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Multiple Scattering Expansions in Several Particle Dynamics

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ABSTRACT

The problem of quantum collisions involving several-particle systems is reviewed within the framework of multiple scattering theory. The basic apparatus of collision theory for nonrelativistic potential problems is first developed, and the Born and eikonal series are introduced. A general analysis is then given of multiple scattering expansions for several-particle problems. We discuss in particular the Born developments. the Faddeev-Watson expansions, the Glauber method and various multiple scattering approaches to the determination of the optical potential. Applications to atomic collision problems and to high-energy hadron-deuteron scattering are discussed at length.

CONTENTS

I. INTRODUCTION

Several particle dynamics is a problem of long-standing interest in physics. While the non-relativistic motion of two particles interacting through a given force is well understood and powerful methods have been developed to deal with situations where a very large number of particles are present, systems containing a few particles have remained difficult to analyze. This is not surprising since in general these systems exhibit all the complexity of the many-body problem. We shall examine in this review some quantum systems of this type from the point of view of collision theory. Thus bound state ("spectroscopic") properties will only be discussed insofar as they influence scattering phenomena.

The theoretical methods which we shall describe to analyze these problems all share a common feature: they may be considered as multiple scattering expansions. Such methods have been very useful in the analysis of atomic, nuclear and "elementary particle" collision processes. It is the purpose of this article to present this approach from a general point of view and to illustrate it on a few selected examples.

In order to introduce some of the concepts involved in multiple scattering expansions within a simple framework, we begin in Section II by a study of the Born and eikonal series in nonrelativistic potential scattering.

Section III is devoted to a general analysis of multiple scattering series for several particle problems. We first discuss the Born and distorted wave Born developments, then the Faddeev-Watson expansions and finally the Glauber "many-body" extensions of the eikonal method. We also give a brief survey of multiple scattering approaches to the determination of the optical potential.

Applications of multiple scattering expansions to atomic collision problems are the subject of Section IV. We first analyze electron-hydrogen collisions, a classic three-body problem. We then discuss several electron-helium scattering processes at intermediate and high (atomic) energies, for which absolute measurements of differential cross sections recently have become available.

Finally, in Section V we consider high-energy hadron-deuteron collisions. These processes lie at the borderline between elementary particle physics and nuclear physics, and have been a locus of fruitful interaction between the two fields. After recalling a few general properties of hadron-nucleus scattering at high energies, we review the applications of Glauber's high-energy diffraction theory to hadron-deuteron collisions. Particular emphasis is given to elastic scattering, for which a comprehensive comparison of theoretical and experimental work is made. We also discuss hadron-deuteron scattering from the point of view of Regge theory. We study the connection between diffraction scattering and Regge poles and then investigate the Regge cut contained in the Glauber eclipse term. We conclude with a brief survey of phenomenological applications.

II. POTENTIAL SCATTERING

1. Basic Formulae

Let us consider the non-relativistic scattering of a spinless particle of mass m by a local potential $V(\underset{\sim}{r})$ of a range a. We denote by $\underset{\sim}{k}_i$ and $\underset{\sim}{k}_f$ the initial and final wave vectors of the particle while θ is the scattering angle between $\underset{\sim}{k}_i$ and $\underset{\sim}{k}_f$. It is also convenient to introduce the "reduced potential" $U(\underset{\sim}{r}) = 2m\ V(\underset{\sim}{r})/\hbar^2$. The energy of the particle $E = \hbar^2 k^2/2m$, where $k = |\underset{\sim}{k}_i| = |\underset{\sim}{k}_f|$ is its wave number. The Hamiltonian describing the system is therefore

$$H = -\frac{\hbar^2}{2m}\nabla^2_{\underset{\sim}{r}} + V(\underset{\sim}{r}) \tag{2.1}$$

We shall call $\psi^{(+)}_{\underset{\sim}{k}_i}$ the stationary scattering eigenstate of H which corresponds to an incident plane wave of momentum $\hbar\underset{\sim}{k}_i$ and exhibits the behavior of an outgoing spherical wave. This wave function satisfies the Lippmann-Schwinger equation

$$\psi^{(+)}_{\underset{\sim}{k}_i}(\underset{\sim}{r}) = \Phi_{\underset{\sim}{k}_i}(\underset{\sim}{r}) + \int G_0^{(+)}(\underset{\sim}{r}, \underset{\sim}{r}')U(\underset{\sim}{r}')\psi^{(+)}_{\underset{\sim}{k}_i}(\underset{\sim}{r}')\, d\underset{\sim}{r}' \tag{2.2}$$

where the incident plane wave is given by

$$\Phi_{\underset{\sim}{k}_i}(\underset{\sim}{r}) = \langle\underset{\sim}{r}\,|\,\underset{\sim}{k}_i\rangle = (2\pi)^{-3/2}\exp(i\underset{\sim}{k}_i\cdot\underset{\sim}{r}). \tag{2.3}$$

The "normalization" convention which we adopt is such that for plane wave states $|\underset{\sim}{g}\rangle$ and $|\underset{\sim}{g}'\rangle$ the orthogonality relation reads

$$<\underset{\sim}{g}'|\underset{\sim}{g}> = \delta(\underset{\sim}{g} - \underset{\sim}{g}'). \tag{2.4}$$

The Green's function $G_o^{(+)}(\underset{\sim}{r}, \underset{\sim}{r}')$ which appears in the Lippmann-Schwinger equation (2.2) is given by

$$G_o^{(+)}(\underset{\sim}{r}, \underset{\sim}{r}') = -(2\pi)^{-3}\int \frac{e^{i\underset{\sim}{K}\cdot(\underset{\sim}{r}-\underset{\sim}{r}')}}{K^2-k^2-i\epsilon}\, d\underset{\sim}{K} \tag{2.5}$$

where the limiting process $\epsilon \to o^+$ is always implied. Explicitly, we have

$$G_o^{(+)}(\underset{\sim}{r}, \underset{\sim}{r}') = -\frac{1}{4\pi}\,\frac{e^{ik|\underset{\sim}{r}-\underset{\sim}{r}'|}}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \tag{2.6}$$

so that the wave function $\psi^{(+)}_{\underset{\sim}{k}_i}$ behaves asymptotically as

$$\psi^{(+)}_{\underset{\sim}{k}_i}(\underset{\sim}{r}) \underset{r\to\infty}{\longrightarrow} (2\pi)^{-3/2}\left[e^{i\underset{\sim}{k}_i\cdot\underset{\sim}{r}} + f\,\frac{e^{ikr}}{r}\right] \tag{2.7}$$

and the elastic scattering amplitude f is given by

$$f = -2\pi^2 <\Phi_{\underset{\sim}{k}_f}|U|\psi^{(+)}_{\underset{\sim}{k}_i}>. \tag{2.8}$$

Here

$$\Phi_{\underset{\sim}{k}_f}(\underset{\sim}{r}) = <\underset{\sim}{r}|\underset{\sim}{k}_f> = (2\pi)^{-3/2}\exp(i\underset{\sim}{k}_f\cdot\underset{\sim}{r}) \tag{2.9}$$

is a plane wave corresponding to the final wave vector $\underset{\sim}{k}_f$, and "normalized" according to the convention (2.4). If the potential is central, we recall that the scattering amplitude (2.8) may also be decomposed in partial waves as

$$f(k,\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)\left[S_\ell(k) - 1\right]P_\ell(\cos\theta) \tag{2.10}$$

where the coefficients $S_\ell(k)$ are the S-matrix elements in the angular momentum representation; they are given in terms of the phase shifts

δ_ℓ by

$$S_\ell(k) = \exp\left[2 i \delta_\ell (k)\right]. \tag{2.11}$$

2. The Born Series

If we elect to solve the Lippmann-Schwinger equation (2.1) by perturbation theory, starting from the "unperturbed" incident plane wave $\Phi_{\underset{\sim}{k}_i}(\underset{\sim}{r})$, we generate the sequence of functions

$$\begin{aligned}
\psi_o(\underset{\sim}{r}) &= \Phi_{\underset{\sim}{k}_i}(\underset{\sim}{r}), \\
\psi_1(\underset{\sim}{r}) &= \Phi_{\underset{\sim}{k}_i}(\underset{\sim}{r}) + \int G_o^{(+)}(\underset{\sim}{r}, \underset{\sim}{r}') \, U(\underset{\sim}{r}') \, \psi_o(\underset{\sim}{r}') \, d\underset{\sim}{r}' \\
&\vdots \\
\psi_n(\underset{\sim}{r}) &= \Phi_{\underset{\sim}{k}_i}(\underset{\sim}{r}) + \int G_o^{(+)}(\underset{\sim}{r}, \underset{\sim}{r}') \, U(\underset{\sim}{r}') \, \psi_{n-1}(\underset{\sim}{r}') \, d\underset{\sim}{r}'
\end{aligned} \tag{2.12}$$

Let us assume for the moment that this sequence converges towards $\psi^{(+)}_{\underset{\sim}{k}_i}$. We may then write the Born series for the scattering wave function, namely

$$\psi^{(+)}_{\underset{\sim}{k}_i} = \sum_{n=1}^{\infty} \varphi_n(\underset{\sim}{r}) \tag{2.13}$$

where $\varphi_o = \psi_o = \Phi_{\underset{\sim}{k}_i}$ and

$$\varphi_n(\underset{\sim}{r}) = \int K_n(\underset{\sim}{r}, \underset{\sim}{r}') \psi_o(\underset{\sim}{r}') \, d\underset{\sim}{r}', \quad n \geq 1 \tag{2.14}$$

with

$$K_1(\underset{\sim}{r}, \underset{\sim}{r}') = G_o^{(+)}(\underset{\sim}{r}, \underset{\sim}{r}') \, U(\underset{\sim}{r}') \tag{2.15}$$

and

$$K_n(\underset{\sim}{r}, \underset{\sim}{r}') = \int K_1(\underset{\sim}{r}, \underset{\sim}{r}'') K_{n-1}(\underset{\sim}{r}'', \underset{\sim}{r}') d\underset{\sim}{r}'', \; n \geq 2. \qquad (2.16)$$

