{\beta_2}_{i_2} \ldots \partial^{\beta_r}_{i_r} \;, \tag{4.70}$$

with

$$Z'^{\alpha_1\ldots\alpha_m}_{i_1\ldots i_r} (D\,\Phi)_{i_1} = 0 \,. \tag{4.71}$$

We may assume that Z' does not contain terms $\sim \partial^{\lambda}_{i_s}$, $s = 1, \ldots, r$. The reasoning of lemma AI tells us that

$$Z'^{\alpha_1\ldots\alpha_m}_{i_1\ldots i_r} = W^{(p)\alpha_1\ldots\alpha_m}_{i_1\ldots i_r, j_1\ldots j_p} (D\Phi)_{j_1} \ldots (D\Phi)_{j_p} \;, \tag{4.72}$$

where

$$\overline{S}[j_1 \ldots j_p]\; W^{(p)\alpha_1\ldots\alpha_m}_{i_1\ldots i_r, j_1\ldots j_p} = 0, \tag{4.73}$$

and

$$S[i_s, j_1\ldots j_p]\; W^{(p)\alpha_1\ldots\alpha_m}_{i_1\ldots i_r, j_1\ldots j_p} = 0\,,\ 1 \le s \le r\,. \tag{4.74}$$

Therefore, $W^{(p)}$ is completely antisymmetric in $i_1, \ldots, i_r$, symmetric in $j_1 \ldots j_p$, and satisfies the constraint, Eq. (4.74). We denote by e the identity element of the symmetric group S_{r+p} on $r+p$ objects, so that

$$e\, W^{(p)\alpha_1\ldots\alpha_m}_{i_1\ldots i_r, j_1\ldots j_p} = W^{(p)\alpha_1\ldots\alpha_m}_{i_1\ldots i_r, j_1\ldots j_p} \,. \tag{4.75}$$

The identity element e, has a resolution in terms of Young operators Y_Γ where Γ's are standard Young Tableaux.[13] We order the $r+p$ induces $i_1, \ldots i_r, j_i \ldots j_p$ in this order. Then the only Young operators which do not annihilate $W_{i_1 \ldots i_r, j_i \ldots j_p}$ with the given symmetries, Eqs. (4.69), (4.73) and (4.74), correspond to the Young tableau Γ_s shown in Figure 1. Therefore, we may write Eq. (4.75) as

$$W_{i_1 \ldots i_r, j_1 \ldots j_p} = \sum_{s=1}^{p} Y_{\Gamma_s} W_{i_1 \ldots i_r, j_1 \ldots j_p} \tag{4.76}$$

where Y_Γ's normalized to be idempotent:

$$Y_{\Gamma_s} Y_{\Gamma_r} = \delta_{s,r} Y_{\Gamma_s} \, . \tag{4.77}$$

Using Eq. (4.76) together with Eq. (4.72) we can write

$$\sum_r (\partial\overline{\omega})_{i_1} \ldots (\partial\overline{\omega})_{i_r} Z'^{\alpha_1 \ldots \alpha_m}_{i_1 \ldots i_r}$$

$$= (D\Phi)_s \frac{\delta}{\delta(\partial\overline{\omega})_s} \Bigg\{ \sum_{r,p} (\partial\overline{\omega})_{\ell_1} \ldots (\partial\overline{\omega})_{\ell_r} (\partial\overline{\omega})_{\ell_{r+1}}$$

$$\times V^{(m,r,p)\alpha_1 \ldots \alpha_m}_{\ell_1 \ldots \ell_r \ell_{r+2}, k_1 \ldots k_{p-1}} (D\Phi)_{k_1} \ldots (D\Phi)_{k_{p-2}} \Bigg\}$$

$$\equiv (D\Phi)_s \frac{\delta}{\delta(\partial\overline{\omega})_s} V^{(m)\alpha_1 \ldots \alpha_m} [\Phi, \omega, \partial\overline{\omega}] \, , \tag{4.78}$$

where $V^{(m,r,p)}$ satisfies the symmetry conditions

$$\overline{A}[\ell_1 \ldots \ell_r \ell_{r+1}] \; V^{(m,r,p)}_{\ell_1 \ldots \ell_{r+1}, k_1 \ldots k_{p-1}} = 0 \,, \tag{4.79}$$

$$\overline{S}[k_1 \ldots k_{p-2}] \; V^{(m,r,p)}_{\ell_1 \ldots \ell_{r+1}, k_1 \ldots k_{p-1}} = 0, \tag{4.80}$$

and, in fact,

$$V^{(m,r,p)}_{i_1 \ldots i_r j_s ; j_1 \ldots \hat{j}_s \ldots j_p} = \left(\frac{p}{r+1}\right) Y_{\Gamma_s} W^{(p)}_{i_1 \ldots i_r, j_1 \ldots j_p} (-)^r . \tag{4.81}$$

Similarly, we may write

$$\sum_r (\partial\overline{\omega})_{i_1} \ldots (\partial\overline{\omega})_{i_r} \left(A[\beta_1 \ldots \beta_r] \; J^{\alpha_1 \ldots \alpha_m}_{\beta_1 \ldots \beta_r} \right) \partial^{\beta_1}_{i_1} \ldots \partial^{\beta_r}_{i_r} L_{\alpha_1} \ldots L_{\alpha_m}$$

$$= (\partial\overline{\omega})_i \, \partial^{\alpha}_i \; \frac{\delta}{\delta L_\alpha} \; U^{(m+1)} [\Phi, \omega, \partial\omega, L] \,, \tag{4.82}$$

where

$$U^{(m+1)} = \sum_{s=2} (\partial\overline{\omega})_{i_2} \ldots (\partial\overline{\omega})_{i_s} \; \frac{1}{m+1} \left(A[\beta\, \beta_2 \ldots \beta_s] \; J^{\alpha_1 \ldots \alpha_m}_{\beta\,\beta_2 \ldots \beta_s} \right)$$

$$\partial^{\beta_2}_{i_2} \ldots \partial^{\beta_s}_{i_s} \; L_{\alpha_1} \ldots L_{\alpha_m} L_\beta \,. \tag{4.83}$$

Likewise, we define

$$V^{(m)}[\Phi, \omega, \partial\omega, L] = V^{(m)\alpha_1 \ldots \alpha_m} [\Phi, \omega, \partial\overline{\omega}] \, L_{\alpha_1} \ldots L_{\alpha_m} \,. \tag{4.84}$$

What we have shown so far is that, given

$$(D\Phi)_i \, \frac{\delta}{\delta\,(\partial\overline{\omega})_i} \; Z^{(m)} = 0 \,, \tag{4.85}$$

there exist $V^{(m)}$ and $U^{(m+1)}$, such that

$$Z^{(m)} = (D\Phi)_i \frac{\delta}{\delta(\partial\overline{\omega})_i} V^{(m)} + (\partial\overline{\omega})_i \, \partial_i^\alpha \frac{\delta}{\delta L_\alpha} U^{(m+1)} . \tag{4.86}$$

Further, note that

$$(D\Phi)_i \frac{\delta}{\delta(\partial\overline{\omega})_i} U^{(m+1)} = 0 \quad , \tag{4.87}$$

as evident from Eq. (4.83).

Now assume that

$$Z^{(n)} = (D\Phi)_i \frac{\delta}{\delta(\partial\overline{\omega})_i} V^{(n)} + (\partial\overline{\omega})_i \, \partial_i^\alpha \frac{\delta}{\delta L_\alpha} \left[U^{(n+1)} + V^{(n+1)}\right],$$

$$(D\Phi)_i \frac{\delta}{\delta(\partial\overline{\omega})_i} U^{(n+1)} = 0 ; \; V^{(m+1)} \equiv 0 . \tag{4.88}$$

Then, by virtue of Eq. (4.65), we have

$$(D\Phi)_i \frac{\delta}{\delta(\partial\overline{\omega})_i} Z^{(n-1)} + (\partial\overline{\omega})_i \, \partial_i^\alpha \frac{\delta}{\delta L_\alpha} \left\{ (D\Phi)_j \frac{\delta}{\delta(\partial\overline{\omega})_j} V^{(n)} \right\} = 0 , \tag{4.89}$$

because

$$\left\{ (\partial\overline{\omega})_i \, \partial_i^\alpha \frac{\delta}{\delta L_\alpha} \right\}^2 = 0 .$$

Equation (4.89) can be written as

$$(D\Phi)_i \frac{\delta}{\delta(\partial\overline{\omega})_i} \left\{ Z^{(n-1)} - (\partial\overline{\omega})_j \, \partial_j^\alpha \frac{\delta}{\delta L_\alpha} V^{(n)} \right\} = 0. \tag{4.90}$$

This is the same equation as Eq. (4.85). Therefore, there exist $V^{(n-1)}$ and $U^{(n)}$ such that

$$Z^{(n-1)} = (D\Phi)_i \frac{\delta}{\delta(\partial\overline{\omega})_i} V^{(n-1)} + (\partial\overline{\omega})_i \partial_i^{\alpha} \frac{\delta}{\delta L_\alpha} [V^{(n)} + U^{(n)}], \qquad (4.91)$$

with

$$(D\Phi)_i \frac{\delta}{\delta(\partial\overline{\omega})} V^{(n)} = 0.$$

Thus, our inductive argument is complete and it terminates when $n - 1 = 0$ is reached. That is

$$Z = \left[(D\Phi)_i \frac{\delta}{\delta(\partial\overline{\omega})_i} + (\partial\overline{\omega})_i \partial_i^{\alpha} \frac{\delta}{\delta L_\alpha}\right] [V + U], \qquad (4.92)$$

where

$$V = \sum_{n=0}^{m} \frac{1}{n!} V^{(n)},$$

$$U = \sum_{n=0}^{m} \frac{1}{n!} U^{(n+1)}.$$

This completes the proof of Eq. (4.60) and, noting the arguments given between Eqs. (4.48) and (4.60), we conclude that all the solutions of the Eq. (4.3) containing equal numbers of ghosts and antighosts and at least one of each can be expressed as $\mathscr{G}'\, F[\Phi, \omega, \bar{\omega}, L]$. Since we had shown earlier that all other solutions with an equal number of $c, \bar{c}$'s can be expressed as, $\{\mathscr{G}' F[\Phi, \omega, \bar{\omega},]\} \oplus \{O^{GI}[\Phi]\}$, the main theorem is proved.[14]

We may now expand $F[\Phi, \omega, \overline{\omega}, L]$ as

$$F[\Phi, \omega, \overline{\omega}, L] = F[\Phi, \omega, \overline{\omega}] - L_\alpha X_\alpha + O(L^2). \qquad (4.93)$$

Then

$$\mathscr{G}' F[\Phi, \omega, \overline{\omega}, L]\Big|_{L=0} \left(\mathscr{G}_0 + \frac{\delta\tilde{S}}{\delta\Phi_i} \frac{\delta}{\delta(\partial\overline{\omega})_i}\right) F[\Phi, \omega, \overline{\omega}] + \frac{\delta\tilde{S}}{\delta\omega_\alpha} X_\alpha[\Phi, \omega, \overline{\omega}]. \qquad (4.94)$$

V. STRUCTURE OF THE RENORMALIZATION MATRIX

In this section we shall prove another important result. It was shown in the last section that the complete set of gauge noninvariant operators that mix with a given set of gauge invariant operators can be expressed as

$$O[\Phi, \omega, \overline{\omega}] = \frac{\delta \tilde{S}}{\delta \Phi_i} \frac{\delta F}{\delta (\partial \overline{\omega})_i} + \frac{\delta \tilde{S}}{\delta \omega_\alpha} X_\alpha + \mathscr{G}_0 F, \tag{5.1}$$

where F and X_α are arbitrary local polynomials of appropriate ghost number and appropriate global and Lorentz transformation properties. It is clear that one can expand the divergence in $\Gamma_N[\Phi, \omega, \overline{\omega}]$ in the basis that consists of all independent linear combinations of the operators $\{O^{GI}[\Phi]\}' \oplus \{O[\Phi, \omega, \overline{\omega}]\}$. It is important to recognize the trivial fact that the matrix elements of the matrix Z, will depend on the basis one has chosen; and therefore any statement about relationships of matrix elements of Z with its eigenvalues are bound to be basis dependent, in general.

In this section,we shall show that mixing with gauge noninvariant operators (and ghost operators) can be made irrelevant in the limited sense that there exists a basis of operators such that when the matrix Z is expressed in this basis, the diagonal submatrix Z^{GI} of Z yields, in fact,the correct eigenvalues corresponding to all eigenvectors of Z which involve gauge invariant operators. The basis is, in fact,

$\{O^{GI}[\Phi]\}' \oplus$ gauge noninvariant operators of the form $O[\Phi, \omega, \overline{\omega}]$ of Eq. (5.1); and the diagonal submatrix Z^{GI} relates the divergences in the insertions of gauge invariant operators which are proportional to $\{O^{GI}[\Phi]\}'$ when expanded in this basis.

To shown this, it is sufficient to show that the divergence in the insertion of a gauge non invariant operator of the form $O[\Phi, \omega, \overline{\omega}]$ of Eq. (5.1) does not contain terms proportional to gauge invariant operators $\{O^{GI}\}'$ when expanded in the above basis.

