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Infrared Behavior of the Reggeon Field Theory for the Pomeron

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Abstract

The infrared structure of Reggeon field theory is investigated using renormalization group methods. The infrared fixed point where only the φ^3 interaction is non-trivial is shown to be stable with respect to all higher order interactions within the context of perturbation theory both at $D=2$ and in the ϵ-expansion. This may imply that the asymptotic behavior of the total cross section is model independent.

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I. Introduction

Renormalization group techniques have recently been used to study the infrared behavior of Reggeon field theories of the Pomeron. For theories with triple Pomeron couplings, the infrared behavior of the theory is governed by the existence of a fixed point which occurs for a finite value of the triple Pomeron coupling.[1] Hence we are led to a consistent picture of the Pomeron at asymptotic energies. The theory has the further advantage that the fixed point behavior may be studied perturbatively in the triple Pomeron coupling, provided that this fixed point value of the triple Pomeron coupling is small enough.

While a Reggeon field theory based on only a triple Pomeron interaction leads to a consistent picture at asymptotic energies, we may not a priori rule out the existence of interactions other than the triple Regge which may modify the earlier picture.[2,3] The purpose of this paper is to study effects of these higher order interactions and determine if they will affect the asymptotic behavior. The basic result is that there will be no modification of the asymptotic behavior so long as the infrared stable fixed point of the renormalization group occurs at zero coupling for all higher Reggeon interactions. We show that this is the case when the fixed point behavior is studied perturbatively in lowest order at $D=2$, and to all orders in the triple Pomeron coupling in the ϵ expansion.

The essence of our discussion is a study of infrared behavior of various operators which might be combined to form a Lagrangian for a Reggeon field theory. The infrared behavior of this model or any model can only be affected by those operators that are not infrared soft; that is, by those operators that don't go to zero in the infrared limit. How an operator will behave in the infrared limit is determined by its effective infrared dimension.[4]

In practice we can only determine the scale of various operators with respect to the free Lagrangian.

II. Formalism

In this section we will discuss the basic formalism of the Reggeon calculus and the relevance of various operator interactions to the infrared behavior of the theory.

In the Reggeon field theory the "energy" variable E is associated with the angular momentum J as $E = 1 - J$ and the momentum $\vec{k}$ with the momentum transfer as $\vec{k}^2 = -t$. The statement that the bare Pomeron trajectory is linear,

$$J = 1 - \Delta_0 + \alpha_0' t,$$

translates into the following nonrelativistic free Lagrangian for our theory

$$\mathcal{L}_0 = \tfrac{1}{2}\psi^* i \overleftrightarrow{\partial}_0 \psi - \alpha_0' \vec{\nabla}\psi^* \cdot \vec{\nabla}\psi - \Delta \psi^* \psi.$$

Now since the free action

$$A_0 = \int d^D x \, dt \, \mathcal{L}_0$$

must be dimensionless we see that the dimensions of various quantities are

$$[\psi] = k^{D/2} \qquad [\alpha_0'] = Ek^{-2} \qquad [\Delta_0] = E \qquad [t] = E^{-1} \qquad [x] = k^{-1}$$

Using dimensional arguments, based on the dimensionality of operators in the free theory above, we can look at various operators as possible choices for interactions and discuss their infrared behavior. An operator with m powers of ψ n powers of the derivative operator $\vec{\nabla}_i \cdot \vec{\nabla}_j$, and p powers of ∂_t integrated over x and t has the following dimension

$$\left[\int dx^D dt (\vec{\nabla}_i \cdot \vec{\nabla}_j)^n (\partial_t)^p \psi^m\right] = [O(m,n,p)] = k^{2n+Dm/2-D} E^{p-1}$$

If we associate with k^{-2} and E^{-1} a given scale length L, we see that the dimension of O is

$$[O(m,n,p)] = L^{1+D/2-Dm/4-n-p}$$

and some typical operators that one might consider for interactions have the following behavior

$$[O(3,0,0)] = L^{1-D/4}, \quad [O(4,0,0)] = L^{1-D/2}, \quad [O(6,0,0)] = L^{1-D}.$$

These dimensions are of course just the inverses of those of the associated coupling constants. If one increases the scale of the theory, the relative importance of various interaction operators changes. In particular, the small momentum behavior (infrared) of the system should be governed by coherent long distance behavior, and only those operators that remain for large L can therefore be relevant. For example, O(6,0,0) goes to zero for $D > 1$. A list of relevant operators in two and four dimensions is given in Table 1.

It is important to emphasize that these operator dimensions are relative to the free theory, and that in the interacting theory these operators may pick up anomalous dimensions in the infrared limit, particularly in two dimensions where O(3,0,0) is singular. The singularity of O(2,0,0) the mass counter-term is a particular problem in these theories,[5] but it does not enter in 1st order. We see that at $D=2$, which is the physical dimension for this problem, only φ^3 and φ^4 interactions can affect the infrared behavior of the theory. That is to say, the study of $\mathcal{L}_0$ with only φ^3 and φ^4 operators is complete with no further room for choice. Now at $D=4$ the situation is different; here only φ^3 appears to be important. Nevertheless, one might still be interested in including a φ^4 interaction, since the physical dimension is two and there we know that φ^4 interactions are relevant. The φ^4 theory is non-renormalizable at $D=4$ and an expansion in ϵ would seem a priori problematic. This is not the case, and only the φ^3 theory turns out to be relevant.

III. Theory of the Pomeron

In this section we give a brief review of the application of the renormalization group to the Reggeon field theory with triple pomeron and four pomeron interactions.[1,2,3] Reggeon field theory is designed to give an equivalent description of Reggeon-Reggeon interactions and to reproduce the correct analyticity and discontinuity relations in the complex angular momentum variables.