It is apparent from Eqs. (2.13) - (2.16), that the Born series is a perturbation series in powers of the interaction potential. Substituting the series (2.13) into the integral representation (2.8), we obtain the corresponding Born series for the scattering amplitude, namely

$$f = \sum_{n=1}^{\infty} \bar{f}_{Bn} \qquad (2.17)$$

where

$$\bar{f}_{Bn} = -2\pi^2 < \Phi_{\underset{\sim}{k}_f} | U G_o^{(+)} U \ldots \; G_o^{(+)} U | \Phi_{\underset{\sim}{k}_i} > \qquad (2.18)$$

is an expression in which the interaction appears n times and the free Green's function (n-1) times. It is worth pointing out that the relation (2.18) gives the term of order n of the Born series in general circumstances, for example when the interaction is complex and even non-local. It is also convenient to define the jth order Born approximation to the scattering amplitude as

$$f_{Bj} = \sum_{n=1}^{j} \bar{f}_{Bn} \, . \qquad (2.19)$$

In order to gain further insight into the physical content of $\bar{f}_{Bn}$, let us analyze Eq. (2.18) in momentum space. Defining (for a local potential)

$$<\underset{\sim}{g}' | U | \underset{\sim}{g}> = <\Phi_{\underset{\sim}{g}'} | U | \Phi_{\underset{\sim}{g}}> = (2\pi)^{-3} \int e^{i(\underset{\sim}{g} - \underset{\sim}{g}') \cdot \underset{\sim}{r}} U(\underset{\sim}{r}) d\underset{\sim}{r} \qquad (2.20)$$

and using the integral representation (2.5) of $G_o^{(+)}$, we secure the relations

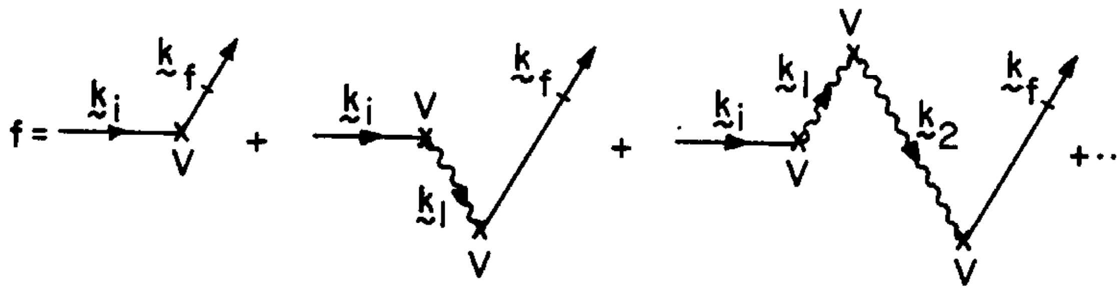
$$\bar{f}_{B1} = f_{B1} = -2\pi^2 <\underset{\sim}{k}_f | U | \underset{\sim}{k}_i> \tag{2.21}$$

and

$$\bar{f}_{Bn} = -2\pi^2 \int d\underset{\sim}{k}_1\, d\underset{\sim}{k}_2 \ldots d\underset{\sim}{k}_{n-1} <\underset{\sim}{k}_f | U | \underset{\sim}{k}_{n-1}> \frac{1}{k^2 - k_{n-1}^2 + i\epsilon}$$

$$\cdot <\underset{\sim}{k}_{n-1} | U | \underset{\sim}{k}_{n-2}> \ldots <\underset{\sim}{k}_2 | U | \underset{\sim}{k}_1> \frac{1}{k^2 - k_1^2 + i\epsilon} <\underset{\sim}{k}_1 | U | \underset{\sim}{k}_i>. \tag{2.22}$$

The Green's function therefore appears as a propagator, while the quantities $\underset{\sim}{k}_1, \underset{\sim}{k}_2, \ldots \underset{\sim}{k}_{n-1}$ are "intermediate momenta". We can thus visualize the Born series by picturing the scattering amplitude as



(2.23)

$$= f_{B1} + \bar{f}_{B2} + \bar{f}_{B3} + \ldots$$

namely as a <u>multiple scattering series</u> in which the projectile interacts repeatedly with the potential V and propagates freely between two such interactions. On the basis of this multiple scattering interpretation we expect that the Born series will converge if the incident particle is sufficiently fast so that it cannot interact many times with the potential and (or) if the potential is weak enough. Detailed studies of the Born series (Jost and Pais, 1951; Kohn, 1954; Zemach and Klein, 1958;

Aaron and Klein, 1960; Davies, 1960; Manning, 1965) indeed confirm these intuitive considerations. In particular

i) For a central potential $V(r)$ less singular than r^{-2} at the origin and decreasing faster than r^{-3} as $r \to \infty$, the Born series always converges at sufficiently high energies.

ii) For a central potential $V(r)$ the Born series converges for all energies if the potential $-|V(r)|$ cannot support any bound state.

Two remarks should be made at this point. Firstly, these convergence conditions for the Born series are <u>sufficient</u> conditions which may be unnecessarily stringent. Secondly, the results quoted above only apply to non-relativistic potential scattering; they may not necessarily be valid for many-body problems and (or) relativistic collisions.

3. The Eikonal Approximation and Eikonal Multiple Scattering Series.

Let us return to the Lippmann-Schwinger equation (2.3). We assume that

$$ka >> 1 \tag{2.24}$$

and that

$$\frac{V_o}{E} = \frac{U_o}{k^2} << 1 \tag{2.25}$$

where V_o is a typical strength of the interaction $V(\underset{\sim}{r})$ and $U_o = 2m\, V_o/\hbar^2$. Since the first, "high wave number" condition (2.24) states that the reduced de Broglie wave length $\lambdabar = k^{-1}$ of the particle is small with respect to the range of the potential, we expect semi-classical methods to be useful in this case. The second condition (2.25) will be referred to as the "high-energy" condition. If these two conditions are satisfied, the eikonal approximation (Moliere, 1947; Glauber, 1953, 1955, 1959; Watson, 1953; Schwinger, 1954; Malenka, 1954; Schiff, 1956; Saxon and Schiff, 1957) may be used to obtain for the scattering wave function $\psi^{(+)}_{\mathbf{k}_i}$ the approximate expression

$$\psi_E(\mathbf{r}) = (2\pi)^{-3/2} \exp\left[i\mathbf{k}_i \cdot \mathbf{r} - \frac{i}{2k}\int_{-\infty}^{z} U(\mathbf{b}, z')dz'\right] \qquad (2.26)$$

where we have adopted a cylindrical coordinate system such that $\mathbf{r} = \mathbf{b} + z\hat{k}_i$, so that the integral is evaluated along a straight line parallel to the incident momentum $\hbar\mathbf{k}_i$. In terms of the potential $V(\mathbf{r})$, we have

$$\psi_E(\mathbf{r}) = (2\pi)^{-3/2} \exp\left[i\mathbf{k}_i \cdot \mathbf{r} - \frac{i}{\hbar v_i}\int_{-\infty}^{z} V(\mathbf{b}, z')dz'\right] \qquad (2.27)$$

where $\mathbf{v}_i = \hbar\mathbf{k}_i/m$ is the incident velocity. We shall not discuss in detail the numerous derivations of the result (2.26). We simply mention that it may be obtained from stationary-phase arguments (Schiff, 1956) or from the fact that the incoming plane wave is modulated by a function which varies slowly over the de Broglie wavelength of the incident particle (Glauber, 1959). Another interesting way of deriving the eikonal wave function (2.26) is to examine the free propagator $G_0^{(+)}$

appearing in the Lippmann-Schwinger equation (2.2) (Malenka, 1954; Schiff, 1956; Byron, Joachain and Mund, 1973). Using its momentum space representation (2.5) and introducing the new variable $\mathbf{Q} = \mathbf{K}-\mathbf{k}_i$, one has

$$G_0^{(+)}(\mathbf{r},\mathbf{r}') = -(2\pi)^{-3} e^{i\mathbf{k}_i\cdot(\mathbf{r},\mathbf{r}')} \int \frac{e^{i\mathbf{Q}\cdot(\mathbf{r}-\mathbf{r}')}}{2\mathbf{k}_i\cdot\mathbf{Q}+Q^2-i\epsilon}\, d\mathbf{Q} \,. \quad (2.28)$$

Returning to the Lippmann-Schwinger equation (2.2) and provided the two conditions (2.24) and (2.25) are satisfied, it is legitimate to "linearize" the denominator of the integrand (i.e., neglect the Q^2 term) and write

$$G_0^{(+)}(\mathbf{r},\mathbf{r}') \simeq -(2\pi)^{-3} e^{i\mathbf{k}_i\cdot(\mathbf{r}-\mathbf{r}')} \int \frac{e^{i\mathbf{Q}\cdot(\mathbf{r}-\mathbf{r}')}}{2\mathbf{k}_i\cdot\mathbf{Q}-i\epsilon}\, d\mathbf{Q} \quad (2.29)$$

The integral on the right of Eq. (2.29) is then readily performed, with the result

$$G_0^{(+)}(\mathbf{r},\mathbf{r}') \simeq \frac{i}{2k} e^{ik(z-z')}\, \delta^2(\mathbf{b}-\mathbf{b}')\, \Theta(z-z') \quad (2.30)$$

where $\mathbf{r} = \mathbf{b} + z\hat{\mathbf{k}}_i$, $\mathbf{r}' = \mathbf{b}' + z'\hat{\mathbf{k}}_i$ and Θ is the step function such that

$$\Theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \quad (2.31)$$

The linearized propagator (2.29) - (2.30), which clearly exhibits forward propagation between successive interactions with the potential, leads directly to the eikonal wave function (2.26). Incidentally, let us remark that the importance of the four-dimensional, relativistic version of the linearized propagator in treating field theoretical problems was recognized by Schwinger (1954) and used recently by several authors

(Chang and Ma, 1969; Abarbanel and Itzykson, 1969; Lévy and Sucher, 1970, Englert et al., 1969) to sum the series of Feynman amplitudes corresponding to large classes of ladder diagrams.