We consider the generating functional of Green's functions

$$W[j, N] = \int [dAdcd\overline{c}] \exp i \Big\{ \tilde{S}[A, c, \overline{c}] - \frac{1}{2} \{f_\alpha[A]\}^2 + j_i A_i + N\, O[A, c, \overline{c}] \Big\}, \tag{5.2}$$

$$O[A, c, \overline{c}] = \frac{\delta \tilde{S}}{\delta A_i} \frac{\delta F[A, c, \overline{c}]}{\delta (\partial \overline{c})_i} + \frac{\delta \tilde{S}}{\delta c_\alpha} X_\alpha [A, c, \overline{c}] + \mathscr{G}_0 F[A, c, \overline{c}]. \tag{5.3}$$

To see if gauge invariant operators appear in the divergence with a single insertion it is sufficient to compute $\Gamma_N[\Phi, \omega = 0, \overline{\omega} = 0]$, i.e., it suffices to consider W as a function of j omitting the dependence on ghost sources. We perform an N-dependent transformation of the integration variables:

$$A_i = A_i' - N \delta F[A, c, \overline{c}] / \delta (\partial \overline{c})_i .$$

$$c_\alpha = c_\alpha' - N X_\alpha [A, c, \overline{c}], \quad \overline{c} = \overline{c}' . \tag{5.4}$$

We also note that F and X_α are local functionals and the Jacobian J of these transformation is

$$\ell n\, J = N \left(\frac{\delta F}{\delta A_i\, \delta(\partial \bar{c})_i} \quad \frac{\delta X_\alpha}{\delta c_\alpha} \right)$$

and $\ell n\, J$ is proportional to $\delta^4(0)$ or derivatives of $\delta^4(x)$ at $x = 0$. We shall use dimensional regularization wherein $\delta^4(0)$ or are interpreted to be zero. Thus $W[j, N]$ is given by

$$W[j,N] = \int [dA'dc'd\bar{c}'] \exp i \left\{ \tilde{S}[A', c', \bar{c}'] - \frac{1}{2}\eta_0 \left[\partial_i^\alpha \left[A_i' - N \frac{\delta F}{\delta(\partial \bar{c})_i} \right] \right]^2 \right.$$

$$\left. + J_i A_i + \mathscr{G}_0 F - J_i N \frac{\delta F}{\delta(\partial \bar{c})_i} \right\} + 0\,(N^2) \quad . \tag{5.5}$$

Omitting primes, we obtain

$$\frac{1}{W[j]} \frac{\delta W[j,N]}{\delta N} \bigg|_{N=0} \equiv \langle O \rangle_j \tag{5.6}$$

$$= \int [dAdcd\bar{c}] \left\{ -j_i \frac{\delta F}{\delta(\partial \bar{c})_i} + \eta_0\, \partial_i^\alpha A_i\, \partial_i^\alpha \frac{\delta F}{\delta(\partial \bar{c})_i} + \mathscr{G}_0 F \right\}$$

$$\exp i \{ \mathscr{L}_{eff} + j_i A_i \}$$

$$\equiv -j_i \left\langle \frac{\delta F}{\delta(\partial \bar{c})_i} \right\rangle_j + \eta_0 \left\langle \partial_j^\alpha A_j\, \partial_i^\alpha \frac{\delta F}{\delta(\partial \bar{c})_i} + \mathscr{G}_0 F \right\rangle_j \quad . \tag{5.7}$$

Next consider,

$$\langle F \rangle_j = \int [dAdcd\bar{c}]\, F[A, c, \bar{c}]\, \exp i \{ \mathscr{L}_{eff}[A, c, \bar{c}] + j_i A_i \} . \tag{5.8}$$

We perform the BRS transformations on the integration variables given in Eq. (2.6) and equate the total change to zero. Noting that the Jacobian of the transformation is one we obtain.

$$0 = \int [dA dc d\bar{c}] \left\{ \mathcal{G}_0 F + \frac{\delta F}{\delta(\partial \bar{c})_i} \eta_0 \partial^\alpha_j A_j \partial^\alpha_i - i j_i D^\alpha_i c_\alpha F \right\} \exp i$$

$$\left\{ \mathcal{L}_{\text{eff}} + j_i A_i \right\} , \tag{5.9}$$

i.e.
$$\left\langle \mathcal{G}_0 F + \eta_0 \partial^\alpha_i \partial^\alpha_j A_j \frac{\delta F}{\delta(\partial \bar{c})_i} \right\rangle_j = i j_i \left\langle D^\beta_i c_\beta F \right\rangle_j . \tag{5.10}$$

Using Eq. (5.10) in Eq. (5.7), we get

$$\left\langle O \right\rangle_j = - j_i \left\langle \frac{\delta F}{\delta(\partial \bar{c})_i} \right\rangle_j + i j_i \left\langle D^\beta_i c_\beta F \right\rangle_j . \tag{5.11}$$

We write

$$- j_i [\phi] = \frac{\delta \Gamma_0}{\delta \phi_i} = \frac{\delta \Gamma_0[\phi]}{\delta \phi_i} - \eta_0 \partial^\alpha_i \partial^\alpha_j \phi_j , \tag{5.12}$$

and thus obtain,

$$\left\langle O \right\rangle_j = \frac{\delta \Gamma_0[\phi]}{\delta \phi_i} \left\{ \left\langle \frac{\delta F}{\delta(\partial \bar{c})_i} \right\rangle_j - i \left\langle D^\alpha_i c_\alpha F \right\rangle_j \right\}$$

$$- \eta_0 \partial^\alpha_j \phi_j \left\langle \partial^\alpha_i \frac{\delta F}{\delta(\partial \bar{c})_i} - \partial^\alpha_i D^\beta_i c_\beta F \right\rangle_j . \tag{5.13}$$

The last term vanishes since

$$\left\langle \partial_i^\alpha \frac{\delta F}{\delta(\partial\overline{c})_i} - \partial_i^\alpha D_i^\beta c_\beta F \right\rangle_j = \left\langle \frac{\delta F}{\delta \overline{c}_\alpha} - i\ (Mc)_\beta F \right\rangle_j$$

$$= \int [dA]\, e^{i\left(\mathcal{L}_0 - 1/2 f_\alpha^2 + j_i A_i\right)} \int [dc d\overline{c}]\, \frac{\delta}{\delta \overline{c}_\alpha} \left[F e^{i\overline{c}Mc} \right] = 0 . \qquad (5.14)$$

Therefore,

$$\left\langle O \right\rangle_j = \frac{\delta \Gamma_0[\phi]}{\delta \phi_i} \left\langle \frac{\delta F}{\delta(\partial\overline{c})_i} - i\ D_i^\alpha c_\alpha F \right\rangle_j \qquad (5.15)$$

$$= \frac{\delta \Gamma_0^{(r)}[\Phi]}{\delta \Phi_i} \left\{ Z^{-1/2} \left\langle \frac{\delta F}{\delta(\partial\overline{c})_i} - i\ D_i^\alpha c_\alpha F \right\rangle_j \right\} \qquad (5.16)$$

$$\equiv \frac{\delta \Gamma_0^{(r)}[\Phi]}{\delta \Phi_i} \mathscr{F}_i [\phi] . \qquad (5.17)$$

$\mathscr{F}_i[\phi]$ is to be renormalized in each loop approximation $\{\mathscr{F}_i[\phi]\}_n^{div}$, the overall divergence in $\mathscr{F}_i[\phi]$ expressed in terms of renormalized fields and parameters, can <u>always</u> be expressed in terms of a complete set of functionals with the same Lorentz and global transformation properties and dimensions as $\mathscr{F}_i[\phi]$. Let this set be $\{F_i^{(p)}[\Phi]\}$.

Then
$$\left\{ \mathscr{F}_i[\phi] \right\}_n^{div} = \sum_p A_p^{(n)} F_i^{(p)} [\Phi] . \qquad (5.18)$$

Thus
$$\left\{ \left\langle 0 \right\rangle_j \right\}_n^{div} = \frac{\delta \mathcal{L}_0[\Phi]}{\delta \Phi_i} \sum_p A_p^{(n)} F_i^{(p)}[\Phi] . \qquad (5.19)$$

But $\delta \mathcal{L}_0/\delta\Phi_i \ \ F_i^{(p)}[\Phi]$ are precisely the ghost independent terms of the gauge non-invariant operators in our basis: $\mathscr{G}' F_i^{(p)} (\partial\overline{c})_i$. It should be noted that, in the set of gauge non-invariant operators $\mathscr{G}' F$, F runs over all functionals of the appropriate kind. Therefore, $\{\langle O\rangle_j\}_n^{div}$ $[\Phi, \omega, \partial\overline{\omega}]$ is expressible as a linear combination of O's in Eq. (5.1), not including O_{GI}'s.

We further note that in Eq. (5.1), functionals $F[\Phi, \omega, \overline{\omega}]$ and $X_\alpha[\Phi, \omega, \overline{\omega}]$ are uncorrelated, so that it is convenient to separate the basis into two parts.

$$\begin{aligned}
&\text{class I} \qquad \text{operators: } \frac{\delta\tilde{S}}{\delta\Phi_i}\ \frac{\delta F}{\delta(\partial\overline{\omega})_i} + \mathscr{G}_0 F \\
&\text{class II GNI operators: } \frac{\delta\tilde{S}}{\delta\omega_\alpha}\ X_\alpha[\Phi, \omega, \overline{\omega}] \ .
\end{aligned} \tag{5.20}$$

[class I operators, in addition to GNI operators contains GI operators which have the form $\delta\mathcal{L}_0/\delta A_i \ \ F_i[A]$. The divergence in the single insertion of a class II operator can be expanded in terms of class II operators only. To see this, consider the generating functional:

$$W[j, \xi, \overline{\xi}, N] = \int [dA\, dc\, d\overline{c}] \exp i \left\{ \tilde{S}[A, c, \overline{c}] - \frac{1}{2}\{f_\alpha\}^2 + j_i A_i + N \frac{\delta\tilde{S}}{\delta c_\alpha} X_\alpha \right\} + \overline{\xi}_\alpha c_\alpha + \overline{c}_\alpha \xi_\alpha \ . \tag{5.21}$$

By performing a transformation on c_α only

$$c_\alpha \rightarrow c_\alpha + N X_\alpha \ , \tag{5.22}$$

and noting that the change in the Jacobian is zero $[X_{\alpha,\alpha} \sim \delta^4(0)]$ in dimentional regularization. We obtain (dropping primes)

$$W[j,\xi,\bar{\xi},N] = \int [dAdcd\bar{c}] \exp i \left\{ \tilde{S}[A,c,\bar{c}] - \frac{1}{2} f_\alpha^2 + j_i A_i + c_\alpha \xi_\alpha + \bar{\xi}_\alpha c_\alpha + \bar{\xi}_\alpha N X_\alpha \right\}$$

$$+ O(N^2) \text{ terms} . \tag{5.23}$$

Thus

$$\left(\frac{1}{W} \frac{\delta W}{\delta N} \right)_{N=0} = \bar{\xi}_\alpha \left\langle X_\alpha \right\rangle_{j,\xi,\bar{\xi}}$$

$$= \frac{\delta \Gamma}{\delta \Omega_\alpha} \left\langle X_\alpha \right\rangle_{j,\xi,\bar{\xi}} = \frac{\delta \Gamma^{(r)}[\Phi,\omega,\bar{\omega}]}{\delta \omega_\alpha} \left\langle \tilde{Z}^{-1/2} X_\alpha \right\rangle_{j\xi\bar{\xi}} , \tag{5.24}$$

$\left\langle X_\alpha \right\rangle_{j,\xi,\bar{\xi}}$ is a functional of $\Phi, \omega, \bar{\omega}$. It is to be renormalized as we renormalized F. [See the discussion following Eq. (5.17).] Thus,

$$\left\{ \Gamma_N[\Phi,\omega,\bar{\omega}] \right\}_n^{\text{div}} = \left\{ \frac{1}{W} \frac{\delta W[j,\xi,\bar{\xi},N]}{\delta N} \Bigg|_{N=0} \right\}_n^{\text{div}}$$

$$= \frac{\delta \tilde{S}[\Phi,\omega,\bar{\omega}]}{\delta \omega_\alpha} \left\{ \left\langle \tilde{Z}^{-1/2} X_\alpha \right\rangle \right\}_n^{\text{div}} \tag{5.25}$$

and is expressible as the sum of class II GNI operators only.

We have thus shown that the matrix Z has the structure:

$$Z = \begin{array}{c|c|c|c|} & \mathrm{GI}' & \mathrm{I} & \mathrm{II} \\ \mathrm{GI}' & Z^{\mathrm{GI}'} & & \\ \hline \mathrm{I} & 0 & & \\ \mathrm{II} & 0 & 0 & \end{array} \tag{5.26}$$

It is obvious that eigenvalues of $Z^{GI'}$ are also the eigenvalues of Z. Furthermore the eigenstates of Z corresponding to these engenvalues are the only eigenstates which involve a gauge invariant operators $\left\{O^{GI}\right\}'$ in this basis. Thus the eigenvalues corresponding to the eigenstates of Z which involve a gauge invariant operator in this basis can be obtained only by computing $Z^{GI'}$. One cannot, however, compute these eigenstates simply by knowing $Z^{GI'}$. In practice, $Z^{GI'}$ can be computed by finding some distinguishing property of $\left\{O^{GI}[\Phi]\right\}'$ that sets it apart from the gauge non-invariant operators with which it mixes. For example in the case of the twist two operators, only the 2-point function of $O^{GI'}[\Phi]$ $\left[\mathrm{F.T.}(\delta O^{GI}[\Phi]/\delta \Phi^{\alpha}_{\mu}\,\delta \Phi^{\alpha}_{\nu})\right]$ contains terms proportional to $g_{\mu\nu}$ while the G.N·I operators with which it mixes do not.

APPENDIX A.

In this appendix we wish to show that a local solution of the equation

$$\mathcal{L}_{0,i}[\Phi]\, P_i^\alpha[\Phi] = 0\,, \tag{A1}$$

has a structure discussed in Lemma A2 below. We begin with a series of definitions and a lemma.

We define

$$D_{ij} = \delta_{\alpha\beta}\,(\partial^2 g_{\mu\nu} - \partial_\mu \partial_\nu)\,\delta^4(x-y)\,, \tag{A2}$$

where $i = (\alpha, \mu, x)$ and $J = (\beta, \nu, y)$. Note that

$$\mathcal{L}_{0,ij}[\Phi = 0] = -D_{ij}\,. \tag{A3}$$

We define the transverse and longitudinal projection operators, T_{ij} and L_{ij} by

$$D_{ij} = \partial_i^2\, T_{ij}\,, \quad \delta_{ij} = T_{ij} + L_{ij}\,, \tag{A4}$$

which satisfy

$$T_{ik}\, T_{kj} = T_{ij}, \quad L_{ik} L_{kj} = L_{ij}\,, \tag{A5}$$

$$T_{ik} L_{kj} = L_{ik}\, T_{kj} = 0\,. \tag{A6}$$

We consider first a simpler case.

<u>Lemma</u> A1: <u>Any functional</u> $P_i^\alpha[\Phi]$ <u>which contains at most</u> n <u>fields</u>, <u>and which satisfies</u>

$$P_i^\alpha[\Phi]\; D_{ij}\,\Phi_j = 0\,, \tag{A7}$$

can be represented as

$$P_i^\alpha[\Phi] = Z^{\alpha\beta}[\Phi]\,\partial_i^\beta + V_{ij}^\alpha[\Phi]\,D_{jk}\,\Phi_k\,, \tag{A8}$$

where

$$V^\alpha_{i_1 i_2}[\Phi] = \sum_{r=1}^{n} V^{\alpha(r)}_{i_1 i_2 \ldots i_{r+1}}[\Phi] \prod_{s=3}^{r+1}\left(D_{i_s k}\,\Phi_k\right)\,, \tag{A9}$$

with

$$\sum_{\{i\}} V^{\alpha(r)}_{i_1 i_2 \ldots i_{r+2}} = 0\,. \tag{A10}$$

The symbol $\sum_{\{i\}}$ denotes sumnation over permutations of the $r+2$ indices $\{i_1, i_2, \ldots, i_{r+2}\}$.