The φ^3 and φ^4 theory of the pomeron is expressed in terms of a "non-relativistic" Lagrangian field theory with the following action

$$A = \int dx^D dt \Big\{ \frac{i}{2} \psi^* \overleftrightarrow{\partial}_o \psi - \alpha'_o \, \vec{\nabla}\psi^* \cdot \vec{\nabla}\psi - \Delta_o \psi^*\psi - i \, \frac{r_o}{2!}\left(\psi^{*2}\psi - \psi^*\psi^2\right)$$

$$- \frac{\lambda_o}{3!}\left(\psi^*\psi^3 + \psi^{*3}\psi\right) - \frac{\lambda_{01}}{(2!)^2} \, \psi^{*2}\psi^2 \Big\}$$

The bare pomeron couplings are assumed constants $(r_0, \lambda_0, \lambda_{01})$. The single particle irreducible (SPI) unrenormalized vertex functions for n incoming and m outgoing pomerons are denoted by $\Gamma^u_{n,m}(\{E,P\})$, and are to lowest order given by

$$i \, \Gamma^u_{1,1}(E,P) = E - \alpha'_o \vec{P}^2 - \Delta_o \, ,$$

$$\Gamma^u_{1,2}(\{E_i, P_i\}) = r_o/(2\pi)^{(D+1)/2} \, ,$$

$$\Gamma^{u}_{1,3}(\{E_i,P_i\}) = -i\ \lambda_o/(2\pi)^{D+1} ,$$

$$\Gamma^{u}_{2,2}(\{E_i,P_i\}) = -i\ \lambda_{01}/(2\pi)^{D+1} .$$

To this order all others are zero.

The renormalization group is used to study the infrared behavior of this theory ($\vec{P} \to 0$, $E \to 0$). The bare couplings are replaced by effective renormalized parameters defined with respect to a given energy or momentum scale. By determining the dependence of the effective parameters on the normalization point, the infrared structure is also determined.

For the purposes of this paper, we take a momentum normalization point. While in principle there are very many normalization points that one can choose, we feel that this one which keeps the P^2 of all Reggeons equal, preserves the basic symmetries of the problem and therefore may be preferable. The renormalized SPI vertex functions are defined by

$$\Gamma^{R}_{n,m}(\{E,P\}) = Z^{(n+m)/2}\ \Gamma^{u}_{n,m}(\{E,P\}),$$

where Z is the wave function renormalization constant.

The renormalized parameter α' and dimensionless coupling constants g, y, and y_1 are determined by the following normalization conditions

$$\partial_E i\ \Gamma^{R}_{1,1}(E,P^2)\Big|_{E=0;\ P^2=k_N^2} = 1,$$

$$\partial_{P^2}\ i\,\Gamma^{R}_{1,1}(E,P^2)\Big|_{E=0;\ P^2=k_N^2} = -\alpha',$$

$$\Gamma^R_{1,2}(E_i,\vec{P}_i)\Big|_{E_i=0;\ \vec{P}_i\cdot\vec{P}_j=\frac{k_N^2}{2}(3\delta_{ij}-\eta_i\eta_j)} = \frac{\alpha'(k_N^2)^{1-D/4}}{(2\pi)^{(D+1)/2}}\, g,$$

$$\Gamma^R_{1,3}(E_i,\vec{P}_i)\Big|_{E_i=0;\ \vec{P}_i\cdot\vec{P}_j=\frac{k_N^2}{3}(4\delta_{ij}-\eta_i\eta_j)} = -i\,\frac{\alpha'(k_N^2)^{1-D/2}}{(2\pi)^{D+1}}\, y,$$

$$\Gamma^R_{2,2}(E_i,\vec{P}_i)\Big|_{E_i=0;\ \vec{P}_i\cdot\vec{P}_j=k_N^2(\delta_{ij}-\frac{1}{4}\eta_i\eta_j+\frac{1}{4})} = -i\,\frac{\alpha'(k_N^2)^{1-D/2}}{(2\pi)^{D+1}}\, y_1,$$

where $\eta_i = 1$ for incoming and -1 for outgoing Reggeons.

The renormalized SPI vertex functions are functions of α', g, y, y_1, and k_N^2. We have assumed that the renormalized intercept occurs at $J = 1$, i.e., $\Gamma^R_{1,1}(0,0) = 0$.

IV. Renormalization Group Discussion

The purpose of introducing renormalized quantities is twofold. By defining renormalized quantities relative to a given momentum scale, we are less sensitive to the high momentum behavior of the theory. A perturbation expansion in the renormalized coupling constant may exist while a perturbation expansion in the unrenormalized coupling may become undefined in the infrared region.

While the renormalized quantities were defined relative to a given normalization point, the unrenormalized quantities are independent of the normalization point. The renormalization group equations tell us how the renormalized quantities change as we change the normalization point. In the φ^3 and φ^4 theory we obtain the following renormalization group equations for the renormalized SPI vertex function

$$\left\{k_N^2 \partial_{k_N^2} + \beta_g \partial_g + \beta_y \partial_y + \beta_{y_1} \partial_{y_1} + \delta\alpha' \partial_{\alpha'}\right\} \Gamma^R_{n,m} = \frac{(n+m)}{2} \gamma \Gamma^R_{n,m}$$

where by dimensional analysis the renormalization group coefficients β_g, β_y, β_{y_1}, δ and γ are functions only of the renormalized coupling constants.