With the eikonal wave function given by Eq. (2.26), we may now return to the integral representation (2.8) and write the eikonal scattering amplitude as

$$f_E(\underset{\sim}{\Delta}) = -\frac{1}{4\pi}\int e^{i\underset{\sim}{\Delta}\cdot\underset{\sim}{r}}\, U(\underset{\sim}{r}) \exp\left[-\frac{i}{2k}\int_{-\infty}^{z} U(\underset{\sim}{b}, z')dz'\right] \quad (2.32)$$

where

$$\underset{\sim}{\Delta} = \underset{\sim}{k}_i - \underset{\sim}{k}_f \quad (2.33)$$

is the wave vector transfer of length $\Delta = 2k \sin(\theta/2)$.

In obtaining the eikonal wave function (2.26), we pointed out that the integration in its phase should be carried out along a straight line parallel to $\underset{\sim}{k}_i$. In fact, since the actual phase of the corresponding semi-classical scattering wave function is evaluated along a curved trajectory, it is reasonable to expect that an improvement on Eq. (2.32) may be achieved by performing the z-integration in the phase along a direction parallel to the bisector of the scattering angle (i.e. perpendicular to $\underset{\sim}{\Delta}$). This suggestion, first made by Glauber (1959) leads directly to the eikonal scattering amplitude

$$f_E = \frac{k}{2\pi i}\int e^{i\underset{\sim}{\Delta}\cdot\underset{\sim}{b}}\left\{e^{i\chi(\underset{\sim}{b})} - 1\right\} d^2\underset{\sim}{b} \quad (2.34)$$

where we work in a cylindrical coordinate system such that

$$\underset{\sim}{r} = \underset{\sim}{b} + z\hat{n} \quad (2.35)$$

and $\hat{n}$ is perpendicular to $\underset{\sim}{\Delta}$. The eikonal phase shift function $\chi(\underset{\sim}{b})$ appearing in Eq. (2.34) is given in terms of the interaction by the simple, linear relationship

$$\chi(\underset{\sim}{b}) = -\frac{1}{2k}\int_{-\infty}^{+\infty} U(\underset{\sim}{b}, z)dz. \qquad (2.36)$$

Defining the quantity

$$\Gamma(\underset{\sim}{b}) = 1 - \exp\left[i\chi(\underset{\sim}{b})\right] \qquad (2.37)$$

we may also rewrite Eq. (2.34) as

$$f_E = \frac{ik}{2\pi}\int e^{i\underset{\sim}{\Delta}\cdot\underset{\sim}{b}}\,\Gamma(\underset{\sim}{b})d^2\underset{\sim}{b}. \qquad (2.38)$$

For potentials which possess azimuthal symmetry, Eq. (2.34) simplifies to

$$f_E = \frac{k}{i}\int_0^\infty J_0(\Delta b)\left\{e^{i\chi(b)} - 1\right\} bdb \qquad (2.39)$$

where $\chi(b)$ is still given by Eq. (2.36). We may also look at this relation from a somewhat different point of view. Indeed, the right-hand side of Eq. (2.39) provides the Fourier-Bessel representation of the <u>exact</u> scattering amplitude, provided that the phase $\chi(b)$ is redefined accordingly. This representation is exact for all energies and angles (Adachi and Kotani, 1965, 1966; Predazzi, 1966; Chadan, 1968). For high-energy, small angle scattering, the phase $\chi(b)$ may be related to the phase shifts δ_ℓ appearing in the partial wave series (2.10). The result is

$$\chi(b) = 2\delta_\ell \qquad (2.40)$$

where b and ℓ are related by $\ell = kb$.

Two important remarks should be made about the eikonal approximation. Firstly, it is equally valid for real and complex potentials. In the latter case the phase shift function $\chi(\underset{\sim}{b})$ becomes complex [see Eq. (2.36)]. Secondly, within its range of validity, the eikonal amplitude satisfies the optical theorem (Glauber, 1959), in contradistinction with the first Born approximation.

By analogy with the Born series, we may define an <u>eikonal multiple scattering series</u> by expanding the quantity $\Gamma(\underset{\sim}{b})$ [see Eq. (2.37)] in powers of the interaction potential. Thus we write

$$f_E = \sum_{n=1}^{\infty} \bar{f}_{En} \tag{2.41}$$

where

$$\bar{f}_{En} = -\frac{ik}{2\pi}\frac{i^n}{n!}\int e^{i\underset{\sim}{\Delta}\cdot\underset{\sim}{b}}[\chi(\underset{\sim}{b})]^n d^2\underset{\sim}{b}. \tag{2.42}$$

In particular, for potentials which possess azimuthal symmetry, Eq. (2.42) reduces to

$$\bar{f}_{En} = -ik\frac{i^n}{n!}\int_0^{\infty} J_0(\Delta b)[\chi(b)]^n \, bdb \tag{2.43}$$

We note that in the case of a real potential the objects $\bar{f}_{En}$ given by Eq. (2.43) are alternatively real and imaginary. As in the case of the Born series [see Eq.(2.19)] we also introduce the quantities

$$f_{Ej} = \sum_{n=1}^{j} \bar{f}_{En}. \tag{2.44}$$

We now investigate the relationship between the Born and eikonal series when ka $\gg$ 1. First of all, it is a simple matter to show that

$$f_{E1} = f_{B1} \tag{2.45}$$

for all momentum transfers (Glauber, 1959). We emphasize that this result only obtains for all angles when the z-axis used in connection with Eq. (2.32) is chosen along a direction perpendicular to $\underset{\sim}{\Delta}$. If, for example, the z-axis is chosen along $\underset{\sim}{k}_i$, then Eq. (2.41) only holds for small scattering angles. In what follows we shall consistently choose $\hat{z}$ perpendicular to $\underset{\sim}{\Delta}$.

Remarkable relationships between the higher terms of the eikonal and Born series have also been noticed recently (Moore, 1970; Byron and Joachain, 1973a). We shall concentrate on real, central potentials and follow the treatment of Byron, Joachain and Mund (1973) who have made a detailed analysis of this problem for a variety of interaction potentials. First of all, we note that Re $\bar{f}_{E2} = 0$ while in general Re $\bar{f}_{B2} \neq 0$; hence there is no analogue of Eq. (2.45) for Re $\bar{f}_{E2}$ and Re $\bar{f}_{B2}$. However, the relationship

$$\lim_{ka \to \infty} \frac{\operatorname{Im} \bar{f}_{B2}(k, \Delta)}{\operatorname{Im} \bar{f}_{E2}(k, \Delta)} = 1 \tag{2.46}$$

holds for <u>all momentum transfers</u> when the interaction has the form of a superposition of Yukawa potentials, namely

$$U(r) = U_o \int_{\alpha_o > o}^{\infty} \rho(\alpha) \frac{e^{-\alpha r}}{r} \, d\alpha. \tag{2.47}$$

For other interactions such as a gaussian potential or a "polarization" potential of the form $U(r) = U_0(r^2 + d^2)^{-2}$ the relation (2.46) only holds for small scattering angles.

The analytical evaluation of the third and higher order terms of the Born and eikonal series is extremely difficult, but detailed studies of such terms suggest that the relations

$$\lim_{ka \to \infty} \frac{\mathrm{Re}\, \bar{f}_{Bn}}{\mathrm{Re}\, \bar{f}_{En}} = 1 \quad (n \text{ odd}) \qquad (2.48a)$$

and

$$\lim_{ka \to \infty} \frac{\mathrm{Im}\, \bar{f}_{Bn}}{\mathrm{Im}\, f_{En}} = 1 \quad (n \text{ even}) \qquad (2.48b)$$

hold for all n and all momentum transfers for potentials of the form (2.47).

The relationships (2.48) have some important consequences. Let us first consider the weak coupling situation such that the condition

$$\frac{|V_0|a}{\hbar v_i} = \frac{|U_0|a}{2k} \ll 1 \qquad (2.49)$$

is added to the inequalities (2.24) and (2.25). In this case the Born series converges rapidly and the relations (2.48), together with the optical theorem imply that the eikonal amplitude f_E gives a consistantly poorer approximation to the exact amplitude than does the second Born approximation f_{B2} (although the eikonal results are nevertheless fairly accurate at all angles). This is due to the fact that for $ka \gg 1$

the exact amplitude may be written for Yukawa-like potentials as

$$f(k,\Delta) = f_{B1}(\Delta) + \underbrace{\left[\frac{A(\Delta)}{k^2} + i\,\frac{B(\Delta)}{k}\right]}_{\bar{f}_{B2}} + \underbrace{\left[\frac{C(\Delta)}{k^2} + i\,\frac{D(\Delta)}{k^3}\right]}_{\bar{f}_{B3}} + \ldots \tag{2.50}$$

while the eikonal amplitude has the structure

$$f_E(k,\Delta) = f_{B1}(\Delta) + \underbrace{i\frac{B(\Delta)}{k}}_{\bar{f}_{E2}} + \underbrace{\frac{C(\Delta)}{k^2}}_{\bar{f}_{E3}} + \ldots \tag{2.51}$$

Therefore neither f_{B2} nor f_{E2} are correct to order k^{-2}. However, since the coefficient A is proportional to U_o^2 while C is proportional to U_o^3, it is clear that for small values of $|U_o|$ the second Born amplitude should be more precise than the eikonal amplitude.

As the coupling increases we expect from the foregoing discussion that the eikonal method should improve steadily. That this is indeed the case may be seen from Fig. 1, which displays the real part of the exact, eikonal and second Born amplitudes for a superposition of two Yukawa potentials of the form

$$U(r) = U_o\left(e^{-r/a} - \rho e^{-2r/a}\right)/r \tag{2.52}$$

with $U_o = -20$, $a = 1$, $\rho = 1.125$ and $ka = 5$. The excellent agreement between the eikonal and exact results, even at large angles, is particularly striking. Similar conclusions may be drawn from Fig. 2, where the corresponding imaginary parts are shown.