Proof of A1: It suffices to consider a monomial

$$P_i^\alpha[\Phi] = P^\alpha_{i, i_1 \ldots i_n}\,\Phi_{i_1}\,\Phi_{i_2} \ldots \Phi_{i_n}\,. \tag{A11}$$

The kernel $P^\alpha_{i, i_1 \ldots i_n}$ is completely symmetric in the n indices $\{i_1, \ldots i_n\}$. Equation (A7) then implies

$$\sum_{\{i_1, \ldots, i_{n+1}\}} P^\alpha_{i, i_1 \ldots i_n}\, D_{i, i_{n+1}} = 0\,, \tag{A12}$$

where the summation is over permutations of the $n+1$ indices $\{i_1, \ldots, i_{n+1}\}$.

We can decompose $P^\alpha_{i, i_1 \ldots i_n}$ with respect to each of its indices using the partition of identity (A4). For example, we write

$$P^{\alpha}_{i,i_1\ldots i_n} = L_{ik}P^{\alpha}_{k,i_1\ldots i_n} + T_{ik}P^{\alpha}_{k,i_1\ldots i_n} . \tag{A13}$$

The first term on the right does not contribute at all in Eq. (A12), and yields the first term on the right of Eq. (A8), $Z^{\alpha\beta}\partial^{\beta}_{i}$ with a suitably defined functional $Z^{\alpha\beta}$. From now we shall assume that $P^{\alpha}_{i,i_1\ldots i_n}$ is transverse with respect to i.

We shall write

$$P^{\alpha}_{i,i_1\ldots i_n} = \sum_{s=0}^{n} \sum_{\{i_1,\ldots,i_n\}} P^{\alpha(s)}_{i,i_1\ldots i_s;i_{s+1}\ldots i_n} \tag{A14}$$

where $P^{\alpha(s)}_{i,i_1\ldots i_s;i_{s+1},\ldots,i_n}$ is transverse with respect to i_j, $1 \le j \le s$, and is longitudinal with respect to i_j, $j \ge s+1$. Substituting Eq. (A14) in Eq. (A12) and setting

$$P^{\alpha(s)}_{j,i_1\ldots i_s;i_{s+1}\ldots i_n} = V^{\alpha(s)}_{j\,j_1 j_2\ldots j_s;\beta_{s+1}\ldots\beta_n}$$

$$\times D_{ji}D_{j_1 i_1}\ldots D_{j_s i_s}\,\partial^{\beta_{s+1}}_{i_{s+1}}\ldots\partial^{\beta_n}_{i_n} , \tag{A15}$$

we obtain

$$\Sigma_{\{i i_1\ldots i_s\}} V^{\alpha(s)}_{j\,j_1 j_2\ldots j_s;\beta_{s+1}\ldots\beta_n}$$

$$\times D_{ji}D_{j_1 i_1}\ldots D_{j_s i_s}\,\partial^{\beta_{s+1}}_{i_{s+1}}\ldots\partial^{\beta_n}_{i_n} = 0 . \tag{A16}$$

For s = 0, we must have

$$V^{\alpha(0)}_{j;\beta_1\ldots\beta_n} = 0 . \tag{A17}$$

For $s \neq 0$, we may set, without loss of generality,

$$\Sigma_{\{j\, j_1 \dots j_s\}} V^{\alpha(s)}_{j\, j_1 \dots j_s; \beta_{s+1} \dots \beta_n} = 0 \quad . \tag{A18}$$

The structure implied by Eqs. (A9) and (A10) follows from (A17) and (A18). This completes the proof of Lemma A1.

For later use, we note the special case when $P^{\alpha}_i [\Phi]$ is restricted to be a local functional. This means that

$$P^{\alpha}_{i, i_1 \dots i_n} = f(\partial_j)\, \delta^4(x_1 - x_\alpha)\, \delta^4(x_{i_1} - x_\alpha) \dots \delta^4(x_{i_n} - x_\alpha), \tag{A19}$$

where $f(\partial_j)$ is a polynomial in differential operators. In general, a functional of the transverse and longitudinal components of the vector field Φ_j:

$$\Phi_j = \Phi^T_j + \partial^{\alpha}_j \Phi^L_\alpha \ ,$$

or

$$\Phi^T_j = (\delta_{ji} - \partial_j \frac{1}{\partial^2} \partial_i)\Phi_i \tag{A20}$$

$$\Phi^L_\alpha \sim \partial^{\alpha}_i \frac{1}{\partial^2} \partial^{\alpha}_j \Phi_j$$

is not a local unless Φ^T_j and Φ^L_α enter the expression (i) through the combination $\Phi^T_j + \partial^{\alpha}_j \Phi^L_\alpha$, or (ii) in the form $\partial^2 \Phi^T_j = D_{ji} \Phi_i$ and $\partial^2 \Phi^L_\alpha = \partial^{\alpha}_i \Phi_i$.

Consider the term in Eq. (A14) which contains the highest power of Φ^T_i, i.e., the term with $s = n$. In this case Eq. (A18) reads

$$\Sigma_{\{j, j_1 j_2 \cdots j_n\}} V^{\alpha(n)}_{j\, j_1 \cdots j_n} = 0 \tag{A21}$$

Suppose now $V^{\alpha(n)}_{j\, j_1 \cdots j_n}$ has a factor $\partial^{-2}_{j_1} \partial^{-2}_{j_2} \cdots \partial^{-2}_{jn}$ (note that $V_{jj_1 \cdots j_n}$ must be symmetric in $j_1, \ldots j_n$). Then Eq. (A21) tells us that $V_{jj_1 \cdots j_n}$ must have a factor ∂^{-2}_{j}, and consequently from Eq. (A15) $P^{\alpha(n)}_{j, i_1 \cdots i_n}$; is not local. Therefore $V^{\alpha(n)}_{jj_1 \cdots j_n}$ must be local. Repeating this argument, and taking into account the points (i) and (ii) above, we find that for a local functional satisfying (A7), $V^{\alpha(r)}_{i_1 \cdots i_{r+1}}$ must be a local functional of $\partial^{\alpha}_{i} \Phi_i$.

We are now prepared to prove the main lemma.

Lemma A2: Any local functional $P^{\alpha}_{i}(\Phi)$ of Φ_j which satisfies:

$$P^{\alpha}_{i} [\Phi] \mathscr{L}_{0,i} [\Phi] = 0 \tag{A22}$$

can be represented as

$$P^{\alpha}_{i}[\Phi] = X^{\alpha\beta}[\Phi]\, D^{\beta}_{i}[\Phi] + Y^{\alpha}_{ij}[\partial\Phi]\, \mathscr{L}_{0,j}[\Phi] \tag{A23}$$

where $X^{\alpha\beta}$ and Y^{α}_{ij} are suitable functionals of Φ_k.

Since $P^{\alpha}_{i}[\Phi]$ is restricted to a local functional we can meaningfully discuss its dimension. Since $\mathscr{L}_{0,i}$ has a definite dimension, we may assume, without loss of generality, that $P^{\alpha}_{i}[\Phi]$ has a definite dimension N. We prove this lemma by means of the previous one.

Proof of A2: We expand $P_i^\alpha[\Phi]$ as

$$P_i^\alpha[\Phi] = \sum_{n=0}^{N} P^\alpha_{i j_1 \cdots j_n} \Phi_{j_1} \cdots \Phi_{j_n} , \tag{A24}$$

and substitute it in Eq. (A22). We equate the coefficient of lowest power of Φ to zero and obtain

$$P_i^\alpha[\Phi = 0]\, D_{ij}\, \Phi_j = 0 . \tag{A25}$$

By the previous lemma, we have

$$P_i^\alpha[\Phi = 0] = X^{\alpha\beta} \partial_i^\beta . \tag{A26}$$

We now define

$$P_i^{\alpha(1)}[\Phi] = P_i^\alpha[\Phi] - X^{\alpha\beta} D_i^\beta[\Phi] . \tag{A27}$$

$P_i^{\alpha(1)}$ is a local functional with dimension N, and begins with first power of Φ, and satisfies

$$P_i^{\alpha(1)}[\Phi]\, \mathscr{L}_{0,i}[\Phi] = 0 . \tag{A28}$$

Obviously this process can be continued. Assume $P_i^{\alpha(n)}$ is a local functional of dimension N and begins with Φ^n, and satisfies

$$P_i^{\alpha(n)}[\Phi]\, \mathscr{L}_{0,i}[\Phi] = 0. \tag{A29}$$

Let the Φ^n term of $P_i^{\alpha(n)}$ be $[P_i^{\alpha(n)}]$. Then

$$[P_i^{\alpha(n)}]\, D_{ij}\, \Phi_j = 0. \tag{A30}$$

Then by Lemma A1, we have

$$[P_i^{\alpha(n)}] = X^{\alpha\beta(n)}[\Phi]\, \partial_i^{\beta} + \sum_{r=1}^{n} V^{\alpha(n,r)}_{ij_1\cdots j_r}[\partial_k^{\beta}\Phi_k] \prod_{s=1}^{r} (D_{j_s \ell}\Phi_\ell) \tag{A31}$$

where $X^{\alpha\beta(n)}$ and $V^{\alpha(n,r)}$ are n^{th} and $(n-r)^{th}$ order functionals of Φ of dimension N - 1 and N -3r, respectively. We then construct

$$P_i^{\alpha(n+1)}[\Phi] = P_i^{\alpha(n)}[\Phi]$$
$$-X^{\alpha\beta(n)}[\Phi]\; D_i^{\beta}[\Phi] - \sum_{r=1}^{n} V_{i\,j_1\cdots j_r}^{\;\alpha(n,r)}[\partial\Phi] \prod_{s=1}^{r} \mathscr{L}_{0,j_s}[\Phi]$$
$$P_i^{\alpha(n+1)}[\Phi]\, \mathscr{L}_{0,i}[\Phi] = 0\,, \tag{A32}$$

which is a local functional of dimension N and begins with Φ^{n+1}.

This process must terminate at some step, because the only solution to Eq. (A30) for a fixed dimension N and large enough n is zero. Suppose that $[P_i^{\alpha(n)}] = 0$ for $n \geq m$. Then $P_i^{\alpha(m)} = 0$.

We can solve the system of equations [A32] for $n = 0, 1, \ldots, m-1$ for $P_i^{\alpha} = P_i^{\alpha\,(0)}$ and find that it has the form

$$P_i^{\alpha}[\Phi] = X^{\alpha\beta}[\Phi]\; D_i^{\beta}[\Phi] + Y_{ij}^{\alpha}[\partial\Phi]\, \mathscr{L}_{0,j}[\Phi].$$

APPENDIX B

Definitions: We define the symmetrizer $S[i_1 \ldots i_p]$ by

$$S[i_1 \ldots i_p]\, f_{i_1 \ldots i_p A} = \frac{1}{p!} \sum_{\{i_1 \ldots i_p\}} f_{i_1 \ldots i_p A}, \tag{B1}$$

where the summation is over permutations of the p indices $\{i_1 \ldots i_p\}$ and A denotes collectively other indices. We define the complement $\bar{S}[i_1 \ldots i_p]$ of $S[i_1 \ldots i_p]$ by

$$\bar{S}[i_1 \ldots i_p] + S[i_1 \ldots i_p] = 1 \quad . \tag{B2}$$

The symmetrizer and its complement are projection operators and commute with $\mathscr{G}$ of Eq. (4.28)

$$\begin{aligned} S^2 &= S, \quad \bar{S}^2 = \bar{S} \;, \\ S\bar{S} &= 0 = \bar{S}S \;, \end{aligned} \tag{B3}$$

$$[\mathscr{G}, S] = 0 = [\mathscr{G}, \bar{S}] \quad . \tag{B4}$$

Lemma BI:

All solutions of

$$\sum_{r=1}^{n} C^{\gamma}_{\eta i_1 \ldots \hat{i}_r \ldots i_n} [\Phi]\; D^{\gamma}_{i_r} = 0 \tag{B5}$$

can be expressed as

$$C^{\gamma_1}_{\eta i_1 \ldots \hat{i}_r[\Phi] \ldots i_n} = \sum_{t=2}^{n-1} E^{(\gamma_1 \ldots \gamma_t)}_{\eta i_1 \ldots \hat{i}_r \underbrace{\hat{i}_p[\Phi] \ldots \hat{i}_q}_{t=1} \ldots i_n} D^{\gamma_2}_{i_p} \ldots D^{\gamma_t}_{i_q} \tag{B6}$$

with

$$S[\gamma_1 \ldots \gamma_t] E^{(\gamma_1 \ldots \gamma_t)} = 0 . \tag{B7}$$

(A hat denotes the absence of that index.)

Proof of BI:

The proof proceeds much the same way as the lemma A II in Appendix A. We can always assume that C^γ has a definite dimensions. We assume that $C^\gamma_{i_1 \ldots \hat{i}_r \ldots i_n}$ begins with a term $\sim \Phi^m$. We denote the Φ^m terms in C^γ by $C^{\gamma(m)}$. Equating the Φ^m terms in Eq. (B5) to zero we obtain

$$\sum_r C^{\gamma(m)}_{\eta i_1 \ldots \hat{i}_r[\Phi] \ldots i_n} \partial^\gamma_{i_r} = 0 . \tag{B8}$$

Now, $C^{\gamma(m)}_{\eta i_1 \ldots i_r \ldots i_n, \{j\}} \equiv C_{\eta i_1 \ldots i_r \ldots i_n, j_1 \ldots j_m}$ can be expanded in powers of momenta associated with $i_1 \ldots i_n$ which are independent. We write

$$C^{\gamma(m)}_{\eta i_1 \ldots \hat{i}_r \ldots i_n} = \bar{C}^{\gamma(m)}_{\eta i_1 \ldots \hat{i}_r \ldots i_n} + \sum_p \partial^\alpha_{i_p} C^{\gamma\alpha(m)}_{i_1 \ldots \hat{i}_r \hat{i}_p \ldots i_n} +$$

$$+ \sum_{pq} \partial^\alpha_{i_p} \partial^\beta_{i_p} C^{\gamma\alpha\beta(m)}_{i_1 \ldots \hat{i}_r \hat{i}_p \hat{i}_q \ldots i_n} + \ldots . \tag{B9}$$

Thus Eq. (B8) implies

$$\bar{C}^{\gamma(m)}_{\eta i_1 \ldots \hat{i}_r \ldots i_n} = 0 \tag{B.10}$$

$$S[\gamma, \alpha]\, C^{\gamma\alpha(m)}_{\eta i_1 \ldots \hat{i}_r \hat{i}_p \ldots i_n} = 0. \tag{B11}$$

$$S[\gamma_1 \ldots \gamma_t]\, C^{\gamma_1 \ldots \gamma_t}_{\eta i_1 \ldots \underbrace{\hat{i}_r \hat{i}_p \ldots \hat{i}_q}_{t} \ldots i_n} = 0, \text{ etc.} \tag{B12}$$

We then form

$$C^{\gamma[m+1]}_{\eta i_1 \ldots \hat{i}_r \ldots i_n} = C^{\gamma}_{\eta i_1 \ldots \hat{i}_r \ldots i_n} - \Big[\sum_p D^{\alpha}_{i_p} C^{\gamma\alpha(m)}_{\eta i_1 \ldots \hat{i}_r \hat{i}_p \ldots i_n}$$

$$+ \sum_{p,q} D^{\alpha}_{i_p} D^{\beta}_{i_q} C^{\gamma\alpha\beta(m)}_{\eta i_1 \ldots \hat{i}_r \hat{i}_p \hat{i}_q \ldots i_n} + \ldots \Big] . \tag{B13}$$

Thus $C^{\gamma[m+1]}$ of Eq. (B13) satisfies Eq. (B5) and begins as terms Φ^{m+1}. This process can be continued, in each step increasing the power of Φ in C^{γ} by one. The process will terminate when the power of Φ exceeds the dimension of C^{γ}. Then Eq. (B13) together with its analogues imply the structure of Eq. (B6) for C^{γ}.