We may study the infrared behavior of the theory by rescaling the energies and momenta $E_i \to \xi^\nu E_i$, $P_i^2 \to \xi P_i^2$ and letting the scale $\xi \to 0$. First let us consider the unrenormalized φ^3 theory. Using dimensional analysis, the SPI vertex functions scale as

$$\Gamma^u_{n,m}(\xi^\nu E_i, \xi^{\frac{1}{2}} P_i, \alpha'_o\ r_o) = \xi^{d_{n,m}} \Gamma^u_{n,m}\left(E_i, P_i, \xi^{1-\nu}\alpha'_o, \xi^{D/4-\nu} r_o\right)$$

where $d_{n,m} = \nu + D(2-n-m)/4$. The choice of $\nu = 1$ allows us to keep the relative energy and momentum scales fixed. In this case, the effect of examining the infrared behavior of the theory is to modify the bare coupling constant, $r_o \to \xi^{D/4-1} r_o$. For the physical number of dimensions, $D = 2$, the bare effective coupling constant increases to infinity as we let $\xi \to 0$. Therefore, a direct analysis of the bare theory becomes useless in the infrared region.

Although the theory considered as a function of the bare coupling is not useful, the theory may become tractable when re-expressed in terms of the renormalized quantities. The theory may remain well defined as a function of the renormalized quantities, even as the effective bare coupling constant tends to infinity.

With this brief introduction, we will proceed to study directly the infrared behavior of our φ^3 and φ^4 theory using the renormalization group. With the help of the above scaling equation for $\nu = 0$ we find that

$$\left\{\xi\,\partial_\xi - \beta_g\,\partial_g - \beta_y\,\partial_y - \beta_{y_1}\,\partial_{y_1} + \alpha'[1-\delta]\partial_{\alpha'}\right.$$

$$\left. + \left[\left(\frac{n+m}{2}\right)\gamma - d_{n,m}\right]\right\}\Gamma^R_{n,m}(E_i,\xi^{\frac{1}{2}}P_i,g,y,y_1,\alpha',k_N^2) = 0.$$

Solving this equation, we can determine the value of the vertex function at momenta $\xi^{\frac{1}{2}}P_i$ in terms of the vertex function at moment

$$\Gamma^R_{n,m}(E_i,\xi^{\frac{1}{2}}P_i,g,y,y_1,k_N^2) = \Gamma^R_{n,m}\left(E_i,P,g(-t),y(-t),y_1(-t),\alpha(-t),k_N^2\right)$$

$$\times\exp\left\{\int_{-t}^{0} dt'\left[d_{n,m} - \frac{(n+m)}{2}\gamma\left(g(t'),y(t'),y_1(t')\right)\right]\right\}$$

where $\xi = e^t$. The characteristic equations that will govern the behavior of coupling constants as functions of the scale ξ or t are

$$\frac{dg}{dt} = -\beta_g\left(g(t),y(t),y_1(t)\right),$$

$$\frac{dy}{dt} = -\beta_y\left(g(t),y(t),y_1(t)\right),$$

$$\frac{dy_1}{dt} = \beta_{y_1}\left(g(t),y(t),y_1(t)\right)$$

and

$$\frac{d\alpha(t)}{dt} = \alpha(t)\left[+1-\delta\left(g(t),y(t),y_1(t)\right)\right],$$

where $g(0) = g$, $y(0) = y$ and $y_1(0) = y_1$. In the infrared limit the theory will be governed by a set of coupling constants that satisfy the above equation in the limit $t\to\infty$, and which remain stable. These coupling constants are determined by the following conditions,

$$\beta_g(\bar{g},\bar{y},\bar{y}_1) = 0 \ ,$$

$$\beta_y(\bar{g},\bar{y},\bar{y}_1) = 0 \ ,$$

$$\beta_{y_1}(\bar{g},\bar{y},\bar{y}_1) = 0 \ ,$$

and a stability condition which will be discussed later. It remains now to determine β_g, β_y, β_{y_1} in some approximation and solve for the fixed point $(\bar{g}, \bar{y}, \bar{y}_1)$ that governs the infrared behavior of the theory.

V. Determination of β's

We will determine the β's in perturbation theory in the one-loop approximation, in D dimensions using the symmetric momentum normalization. The calculations are relatively straightforward.

From the graphs in Fig.1a, $\Gamma^u_{1,1}$ is found to be

$$i\,\Gamma^u_{1,1}(E,P^2) = E - \alpha'_o P^2 - \frac{r_o^2}{(\alpha'_o)^{D/2}} \left(\frac{\pi}{2}\right)^{D/2} \frac{\Gamma(1-D/2)}{2\,(2\pi)^D} \left(\frac{\alpha'_o P^2}{2} - E\right)^{D/2-1}$$

and therefore

$$Z^{-1} = 1 - g_o^2\,\frac{4}{\epsilon}\,K\,2^{\epsilon/2}$$

and

$$\alpha'(k^2) = \alpha'_o\left(1 + g_o^2\,\frac{2}{\epsilon}\,K\,2^{\epsilon/2}\right) ,$$

where $\epsilon = 4-D$,

$$K = \left(\frac{\pi}{2}\right)^{D/2} \frac{\Gamma(3-D/2)}{4\,(2\pi)^D}$$

and

$$g_o = \frac{r_o (k_N^2)^{D/4-1}}{\alpha'_o} .$$

From the one particle irreducible graphs, shown in Fig.1b, we calculate $\Gamma^R_{1,2}$ and g, the renormalized coupling constant. We find

$$g = g_o \left\{1 - g_o^2 \frac{4K\, 2^{\epsilon/2}}{\epsilon} \left(4I_D(1,1,1) - 1\right) + \frac{4K}{\epsilon} \frac{2^{\epsilon/2-1}}{1 - D/2} (2y_0 + y_{01})\right\},$$

where

$$y_0 = \frac{\lambda_0 (k_N^2)^{D/2-1}}{\alpha_o'} \quad , \quad y_{01} = \frac{\lambda_{01} (k_N^2)^{D/2-1}}{\alpha_o'}$$

and

$$I_D(a,b,c) = \int_o^1 \frac{d\beta_1 d\beta_2 \delta(1-\beta_1-\beta_2)}{(a\beta_1 + b\beta_2 + c\beta_1\beta_2)^{2-D/2}}$$

is tabulated in Table II.