Let us now comment briefly on the strong coupling situation, for which $|V_0|/E>1$. In this case the Born series is useless. On the other hand, and despite the fact that the condition (2.25) is violated, the eikonal approximation is still quite accurate at small angles for a variety of interaction potentials. This is illustrated in Fig. 3 and 4 where the real and imaginary parts of the exact and eikonal amplitudes are displayed for an interaction of the type (2.52) with $U_0 = -20$, $a = 1$, $\rho = 1.125$ and $ka=2$. These results, together with the large angle agreement found above in the intermediate coupling situation, strongly suggest that the traditional criteria for the validity of the eikonal approximation (Glauber 1959) are sufficient conditions which may be too restrictive.

We shall not attempt to discuss here various other forms of the eikonal approximation (Saxon and Schiff, 1957; Blankenbecler and Goldberger, 1962; Feshbach, 1967; Schiff, 1968; Wilets and Wallace, 1968; Lévy and Sucher, 1969; Abarbanel and Itzykson, 1969; Moore, 1970; Wallace, 1971, 1973; Baker, 1972). We note, however, that the Glauber form which we have discussed above is probably the simplest eikonal approximation, a feature which is very important when one wants to generalize the method to many-body collisions.

Finally, we note that the derivation of the eikonal scattering amplitude (2.34) may be generalized to relativistic collisions and does not require the existence of a potential to describe the collision process, although an optical potential can always be found to describe the scattering in the eikonal approximation (Glauber, 1959, Omnès, 1965, see also Section III.4). Moreover, for high-energy, small angle scattering, the basic formula (2.34) is valid in the laboratory system as well as in the center of mass system (Franco and Glauber, 1966). The only modifications are that the center of mass wave vectors $\underset{\sim}{k}_i$ and $\underset{\sim}{k}_f$ must now be replaced by the corresponding laboratory quantities $\underset{\sim}{k}$ and $\underset{\sim}{k}'$, while $\underset{\sim}{\Delta} = \underset{\sim}{k}_i - \underset{\sim}{k}_f$ is replaced by $\underset{\sim}{q} = \underset{\sim}{k} - \underset{\sim}{k}'$. Of course the magnitude of $\underset{\sim}{k}'$ is now smaller than that of $\underset{\sim}{k}$ because of recoil effects, but these effects are small for scattering near the forward direction and can be minimized by interpreting the quantity $(-q^2)$ as the Mandelstam variable t, namely the square of the four-momentum transfer of the collision.

III. SEVERAL PARTICLE PROBLEMS

1. The Born Series and the Distorted-Wave Born Series

Let us consider a general quantum collision process $a \to b$ for which we denote the S-matrix element by $\langle b|S|a\rangle$. The theoretical analysis is conveniently carried out in terms of the $\mathcal{T}$-matrix elements such that (see for example Goldberger and Watson, 1964)

$$\langle b|S|a\rangle = \delta_{ba} - 2\pi i\ \delta(E_b - E_a)\ \langle b|\mathcal{T}|a\rangle . \qquad (3.1)$$

It will sometimes prove convenient to use a somewhat more explicit notation and write $a \equiv (i, \alpha)$ and $b \equiv (f, \beta)$ where i and f are "arrangement channel" indices and α and β denote respectively the state of the system in the initial and final channel. Thus in the initial channel the total Hamiltonian of the system may be decomposed as

$$H = H_i + V_i \tag{3.2}$$

where V_i is the interaction between the two colliding particles and the channel Hamiltonian H_i describes these particles when they are far apart and do not interact. We then have

$$H_i \Phi_a = E_a \Phi_a \tag{3.3}$$

where Φ_a is the corresponding free state vector. Similarly, in the final channel,

$$H = H_f + V_f \tag{3.4}$$

with

$$H_f \Phi_b = E_b \Phi_b . \tag{3.5}$$

We also introduce the Green's operators

$$G^{(\pm)}(E) = (E - H \pm i\epsilon)^{-1}, \tag{3.6a}$$

$$G_i^{(\pm)}(E) = (E - H_i \pm i\epsilon)^{-1}, \tag{3.6b}$$

and

$$G_f^{(\pm)}(E) = (E - H_f \pm i\epsilon)^{-1}. \tag{3.6c}$$

More generally, if c is any arrangement channel index such that $H = H_c + V_c$, we have

$$G_c^{(\pm)} = (E - H_c \pm i\epsilon)^{-1} \tag{3.6d}$$

Direct collisions are characterized by the fact that the channel Hamiltonians are the same in the initial and final states. Writing

$H_i = H_f = H_d$ and $V_i = V_f = V_d$ in this case, we also define

$$G_d^{(\pm)} = (E - H_d \pm i\epsilon)^{-1}. \tag{3.6e}$$

Finally, we shall denote by H_o the kinetic energy operator of the entire system (i.e., the Hamiltonian H from which the total interaction V has been removed). The corresponding free Green's operator is then

$$G_0^{(\pm)} = (E - H_0 \pm i\epsilon)^{-1}. \tag{3.6f}$$

Let us now examine the "on the energy shell" transition matrix elements $<b|\mathcal{T}|a>$ appearing in Eq. (3.1). It is convenient to factor out a momentum-conserving delta function and to introduce the reduced T-matrix elements T_{ba} such that $<b|\mathcal{T}|a> = \delta(\underset{\sim}{P}_b - \underset{\sim}{P}_a)\, T_{ba}$. Then (Goldberger and Watson, 1964)

$$T_{ba} = <\Phi_b | V_f | \Psi_a^{(+)}> \tag{3.7a}$$

or

$$T_{ba} = <\Psi_b^{(-)} | V_i | \Phi_a> \tag{3.7b}$$

where the state vectors $\Psi_a^{(+)}$ and $\Psi_b^{(-)}$ are such that

$$\Psi_a^{(+)} = \Phi_a + G^{(+)} V_i\, \Phi_a, \tag{3.8a}$$

$$\Psi_b^{(-)} = \Phi_b + G^{(-)} V_f\, \Phi_b \tag{3.8b}$$

and satisfy the Lippmann-Schwinger equations

$$\Psi_a^{(+)} = \Phi_a + G_i^{(+)} V_i\, \Psi_a^{(+)}, \tag{3.9a}$$

$$\Psi_b^{(-)} = \Phi_b + G_f^{(-)} V_f\, \Psi_b^{(-)}. \tag{3.9b}$$

We note that for any arrangement channel c the Green's operators $G^{(\pm)}$ satisfy the relations

$$G^{(\pm)} = G_c^{(\pm)} + G_c^{(\pm)} V_c G^{(\pm)} \tag{3.10a}$$

or

$$G^{(\pm)} = G_c^{(\pm)} + G^{(\pm)} V_c G_c^{(\pm)} \tag{3.10b}$$

which are the Lippmann-Schwinger equations for the full Green's operator. We also define the transition operators

$$U_{fi} = V_f + V_f G^{(+)} V_i, \tag{3.11a}$$

$$\overline{U}_{fi} = V_i + V_f G^{(+)} V_i, \tag{3.11b}$$

and

$$T = V + V G^{(+)} V. \tag{3.11c}$$

These operators satisfy the Lippmann-Schwinger equations

$$U_{fi} = V_f + U_{fi} G_i^{(+)} V_i, \tag{3.12a}$$

$$\overline{U}_{fi} = V_i + V_f G_f^{(+)} \overline{U}_{fi}, \tag{3.12b}$$

and

$$T = V + V G_0^{(+)} T \tag{3.12c}$$

$$= V + T G_0^{(+)} V. \tag{3.12d}$$

On the energy shell $E_a = E_b$ we have $<\Phi_b | V_i | \Phi_a> = <\Phi_b | V_f | \Phi_a>$ and therefore, from Eq. (3.7)

$$T_{ba} = <\Phi_b | U_{fi} | \Phi_a> = <\Phi_b | \overline{U}_{fi} | \Phi_a>. \tag{3.13}$$

A variety of Born series expansions for the transition matrix element T_{ba} may be obtained by solving the various Lippmann-Schwinger equations written above by successive iterations. For example, we may first solve for the full Green's operator $G^{(\pm)}$ from Eqs. (3.10) and write

$$G^{(\pm)} = G_c^{(\pm)} + G_c^{(\pm)} V_c G_c^{(\pm)} + G_c^{(\pm)} V_c G_c^{(\pm)} V_c G_c^{(\pm)} + \ldots \qquad (3.14)$$

Then, upon substitution in Eqs. (3.8) and then in Eqs. (3.7) we find the Born development

$$T_{ba} = \langle \Phi_b | V_i \text{ (or } V_f) + V_f G_c^{(+)} V_i + V_f G_c^{(+)} V_c G_c^{(+)} V_i + \ldots | \Phi_a \rangle . \qquad (3.15)$$

The first Born approximation consists in retaining only the first term of this expansion, namely

$$T_{ba}^{B1} = \langle \Phi_b | V_i | \Phi_a \rangle = \langle \Phi_b | V_f | \Phi_a \rangle . \qquad (3.16)$$

For direct collisions it is natural to choose the propagator $G_c^{(+)}$ so that it coincides with the Green's operators $G_d^{(+)}$ defined by Eq. (3.6e). The corresponding Born series then reads

$$T_{ba} = \langle \Phi_b | V_d + V_d G_d^{(+)} V_d + V_d G_d^{(+)} V_d G_d^{(+)} V_d + \ldots | \Phi_a \rangle . \qquad (3.17)$$

Little is known about the mathematical properties of the Born series (3.15). For direct collisions the conditions of convergence of the series (3.17) are probably similar to those discussed in Section II.2 for potential scattering. For example, the Born series (3.17) may well be convergent for non-relativistic direct processes at sufficiently high colliding energies; this will be illustrated in Section IV.1. On the other hand, when rearrangement collisions occur some particles are transferred between the colliding systems during the reaction, so that $V_i \neq V_f$. The question of the convergence of the Born series (3.15) in this case has been investigated by several authors (see for example Aaron, Amado and Lee, 1961; Weinberg, 1963a, b; 1964a, b, c; Bransden, 1965, 1969; Rubin, Sugar and Tiktopoulos, 1966, 1967a, b; Dettman and Leibfried, 1968, 1969). At low energies the series diverges, and even at high energies its convergence is doubtful. Recently, however, Dettman and Leibfried (1969) have pointed out that for rearrangement processes occuring in three-body systems, and for a wide class of potentials, the energy variation of the T-matrix element is given correctly at high energies by the first two terms of the Born series. It is interesting to note in this context that variational methods of the Schwinger type (Lippmann and Schwinger, 1950; Lippmann, 1956; Joachain, 1965) also involve in lowest order the first and second order terms of the Born series. Whether the Born series itself converges or is perhaps semi-convergent (asymptotic) is still an open question.