Lemma BII:

Suppose a local functional $P_i[\Phi, \omega]$ satisfies

$$\mathscr{G}(P_i[\Phi,\omega]) = \omega_{\alpha} X^{\alpha}[\Phi, \omega]\; D^{\gamma}_i[\Phi] \tag{B14}$$

where P_i may carry additional indices. Then there exists a local functional $Z^{\beta}[\Phi, \omega]$ such that

$$\mathscr{G}(P_i[\Phi,\omega] - Z^\beta[\Phi,\omega]\, D_i^\beta[\Phi]) = 0 \;. \tag{B15}$$

Proof of BII: Eq. (B14) implies:

$$\mathscr{G}(\omega_\alpha X^{\alpha\gamma}[\Phi,\omega])\, D_i^\gamma = 0 \;. \tag{B16}$$

We multiply Eq. (B11) by ∂_i^σ. Since $\partial_i^\sigma D_i^\gamma = M^{\sigma\gamma}[\Phi]$ is invertible, we have

$$\mathscr{G}(\omega_\alpha X^{\alpha\gamma}) = 0 \;. \tag{B17}$$

We can now apply the Dixon-Taylor lemma to Eq. (B17). This together with the discussion of locality in Appendix C, implies

$$\omega_\alpha X^{\alpha\gamma} = \mathscr{G}(Z^\gamma) \;, \tag{B18}$$

for some local $Z^\gamma[\Phi,\omega]$. Now since $\mathscr{G}(D_i^\gamma) = 0$,

$$\mathscr{G}[Z^\beta[\Phi,\omega]\, D_i^\beta[\Phi]] = \omega_\alpha X^{\alpha\gamma} D_i^\gamma \tag{B19}$$

QED.

Lemma BIII:

Given the equation

$$\mathscr{G}\left[\sum_{r=1}^{n} C^\gamma_{i_1\ldots i_r\ldots i_n}[\Phi,\omega]\, D^\gamma_{i_r}\right] = 0 \;, \tag{B20}$$

we can always redefine $C^\gamma_{i_1\ldots\hat{i}_r\ldots i_n}$ without altering $\left[\sum_{r=1}^{n} C^\gamma_{i_1\ldots\hat{i}_r\ldots i_n} D^\gamma_{i_r}\right]$ such that

$$\mathscr{G}\left[C^\gamma_{i_1\ldots\hat{i}_r\ldots i_n}[\Phi,\omega]\right] = 0, \tag{B21}$$

where C's are local functionals and may carry additional indices.

Proof of BIII: Eq. (B20) implies

$$\sum_{r=1}^{n} \mathscr{G}(C^{\gamma}_{i_1 \ldots \hat{i}_r \ldots i_n}) D^{\gamma}_{i_r} = 0 \ . \tag{B22}$$

Lemma BI as applied to Eq. (B22) implies

$$\begin{aligned} \mathscr{G}(C^{\gamma_1}_{i_1 \ldots \hat{i}_r \ldots i_n}) &= \sum_{t=2} E^{\gamma_1 \ldots \gamma_t}_{i_1 \ldots \underbrace{\hat{i}_r \hat{i}_p \ldots \hat{i}_q}_{t} \ldots i_n} \\ &\quad \times D^{\gamma_2}_{i_p} \ldots D^{\gamma_t}_{i_q} \ , \end{aligned} \tag{B23}$$

with

$$S[\gamma_1 \ldots \gamma_t] E^{\gamma_1 \ldots \gamma_t} = 0 \ . \tag{B24}$$

Here and henceforth we suppress suffixes whenever inessential.

We define

$$\begin{aligned} E'^{\gamma_1 \ldots \gamma_t} &= S[\gamma_1 \ldots \gamma_{t-1}] \bar{S}[\gamma_1 \ldots \gamma_t] \\ &\times \sum_{p=0}^{n-t} E^{\gamma_1 \ldots \gamma_{t+p}} D^{\gamma_{t+1}} \ldots D^{\gamma_{t+p}} \end{aligned} \tag{B25}$$

with

$$S[\gamma] = 1.$$

Then, $E'^{\gamma_1 \ldots \gamma_t}$ satisfies

$$\bar{S}[\gamma_1 \ldots \gamma_r]\, E'^{\gamma_1 \ldots \gamma_t} = 0, \quad 2 \leq r < t, \; t > 2 \;, \tag{B26}$$

because

$$\bar{S}\,[\gamma_1 \ldots \gamma_r]\, S\,[\gamma_1 \ldots \gamma_t] =$$

$$= S[\gamma_1 \ldots \gamma_t]\, \bar{S}\,[\gamma_1 \ldots \gamma_r] = 0, \text{ for } r < t. \tag{B27}$$

Further, using

$$\sum_{r=2}^{n} S[\gamma_1 \ldots \gamma_{r-1}]\, \bar{S}\,[\gamma_1 \ldots \gamma_r]$$

$$= \bar{S}\,[\gamma_1 \ldots \gamma_n] \tag{B28}$$

and Eq. (B24), we find that

$$\sum_{t=2}^{n} E'^{\gamma_1 \ldots \gamma_t} D^{\gamma_2} \ldots D^{\gamma_t} = \sum_{t=2}^{n} E^{\gamma_1 \ldots \gamma_t} D^{\gamma_2} \ldots D^{\gamma_t} \,. \tag{B29}$$

Thus, Eq. (B23) implies

$$\mathscr{G}(C^{\gamma_1}_{i_1 \ldots \hat{i}_r \ldots i_n}) = \left(\sum_{t=2}^{n} E'^{\gamma_1 \gamma_2 \ldots \gamma_t} D^{\gamma_3} \ldots D^{\gamma_t} \right) D^{\gamma_2} \,, \tag{B30}$$

or

$$\bar{S}[\gamma_1 \gamma_2] \;\; \mathscr{G} \sum_{t=2}^{n} \left(E'^{\gamma_1 \gamma_2 \ldots \gamma_t} D^{\gamma_3} \ldots D^{\gamma_t} \right) = \mathscr{G}(E'^{\gamma_1 \gamma_2}) = 0 \;, \tag{B31}$$

where use has been made of Eq. (B26). Repeating this process we find that

$$\mathscr{G}(E'^{\gamma_1 \ldots \gamma_t}) = 0, \quad 2 \leq t \leq n \quad . \tag{B32}$$

By the Dixon-Taylor lemma, therefore, there exist $F^{\gamma_1 \ldots \gamma_t}$, such that

$$E'^{\gamma_1 \ldots \gamma_t} = \mathscr{G}\, F^{\gamma_1 \ldots \gamma_t} , \tag{B33}$$

with

$$S[\gamma_1 \ldots \gamma_t]\, F^{\gamma_1 \ldots \gamma_t} = 0 . \tag{B34}$$

We can write Eq. (B23) as

$$\mathscr{G}[\, C'^{\gamma_1}_{i_1 \ldots \hat{i}_r \ldots i_t}] = 0 \tag{B35}$$

with

$$\begin{aligned} C'^{\gamma_1}_{i_1 \ldots \hat{i}_r \ldots i_t} &= C^{\gamma_1}_{i_1 \ldots \hat{i}_r \ldots i_n} \\ &- \sum_{t=2}^{n} F^{\gamma_1 \ldots \gamma_t}_{i_1 \ldots \hat{i}_r \hat{i}_p \ldots \hat{i}_q \ldots i_n} D^{\gamma_2}_{i_p} \ldots D^{\gamma_t}_{i_q} , \end{aligned} \tag{B36}$$

and

$$\sum_{r=1}^{n} C'^{\gamma}_{i_1 \ldots \hat{i}_r \ldots i_n} D^{\gamma}_{i_r} = \sum_{r=1}^{n} C^{\gamma}_{i_1 \ldots \hat{i}_r \ldots i_n} D^{\gamma}_{i_r}$$

on account of Eq. (B34) Q. E. D.

<u>Lemma BIV:</u>

Let a set of local functionals $B^{\rho\alpha}[\Phi]_{i_1 \ldots \hat{i}_r \ldots i_n}$ $(1 \leq r \leq n)$ and $A^{\rho}[\Phi]_{i_1 \ldots i_n k}$ satisfy

$$\sum_{r=1}^{n} B^{\rho\alpha}_{i_1\ldots\hat{i}_r\ldots i_n} D^{\alpha}_{i_r} = A^{\rho}_{i_1\ldots i_n k} \mathscr{L}_{0,k} \ . \tag{B37}$$

Then there exist local functionals $C^{\rho\alpha}_{i_1\ldots\hat{i}_r\ldots i_n k}[\Phi]$ such that

$$\sum_{r=1}^{n} B^{\rho\alpha}_{i_1\ldots\hat{i}_r\ldots i_n} D^{\alpha}_{i_r} = \sum_{r=1}^{n} C^{\rho\alpha}_{i_1\ldots\hat{i}_r\ldots i_n k} \mathscr{L}_{0,k} D^{\alpha}_{i_r} \ . \tag{B38}$$

A, B and C may carry additional indices.

Proof of B IV:

Since $\mathscr{L}_{0,k}$ and D^{α}_{i} have definite dimensions we may assume that A and B have also. We assume that A begins with a term Φ^{m-1} and B with Φ^{m}. We expand Φ^{m-1} terms in A and Φ^{m} terms in B in powers of independent momenta associated with the points $(i_1\ldots\hat{i}_r\ldots i_n)$:

$$A^{\rho(m-1)}_{i_1\ldots i_n k} = \bar{A}^{\rho(m-1)}_{i_1\ldots i_n k} + \sum \partial^{\alpha}_{i_p} \bar{A}^{\rho\alpha(m-1)}_{i_1\ldots\hat{i}_r\hat{i}_p\ldots i_n} + \sum_{pq} \partial^{\alpha}_{i_p} \partial^{\beta}_{i_q} A^{\rho\alpha\beta(m-1)}_{i_1\ldots\hat{i}_r\hat{i}_p\hat{i}_q\ldots i_n}$$

$$+ \ldots \ , \tag{B39}$$

$$B^{\rho(m)}_{i_1\ldots\hat{i}_r\ldots i_m} = \bar{B}^{\rho(m)}_{i_1\ldots\hat{i}_r\ldots i_n} + \sum_{p} \partial^{\beta}_{i_p} B^{\rho\beta(m)}_{i_1\ldots\hat{i}_r\hat{i}_p\ldots i_n} +$$

$$+ \sum_{p,q} \partial^{\beta}_{i_p} \partial^{\gamma}_{i_q} B^{\rho\beta\gamma(m)}_{i_1\ldots\hat{i}_r\hat{i}_p\hat{i}_q\ldots i_n} \ . \tag{B40}$$

$$+ \ldots \ .$$

We now compare the coefficients of $\mathfrak{z}^m$ in Eq. (B37) and thus obtain,

$$\sum_{r=1}^{n} B^{\rho\alpha}_{i_1 \ldots \hat{i}_r \ldots i_n, \{j\}} \, \partial^{\alpha}_{i_r} = \sum_{s=1}^{m} A^{\rho}_{i_1 \ldots i_n k, j_1 \ldots \hat{j}_s \ldots j_m} D_{kj_s} , \tag{B41}$$

with
$$B^{\rho\alpha}_{i_1 \ldots \hat{i}_r \ldots i_n, \{j\}} \equiv B^{\rho\alpha}_{i_1 \ldots \hat{i}_r \ldots i_n, j_1 \ldots j_m} . \tag{B42}$$

We substitute the expansions of Eqs. (B39) and (B40) in Eq. (B41) and compare coefficients of various independent polynomials of momenta and obtain a set of equations:

$$\sum_s \bar{A}^{\rho}_{i_1 \ldots i_n k, j_1 \ldots \hat{j}_s \ldots j_m} D_{kj_s} = 0 ,$$

$$\sum_s A^{\rho\alpha}_{i_1 \ldots \hat{i}_p \ldots i_n k, j_1 \ldots \hat{j}_s \ldots j_m} D_{kj_s} = \bar{B}^{\rho\alpha}_{i_1 \ldots \hat{i}_p \ldots i_n, \{j\}}$$

$$\ldots\ldots$$

$$\sum_s A^{\rho\alpha_1 \ldots \alpha_t}_{i_1 \ldots \underbrace{\hat{i}_p \ldots \hat{i}_q}_{t} \ldots i_n k, j_1 \ldots \hat{j}_s \ldots j_m} D_{kj_s} = (t!) S[\alpha_1 \ldots \alpha_t] B^{\rho\alpha_1 \ldots \alpha_t}_{i_1 \ldots \underbrace{\hat{i}_p \ldots \hat{i}_q}_{t} \ldots i_n, \{j\}}$$

etc. (B43)

We then define

$$(m-1)! \, C^{\rho\alpha(m-1)}_{i_1 \ldots \hat{i}_r \ldots i_n k} = [A^{\rho\alpha(m-1)}_{i_1 \ldots \hat{i}_r \ldots i_n k, j_1 \ldots j_{m-1}} +$$

$$+ \sum_p \partial^{\beta}_{i_p} A^{\rho\alpha\beta(m-1)}_{i_1 \ldots \hat{i}_r \hat{i}_p \ldots i_n k, j_1 \ldots j_{m-1}} +$$