From the one particle irreducible graph in Fig.1c, we calculate $\Gamma^R_{1,3}$ and the renormalized coupling constant y.

$$y = \tilde{y}(g_0) + y_0 + y_0 g_0^2 \frac{K\, 2^{\epsilon/2}}{\epsilon} \left[6 - 24 I_D(1,1,4/3) - 48 I_D(1,4/3,1)\right]$$

$$- y_{01} g_0^2 \frac{K\, 2^{\epsilon/2}}{\epsilon} \left[24 I_D(1,4/3,1)\right] - y_0 y_{01} \frac{16K}{\epsilon(2-D)} \left(\frac{3}{2}\right)^{\epsilon/2},$$

where $\tilde{y}(g_0)$ denotes the contribution to y from the pure φ^3 theory.

Finally, the two-to-two graphs in Fig.1d give $\Gamma^R_{2,2}$ and therefore y_1

$$y_1 = \tilde{y}_1(g_0) + y_{01} + g_0^2 y_{01} \frac{K\, 2^{\epsilon/2}}{\epsilon} \left[6 - 32 I_D(2,1,1) - 32 I_D(1,1,1)\right]$$

$$- g_0^2 y_0 \frac{K\, 2^{\epsilon/2}}{\epsilon} 64 I_D(1,1,1) - \frac{8K}{\epsilon(2-D)} \left(y_{01}^2 + 2y_0^2\, 2^{\epsilon/2}\right),$$

where $\tilde{y}_1(g_0)$ is the effective two-to-two coupling that one would obtain in a pure φ^3 theory. We note that both $\tilde{y}(g_0)$ and $\tilde{y}_1(g_0)$ are of order g_0^4, and come from box graphs that we will not have to calculate.

We can now calculate the β's. First, however, let us make a change of variable, and define a matrix notation

$$h_1 = y-\tilde{y}(g_0) \qquad h_2 = y_1-\tilde{y}_1(g_0) \qquad h_3 = g$$

$$h_1^o = y_0 \qquad h_2^o = y_{01} \qquad h_3^o = g_0$$

with an operator matrix of coefficients M,

$$h = M\, h_0$$

and

$$h_0 = M^{-1} h \ .$$

Thus

$$\beta_{\mathbf{h}} = k_N^2 \partial_{k_N^2} h = M' h_0 + M\, C\, h_0$$

where

$$C = \begin{pmatrix} 1-\varepsilon/2 & 0 & 0 \\ 0 & 1-\varepsilon/2 & 0 \\ 0 & 0 & -\varepsilon/4 \end{pmatrix} .$$

On substituting h_0 as a function of h, we find

$$\beta_h = (M'M^{-1} + M C M^{-1})h.$$

Carrying this through in perturbation theory, we get

$$\beta_g = -\varepsilon/4\ g + g^3 K\, 2^{\varepsilon/2+1}\left(4 I_D(1,1,1)-1\right) + \frac{K\, 2^{\varepsilon/2+1}}{\varepsilon}\, g(2h_1 + h_2),$$

$$\beta_{h_1} = (1-\varepsilon/2)h_1 + g^2 h_1 K\, 2^{\varepsilon/2}[12 I_D(1,1,4/3) + 24 I_D(1,4/3,1)-3]$$
$$+\ g^2 h_2 K\, 2^{\varepsilon/2} 12 I(1,4/3,1) + h_1 h_2 \frac{8K}{\varepsilon}\left(\frac{3}{2}\right)^{\varepsilon/2}$$

and

$$\beta_{h_2} = (1-\varepsilon/2)h_2 + g^2 h_1 K\, 2^{\varepsilon/2} 32 I_D(1,1,1)$$

$$+ g^2 h_2 K\, 2^{\varepsilon/2} \left(16 I_D(2,1,1) + 16 I_D(1,1,1) - 3\right) + \frac{K}{\varepsilon}\,(2h_1^2\, 2^{\varepsilon/2} + h_2^2).$$

With the β's in this form, the fixed point of the theory is easy to obtain. Clearly, for $\varepsilon \neq 0$

$$\bar{h}_1 = 0,\ \bar{h}_2 = 0\,,\ \ \overline{g^2} = \frac{\varepsilon/4}{2K^{\varepsilon/2}\left(4 I_D(1,1,1)-1\right)}$$

is a fixed point of the theory, and $\overline{g^2}$ has the same value that one would obtain in a φ^3 theory with a symmetric momentum normalization. Therefore the fixed point occurs when we have all φ^4 couplings zero and the φ^3 coupling is given by the fixed point in the pure φ^3 theory. This is, of course, only true away from the singularities in the β's. Furthermore, we must investigate whether the fixed point is stable. It is entirely possible that while the φ^4 coupling has left the location of the fixed point alone it could have destabilized it. Note that we assume $\varepsilon \neq 0$ here. The limit $\varepsilon \to 0$ will be treated in a later section.