Distorted wave Born series are obtained by a simple application of the two-potential scattering formalism (Gell-Mann and Goldberger, 1953). Let us assume that the interaction potentials V_i and V_f may be split as

$$V_i = U_i + W_i \tag{3.18a}$$

$$V_f = U_f + W_f \tag{3.18b}$$

and more generally, in any arrangement channel c,

$$V_c = U_c = W_c \ . \tag{3.18c}$$

We define the new Hamiltonians $\overline{H}_c = H_c + U_c$, together with the Green's operators $\overline{G}_c^{(\pm)} = (E - \overline{H}_c \pm i\epsilon)^{-1}$, and assume that the distorted waves

$$\chi_a^{(+)} = \Phi_a + \overline{G}_i^{(+)} U_i \Phi_a \tag{3.19a}$$

and

$$\chi_b^{(-)} = \Phi_b + \overline{G}_f^{(-)} U_f \Phi_b \tag{3.19b}$$

are known. The T-matrix elements (3.7) are then given by (Gell-Mann and Goldberger, 1953; Gerjuoy, 1958)

$$T_{ba} = \langle \chi_b^{(-)} | (V_i - W_f) | \Phi_a \rangle + \langle \chi_b^{(-)} | W_f | \Psi_a^{(+)} \rangle \tag{3.20a}$$

or

$$T_{ba} = \langle \Phi_b | (V_f - W_i) | \chi_a^{(+)} \rangle + \langle \Psi_b^{(-)} | W_i | \chi_a^{(+)} \rangle \tag{3.20b}$$

with

$$\Psi_a^{(+)} = \chi_a^{(+)} + G^{(+)} W_i \chi_a^{(+)} \tag{3.21a}$$

and

$$\Psi_b^{(-)} = \chi_b^{(-)} + G^{(-)} W_f \chi_b^{(-)} . \tag{3.21b}$$

The two-potential formulae (3.20) simplify when the distorting potentials U_i and U_f cannot induce the transition $a \to b$ considered. This may happen for example if the interactions U_i and U_f only generate elastic scattering and the transition $a \to b$ is an inelastic process or a rearrangement collision. In this case the first term on the right of Eqs (3.20) vanishes, so that

$$T_{ba} = <\chi_b^{(-)} | W_f | \Psi_a^{(+)}> \tag{3.22a}$$

or

$$T_{ba} = <\Psi_b^{(-)} | W_i | \chi_a^{(+)}>. \tag{3.22b}$$

If we wish to treat exactly the interactions U_i and U_f but elect to use perturbation theory to handle the interactions W_i and W_f we generate the distorted wave Born series. For example, using the fact that

$$G^{(\pm)} = \overline{G}_c^{(\pm)} + \overline{G}_c^{(\pm)} W_c \overline{G}_c^{(\pm)} + \overline{G}_c^{(\pm)} W_c \overline{G}_c^{(\pm)} W_c \overline{G}_c^{(\pm)} + \ldots \tag{3.23}$$

we see that Eqs. (3.22) yield, with the help of (3.21),

$$T_{ba} = <\chi_b^{(-)} | W_i (\text{or } W_f) + W_f \overline{G}_c^{(+)} W_i + W_f \overline{G}_c^{(+)} W_c \overline{G}_c^{(+)} W_i + \ldots | \chi_a^{(+)}>. \tag{3.24}$$

The first term of this expansion gives the <u>distorted wave Born approximation</u> (DWBA), namely

$$T_{ba}^{DWBA} = <\chi_b^{(-)} | W_i | \chi_a^{(+)}> = <\chi_b^{(-)} | W_f | \chi_a^{(+)}> . \qquad (3.25)$$

With a suitable choice of distorting potentials U_i and U_f this formula may improve significantly over the first Born approximation (3.16), at least for direct collisions. For rearrangement processes the situation is considerably more involved. As for the Born series (3.15), the convergence of the distorted wave Born series (3.24) is again doubtful in this case (Greider and Dodd, 1966; Dodd and Greider, 1966).

A simple but physically reasonable interpretation may be given of Eq. (3.25). Let us imagine for example that the transition a → b is a process of the type A + B → C + D (see Fig. 5). We see that the two particles A and B first feel the initial state interaction U_i (embodied in $\chi_a^{(+)}$), then interact once through W_i (or W_f) and finally experience the final state interaction U_f while emerging from the collision. Since U_i and U_f are treated exactly we note that the particles are allowed to interact repeatedly through the distorting potentials.

The DWBA formula (3.25) has been used extensively in atomic and nuclear physics (see for example Mott and Massey, 1965; Tobocman, 1961). It also provided an intuitive starting point for the various high-energy absorption models (Sopkovich, 1962; Gottfried and Jackson, 1964; Durand and Chiu, 1964, 1965; Jackson, 1965). We shall discuss in Sections IV and V a few applications of the eikonal DWBA approximations, in which the distorted waves $\chi_a^{(+)}$ and $\chi_b^{(-)}$ appearing in Eq. (3.25) are

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obtained with the help of the eikonal approximation.

2. The Faddeev-Lovelace-Watson Expansion

In this paragraph we shall study a non-relativistic three-body system such that the particles 1, 2, 3 interact by means of two-body interactions. We shall denote by $V^1 \equiv V_{23}$ the potential acting between the particles 2 and 3, while $V^2 \equiv V_{13}$ acts between 1 and 3 and $V^3 \equiv V_{12}$ between 1 and 2. The total Hamiltonian of the system is then

$$H = H_0 + V \tag{3.26}$$

where H_0 is the kinetic energy operator and

$$V = \sum_{i=1}^{3} V^i \tag{3.27}$$

We shall also need the Hamiltonian describing two particles interacting while the third one is free, namely

$$H_i = H_0 + V^i \tag{3.28}$$

and we define the operators

$$V_i = V - V^i \tag{3.29}$$

corresponding to the interactions in which particle i participates. (For example: $V_1 = V - V^1 = V_{12} + V_{13}$.) The Green's operators corresponding to H, H_i and H_0 are defined respectively by Eqs. (3.6a), (3.6b) and (3.6f).

The two-body T-matrices are given by

$$T_i = V^i + V^i G_i^{(+)} V^i \tag{3.30a}$$

$$= V^i + T_i G_0^{(+)} V^i \tag{3.30b}$$

$$= V^i + V^i G_0^{(+)} T_i . \quad (3.30c)$$

We also note that

$$G_i^{(\pm)} = G_0^{(\pm)} + G_0^{(\pm)} V^i G_i^{(\pm)} \quad (3.31a)$$

$$= G_0^{(\pm)} + G_i^{(\pm)} V^i G_0^{(\pm)} \quad (3.31b)$$

$$= G_0^{(\pm)} + G_0^{(\pm)} T_i G_0^{(\pm)} \quad (3.31c)$$

and

$$G_i^{(+)} V_i = G_0^{(+)} T_i . \quad (3.31d)$$

We shall describe the various possible modes of fragmentation of the three-body system by indices i, f which take on the values 0, 1, 2, 3. Thus i = 0 corresponds to three free particles in the initial state, i = 1 means that initially the particle 1 is free and the pair (2, 3) is bound, etc. A collision process a → b is then described by the reduced transition matrix T_{ba}, given by Eqs. (3.7) and (3.13). More explicitly, we shall write (on the energy shell)

$$T_{ba} \equiv T_{f\beta, i\alpha} = \langle \Phi_{f\beta} | U_{fi} | \Phi_{i\alpha} \rangle = \langle \Phi_{f\beta} | \overline{U}_{fi} | \Phi_{i\alpha} \rangle \quad (3.32)$$

where the indices α and β contain additional information on the momenta, spin, bound states, etc. of the initial and final states considered. Moreover, the operators U_{fi} and $\overline{U}_{fi}$ are given respectively by Eqs. (3.11a) and (3.11b) with $V_i = V - V^i$ and $V_f = V - V^f$.

Before we turn to the problem of obtaining multiple scattering expansions for the operators $U_{fi}^{(-)}$, we briefly recall the work of Faddeev (1960, 1961, 1962) who writes the operator $T = V + V G^{(+)} V$ as

$$T = T^{(1)} + T^{(2)} + T^{(3)} \quad (3.33)$$

where $T^{(1)}$ represents the sum of all contributions to T in which the particles 2 and 3 interact last. The objects $T^{(i)}$ then satisfy the Faddeev equations

$$\begin{pmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} + \begin{pmatrix} O & T_1 & T_1 \\ T_2 & O & T_2 \\ T_3 & T_3 & O \end{pmatrix} G_0^{(+)} \begin{pmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{pmatrix} \tag{3.34}$$

which exhibit much better mathematical properties than the Lippmann-Schwinger equations (3.12). The Faddeev approach to the three-body problem has immediately attracted a great deal of attention (see for example Lovelace, 1964 a,b; Weinberg, 1964a; Omnès, 1964; Rosenberg, 1964) and possible applications to a number of nuclear and atomic problems have been investigated (a list of references may be found in Watson and Nultall, 1967 and Chen and Joachain, 1971). Extensions of the Faddeev equations to relativistic three-body problems have also been proposed (Alessandrini and Omnès, 1965; Freedman, Lovelace and Namysłowski, 1966, Blankenbecler and Sugar, 1966).