(contd.)

$$+\sum_{p,q} \partial^{\beta}_{i_p} \partial^{r}_{i_q} A^{\rho\alpha\beta\gamma}_{i_1\ldots\hat{i}_r\hat{i}_p\hat{i}_q\ldots i_n k, j_1\ldots j_{m-1}} + \ldots \Big] \left(\prod_{s=1}^{m-1} \Phi_{j_s} \right). \quad \text{(B44)}$$

From Eqs. (B43) it follows that

$$\sum_{r=1}^{n} B^{\rho\alpha(m)}_{i_1\ldots\hat{i}_r\ldots i_n, \{j\}} \partial^{\alpha}_{i_r} = \sum_{r=1}^{n} C^{\rho\alpha(m-1)}_{i_1\ldots\hat{i}_r\ldots i_n k} \partial^{\alpha}_{i_r} D_{kj} \Phi_j . \quad \text{(B45)}$$

We define,

$$A'^{\rho[m]}_{i_1\ldots i_n k} = A^{\rho}_{i_1\ldots i_n k} - \sum_{r=1}^{n} C^{\rho\alpha(m-1)}_{i_1\ldots\hat{i}_r\ldots i_n k} D^{\alpha}_{i_r} \quad \text{(B46)}$$

and

$$B'^{\rho\alpha[m+1]}_{i_1\ldots\hat{i}_r\ldots i_n} = B^{\rho\alpha}_{i_1\ldots\hat{i}_r\ldots i_n} - C^{\rho\alpha(m-1)}_{i_1\ldots\hat{i}_r\ldots i_n k} \mathscr{L}_{0,k} . \quad \text{(B47)}$$

It is clear that $A'^{\rho[m]}$ and $B'^{\rho\alpha[m+1]}$ may contain terms which go as Φ^{m-1} and Φ^{m} respectively. But they satisfy

$$A'^{\rho[m]}_{i_1\ldots i_n k} D_{kj} \Phi_j = 0 + O[\Phi^{m+1}] \quad \text{(B48)}$$

and

$$\sum_{r=1}^{n} B'^{\rho\alpha[m+1]}_{i_1\ldots\hat{i}_r\ldots i_n} \partial^{\alpha}_{i_r} = 0 + O[\Phi^{m+2}] \quad \text{(B49)}$$

From the proof of Lemma AI and AII, we know that given Eq. (B48) we can always construct a local functional $H^{\rho}_{i_1\ldots i_n k}[\Phi]$ such that

$$H^{\rho}_{i_1 \ldots i_n k} \mathscr{L}_{0,k} = 0 \tag{B50}$$

$$H^{\rho}_{i \ldots i_n k} = A^{\rho[m]'}_{i_1 \ldots i_n k} + O[\Phi^m] \quad . \tag{B51}$$

From the proof of lemma (BI) we know that there exists a local functional $J^{\rho\alpha}_{i_1 \ldots \hat{i}_r \ldots i_n k}[\Phi]$ such that

$$\sum_r J^{\rho\alpha}_{i_1 \ldots \hat{i}_r \ldots i_n k} D^{\alpha}_{i_r} = 0 \tag{B52}$$

and

$$J^{\rho\alpha}_{i_1 \ldots \hat{i}_r \ldots i_n k} = B^{\rho\alpha}_{i_1 \ldots \hat{i}_r \ldots i_n} + O[\Phi^{m+1}] \ . \tag{B53}$$

Then we form,

$$A^{\rho[m]} = A^{\rho[m]'} - H^{\rho} \sim O[\Phi^m] \tag{B54}$$

$$B^{\rho\alpha[m+1]} = B^{\rho\alpha[m+1]'} - J^{\rho} \sim O[\Phi^{m+1}] \tag{B55}$$

which satisfy Eq. (B37) viz.

$$A^{\rho[m]}_{i_1 \ldots i_n k} \mathscr{L}_{0,k} = \sum_r B^{\rho\alpha[m+1]}_{i_1 \ldots \hat{i}_r \ldots i_n} D^{\alpha}_{i_r} \ . \tag{B56}$$

This process can obviously be repeated. Each step increases the power of Φ in A and B by one; th while their dimension remain constant. The process terminates when the power of Φ exceeds their dimension. Structure of Eq. (B47) and its analogues lead us to the statement of Eq. (B38).

We consider a local polynomial functional $Q_{iA}[\Phi, \omega]$. Here, A denotes the additional indices (if any) collectively. Let the power of ω in Q_{iA} be p. Let $D = \dim(Q_{iA})$. We define $\dim'(Q_{iA}) = D - p$.

Lemma B V: Let

$$(\mathcal{G} Q_{iA}) \mathcal{L}_{0,i} = 0 . \tag{B57}$$

Then there exists a local functional $P_{iA}[\Phi, \omega]$ such that

$$\mathcal{G}(Q_{iA} + P_{iA}) = 0 , \tag{B58}$$

$$P_{iA} \mathcal{L}_{0,i} = 0 . \tag{B59}$$

We note that Q_{iA} and P_{iA} have the same power of ω and the same dimensions.

In other words, given Eq. (B57) one can redefine Q_{iA} such that $\mathcal{G} Q_{iA} = 0$ without altering $Q_{iA} \mathcal{L}_{0,i}$.

We shall prove the lemma by induction. We shall omit the additional indices A whenever unessential.

Proof of B V:

According to lemma A II,

$$\mathcal{G} Q_{iA} = X^{\gamma}_{A} D^{\gamma}_{i} + Y_{ijA} \mathcal{L}_{0,j} , \tag{B60}$$

with

$$Y_{ij} = \sum_{r=0}^{n} Y^{(r)}_{ij\, i_1 \ldots i_r} \mathscr{L}_{0,i_1} \cdots \mathscr{L}_{0,i_r} , \tag{B61}$$

with

$$S[i_1 \ldots i_{r+2}]\; Y^{(r)}_{i_1 \ldots i_{r+2}} = 0 . \tag{B62}$$

(i) Suppose that $\dim'(Q_{iA}) < 3$. Then $Y_{ij} = 0$ and the proposition is true for $\dim'(Q_{iA}) < 3$ by virtue of lemma B II.

(ii) Let us assume that the proposition is true for $\dim'(Q_{iA}) < N$, for all possible A and all possible $p \geq 0$.

We wish now to prove it for $\dim'(Q_{iA}) = N$. From Eq. (B60)

$$(\mathscr{G} X_\gamma) D^\gamma_i = -\mathscr{G}(Y_{ij}) \mathscr{L}_{0,j} . \tag{B63}$$

According to lemma BIV, there exists a local functional W_{rj} such that

$$\mathscr{G} X_\gamma = W_{\gamma j} \mathscr{L}_{0,j} \tag{B64}$$

or

$$\mathscr{G}(W_{\gamma j}) \mathscr{L}_{0,j} = 0 . \tag{B65}$$

Since $\dim'(W_{\gamma j}) = \dim(X'_{\gamma j}) - 3 = N - 4$, we can apply the proposition of this lemma to Eq. (B65) to conclude that we can always redefine (if necessary) $W_{\gamma j}$ without altering $W_{\gamma j} \mathscr{L}_{0,j}$ such that

$$\mathscr{G}(W_{\gamma j}) = 0 . \tag{B66}$$

We can apply the Dixon-Taylor lemma and write

$$W_{\gamma j} = \mathscr{G} X_{\gamma j} \quad . \tag{B67}$$

Therefore from Eqs. (B64) and (B67),

$$\mathscr{G}(X_\gamma - X_{\gamma j} \mathscr{L}_{0,j}) = 0$$

or

$$X_\gamma = X_{\gamma j} \mathscr{L}_{0,j} + X^0_\gamma \tag{B68}$$

with

$$\mathscr{G} X^0_\gamma = 0 \quad . \tag{B69}$$

Substituting Eq. (B68) in Eq. (B60), we obtain,

$$\mathscr{G} Q_i = X^0_\gamma D^\gamma_i + (X_{\gamma j} D^\gamma_i + Y_{ij}) \mathscr{L}_{0,j} \tag{B70}$$

where from Eq. (B69)

$$\mathscr{G}(X^0_\gamma) D^\gamma_i = 0 \quad . \tag{B71}$$

By lemma (BII), there exists a local functional T^0_γ, such that

$$X^0_\gamma = \mathscr{G}(T^0_\gamma) \tag{B72}$$

or

$$\mathscr{G}(Q_i - T^0_\gamma D^\gamma_i) = (X_{\gamma j} D^\gamma_i + Y_{ij}) \mathscr{L}_{0,j} \tag{B73}$$

and thus

$$\mathscr{G}(X_{\gamma_j} D^\gamma_i + Y_{ij}) \mathscr{L}_{0,j} = 0. \tag{B74}$$

Further $\dim'(X_{\gamma_j} D^\gamma_i + Y_{ij}) = N - 3$. Therefore we may apply the proposition to conclude that there exists a local functional P_{ij} such that

$$P_{ij}\mathscr{L}_{0,j} = 0 \tag{B75}$$

and

$$\mathscr{G}(X_{\gamma j} D^{\gamma}_{i} + Y_{ij} + P_{ij}) = 0 \ . \tag{B76}$$

From lemma AII we may express P_{ij} as

$$P_{ij} = X'_{\gamma i} D^{\gamma}_{j} + \sum_{r=1} P^{(r)}_{ij\,k_1 \ldots k_r} \mathscr{L}_{0,k_1} \cdots \mathscr{L}_{0,k_r} \tag{B77}$$

with

$$S[i_1 \ldots i_{r+2}]\ P^{(r)}_{i_1 \ldots i_{r+2}} = 0 \ . \tag{B78}$$

We define

$$Y'^{(r)}_{i_1 i_2 \ldots i_{r+2}} = Y^{(r)}_{i_1 i_2 \ldots i_{r+2}} + P^{(r)}_{i_1 i_2 \ldots i_{r+2}} \ . \tag{B79}$$

Then on account of Eqs. (B78) and (B62)

$$S[i_1 \ldots i_{r+2}]\ Y'^{(r)}_{i_1 \ldots i_{r+2}} = 0 \ , \tag{B80}$$

and

$$Y_{ij}\mathscr{L}_{0,j} = Y'_{ij}\mathscr{L}_{0,j} \equiv \sum_{r=1}^{n} Y'^{(r)}_{ij\,i_1 \ldots i_r} \mathscr{L}_{0,i_1} \cdots \mathscr{L}_{0,i_r} \mathscr{L}_{0,j} \ . \tag{B81}$$

We may thus write Eq. (B76) as

$$\mathscr{G}(X_{\gamma j} D^{\gamma}_{i} + X'_{\gamma i} D^{\gamma}_{j} + Y'_{ij}) = 0 \ . \tag{B82}$$

We now define

$$C^{(0)}_{ij} = \bar{S}[ij]\,(X_{\gamma j} D^{\gamma}_{i} + X'_{\gamma i} D^{\gamma}_{j} + Y'_{ij})$$

$$= \frac{1}{2}[X_{\gamma j} D^{\gamma}_{i} + X'_{\gamma i} D^{\gamma}_{j} - X_{\gamma j} D^{\gamma}_{i} - X'_{\gamma i} D^{\gamma}_{j}] + Y^{(0)}_{ij} + \sum_{r=1}^{n} \bar{S}[ij]\, Y^{(r)}_{ij i_1 \ldots i_r} \mathscr{L}_{0,i} \cdots \mathscr{L}_{0,i} \tag{B83}$$

and from Eq. (B82) it satisfies

$$\mathscr{G}(C^{(0)}_{ij}) = 0 \qquad\qquad S[ij]\; C^{(0)}_{ij} = 0 \;. \tag{B84}$$

By the Dixon-Taylor lemma, there exists a local functional $U^{(0)}_{ij}$ such that

$$C^{(0)}_{ij} = \mathscr{G} U^{(0)}_{ij} \qquad\qquad S[ij]\; U^{(0)}_{ij} = 0 \;. \tag{B85}$$

Using Eqs. (B85), (B83) in Eq. (B73), it may be written as

$$\mathscr{G}[Q_i - T^{0}_{\gamma} D^{\gamma}_{i} - U^{(0)}_{ij} \mathscr{L}_{0,j}] = \{\frac{1}{2}[X_{\gamma j} + X'_{\gamma j}]\, D^{\gamma}_{i} + (i \longleftrightarrow j)] + S[ij]\; Y'_{ij}\} \mathscr{L}_{0,j}$$

$$= \{X''_{\gamma j} D^{\gamma}_{i} + X''_{\gamma i} D^{\gamma}_{j} + S[ij]\; Y'_{ij}\} \mathscr{L}_{0,j} \tag{B86}$$

with

$$X''_{\gamma j} = \frac{1}{2}(X_{\gamma j} + X'_{\gamma j}) \;. \tag{B87}$$

Further from Eq. (B82)

$$\mathscr{G}\{X''_{\gamma j} D^{\gamma}_{i} + X''_{\gamma i} D^{\gamma}_{j} + S[ij]\; Y'_{ij}\} =$$

$$S[ij]\mathscr{G}[X_{\gamma j} D_{i} + X_{\gamma i} D_{j} + Y'_{ij}] = 0 \;. \tag{B88}$$

We note that $S[ij]\; Y'_{ij}$ contains at least one factor of $\mathscr{L}_{0,k}$, since

$S[ij]\, Y^{(0)}_{ij} = 0$. Let

$$S[ij]\; Y'_{ij} = -Z_{ijk}\mathscr{L}_{0,k} . \tag{B89}$$

Thus Eq. (B88) can be written as

$$\mathscr{G}(X''_{\gamma j})D^{\gamma}_{i} + \mathscr{G}(X''_{\gamma i})D^{\gamma}_{j} = \mathscr{G}(Z_{ijk})\mathscr{L}_{0,k} . \tag{B90}$$

Applying lemma BIV to Eq. (B90) we learn that there exist local functionals $C_{\gamma jk}$ such that

$$\mathscr{G}(X''_{\gamma j})D^{\gamma}_{i} + \mathscr{G}(X''_{\gamma i})D^{\gamma}_{j} = (C_{\gamma jk}D^{\gamma}_{i} + C_{\gamma ik}D^{\gamma}_{j})\mathscr{L}_{0,k} = Z_{ijk}\mathscr{L}_{0,k}. \tag{B91}$$

Thus

$$\mathscr{G}(C_{\gamma jk}\mathscr{L}_{0,k})D^{\gamma}_{i} + \mathscr{G}(C_{\gamma ik}\mathscr{L}_{0,k})D^{\gamma}_{j} = 0 . \tag{B92}$$