In the physical case $D = 2$, the β's become

$$\beta_g = g\left[-\tfrac{1}{2} + 2.443\,\frac{g^2}{8\pi} + \frac{h_1}{8\pi} + \tfrac{1}{2}\,\frac{h_2}{8\pi}\right],$$

$$\frac{\beta_{h_1}}{8\pi} = \frac{g^2}{8\pi}\left[12.527\,\frac{h_1}{8\pi} + 4.540\,\frac{h_2}{8\pi}\right] + 1.5\,\frac{h_1}{8\pi}\,\frac{h_2}{8\pi},$$

$$\frac{\beta_{h_2}}{8\pi} = \frac{g^2}{8\pi}\left[13.772\,\frac{h_1}{8\pi} + 10.372\,\frac{h_2}{8\pi}\right] + 2\left[\frac{h_1}{8\pi}\right]^2 + .5\left[\frac{h_2}{8\pi}\right]^2.$$

We can only hope that perturbation theory in the one loop approximation gives a reasonable approximation to the β's. With this proviso, we can solve for the effective infrared coupling

$$\frac{\bar{g}^2}{8\pi} = .2046.$$

We note that this number is slightly different from that obtained with an energy normalization, which is as it should be since different normalization points represent different orderings of the perturbation theory, and therefore the lowest order approximations understandably give different numbers. The fact that the numbers are nearly equal might be taken as an indication of a rapid convergence of the theory. We should note that higher order perturbation calculations in the φ^3 theory in the ϵ-expansion have been shown to be slowly convergent.[6] It is not clear however that this will apply also to $D = 2$. The higher order calculations at $D = 2$ are not straightforward because of problems with the mass counter term.[5,7] Therefore the accuracy of lowest order perturbation expansion remains an open question.[7]

Let us now turn to the question of stability of the solution that we have just discussed. The standard technique for studying stability is to linearize the equation about a fixed point, i.e.,

$$-\frac{dg}{dt} = \beta_g = \lambda(g-\bar{g}) + c_1h_1 + c_2h_2$$

$$-\frac{dh_1}{dt} = \beta_{h_1} = d_1h_1 + d_2h_2$$

$$-\frac{dh_2}{dt} = \beta_{h_2} = e_1h_1 = e_2h_2$$

Solving these equation, we find

$$g(t)-\bar{g} = a_1 e^{-\lambda t} - a_2(c_1 d_2 - c_2 d_1 + c_2\lambda_2)e^{-\lambda_2 t} - a_3(c_1 d_2 - c_2 d_1 + c_2\lambda_3)e^{-\lambda_3 t},$$

$$h_1(t) = a_2 d_2(\lambda-\lambda_2)e^{-\lambda_2 t} + a_3 d_2(\lambda-\lambda_3)e^{-\lambda_3 t}$$

and

$$h_2(t) = -a_2(d_1-\lambda_2)(\lambda-\lambda_2)e^{-\lambda_2 t} - a_3(d_1-\lambda_3)(\lambda-\lambda_3)e^{-\lambda_3 t},$$

where the a's are determined from the initial conditions at $t=0$. The eigenvalues λ_2, λ_3 are given by

$$2\lambda_{3,2} = +(e_2+d_1) \pm [(e_2-d_1)^2 + 4e_1 d_2]^{\frac{1}{2}}.$$

Stability requires all the eigenvalues to be positive

$$\lambda, \lambda_2, \lambda_3 > 0.$$

For the physical dimension $D=2$ we find that

$$\lambda = .50, \qquad \lambda_2 = .71, \qquad \lambda_3 = 3.98,$$

all of which are positive and therefore our solution is stable. Actually, the theory we are considering is an expansion in g^2, h_1 and h_2, and the eigenvalue associated with the true expansion parameter g^2 is $\lambda_{g^2} = 1$. It appears, therefore, that $g^2-\bar{g}^2$, h_1 and h_2 approach their critical values at the same rate, indicating that the presence of h_1 and h_2 couplings will affect the approach of our theory to its scaling behavior, that is, the secondary terms in the cross section. These linearized equations can also be given a physical interpretation of some interest. Consider the linear terms in the original expression that we had for the β's

$$- \frac{dg}{dt} = -\epsilon/4 \; g + \ldots$$

$$- \frac{dh_1}{dt} = (1-\epsilon/2)h_1 + \ldots$$

$$- \frac{dh_2}{dt} = (1-\epsilon/2)h_2 + \ldots$$

The coefficients of the linear terms are essentially minus the dimensionality of the various operators with respect to the free theory. Thus the dimensionality of O(3,0,0) is 1/2 for $D = 2$ ($\epsilon = 3$), and zero for $D = 4$ ($\epsilon = 0$), while O(4,0,0) has the dimensionality zero at $D = 2$ and -1 at $D = 4$, in agreement with Table I. Now in the interacting theory, at the fixed point we find that various linear combinations of $g-\bar{g}$, h_1, and h_2, which can easily be determined from the above equations, satisfy similar linear equations with coefficients λ, λ_2 and λ_3. Thus in the interacting theory in the infrared limit, linear combinations of various operators have definite dimensions. In particular, there are two combinations of O(4,0,0) operators which have dimensions -.71 and -3.98, and there is a combination of O(3,0,0) and the two O(4,0,0) operators that has dimension -.5. Note that these dimensions have changed considerably from the free dimension indicating some danger in relying on the dimension of the operators with respect to the free theory to determine what operators are relevant in the infrared limit. It is also interesting to note that the approach to scaling is determined by the operator that is least soft in the infrared limit.

In the ϵ-expansion however things are considerably nicer since the dimensionality of operators only changes from the dimension with respect to the free theory by a perturbation of order ϵ. Thus the argument of the relevancy of various operators with respect to the free Hamiltonian is assured for small ϵ.