A slightly different version of the Faddeev equations, derived by Lovelace (1964 a) involves the operators $U_{fi}^{(-)}$ which lead directly to the transition matrix elements (3.32) for a process $i\alpha \rightarrow f\beta$. The result is

$$U_{fi} = V_f + \sum_{k\neq i} U_{fk} \; G_0^{(+)} T_k \tag{3.35a}$$

and

$$\overline{U}_{fi} = V_i + \sum_{k\neq f} T_k \, G_0^{(+)} \overline{U}_{ki} \tag{3.35b}$$

where i, f, k = 1, 2, 3. The case i = f = 0 may also be included by defining $T_o = 0$. For example, if i = 1, i.e. the particle 1 is incident on the bound pair (2, 3), we find that Eq. (3.35b) becomes

$$\begin{aligned}
\overline{U}_{11} &= V_1 + T_2 \, G_0^{(+)} \overline{U}_{21} + T_3 \, G_0^{(+)} \overline{U}_{31} \\
\overline{U}_{21} &= V_1 + T_1 \, G_0^{(+)} \overline{U}_{11} + T_3 \, G_0^{(+)} \overline{U}_{31} \\
\overline{U}_{31} &= V_1 + T_1 \, G_0^{(+)} \overline{U}_{11} + T_2 \, G_0^{(+)} \overline{U}_{21}
\end{aligned} \tag{3.36}$$

We note that the matrix kernel of the Lovelace equations (3.36) is just the transpose of the Faddeev kernel appearing in (3.34) so that all the mathematical properties of the Faddeev kernel apply equally well to the Lovelace kernel. In contrast with the Faddeev equations, however, the Lovelace equations involve interaction potentials. Nevertheless, a simple modification of the Lovelace formalism yields also equations which do not include any direct reference to potentials (Alt, Grassberger and Sandhas, 1967). A comparison between the Faddeev and the Lovelace-Alt approaches to the three-body problem has been made recently by Osborn and Kowalski (1971). It is also worth pointing out that the Faddeev or Lovelace equations are closely related to Watson's multiple scattering equations (Watson, 1953, 1956, 1957; see also Goldberger and Watson, 1964; Watson and Nuttall, 1967). We shall return to this point below.

Let us now investigate how to obtain multiple scattering expansions for various three-body processes (Ekstein, 1956; Rosenberg, 1964; Queen, 1964, 1966; Bransden, 1965; Sloan, 1967, 1968; Chen and Joachain, 1971). In what follows we shall concentrate on the intermediate and high energy regions such that the relative kinetic energy of the incident particle 1 by respect to the target (2, 3) is large compared to the binding energy of that target ("weak binding" condition).

We start with the case f = 1 (elastic and inelastic direct processes) and return to the Lovelace equations (3.36). A simple iteration of these equations gives

$$\overline{U}_{11} = V_1 + T_2 G_0^{(+)} V_1 + T_3 G_0^{(+)} V_1 + T_2 G_0^{(+)} T_1 G_0^{(+)} V_1 + T_2 G_0^{(+)} T_3 G_0^{(+)} V_1 + T_3 G_0^{(+)} T_1 G_0^{(+)} V_1 + T_3 G_0^{(+)} T_3 G_0^{(+)} V_1 + \dots \tag{3.37}$$

Then, using the fact that $V - V^1 = V^2 + V^3$ and eliminating the potentials in favor of the two-body T-matrices by repeated use of Eq. (3.30b), we find that

$$\begin{aligned}\overline{U}_{11} = {} & T_2 + T_3 + T_2 G_0^{(+)} T_3 + T_3 G_0^{(+)} T_2 + T_2 G_0^{(+)} T_1 G_0^{(+)} T_2 \\ & + T_2 G_0^{(+)} T_3 G_0^{(+)} T_2 + T_2 G_0^{(+)} T_1 G_0^{(+)} T_3 + T_3 G_0^{(+)} T_1 G_0^{(+)} T_2 \\ & + T_3 G_0^{(+)} T_1 G_0^{(+)} T_3 + T_3 G_0^{(+)} T_2 G_0^{(+)} T_3 + \dots\end{aligned} \tag{3.38}$$

Similar expansions may be found for the case of rearrangement collisions. For example, when f = 3, i.e. for a process $1 + (2,3) \to (1,2) + 3$

we obtain from Eqs. (3.36)

$$\overline{U}_{31} = V^3 + T_2 + T_1 G_0^{(+)} T_2 + T_1 G_0^{(+)} T_3 + T_2 G_0^{(+)} T_3 + \ldots \qquad (3.39)$$

For three-body break-up collisions 1 + (2, 3) → 1 + 2 + 3, we must first include the channels i = f = 0 in the Lovelace equations (with $T_0 = 0$). Then

$$\overline{U}_{01} = T_2 + T_3 + T_1 G_0^{(+)} T_2 + T_1 G_0^{(+)} T_3 + T_2 G_0^{(+)} T_3 + T_3 G_0^{(+)} T_2 + \ldots \qquad (3.40)$$

The rules for obtaining higher-order multiple scattering terms in the expansions (3.38) - (3.40) are easily derived

(i) Start from the right with a two-body T-matrix for any of the pairs which participates in the initial interaction $V_i = V - V^i$

(ii) Write $G_0^{(+)}$ and T_i alternatively, avoiding the repetition of adjacent indices

(iii) Terminate to the desired order with a two-body T-matrix for any of the pairs which participate in the final interaction $V_f = V - V^f$.

The multiple scattering expansions (3.38) - (3.40) have been obtained by using the operators $\overline{U}_{fi}$. Similar expansions may of course be written down by making use of the operators U_{fi} . For direct collisions one finds again (3.38), while the new rearrangement and break-up series are respectively

$$U_{31} = V^1 + T_2 + T_1 G_0^{(+)} T_2 + T_1 G_0^{(+)} T_3 + T_2 G_0^{(+)} T_3 + \ldots \qquad (3.41)$$

and

$$U_{01} = V^1 + T_2 + T_3 + T_1 G_0^{(+)} T_2 + T_1 G_0^{(+)} T_3 + T_2 G_0^{(+)} T_3 + T_3 G_0^{(+)} T_2 + \ldots \quad (3.42)$$

By comparing Eqs. (3.40) and (3.42) we expect that the interaction V^1 should not contribute to the break-up transition matrix element. This is easily verified, since $\langle \Phi_{0\beta} | V^1 | \Phi_{1\alpha} \rangle = \langle \Phi_{0\beta} | H_1 - H_0 | \Phi_{1\alpha} \rangle = 0$.

Let us comment briefly on the multiple scattering expansions which we have generated. First of all, it is a simple matter to verify that these expansions may also be obtained from the Watson multiple scattering equations which (in the weak binding limit) read in this case

$$\Psi_{1\alpha}^{(+)} = \Phi_{1\alpha} + \sum_{j=2}^{3} G_1^{(+)} T_j \varphi_j . \quad (3.43a)$$

Here the effective waves φ_j are given by

$$\varphi_j = \Phi_{1\alpha} + \sum_{k \neq j} G_1^{(+)} T_k \varphi_k \qquad (k=2, 3) \quad (3.43b)$$

and may be readily expressed in terms of the free Green's operator $G_0^{(+)}$ by using Eqs. (3.31). We also note that the Faddeev-Lovelace-Watson expansions (3.38) - (3.42) are rearrangements of the Born series (3.15). However, in contrast with the Born development, and except for bare-potential first terms, there are no disconnected terms (i.e. contributions such that two particles interact while the third one remains undisturbed) in the Faddeev-Lovelace-Watson expansions. Hence these expansions should exhibit a better convergence behavior than the Born series, especially for rearrangement collisions.

If we write approximately (for weak coupling situations) $T_i \simeq V^i$ and limit our expansions (3.38) - (3.42) to the first order in the interaction potentials, we recover the first Born approximation (3.16) for the transition matrix element. However, this "derivation" of first Born results only holds in the weak coupling limit, an approximation which is not on firm grounds for rearrangement processes, even at high energies. Indeed, the interaction potentials involved in such collisions must act long enough to bind new particles, so that the approximation $T_i \simeq V^i$ is unlikely to be correct.

Let us return to the multiple scattering expansion (3.38) for direct (elastic or inelastic) scattering. At sufficiently high energies a useful approximation consists in keeping only the first order terms of this series, so that the corresponding transition matrix element reads

$$T_{ba} = T_{1\beta,1\alpha} \simeq \langle \Phi_{1\beta} | T_2 + T_3 | \Phi_{1\alpha} \rangle \tag{3.44}$$

and we recover the impulse approximation (Fermi, 1936; Chew 1950; Chew and Wick, 1952; Ashkin and Wick, 1952; Chew and Goldberger, 1952) for the process considered. We note that the two-body T-matrices T_2 and T_3 describe the scattering of the incident particle 1 by the two target particles 2 and 3 as if those particles were free. The effect of the interaction $V^1 = V_{23}$ between the two target particles appears only in higher order terms of the series (3.38).

As a final remark, we note that the Faddeev-Lovelace-Watson expansions presented in this section may be generalized to systems with more than three particles. The three-body system considered here was only selected as the obvious prototype of many-body scattering.

3. The Eikonal Approximation for Many-Body Collisions

The extension of the eikonal approximation to many-body scattering problems was first proposed by Glauber (1953, 1955, 1959, 1960, 1967, 1969) in connection with high-energy, small angle hadron-nucleus collisions. The resulting high-energy diffraction theory is in fact a generalization of the classical Fraunhofer diffraction theory (see for example Born and Wolf, 1964).