We apply lemma BIII to Eq. (B92) to conclude that we can choose $C_{\gamma jk}\mathscr{L}_{0,k}$ such that

$$\mathscr{G}(C_{\gamma jk})\mathscr{L}_{0,k} = 0 \tag{B93}$$

without altering $(C_{\gamma jk}D^{\gamma}_{i} + C_{\gamma ik}D^{\gamma}_{j})\mathscr{L}_{0,k}$. Further $\dim'(C_{\gamma jk}) = \dim'(X_{\gamma j}) = N-3$. Thus we may further apply the proposition to conclude that we may redefine $C_{\gamma jk}$ further so that

$$\mathscr{G}(C_{\gamma jk}) = 0 \tag{B94}$$

with $C_{\gamma jk}\mathscr{L}_{0,k}$ unchanged. By the Dixon-Taylor lemma, there exists a local functional $W_{\gamma jk}$ such that

$$C_{\gamma jk} = \mathscr{G}(W_{\gamma jk}) \quad . \tag{B95}$$

We may write Eq. (B91) as,

$$\mathscr{G}[\, W_{\gamma jk} D_i^{\gamma} + W_{\gamma ik} D_j^{\gamma} - Z_{ijk} \,]\mathscr{L}_{0,k} = 0 \quad . \tag{B96}$$

Since $\dim' [\, W_{\gamma jk} D_i^{\gamma} + (i \longleftrightarrow j) - Z_{ijk}] = \dim \mathscr{I}(C_{\gamma jk}) = N-3$, we may apply the proposition to conclude that there exists a local functional R_{ijk} such that

$$\mathscr{G}[\, W_{\gamma jk} D_i^{\gamma} + W_{\gamma ik} D_j^{\gamma} - Z_{ijk} - R_{ijk}] = 0, \tag{B97}$$

with

$$R_{ijk} \mathscr{L}_{0,k} = 0. \tag{B98}$$

From lemma AII

$$R_{ijk} = -R'_{\gamma_{ij}} D_k^{\gamma} + \sum_{r=1} R^{(r+1)}_{ijk\, i_1 \ldots i_r} \mathscr{L}_{0,1} \ldots \mathscr{L}_{0,i} , \tag{B99}$$

with

$$S[\, i_3 \ldots i_{r+3}]\; R^{(r)}_{i_1 \ldots i_{r+3}} = 0 \,. \tag{B100}$$

We define

$$Z'_{ijk} = Z_{ijk} + \sum_{r=1} R^{(r+1)}_{ijk\; i_1 \ldots i_r} \mathscr{L}_{0,i_1} \ldots \mathscr{L}_{0,i_r} \quad . \tag{B101}$$

From the discussion under Eq. (B78), it is clear that we may define $Y'^{(r)}_{i_1 \ldots i_{r+2}}$ $(r \geq 1)$ so that

$$S[ij]\, Y''_{ij} = -Z'_{ijk}\mathscr{L}_{0,k} = Z_{ijk}\mathscr{L}_{0,k} \quad , \tag{B102}$$

while $Y''_{i_1 \ldots i_{r+2}}$ still satisfy

$$S[i_1 \ldots i_{r+2}]\, Y''^{(r)}_{i_1 \ldots i_{r+2}} = 0 \ . \tag{B103}$$

We thus have

$$\mathscr{G}[W_{\gamma jk}\, D^\gamma_i + W_{\gamma ik}\, D^\gamma_j + R'_{\gamma ij}\, D^\gamma_k - Z'_{ijk}] = 0 \quad . \tag{B104}$$

We further note that Eq. (B91) implies, with the help of Eq. (B95)

$$\mathscr{G}[\,(X''_{\gamma j} - W_{\gamma jk}\mathscr{L}_{0,k})\, D^\gamma_i + (i \longleftrightarrow j)\,] = 0 \ ,$$

i.e.,

$$\mathscr{G}[(X''_{\gamma j} - W_{\gamma jk}\mathscr{L}_{0,k})\mathscr{L}_{0,j}\,]\, D^\gamma_i = 0. \tag{B105}$$

We may, now, express Eq. (B86) as

$$\mathscr{G}[Q_i - T^0_\gamma D^\gamma_i - U^{(0)}_{ij}\mathscr{L}_{0,j}\ ] = (X''_{\gamma j} - W_{\gamma jk}\mathscr{L}_{0,k})\mathscr{L}_{0,j}\, D^\gamma_i$$

$$+ [\,W_{\gamma jk}\, D^\gamma_i + W_{\gamma ik}\, D^\gamma_j + R'_{\gamma ij}\, D^\gamma_k - Z'_{ijk}]\mathscr{L}_{0,j}\mathscr{L}_{0,k} \ . \tag{B106}$$

On account of Eq. (B105) we may write

$$(X''_{\gamma j} - W_{\gamma jk}\mathscr{L}_{0,k})\mathscr{L}_{0,j}\, D^\gamma_i = \mathscr{G}(T^{(1)}_\gamma)\, D^\gamma_i \ . \tag{B107}$$

Further Eq. (B104) yields

$$\mathscr{G}[\bar{S}[ijk]\{W_{\gamma jk}D^{\gamma}_{i} + W_{\gamma ik}D^{\gamma}_{j} + R'_{\gamma ij}D^{\gamma}_{k} - Z'_{ijk}\}] = 0 . \tag{B108}$$

Thus there exists a local functional $U^{(1)}_{ijk}$ such that

$$S[ijk]\{W_{\gamma jk}D^{\gamma}_{i} + W_{\gamma ik}D^{\gamma}_{j} + R'_{\gamma ij}D^{\gamma}_{k} - Z'_{ijk}\} = \mathscr{G}(U^{(1)}_{ijk}) \tag{B109}$$

with

$$\bar{S}[ij]\,U^{(1)}_{ijk} = 0 , \qquad S[ijk]\,U^{(1)}(ijk) = 0 . \tag{B110}$$

Using Eq. (B107) and (B109) in Eq. (B106) we get

$$\begin{aligned}
&\mathscr{G}[Q_i - (T^0_{\gamma} + T^1_{\gamma})D^{\gamma}_{i} - U^0_{ij}\mathscr{L}_{0,j} - U^{(1)}_{ijk}\mathscr{L}_{0,j}\mathscr{L}_{0,k}] \\
&= \{S[ijk]\,X'''_{\gamma jk}\,D^{\gamma}_{i} - S[ijk]\,Z'_{ijk}\}\mathscr{L}_{0,k}\mathscr{L}_{0,j} \\
&= \{S[ijk]\,X'''_{\gamma jk}D^{\gamma}_{i} + S[ijk]\sum_{r=1}^{n} Y^{(r+1)}_{ijk\;i_1\ldots i_r}\mathscr{L}_{0,i_1}\cdots\mathscr{L}_{0,i_r}\} \\
&\quad\times\mathscr{L}_{0,j}\mathscr{L}_{0,k}
\end{aligned} \tag{B111}$$

where

$$S[ijk]\,X''_{\gamma jk}\,D^{\gamma}_{i} \equiv S[ijk]\{W_{\gamma jk}D^{\gamma}_{i} + W_{\gamma ik}D^{\gamma}_{j} + R'_{\gamma ij}D^{\gamma}_{k}\} \tag{B112}$$

and

$$\mathscr{G}\{S[ijk]\,X''_{\gamma jk}D^{\gamma}_{i} + S[ijk]\sum_{r=1}^{n} Y'^{(r+1)}_{ijk\,i_1\ldots i_r}\mathscr{L}_{0,i_1}\cdots\mathscr{L}_{0,i_r}\} = 0 \tag{B113}$$

This process can clearly be continued to the end with the help of lemmas BIII, BIV. In the end one would obtain

$$\mathscr{G}[\, Q_i - \sum_{r=0}^{n+1} T_\gamma^{(r)} D_i^\gamma - \sum_{r=0}^{n} U^{(r)}_{i\,i_2 \ldots i_{r+2}} \mathscr{L}_{0,i_2} \ldots \mathscr{L}_{0,i_{r+2}} \,] = 0 \tag{B114}$$

$$S[\, i_1 \ldots i_{r+2} \,]\; U^{(r)}_{i_1 \ldots i_{r+2}} = 0 \, . \tag{B115}$$

Further,

$$\left[\sum_{r=0}^{n+1} T_\gamma^{(r)} D_i^\gamma - \sum_{r=0}^{n} U^{(r)}_{i\,i_2 \ldots i_{r+2}} \mathscr{L}_{0,i_2} \ldots \mathscr{L}_{0,i_{r+2}} \right] \mathscr{L}_{0,i} = 0 \tag{B116}$$

Q. E. D.

APPENDIX C

In Appendix B we have applied the Dixon Taylor lemma to the operator $\mathscr{G}'$ of Eq. (4.28). One may apply the lemma since the Eq. (4.11) hold for $A = \omega_\eta \frac{\delta}{\delta\Phi_\eta^L}$ when X is proportional to at least one ghost field ω. Here we have decomposed $\mathscr{G}$ as

$$\mathscr{G} = A + gB; \quad \mathscr{G}^2 = 0 \quad . \tag{C1}$$

However, because of the presence of the derivative with respect to Φ_η^L in A, one does not know, as mentioned in Sec. IV, whether X of Eq. (4.11) can be chosen to be local given that Y is local. It should be emphasized that there are instances in which the Dixon Taylor construction can be used to construct J of Eq. (4.10), however, it is impossible to choose J to be local even though H of Eq. (4.9) is local. We shall show that <u>in the cases of our interest</u> in Appendix B, it is possible to choose the solution "J" to be local by showing that in the cases of our interest the Eq.

$$BY = 0 \qquad Y \text{ local} \tag{C2}$$

implies that Y may be expressed as

$$Y = BX + AW \tag{C3}$$

for some local X and W. [In particular W may be zero, which will be the case in all applications of Appendix B] so that the Dixon Taylor construction can be carried out "backwards". This way the locality of the solution is clear. We consider the Eq. (C2) where Y carried a free index say l. The following relations are easily varified.

$$B(\Phi_i) = 0 \;; \qquad B(t^{\alpha}_{ij}) = 0; \qquad B(f^{\alpha\beta\gamma}) = 0 \,. \tag{C4}$$

The last two imply

$$B(g_{\mu\nu}) = 0. \tag{C5}$$

Further,

$$B(\delta_{ij}) = 0 \qquad B(\delta_{\alpha\beta}) = 0 \,. \tag{C6}$$

Also, we define

$$B(\delta^4(x-y)) = 0 \,. \tag{C7}$$

However,

$$B(\partial^{\alpha}_{i}) = g\,(\omega\,\partial)_j\, t^{\alpha}_{ij} \neq 0; \qquad (\omega\partial)_j \equiv \omega_{\alpha}\partial^{\alpha}_{j} \,, \tag{C8}$$

and

$$B(\omega\,\alpha) = \frac{1}{2}\, g\, f_{\alpha\beta\gamma}\,\omega_{\beta}\,\omega_{\gamma} \neq 0 \,, \tag{C9}$$

nevertheless,

$$B(\omega\,\partial)_i = 0 \,. \tag{C10}$$

Without loss of generality we may consider $Y(\omega, \Phi)$ which involves a fixed number of ω and Φ

$$Y_\ell\left[\omega, \Phi\right] = \omega_{\alpha_1} \cdots \omega_{\alpha_n} Y^{\alpha_1 \cdots \alpha_n}_{\ell, k_1 \cdots k_m} \Phi_{k_1} \cdots \Phi_{k_n} \quad . \tag{C11}$$

Then Eq. (C2) together with $B(\Phi_i) = 0$ implies

$$B\left[\omega_{\alpha_1} \cdots \omega_{\alpha_n} Y^{\alpha_1 \cdots \alpha_n}_{\ell, k_1 \cdots k_m}\right] = 0 \quad . \tag{C12}$$

Thus, it is sufficient to consider Y not containing fields Φ but carrying an arbitrary number of indices. Now, the operator B does not alter the number of derivatives in a functional. Hence we may assume that Y contains a fixed number of derivatives.

We shall first consider the case in which Y does not contain any derivatives. The considerations involved in this case will be useful in the case in which Y contains derivatives. In this case $Y^{\alpha_1 \cdots \alpha_n}_{\ell, k_1 \cdots k_m}$ consists only of the B-invariants, so that Eq. (C12) implies

$$B\left[\omega_{\alpha_1} \cdots \omega_{\alpha_n}\right] Y^{\alpha_1 \cdots \alpha_n} = 0 \tag{C13}$$

where we have suppressed the extra free indices. Thus,

$$f_{\eta\, \alpha_n \alpha_{n+1}} \omega_{\alpha_1} \cdots \omega_{\alpha_{n+1}} Y^{\alpha_1 \cdots \alpha_{n+1}\, \eta} = 0. \tag{C14}$$

Thus the totally antisymmetric part of $f_{\eta \alpha_n \alpha_{n+1}} Y^{\alpha_1 \cdots \alpha_{n-1}\, \eta}$ vanishes. It consists of a sum of $\binom{n+1}{2} = \frac{n(n+1)}{2}$ terms, since the above is already antisymmetric in $(\alpha_1 \cdots \alpha_{n-1})$ and in (α_n, α_{n+1}). We write one of these terms as

$$f_{\eta\alpha_n\alpha_{n+1}} Y^{\alpha_1\ldots\alpha_{n-1}\eta} = -f_{\eta\alpha_{n-1}\alpha_n} Y^{\alpha_{n+1}\alpha_1\ldots\alpha_{n-1}\eta} + \ldots .$$

$$-f_{\eta\alpha_1\alpha_2} Y^{\alpha_3\ldots\alpha_n\alpha_{n+1}\eta} + \ldots . \qquad \text{(C15)}$$

where we have shown two typical terms on the r.h.s. Since Y does not contain derivatives, each term contains a product of same δ functions which can be extracted. Thus in Eq. (10) we shall assume that the terms have no space-time dependence. We multiply Eq. (C15) by $f_{\xi\alpha_n\alpha_{n+1}}$ and use,

$$f_{\eta\alpha_n\alpha_{n+1}} f_{\xi\alpha_n\alpha_{n+1}} = \delta_{\eta\xi} C_2 \quad , \qquad \text{(C16)}$$

so that,

$$C_2 Y^{\alpha_1\ldots\alpha_{n-1}\xi} = -f_{\xi\sigma\lambda} f_{\eta\alpha_{n-1}\sigma} Y^{\lambda\alpha_1\ldots\alpha_{n-2}\eta} + \ldots$$

$$-f_{\eta\alpha_1\alpha_2} f_{\xi\sigma\lambda} Y^{\alpha_3\ldots\alpha_{n-1}\sigma\lambda\eta} + \ldots \qquad \text{(C17)}$$

and thus

$$C_2 \omega_{\alpha_1} \ldots \omega_{\alpha_{n-1}} \omega_\xi Y^{\alpha_1\ldots\alpha_{n-1}\xi} = -(f_{\xi\sigma\lambda} f_{\eta\alpha_{n-1}\sigma} \omega_{\alpha_{n-1}} \omega_\xi) Y^{\lambda\alpha_1\ldots\alpha_{n-2}\eta} \omega_{\alpha_1} \ldots \omega_{\alpha_{n-2}}$$

$$+ \ldots - f_{\eta\alpha_1\alpha_2} \omega_{\alpha_1} \omega_{\alpha_2} f_{\xi\sigma\lambda} Y^{\alpha_3\ldots\alpha_{n-1}\sigma\lambda\eta} \omega_\xi \omega_{\alpha_3} \ldots \omega_{\alpha_{n-1}} + \ldots . \qquad \text{(C18)}$$