D = 4 Dimension and the ε-Expansion

There are several interesting points about our φ^3 and φ^4 theory near or at 4 dimensions. Consider the renormalizability of the theory. The superficial degree of divergence δ_s is

$$\delta_s = V_3(D/2-2) + V_4(D-2) - D/2\ \xi + D+2$$

where V_3 and V_4 are the number and three- and four-particle vertices, and ξ is the number of external lines. At $D=4$ the superficial degree of divergence grows like twice the number of four-particle vertices; that is, the theory is nonrenormalizable at $D=4$. We learned earlier, however, that the fixed point coupling of the φ^4 occurs at zero coupling; therefore to study the scaling behavior we need only study the theory to the lowest order in the φ^4 coupling, and we will be left with a finite number of subtraction constants. Calculating the β's from our previous expansion we find

$$\beta_g = g\left(-\epsilon/4 + \frac{3}{2}\left(\frac{g}{8\pi}\right)^2 + \frac{1}{\epsilon}\left[\frac{h_1}{(8\pi)^2} + \frac{1}{2}\frac{h_2}{(8\pi)^2}\right]\right)$$

$$\frac{\beta_{h_1}}{(8\pi)^2} = (1-\epsilon/2)\frac{h_1}{(8\pi)^2} + 8.25\left(\frac{g}{8\pi}\right)^2\frac{h_1}{(8\pi)^2} + 3\left(\frac{g}{8\pi}\right)^2\frac{h_2}{(8\pi)^2} + \frac{2}{\epsilon}\frac{h_1}{(8\pi)^2}\frac{h_2}{(8\pi)^2}$$

$$\frac{\beta_{h_2}}{(8\pi)^2} = (1-\epsilon/2)\frac{h_2}{(8\pi)^2} + 7.25\left(\frac{g}{8\pi}\right)^2\frac{h_2}{(8\pi)^2} + 8\left(\frac{g}{8\pi}\right)^2 h_1 + \frac{2}{\epsilon}\frac{h_1^2}{(8\pi)^4} + \frac{1}{\epsilon}\frac{h_2^2}{(8\pi)^4}$$

Notice that to first order in h_1 and h_2, β_g has a pole at $\epsilon = 0$ while β_{h_1} have poles in higher order. This can be viewed from two points of view. First, all poles in ε result from ultraviolet divergences in transverse momentum integrals with the quartic coupling present, which also lead to the renormalizability problems of the φ^4 theory at $\epsilon = 0$.

Physically, large transverse momentum components have no place in determining the behavior of a Reggeon field theory which purports to calculate the behavior of Green's functions only near $j = 1$ and $p_\perp^2 = 0$. One may simply note that as $p_\perp^2 \to \infty$, staying on the bare trajectory defined by $j = \alpha_0 - \alpha_0' p_\perp^2$ leads to $j \to -\infty$. Indeed, as $p_\perp^2 \to \infty$, one eventually approaches a fixed angle limit which has no place in the Reggeon calculus. A resolution of this problem lies in the initial assumption of no $p_\perp^2$ damping in bare couplings. One can assume that some underlying cutoff Λ is in fact present in the original theory. The cutoff φ^4 theory is then renormalizable, of course, and the poles in ϵ and in the β's will be replaced by some function of Λ. One is still obliged, however, to demonstrate that the scaling results are independent of Λ[8]; we shall do this.

Secondly, we recall that there are several operators in addition to φ^4 which scale like L^{-1} which we have not included. If we were to include all these operators as interactions then, to first order in perturbation theory, the renormalization of these operators, which are complete to this order in L, would remove all $1/\epsilon$ from all β's (which would be greatly expanded in number). This of course is another way of handling the renormalization difficulties of this theory in four dimensions.

In either case, the infrared behavior of the theory is unaffected. The point is that the infrared and ultraviolet behavior of the theory are decoupled and the fact that we have included a set of operator interactions that behave badly in the ultraviolet region simply doesn't matter. The asymptotic behavior of the theory is completely determined by the fixed point. This is to be determined by the theory to all orders in the φ^3 interaction, but only to first order in the φ^4 interactions.

We can further illustrate this point by considering the ϵ-expansion in somewhat more detail. In particular, the $1/\epsilon$ factors in the β's cannot affect the fixed point. For simplicity, we will only include one φ^4 coupling in the following argument.

The most general form for the renormalized cubic coupling g to first order in y_0 and all orders in g_0 is

$$g = \tilde{g}(g_0) + \frac{y_0 g_0}{\varepsilon} f_0(g_0{}^2, \varepsilon)$$

where, as before, $\tilde{g}(g_0)$ is the renormalized cubic coupling with $y_0 = 0$. It is easily seen that the most singular behavior of f_0 is $f_0(g_0{}^2/\varepsilon)$. This follows from low order perturbation theory results and the fact that the most singular insertion in a graph is a bubble which produces a factor $(g_0{}^2/\varepsilon)[1+0(\varepsilon)]$ and does not change any convergence properties of integrals. We shall display only the most singular behavior of the arguments of all functions.

Now we have (c.f. Eq.(68) of the second paper in Ref.6)

$$g_0 = \tilde{g}\ f_1(\tilde{g}^2/\varepsilon)$$

where $f_1(x) = 1 + 0(x)$. This produces

$$g = \tilde{g} + \frac{y_0 \tilde{g}}{\varepsilon} f_2(\tilde{g}^2/\varepsilon)$$

where $f_2(x) = f_1(x) f_0\left(x\ f_1{}^2(x)\right)$. A similar expression is obtained for the renormalized quartic coupling to lowest order in y_0

$$y = \tilde{y}(g_0) + y_0\ f_3^{(0)}(g_0{}^2/\varepsilon)$$

which becomes

$$y = \tilde{y}[\tilde{g}\ f_1(\tilde{g}^2/\varepsilon)] + y_0\ f_3(\tilde{g}^2/\varepsilon)$$

where $f_3(x) = f_3^{(0)}\left(x\ f_1{}^2(x)\right)$. Here,

$$\tilde{y}(\tilde{g}\ f_1) = c\tilde{g}^4 + 0(\tilde{g}^6/\varepsilon)$$

is the induced renormalized quartic coupling at $y_0 = 0$. c is finite at $\varepsilon = 0$, as can easily be checked.