Consider a fast point particle A incident on a target B which contains N scatterers. We assume that the motion of the target particles is slow compared to that of the projectile and that the incident particle interacts with the target scatterers via two-body spin-independent interactions. The Glauber scattering amplitude for a small angle direct collision leading from an initial target state $|0\rangle$ to a final state $|m\rangle$ is given in the center of mass system by

$$F^G_{m0} = \frac{k_i}{2\pi i}\int d^2\underset{\sim}{b}\, e^{i\underset{\sim}{\Delta}\cdot\underset{\sim}{b}} \langle m|\, [\, e^{i\chi^G_{tot}(\underset{\sim}{b}, \underset{\sim}{b}_1, \ldots \underset{\sim}{b}_N)} - 1]\,|0\rangle, \qquad (3.45)$$

the corresponding differential cross section being $d\sigma_{m0}/d\Omega$ = $(k_f/k_i)\,|F^G_{m0}|^2$. Here $\underset{\sim}{\Delta} = \underset{\sim}{k}_i - \underset{\sim}{k}_f$ is the center of mass wave vector transfer, while

$$\underset{\sim}{r} = \underset{\sim}{b} + z\,\hat{z}$$

is the initial relative coordinate and

$$\underset{\sim}{r}_j = \underset{\sim}{b}_j + z_j\,\hat{z} \qquad (3.47)$$

are the coordinates of the target particles (relative to the target center of mass). The z-axis may be chosen along $\underset{\sim}{k}_i$ for small angle collisions, but we shall also consider other choices below [see the discussion preceeding Eq. (2.34)]. The total Glauber phase shift function

$$\chi^G_{tot}(\underset{\sim}{b}, \underset{\sim}{b}_1, \ldots \underset{\sim}{b}_N) = \sum_{j=1}^{N} \chi_j(\underset{\sim}{b}-\underset{\sim}{b}_j) \qquad (3.48)$$

is just the sum of the phase shifts χ_j contributed by each of the target scatterers as the wave representing the incident particle progresses through the target system. We note that if the elementary interactions between the incident particle and the target particles are genuine two-body problems (such as in non-relativistic electron-atom collisions) the phase shift functions χ_j are purely real. On the contrary, if these elementary interactions may lead to several final channels (such as $\pi + N \rightarrow A_1 + N$, where N is a nucleon of a target nucleus) the phase shift functions χ_j are complex.

The crucial property of phase shift additivity, expressed by Eq. (3.48) is clearly a direct consequence of the one-dimensional nature of the relative motion, together with the neglect of three-body forces, target scatterer motions, and longitudinal momentum transfer.

Another important remark concerning Eq. (3.45) is that it applies only to collisions for which the energy transfer ΔE is small compared with the incident particle energy E_i. This is true for elastic collisions and for "mildly" inelastic ones in which the target is excited or perhaps breaks up. It is not true for "deeply" inelastic collisions in which the nature of the incident or target particles is modified or the number of particles is altered during the collision. We shall leave aside such processes in what follows and comment briefly on them in Section V. It is also worth noting that if we neglect recoil effects, which are small near the forward direction, we may write the Glauber scattering amplitude in the laboratory system as

$$F^G_{m0} = \frac{k}{2\pi i}\int d^2\underset{\sim}{b}\, e^{i\underset{\sim}{q}\cdot\underset{\sim}{b}} \langle m | [e^{i\chi^G_{tot}(\underset{\sim}{b}, \underset{\sim}{b}_1, \ldots \underset{\sim}{b}_N)} - 1] | 0 \rangle \qquad (3.49)$$

where $\underset{\sim}{q} = \underset{\sim}{k} - \underset{\sim}{k}'$ is now the laboratory wave vector transfer, and we have denoted the initial and final laboratory wave numbers by $\underset{\sim}{k}$, and $\underset{\sim}{k}'$ respectively. This last expression is more convenient to analyze high-energy hadron-nucleus collisions, since we want the nuclei to remain non-relativistic and we also wish to compare directly hadron-nucleus cross sections with those on free nucleons [see Section V]. Defining the quantity (Glauber, 1959)

$$\Gamma_{tot}(\underset{\sim}{b}, \underset{\sim}{b}_1, \ldots \underset{\sim}{b}_N) = 1 - \exp [i\chi^G_{tot}(\underset{\sim}{b}, \underset{\sim}{b}_1, \ldots \underset{\sim}{b}_N)] \qquad (3.50)$$

we see that Eq. (3.49) becomes

$$F_{m0}^{G} = \frac{ik}{2\pi}\int d^2\underset{\sim}{b}\; e^{i\underset{\sim}{q}\cdot\underset{\sim}{b}} \langle m \,|\, \Gamma_{tot}(\underset{\sim}{b}, \underset{\sim}{b}_1, \ldots \underset{\sim}{b}_N) \,|\, 0 \rangle . \tag{3.51}$$

Introducing the quantities

$$\Gamma_j\,(\underset{\sim}{b}-\underset{\sim}{b}_j) = 1 - \exp\,[\, i\chi_j(\underset{\sim}{b}-\underset{\sim}{b}_j)] \tag{3.52}$$

Glauber now writes

$$\Gamma_j(\underset{\sim}{b}-\underset{\sim}{b}_j) = 1 - \prod_{j=1}^{N} [1-\Gamma_j(\underset{\sim}{b}-\underset{\sim}{b}_j)] \tag{3.53}$$

or

$$\Gamma_{tot} = \sum_j \Gamma_j - \sum_{j\neq \ell} \Gamma_j\,\Gamma_\ell + \ldots + (-)^{N-1} \prod_{j=1}^{N} \Gamma_j . \tag{3.54}$$

This last equation, when substituted in Eq. (3.51), leads directly to an interpretation of the collision in terms of a multiple scattering expansion involving the incident particle and the various target scatterers. The term linear in Γ_j on the right-hand side of Eq. (3.54) accounts for the "single scattering" (impulse) contribution to the scattering amplitude, whereas the next terms provide double, triple, ... scattering corrections. We note that the order of the multiple scattering can at most be N, reflecting the fact that the scattering is focused in the forward direction.

It is important to realize that the above generalization of the eikonal method makes no reference to interaction potentials: only the two-body phase shift functions χ_j (or the functions Γ_j) must be known in order to calculate Γ_{tot}. This fact makes the Glauber formula (3.49) particularly useful to analyze high-energy hadron-nucleus scattering, as we shall illustrate in Section V.

If the basic two-body interactions are known, as in atomic physics, we can actually gain further insight by obtaining the eikonal scattering amplitude in terms of these interaction potentials. For example, if we consider the non-relativistic scattering of a charged, "elementary" particle (i.e. a particle which does not exhibit any internal structure in the collision considered) by an atom, and if we work in the center of mass system, we may write the full eikonal wave function as a direct generalization of the expression (2.27) namely

$$\Psi_E(\underset{\sim}{r}, X) = (2\pi)^{-3/2} \exp [i\underset{\sim}{k}_i \cdot \underset{\sim}{r} - \frac{i}{\hbar v_i} \int_{-\infty}^{z} V_i(\underset{\sim}{b}, z', X) dz'] \; \psi_o(X) \tag{3.55}$$

Here $\underset{\sim}{r}$ is the initial relative coordinate, $v_i = \hbar k_i / M_i$ is the initial relative velocity (with M_i the reduced mass in the initial channel), X denotes collectively the target coordinates, and $\psi_o(X)$ is the initial bound-state wave function of the target. The potential V_i is the full initial channel interaction between the incident particle and all the particles in the target. The corresponding transition matrix element is then given by Eq. (3.7a) in which the exact state vector $\Psi_a^{(+)}$ is replaced by Ψ_E. A similar expression may also be obtained from Eq. (3.7b). For a direct collision process ($V_i = V_f = V$) leading to a final target state $|m>$, we may write more explicitly the many-body eikonal scattering amplitude as

$$F_{m0} = - \frac{M_i}{2\pi\hbar^2} \int d^2b dz \, e^{i\Delta \cdot r} <m | V(b, z, X) \exp [- \frac{i}{\hbar v_i} \int_{-\infty}^{z} V(b, z', X)dz'] | 0> \tag{3.56}$$

For elastic scattering processes such that $|k_i| = |k_f| = k$, and if we choose the z-axis to be perpendicular to the momentum transfer, we may perform the z-integral in Eq. (3.56) to obtain the Glauber result [see Eq. (3.45) with m = 0]

$$F_{el}^{G} = \frac{k}{2\pi i} \int d^2b \, e^{i\Delta \cdot b} <0 | [e^{i\chi_{tot}^{G}(b, b_1, \dots b_N)} - 1] | 0> \tag{3.57}$$

with

$$\chi_{tot}^{G}(b, b_1, \dots b_N) = - \frac{1}{\hbar v_i} \int_{-\infty}^{+\infty} V(b, z, X)dz. \tag{3.58}$$

However, for inelastic (direct) processes the Glauber scattering amplitude (3.45) can only be obtained from Eq. (3.56) by neglecting the longitudinal momentum transfer, since Δ now lies along k_i in the case of forward scattering (choosing the z-axis perpendicular to Δ would therefore be rather unnatural in this case). This neglect of the longitudinal momentum transfer is not too serious for mildly inelastic hadron-nucleus collisions at high energies, but it leads to undesirable features in atomic collisions. We shall return to this point in Section IV. In this case it is more appropriate to return to the more general eikonal expression (3.56).

Instead of generating a multiple scattering expansion in terms of the quantities Γ_j (which in turn, as we shall see in Section V, may be obtained from the two-body scattering amplitudes describing the scattering of the incident particle by the j^{th} scatterer), we may also write from Eq. (3.45) another multiple scattering series which is more closely related to the one we have analyzed in connection with potential scattering (see Section II.3). Limiting ourselves to elastic scattering, we write the Glauber scattering amplitude (3.57) as

$$F_{el}^{G} = \sum_{n=1}^{\infty} \overline{F}_{Gn} \qquad (3.59)$$

where

$$\overline{F}_{Gn} = \frac{k}{2\pi i} \frac{i^n}{n!} \int d^2\underset{\sim}{b}\, e^{i\underset{\sim}{\Delta}\cdot\underset{\sim}{b}} <0 \mid [\chi_{tot}^{G}(\underset{\sim}{b}, \underset{\sim}{b}_1, \ldots \underset{\sim}{b}^{G})]^n \mid 0> \qquad (3.60)$$

We shall also denote by F_{Gn} the sum of the first n terms of the series (3.59). Thus

$$F_{Gn} = \sum_{j=1}^{n} \overline{F}_{Gj}. \qquad (3.61)$$

With the choice of z-axis which we have adopted ($\hat{z}$ perpendicular to $\underset{\sim}{\Delta}$), it is a simple matter to see that for all scattering angles

$$F_{G1} = F_{B1} \qquad (3.62)$$

where F_{B1} is the corresponding first Born scattering amplitude. Higher terms of the Glauber series (3.59) and of the Born series will be examined in Section IV for electron-atom collisions.