In the first term on the r.h.s. of Eq. (C18), we use,

$$\omega_{\alpha_{n-1}}\omega_\xi f_{\xi\sigma\lambda} f_{\eta\alpha_{n-1}\sigma} = \frac{1}{2}\,\omega_{\alpha_{n-1}}\omega_\xi \left\{ f_{\xi\sigma\lambda} f_{\eta\alpha_{n-1}\sigma} - (\xi \longleftrightarrow \alpha_{n-1}) \right\}$$

$$= \frac{1}{2}\,\omega_{\alpha_{n-1}}\omega_\xi\, f_{\alpha_{n-1}\xi\kappa}\, f_{\lambda\kappa\eta} \tag{C19}$$

and thus we can write Eq. (C18) as

$$\omega_{\alpha_1}\cdots\omega_{\alpha_n}\, Y^{\alpha_1\ldots\alpha_{n-1}} = \frac{1}{2}\, f_{\gamma\delta\kappa}\,\omega_r\omega_\delta$$

$$\times\left\{ A_1 f_{\kappa\lambda\eta}\, Y^{\lambda\alpha_1\ldots\alpha_{n-2}\eta}_{\omega_{\alpha_1}\cdots\omega_{\alpha_{n-2}}} + A_2\, f_{\xi\sigma\lambda}\, Y^{\kappa\alpha_1\ldots\alpha_{n-3}\sigma\lambda\eta}_{\omega_\xi\,\omega_{\alpha_1}\cdots\omega_{\alpha_{n-3}}} \right\} \tag{C20}$$

where A_1 and A_2 are some numbers depending on n and C_2.

Even though the argument following Eq. (C14) does not go through in the case when n equals the number of generators of the group, the relation of the form of Eq. (C20) still holds (with $A_2 = 0$). This can be seen by explicitly working out the right hand side of Eq. (C20) for this case. As we shall see later, we shall need the relation only for the case of SU(2) (i.e., 3 ghosts) in which case we see that

$$\frac{1}{2}\,\epsilon_{\gamma\delta\kappa}\,\omega_\gamma\omega_\delta\,\epsilon_{\lambda\kappa\eta}\, Y^{\lambda\alpha\eta}\,\omega_\alpha$$

$$= \frac{1}{2}\,\omega_\gamma\omega_\delta\omega_\alpha \left\{ -\delta_{\gamma\lambda}\,\delta_{\delta\eta} + \delta_{\gamma\eta}\,\delta_{\delta\lambda} \right\} Y^{\lambda\alpha\eta}$$

$$= Y^{\alpha\beta\gamma}\,\omega_\alpha\,\omega_\beta\omega_\gamma \quad . \tag{C21}$$

Using Eq. (C20) we define an $X^{\kappa\alpha_1\ldots\alpha_{n-2}}$ by,

$$Y^{\alpha_1\ldots\alpha_n}\,\omega_{\alpha_1}\ldots\omega_{\alpha_n} \equiv \frac{1}{2} f_{\gamma\delta\kappa}\,\omega_\gamma\,\omega_\delta\, X^{\kappa\alpha_1\ldots\alpha_{n-2}}\,\omega_{\alpha_1}\ldots\omega_{\alpha_{n-2}} \tag{C22}$$

$X^{\kappa\alpha_1\ldots\alpha_{n-2}}$ is clearly antisymmetric in its last (n-2) indices. Therefore it can be expressed as a sum of two terms each having either one of the permutation symmetries of Young Tableaux in Fig. 2. We write

$$X = X' + X'' \tag{C23}$$

where X' is totally antisymmetric in $(\kappa\alpha_1\ldots\alpha_{n-2})$. Substituting the decomposition of Eq. (C23) in Eq. (C22), we get a corresponding decomposition for Y:

$$Y = Y' + Y'' \ . \tag{C24}$$

It is clear that Y' may be expressed as,

$$Y'^{\alpha_1\ldots\alpha_n}\,\omega_{\alpha_1}\ldots\omega_{\alpha_n} \simeq B(\omega_\kappa X^{\kappa\alpha_1\ldots\alpha_{n-2}}\,\omega_{\alpha_1}\ldots\omega_{\alpha_{n-2}}) \tag{C25}$$

where $\simeq$ means equal within a numerical factor, and that

$$BY'' = 0 \ . \tag{C26}$$

Thus showing that $BY = 0 \Rightarrow Y = BX$ (X local), in our cases of interest amounts to examining B-invariants of the form Y'' [related to X'' with the permutation symmetry of the Young Tableau of Fig. (2b) by by Eq. (C22)] and showing that such B-invariants do not occur in the cases of our interest. We define such B-invariants as anomalous invariants. The anomalous invariants cannot be expressed as BZ since the X-function of Eq. (C22) for BZ for any Z can always be chosen to be antisymmetric (and in fact proportional to Z). Since the classification of B-invariants depends entirely on the indices of Y contracted with ghosts; it follows that,

(i) if $Y_i[\omega, \Phi]$ is an anomalous invariant, so is $Y_i[\Phi]\, T_i[\Phi]$ where $T_i[\Phi]$ is a local functional not containing derivatives.

(ii) if $Y^{(1)}$ and $Y^{(2)}$ are anomalous invariants so is $Y^{(1)} + Y^{(2)}$.

(iii) If $Y_i[\omega, \Phi]$ is an anomalous invariant, and $Y_i[\omega, \Phi] = K_{ij}[\omega, \Phi]\, L_j[\Phi]$ where L is local, $K_{ij}[\omega, \Phi]$ is also an anomalous invariant, [i.e., the part of K which contributes to $K_{ij} L_j$] .

Let us therefore consider Eq. (C22) where Y is an anomalous invariant. Using B(Y) = 0, we get,

$$Z_{\kappa\alpha_1 \ldots \alpha_{n-2}} X^{\kappa\alpha_1 \ldots \alpha_{n-2}} = 0 \ , \tag{C27}$$

with,

$$Z_{\kappa\alpha_1 \ldots \alpha_{n-2}} = A[\alpha_1 \ldots \alpha_{n-2}] \{ f_{\kappa\gamma\delta}\, \omega_\gamma\, \omega_\delta\, f_{\alpha_1 \xi \eta}\, \omega_\xi\, \omega_\eta\, \omega_{\alpha_2} \ldots \omega_{\alpha_{n-2}} \} \tag{C28}$$

and thus Z as well as X has the permutation symmetry of the Young Tableau of Fig. (2b). Further for n = 3, Z satisfies

$$\delta^{\alpha\kappa} Z_{\kappa\alpha} = 0 \tag{C29}$$

on account of the Jacobi Identity.

An examination of the lemmas BIII and BV shows that, we have applied the Dixon Taylor lemma for $\mathscr{G}$ in the cases when H of Eq. (4.8) contains 2 or 3 or 4 ghosts. We shall therefore need to examine only these possibilities.

(i) Two ghosts: Here, Eq. (C22) gives

$$\omega_\alpha \omega_\beta Y^{\alpha\beta} = \frac{1}{2} f_{\kappa\gamma\delta} \omega_\gamma \omega_\delta X^\kappa = B(\omega_\kappa X^\kappa)$$

without exceptions. Thus there are no anomalous invariants containing two ghosts.

(ii) Three ghosts: Eq. (C27) and (C29) imply that the traceless symmetric part of $X^{\kappa\alpha}$ (which is already given to be symmetric in (κ, α)) vanishes. Thus $X^{\kappa\alpha}$ must be proportional to $\delta^{\kappa\alpha}$. This argument does not apply to the case of the group SU(2) because there Z vanishes identically. But in this case, it is clear that only the part of $X^{\kappa\alpha}$ proportional to $\delta^{\kappa\alpha}$ contributes to $f_{\kappa\gamma\delta}\, \omega_\gamma \omega_\partial X^{\kappa\alpha} \omega_\alpha$. Thus such anomalous B-invariants have the disconnected form

$$f_{\alpha\beta\gamma}\, \omega_\alpha \omega_\beta \omega_\gamma X[\Phi]$$

where X carries all other indices that Y does.

(iii) four ghosts: Here, we shall not need to know the form of the anomalous invariants. We have used the Dixon Taylor lemma after Eq. (B.32) where E′ contains 4 ghosts. We shall show that E′ can be chosen not to contain anomalous invariants. We note that in Eq. (B.23) the terms containing no derivatives on the r.h.s. arise entirely from the action of B on the terms in C not containing derivatives, since C is local. Thus this term on the r.h.s. of Eq. (B.32) are expressible as BZ and by definition do not contain anomalous invariants. Therefore it is clear that terms in E not containing derivatives may be chosen not to contain anomalous invariants. Since E′ are a linear combination of product of E and D_i^{γ}'s, it is clear that terms with no derivatives in E′ do not contain anomalous invariants, if such terms in E do not.

Therefore, we need only to worry about the application of the lemma to terms containing 3 ghosts. Application to Eq. (B.66) is a typical example which was repeated at each subsequent stage in Appendix B.

We note that in Eq. (B.64), X_γ is a local functional containing two ghosts. Thus, terms in $W_{\gamma j}\mathcal{L}_{0,j}$ not containing derivatives do not contain the anomalous invariants as argued in the discussion in the case of 4 ghosts. Therefore, terms in $W_{\gamma j}$ not containing derivatives may be assumed not to contain anomalous invariants. The question, then, is whether one needs to add such terms to $W_{\gamma j}$ so as to make

Eq. (B66) valid. This will be so if

$$\mathscr{G}(W_{rj}[\Phi]) = -\mathscr{G}(f_{\alpha\beta\gamma}\omega_\alpha\omega_\beta\omega_\gamma X[\Phi])$$

$$= -f_{\alpha\beta\gamma}\,\omega_\alpha\omega_\beta\omega_\gamma\,\mathscr{G}(X[\Phi])\ . \tag{C30}$$

On the right hand side, $X[\Phi]$ is a local functional with no derivatives in it. Thus,

$$\mathscr{G}(X[\Phi]) = AX[\Phi] = \omega_\eta\,\partial^\eta_i\,X_{j\,i}[\Phi]\ .$$

Then Eq. (C. 30) becomes

$$\mathscr{G}(W_{rj}[\Phi]) = -f_{\alpha\beta\gamma}\,\omega_\alpha\omega_\beta\omega_\gamma\omega_\eta\,\partial^\eta_i X_{,i}\,[\Phi]\ . \tag{C30a}$$

The terms in the l. h. s. of Eq. (C30a) which contain one derivative can come from either the action of A on terms containing no derivatives ($\equiv \omega_\alpha\omega_\beta\omega_\gamma\,U^{\alpha\beta\gamma}[\Phi]$) or the action on B on terms containing one derivative ($\equiv \omega_\alpha\omega_\beta\omega_\gamma\,V_k^{\alpha\beta\gamma\,\delta}\,\partial^\delta_k$). Thus we have to see if the following equation is possible:

$$-\omega_\alpha\omega_\beta\omega_\gamma\omega_\eta\,\partial^\eta_i\,U^{\alpha\beta\gamma}_{,i}[\Phi] + B(\omega_\alpha\omega_\beta\omega_\gamma)V_k^{\alpha\beta\gamma\,\delta}[\Phi]\,\partial^\delta_k$$

$$+\,\omega_\alpha\omega_\beta\omega_\gamma\,V_k^{\alpha\beta\gamma\,\delta}\,B(\partial^\delta_k) = -f_{\alpha\beta\gamma}\,\omega_\alpha\omega_\beta\omega_\gamma\omega_\eta\,\partial^\eta_i\,X_{,i}\ . \tag{C31}$$

From Eq. (C. 31), it is clear that ∂^δ_k must be contracted with a ghost. Then letting,

$$\omega_\alpha \omega_\beta \omega_\gamma V_k^{\alpha\beta\gamma\delta} \partial_k^\delta = \omega_\alpha \partial_i^\alpha \omega_\beta \omega_\gamma V_i^{\beta\gamma}[\Phi],$$

we get

$$-\omega_\alpha \omega_\beta \omega_\gamma \omega_\eta \partial_i^\eta U_{,i}^{\alpha\beta\gamma}[\Phi] + \omega_\alpha \partial_i^\alpha B(\omega_\beta \omega_\gamma) V_i^{\beta\gamma}[\Phi]$$

$$= -f_{\alpha\beta\gamma} \omega_\alpha \omega_\beta \omega_\gamma \omega_\eta \partial_i^\eta X_{,i} \quad .$$

But this is impossible, since $U^{\alpha\beta\gamma}$ does not contain a factor of the form $f_{\alpha\beta\gamma}$.