We now calculate the β functions. Calling $\tilde{\beta}_g(\tilde{g})$ and $\tilde{\beta}_y(\tilde{g})$ the unmodified ϕ^3 functions obtained by applying $k_N^2 \partial_{k_N^2}$ to $\tilde{g}$ and $\tilde{y}$ and using the chain rule $k_N^2 \partial_{k_N^2} \tilde{y}(g_0) = [\partial_{\tilde{g}} \tilde{y}(\tilde{g}\ f_1)][k_N^2 \partial_{k_N^2} \tilde{g}(g_0)]$ we get

$$\tilde{\beta}_y(\tilde{g}) = \tilde{\beta}_g(\tilde{g})[4c\ \tilde{g}^3 + O(\tilde{g}^5/\epsilon)].$$

The β functions for the full theory defined by applying $k_N^2 \partial_{k_N^2}$ to g and y are

$$\beta_g(g,y_0) = \tilde{\beta}_g(\tilde{g}) + \frac{y_0 \tilde{g}}{\epsilon} f_2(\tilde{g}^2/\epsilon)[1 + O(\epsilon)],$$

$$\beta_y(g,y_0) = \tilde{\beta}_y(\tilde{g}) + y_0\left(1 - \frac{\epsilon}{2}\right) f_3(\tilde{g}^2/\epsilon) + y_0 \tilde{g}^2 f_4(\tilde{g}^2/\epsilon),$$

where $f_4(x) = -\frac{1}{2} f_1^{\,2}(x)[df_3^{(0)}(y)/dy]$ evaluated at $y = xf_1^{\,2}(x)$. In lowest order in $\tilde{g}$ these results agree with those found in the one loop approximation.

Consider for illustration the $O(\epsilon^2)$ terms in the critical exponents. In ϕ^3 theory, they result from a vanishing of $\tilde{\beta}_g(\tilde{g})$ through $O(\epsilon^{\frac{5}{2}})$; i.e., $\tilde{\beta}_g(\tilde{g}) = O(\epsilon^{\frac{7}{2}})$.[6] This leads to $\bar{g}^2 = \bar{g}_1\ \epsilon + \bar{g}_2\ \epsilon^2$. However this implies that $\tilde{\beta}_y(\tilde{g}) = O(\epsilon^5)$. The second and third terms of β_y are of $O(y_0)$ and $O(y_0\epsilon)$, respectively. Cancellation of the two lowest order terms in ϵ in β_y then implies $\bar{y}_0 = O(\epsilon^5)$. This is a full three powers in ϵ above the result implied for the <u>induced</u> quartic coupling, and is necessary for our result. Having $\bar{y}_0$ only of $O(\epsilon^2)$ would result in the second term of β_g being of $O(\epsilon^{\frac{3}{2}})$ or of $O\left(\epsilon^{\frac{5}{2}}\psi(\Lambda)\right)$ if the pole in ϵ were replaced by some function ψ of a cutoff Λ. Since β_g must vanish in $O(\epsilon^{\frac{5}{2}})$, the fixed point coupling $\bar{g}$ would evidently depend on details of the quartic theory. In fact, there is no such problem.

The "singular" term in β_g is actually of $O(\epsilon^{9/2})$ and does not disturb the ϕ^3 results. Clearly, the same argument goes through in any finite order, proving universality in the ϵ-expansion in the mixed theory. In lowest order, where $f_{2,3,4}$ are constants, the solution is easily seen to be stable for small ϵ.

For the case of two φ^4 interactions discussed earlier we find the fixed point is unchanged from the pure φ^3 theory in accordance with our remarks on the ϵ-expansion to lowest order. That is

$$\bar{h}_1 = 0 \qquad \bar{h}_2 = 0 \qquad \left(\frac{\bar{g}}{8\pi}\right)^2 = \epsilon/12$$

The eigenvalues calculated as before are

$$\lambda = \epsilon/4 \qquad \lambda_2 = 1 - .03\epsilon \qquad \lambda_3 = 1 + 1.6\epsilon$$

We see that these λ are all positive, indicating the stability of the critical point. Furthermore, λ is the smallest eigenvalue for small ϵ and therefore the approach to scaling will be governed by the φ^3 coupling. Furthermore, in accordance with our discussion of the infrared dimension of operators, we see that they only differ from the free dimension by perturbation of order ϵ, that is, the infrared dimension of $O(3,0,0)$ is $-\epsilon/4$ and the dimension of the two $O(4,0,0)$ operators are respectively $-1 + .03\epsilon$ and $-1 - 1.6\epsilon$.

Conclusion

We have argued that the infrared behavior of the Reggeon field theory is governed by the dimensionality of possible interactions, and that therefore the infrared behavior is totally insensitive to the details of the bare theory with the exception of the complete omission of a φ^3 interaction. We have explicitly shown this to be the case for a theory with cubic and quartic couplings calculated to one loop approximation in both two dimensions and in the ϵ-expansion. We have also shown that the fixed point depends only on the φ^3 interaction, and remains unaffected to all orders in the ϵ-expansion. At $D=2$ the situation is similar, the fixed point is totally determined by the φ^3 interaction and can be calculated to all orders independent of the other interactions (assuming, of course, that there is a fixed point at all in higher orders).

We have not discussed other possible fixed points of the φ^3 and φ^4 interactions; they will be discussed elsewhere.[9]

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Figure and Table Captions

Figure 1. The one particle irreducible, one loop approximation graphs that contribute to the Green's functions (a) $\Gamma_{1,1}$, (b) $\Gamma_{1,2}$, (c) $\Gamma_{1,3}$, (d) $\Gamma_{2,2}$.

Table I. Scaling behavior in 2 and 4 dimensions of operators O(m,n,p).

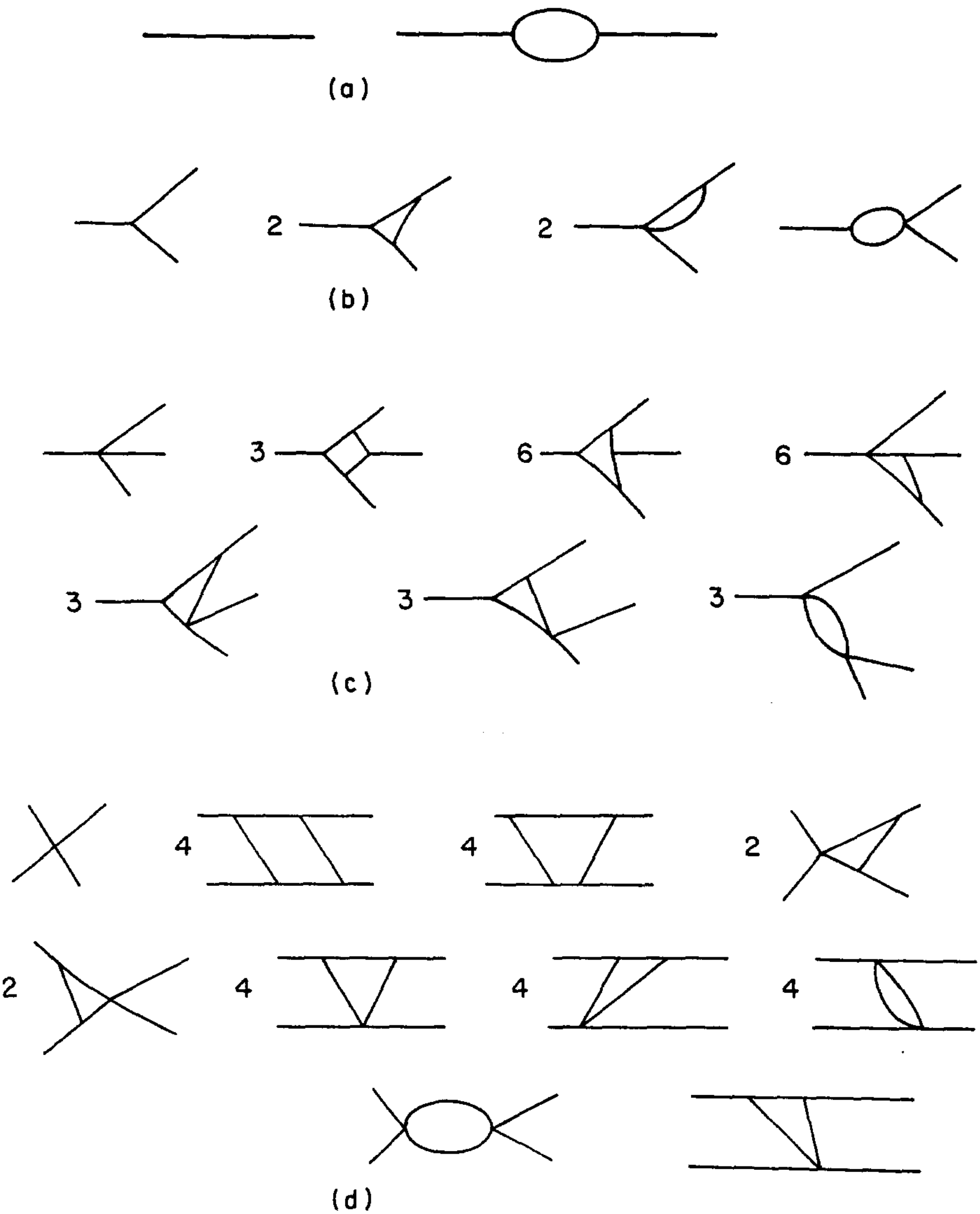
Table II. Values of the integral $I_D(a,b,c)$.

Operators	D = 2	D = 4
0(2,0,0)	L^1	L^1
0(2,1,0)	L^0	L^0
0(2,0,1)	L^0	L^0
0(3,0,0)	$L^{\frac{1}{2}}$	L^0
0(4,0,0)	L^0	L^{-1}
0(5,0,0)	$L^{-\frac{1}{2}}$	L^{-2}
0(4,1,0)	L^{-1}	L^{-2}
0(4,0,1)	L^{-1}	L^{-2}
0(3,1,0)	$L^{-\frac{1}{2}}$	L^{-1}
0(3,0,1)	$L^{-\frac{1}{2}}$	L^{-1}

Table I

$I_D(a,b,c)$				
a	b	c	D	$I_D(a,b,c)$
1	1	1	2	$2/\sqrt{5}\ \ln\sqrt{5} + 1/\sqrt{5} - 1 = .8608$
1	1	4/3	2	$4/3\ \ln 3 = .8240$
1 4/3	4/3 1	1 1	2 2	$3/2\sqrt{13}\ [\ln\sqrt{13} + 1/\sqrt{13} - 1 + \ln\sqrt{13} + 2/\sqrt{13} - 2 = .7573$
2 1	1 2	1 1	2 2	$1/2\sqrt{2}\ \ln\sqrt{2} + 1/\sqrt{2} - 1 = .6232$
a	b	c	4	1

Table II


(a)
2
2
(b)
3
6
6
3
3
3
(c)
4
4
2
2
4
4
4
(d)