Next, we consider the case when Y contains derivatives. Let Y contain exactly r derivatives at least p $(0 \leq p \leq r)$ of which are contracted with ω in the form $\omega_\alpha \partial_i^\alpha$. BY, therefore, consists of a sum of terms containing $(n+1)$ ghosts, r derivatives at least p of which are contracted with ω's. We write

$$Y_{\ell,}^{\alpha_1 \ldots \alpha_n} \omega_{\alpha_1} \ldots \omega_{\alpha_n} = \omega_{\alpha_1} \ldots \omega_{\alpha_{n-p}} (\omega \cdot \partial)_{j_1} \ldots (\omega \cdot \partial)_{j_p} \partial_{k_1}^{\beta_1} \ldots \partial_{k_{r-p}}^{\beta_{r-p}}$$

$$Y_{\ell, j_1 \ldots j_p \, k_1 \ldots k_{r-p}}^{\prime \, \alpha_1 \ldots \alpha_{n-p}, \, \beta_1 \ldots \beta_{r-p}} + \ldots . \qquad \text{(C32)}$$

The terms in BY containing precisely p derivatives contracted with ghosts come from the action of B on $(\omega_{\alpha_1} \ldots \omega_{\alpha_{n-p}})$ only. These terms must separately vanish. Thus

$$0 = B(\omega_{\alpha_1} \ldots \omega_{\alpha_{n-p}})(\omega\cdot\partial)_{j_1} \ldots (\omega\cdot\partial)_{j_p} \, \partial_{k_1}^{\beta_1} \ldots \partial_{r-p}^{\beta_{r-p}} \, Y'^{\alpha_1 \ldots \alpha_{n-p}, \beta_1 \ldots \beta_{r-p}}_{\ell, j_1 \ldots j_p, k_1 \ldots k_{r-p}} \tag{C33}$$

We note that the momenta carried by all free space-time points except ℓ are independent. Further $Y'_{\ell, j_1 \ldots j_p}$ is totally antisymmetrized in $(j_1 \ldots j_p)$. Using these and comparing coefficients of independent moment a, we find,

$$B(\omega_{\alpha_1} \ldots \omega_{\alpha_{n-p}}) Y'^{\alpha_1 \ldots \alpha_{n-p}, \beta_1 \ldots \beta_{r-p}}_{\ell, j_1 \ldots j_p, k_1 \ldots k_{r-p}} = D \tag{C34}$$

which is precisely the equation we solved earlier; with the result that (we suppress some indices here)

$$(\omega_{\alpha_1} \ldots \omega_{\alpha_{n-p}} Y^{\alpha_1 \ldots \alpha_{n-p}, \ldots}_{\ell, \ldots \ldots}) = B(\omega_{\alpha_1} \ldots \omega_{\alpha_{n-p-1}} X'^{\alpha_1 \ldots \alpha_{n-p-1}}_{\ell, \ldots}) .$$

We now construct

$$Y_\ell^{(1)} = Y_\ell - B \;\; \omega_{\alpha_1} \ldots \omega_{\alpha_{n-p-1}} X'^{\alpha_1 \ldots \alpha_{n-p-1}, \beta_1 \ldots \beta_{r-p}}_{\ell, j_1 \ldots j_p, k_1 \ldots k_{r-p}} \, \partial_{k_1}^{\beta_1} \ldots \partial_{k-p}^{\beta_{r-p}}$$
$$(\omega\cdot\partial)j_1 \ldots (\omega\cdot\partial)j_p \tag{C35}$$

Now, $Y_\ell^{(1)}$ satisfies $BY_\ell^{(1)} = 0$ and contains at least $(p + 1)$ derivatives contracted with ghosts. This process can be obviously continued until the end.[15]

Finally, we prove the locality of S_i of Eq. (4.22) which together with proof given by Dixon and Taylor in Ref. (5) completes the proof of the statement of Eq. (4.43). Here we use the Dixon-Taylor lemma for functionals with one ghost. We thus have to show that

$$B(\omega_\alpha Y_i^\alpha[\Phi]) = 0 \tag{C.36}$$

implies that for some local $X_i^\alpha[\Phi]$ and $Z_i^\alpha[\Phi]$,

$$\omega_\alpha Y_i^\alpha[\Phi] = B(X_i^\alpha[\Phi]) + A(Z_i^\alpha[\Phi]) . \tag{C37}$$

From the discussion given earlier in this Appendix, we need to consider Y_i which is independent of Φ, carrying an arbitrary number of additional indices in

$$B[\omega_\alpha Y_i^\alpha] = 0 . \tag{C38}$$

Now, Y_i^α must contain at least one derivative for otherwise

$$B[\omega_\alpha Y_i^\alpha] = \frac{1}{2} g f_{\alpha\beta\gamma} \omega_\beta \omega_\gamma Y_i^\alpha = 0 \tag{C39}$$

implies $Y_i^\alpha \equiv 0$. Further one of the derivatives in Y_i^α must be contracted with ω_α in the form $(\omega_\alpha \partial_m^\alpha)$ for if we write

$$\omega_\alpha Y_i^\alpha = \omega_\alpha \partial_m^\alpha Y'_{mi}[\partial] + \omega_\alpha \partial_{k_1}^{\beta_1} \cdots \partial_{k_n}^{\beta_n} Y''^{\alpha,\beta_1\ldots\beta_n}_{i,k_1\ldots k_n} \tag{C40}$$

where the last term does not contain a factor $\omega_\alpha \partial_m^\alpha$, then comparing coefficients of terms in Eq. (C.38) not containing such a factor, we get

$$\frac{1}{2}\, g f_{\alpha\beta\gamma}\omega_\beta\omega_\gamma \left[\partial^{\beta_1}_{k_1}\cdots\partial^{\beta_n}_{k_n}\; Y''^{\alpha,\beta_1\ldots\beta_n}_{i,k_1\ldots k_n}\right] = 0 \tag{C41}$$

showing that the last term in Eq. (C.40) vanishes. Let us first consider the case in which Y'_{mi} does not contain derivatives. Then, remembering that Y'_{mi} is a local functional either[15]

$$Y'_{mi,\ldots} = t^{\gamma}_{mp}\, Y'''_{pi,\ldots.} \tag{C42}$$

or

$$\omega_\alpha Y^{\alpha}_{i}[\Phi] \equiv \omega_\alpha \partial^{\alpha}_{m} Y'_{mi}[\Phi] = \omega_\alpha\, \partial^{\alpha}_{m}\{\Phi_m(\Phi_j\Phi_j)^p F_i[\Phi]\,\delta^4(\ldots)\} \tag{C43}$$

where in $F[\Phi]$ all Φ are contracted with f's or t's. In the latter case we may write

$$\omega_\alpha Y^{\alpha}_{i}[\Phi] = -\omega_\alpha\, \partial^{\alpha}_{m}(\Phi_j\Phi_j)^{p+1}\, \frac{\delta F_i[\Phi]}{\delta\Phi_m}\, \delta^4(\ldots)$$

$$+ A\{\frac{1}{2(p+1)}(\Phi_j\Phi_j)^{p+1} F_i[\Phi]\,\delta^4(\ldots)\} \tag{C44}$$

where, the 1st term on the r.h.s. is itself a B-invariant and has the form of the term in Eq. (C.42). Thus we need consider Y'_{mi} of the form of the r.h.s. of Eq. (C.42) only. Then,

$$\omega_\alpha\partial^{\alpha}_{m} Y'_{mi} = \omega_\alpha\partial^{\alpha}_{m} t^{\gamma}_{mp} Y'''_{pi}$$

$$= \frac{1}{g}\, B(\partial^{\gamma}_{p} Y'''_{pi}) . \tag{C45}$$

Equations (C.44) and (C.45) prove the result for this case. Finally, in the case when Y'_{mi} contains derivatives, we write

$$\omega_\alpha Y_i^\alpha = \omega_\alpha \partial_m^\alpha \partial_{j_1}^{\beta_1} \dots \partial_{j_n}^{\beta_n} Y_{mij_1\dots j_n}^{\beta_1\dots\beta_n} , \tag{C46}$$

where $Y_{j_1\dots j_n}^{\beta_1\dots\beta_n}$ has the appropriate symmetry. Then $B(\omega_\alpha Y_i^\alpha) = 0$ implies

$$\omega_\alpha \partial_m^\alpha \; \omega_\gamma \partial_k^\gamma t_{kj_1}^{\beta_1} \partial_{j_2}^{\beta_2} \dots\dots \partial_{j_n}^{\beta_n} Y_{mi,j_1\dots j_n,\dots\dots}^{\beta_1\dots\beta_n} = 0 \tag{C47}$$

i.e.,

$$t_{kj_1}^{\beta_1} Y_{mi,j_1\dots j_n}^{\beta_1\dots\beta_n} - (k \longleftrightarrow m) = 0 . \tag{C48}$$

It is easy to verify that such a relation necessarily requires that the group index in m is attached to a $t_{m\ell}^{\eta}$. Then Eq. (C46) can be written as

$$\omega_\alpha Y_i^\alpha = \omega_\alpha \partial_m^\alpha t_{m\ell}^{\eta} \partial_{j_1}^{\beta_1} \dots\dots \partial_{j_n}^{\beta_n} K_{i,j_1\dots j_n \ell}^{\beta_1\dots\beta_n \eta} \tag{C49}$$

Then Eq. (C48) implies that $K_{i,j_1\dots j_n \ell}^{\beta_1\dots\beta_n \eta}$ is symmetric under the interchange of $(\beta_r, j_r) \longleftrightarrow (\eta\, \ell)$. Thus we may write

$$\omega_\alpha Y_i^\alpha = B(\partial_\ell^\eta \partial_{j_1}^{\beta_1} \dots \partial_{j_n}^{\beta_n} K_{i,j_1\dots j_n \ell}^{\beta_1\dots\beta_n \eta}) \tag{C50}$$

proving the result.

REFERENCES and FOOTNOTES

[1]For a discussion of renormalization of gauge theories, see B. W. Lee and J. Zinn-Justin D5, 3121 (1972), D5, 3137 (1972), D5, 3155 (1972), D7, 1049 (1973). G. 't Hooft and M. Veltman, Nucl. Phys. B

B. W. Lee, Phys. Rev. D9, 938 (1974).

[2]H. Georgi and D. Politzer, Phys. Rev. D9 , 416 (1973).

[3]D. Gross and F. Wilczek, Phys. Rev. D8, (1973); Phys. Rev. D9, 980 (1974).

[4]Faddeev and N. Popov, Phys. Letters. 25B, 29 (1967).

[5]J. A. Dixon and J. C. Taylor, Nucl. Phys. B78 , 552 (1974) and University of Oxford preprint (74-74). We, however, disagree with their statement that the case of $F^\alpha_{\mu\nu} F^\alpha_{\mu\nu}$ is exceptional.

[6]H. Kluberg-Stern and J. B. Zuber, Saclay Preprint DPh-T/74/56, to be published in Phys. Rev. D.

We, however, believe that the language used in the conclusion of their discussion of renormalization of $F^\alpha_{\mu\nu} F^\alpha_{\mu\nu}$ is incorrect. As emphasized in Sec. V, the question is not whether the diagonal element gives the correct eigenvalue in an arbitrarily chosen ["Simple"] basis, but rather the question is whether there is a basis in which it is true and if so, how the basis is characterized.

[7] H. Kluberg-Stern and J. B. Zuber, Saclay Preprint DPh-T/75/28.

[8] S. Sarkar, Nucl. Physics B82, 447 (1974); S. Sarkar and H. Strubbe, CERN Preprint TH/1929 (1974).

[9] B. W. Lee, Phys. Rev. D9, 938 (1974).

[10] J. Zinn-Justin, Lectures given at "International Summer Institute for Theoretical Physics," Bonn 1974.

[11] C. Becchi, A. Rouet and R. Stora, CPT Preprint, Marseille 1974, "Renormalization of the Abelian Higgs-Kibble Model.

[12] Here 'solution' does not refer to the solution of Eq. (1.2) but to the solution obtained by the Dixon-Taylor construction.

[13] See, for example, M. Hamermesh, Group Theory (Addison-Wesley, Reading, Mass. 1962), p. 244.

[14] There may be gauge invariant operators which may be expressible in this form ($\mathcal{G}'F$). They are expressible as $\frac{\delta \mathcal{L}_0}{\delta A_i} F_i[A]$ where F_i is a covariant local functional, i.e. $\mathcal{G} F_i = 0$. We thank Dr. Zuber for pointing out this to us. In the case of twist two operators, however, there are no such gauge invariant operators.

[15] Since we are interested in solving $\mathcal{G} W = 0$ i.e. $(A + gB)\, W = 0$, only those B-invariants which can occur in a $\mathcal{G}$ invariant W are of interest to us.

FIGURE CAPTIONS

Fig. 1 Young tableaux Γ_s, $s = 1, \ldots, m$ whose Young operators do not annihilate $W_{i_1 \ldots i_r, j_1 \ldots j_p}$.

Fig. 2 Young Tableaux for the permutation symmetries of X' and X'' of Eq. (C.23).

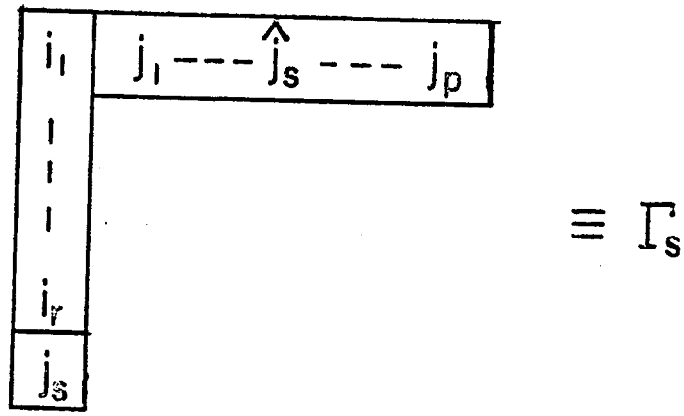

Fig. 1

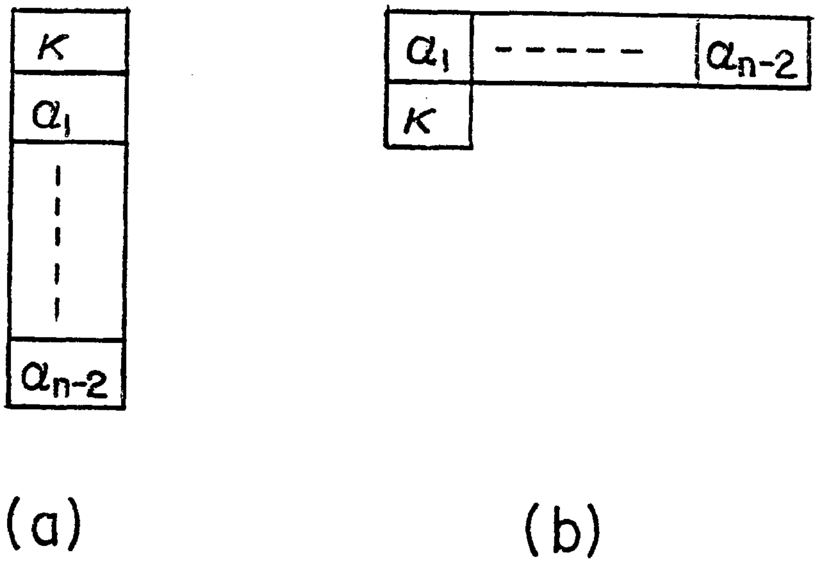

Fig. 2