We have so far studied the many-body generalization of the eikonal method proposed by Glauber. Various attempts at deriving or improving

Glauber's method by starting from the multiple scattering formalism (Goldberger and Watson, 1964; Kerman, McManus and Thaler, 1959) have been made by several authors (Czyz and Maximon, 1968, 1969; Remler, 1968, 1971; Feshbach and Hüfner, 1970; Tarasov and Tseren, 1970; Kelly, 1971, Eisenberg, 1972; Karlsson and Namysłowski, 1972; Namysłowski, 1972a). The Glauber result (3.45) may also be viewed as an eikonal approximation to a model proposed by Chase (1956), in which the target particles are frozen in a given configuration (Mittleman, 1970). Osborn (1970) has used the Faddeev equation to suggest a way of unitarizing the inpulse approximation and obtaining Glauber-type results without the eikonal approximation. A comparison of the Born and Faddeev-Lovelace-Watson expansions with the Glauber theory has also been made for various atomic and nuclear processes. We shall return to these questions in Sections IV and V. In particular, we shall see in Section IV. 1 that the combined use of the eikonal approximation and the Born series (such that higher order Born terms are calculated by means of the eikonal approximation) yields very encouraging results for elastic electron-atom scattering at intermediate energies (Byron and Joachain, 1973b).

Many-body collisions may also be studied by using the eikonal approximation together with the optical model formalism. For elastic collisions one first tries to obtain an optical potential which is subsequently "eikonalized." The optical model concept may also be used within the framework of the eikonal DWBA approximation to study inelastic collisions. The basic problem in this approach is the determination of optical potentials, a question which we now briefly review from the point of view of multiple scattering theory.

4. Multiple Scattering Approach to the Optical Potential.

The earlier applications of the optical model method were made to the analysis of the propagation of light through a refractive medium. In this case the use of a complex refractive index is in fact equivalent to the introduction of an optical potential (see for example Lax, 1951). A generalization of the optical model idea was made by Ostrofsky, Breit and Johnson (1936) to the study of α-decay of nuclei, while Bethe (1940) introduced the concept of an optical potential model for low-energy nuclear collisions. The description of high-energy nuclear collisions within the optical model formalism was initiated by Serber et al. (Serber, 1947; Fernbach, Serber and Taylor, 1949) who first described nucleon-nucleus collisions in terms of nucleon-nucleon scattering. Their multiple scattering analysis led to the conclusion that particles should move more or less freely through nuclear matter at high energies. This fact was verified qualitatively by experiment, and led to a reassessment of the optical model for low-energy nuclear scattering (see for example Le Levier and Saxon, 1952, Feshbach, Porter and Weisskopf, 1954).

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Following the work of Serber et al., several attempts were made to derive the optical model from first principles (Francis and Watson, 1953; Riesenfeld and Watson, 1956; Feshbach, 1958, 1962, Kerman, McManus and Thaler, 1959; Glauber, 1959). We shall summarize here the multiple scattering derivations of Watson et al. (Goldberger and Watson. 1964; Fetter and Watson, 1965) and of Glauber (1959).

Let us assume first that the incident particle is distinct from each of the N scatterers in the target. We write the total Hamiltonian of the system as $H = H_d + V$, where the direct arrangement channel Hamiltonian H_d includes the kinetic energy of the colliding particles and the internal target Hamiltonian, while V is the full interaction between the incident particle and the target system. Thus $V = \sum_{j=1}^{n} v_j$, where v_j is the interaction between the beam particle and the j th target scatterer. Assuming that the target is initially in the state $|0\rangle$, we call $\Psi_{c,a}^{(+)}$ that part of the complete state vector $\Psi_a^{(+)}$ corresponding to coherent (elastic) scattering. That is

$$\Psi_{c,a}^{(+)} = \Pi_0 \Psi_a^{(+)} \tag{3.63}$$

where Π_0 is a projection operator onto the state $|0\rangle$. We may therefore introduce formally an optical potential operator V_{opt} such that

$$\Psi_{c,a}^{(+)} = \Phi_a + G_d^{(+)} V_{opt} \Psi_{c,a}^{(+)} \tag{3.64}$$

with $G_d^{(+)} = (E - H_d + i\epsilon)^{-1}$. Thus the optical potential is defined as an operator which, through the Lippmann-Schwinger equation (3.64) [or the corresponding one for the elastic T-matrix] leads to the exact transition amplitude for elastic scattering of the incident particle by the target.

Following the method of Watson et al. (Goldberger and Watson, 1964, Fetter and Watson, 1965) one may introduce an operator F defined by

$$\Psi_a^{(+)} = F\,\Psi_{c,a}^{(+)} \tag{3.65}$$

in terms of which the optical potential, which does not depend on the internal coordinates of the target, is given by

$$V_{opt} = \langle 0 | VF | 0 \rangle \tag{3.66}$$

The operator F satisfies the Lippmann-Schwinger equation

$$F = 1 + G_d^{(+)} (1 - \Pi_0)\, VF \tag{3.67}$$

which can be solved by successive iterations. In this way one generates for F a Born series in powers of the interaction V, namely

$$F = 1 + G_d^{(+)} (1 - \Pi_0)\, V + \ldots \tag{3.68}$$

which, substituted into Eq. (3.66) yields

$$V_{opt} = \langle 0 | V | 0 \rangle + \langle 0 | V\, G_d^{(+)} (1 - \Pi_0)\, V | 0 \rangle + \ldots \tag{3.69}$$

As an illustration of the use of Eq. (3.69), let us consider the non-relativistic elastic scattering of an "elementary" particle A of charge Q by a neutral atom B having Z electrons (Mittleman and Watson, 1959, 1960; Mittleman, 1961, 1965). We treat the collision in the center of mass system, using the relative coordinate $\underset{\sim}{r}$ which joins the position of the atomic nucleus (which we assume to coincide with the center of mass of the atom) to that of the particle A. We also denote by $\underset{\sim}{r}_j$ (j = 1, 2, ... Z) the vectors which determine the positions of the atomic electrons. The relative kinetic energy operator is $K = -\hbar^2\nabla_r^2/2M$, where M is the reduced mass of the two colliding particles A and B. We assume that the internal target Hamiltonian h of the atom is such that $h|n\rangle = w_n|n\rangle$, the atom being in the state $|0\rangle$ before and after the collision. The interaction V is the sum of the individual interactions of the incident particle A with the (Z + 1) particles of the target. Neglecting all but Coulomb interactions, we have

$$V = \frac{ZeQ}{r} + \sum_{j=1}^{Z} \frac{-eQ}{|\underset{\sim}{r}-\underset{\sim}{r}_j|} . \tag{3.70}$$

We also ignore the possible effects of the Pauli principle between the incident and target particles, for the moment.

The first term on the right of Eq. (3.69) is simply the <u>static</u> potential $\langle 0|V|0\rangle$. With the help of Eq. (3.70), we see that in the case considered here the first approximation to the optical potential is given by

$$V^{(1)}(\underset{\sim}{r}) = \langle 0|V|0\rangle = \frac{ZeQ}{r} - Qe\sum_{j=1}^{Z} \langle 0| \frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}_j|} |0\rangle . \tag{3.71}$$

This expression may be readily evaluated for simple atoms or when an independent particle model (such as the Hartree-Fock method) is used to describe the state $|0\rangle$ of the target. The static potential (3.71) has been used frequently to describe the elastic scattering of charged particles by atoms. It is worth noting, however, that this potential can at most give a qualitative account of the scattering. At low incident energies it does not take into account the distortion of the atom (in addition to neglecting the effects of the Pauli principle if the incident particle is an electron or an ion identical to the nucleus of the target atom). At energies above the excitation and ionization thresholds, the static potential is also unreliable because it is real and therefore does not account for the removal of incident particles from the initial channel.

The second term on the right of Eq. (3.69) may be written as

$$V^{(2)} = \sum_{n\neq 0} \frac{\langle 0|V|n\rangle \langle n|V|0\rangle}{E - K - (w_n - w_0) + i\epsilon} \tag{3.72}$$

where the summation runs over all the intermediate states of the target and $E = \hbar^2 k^2/2M$ is the incident relative kinetic energy. A detailed study of the expression (3.72) has been made by Mittleman and Watson (1959, see also Goldberger and Watson, 1964). In particular, Mittleman and Watson analyzed the <u>adiabatic approximation</u>, which consists in neglecting the kinetic energy variation in the expression (3.72). Then $V^{(2)} \simeq V^{(2)}_{ad}$, where the (local and real) adiabatic potential $V^{(2)}_{ad}$ may be shown to behave

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at large distances as

$$V_{ad}^{(2)} \xrightarrow[r\to\infty]{} -\frac{\bar{\alpha} Q^2}{2r^4}, \qquad (3.73)$$

with $\bar{\alpha}$ being the atomic polarizability. A convenient phenomenological parameterization of $V_{ad}^{(2)}$ is then given by the Buckingham polarization potential (Buckingham, 1937)

$$V_p(r) = -\frac{\bar{\alpha} Q^2}{2(r^2+d^2)} \qquad (3.74)$$

where d is a cut-off parameter. The adiabatic approximation has been shown by Mittleman and Watson (1959) to improve with decreasing incident energies and increasing values of Z.

Another approximate expression for the second order term $V^{(2)}$, which has proved to be useful for intermediate and high incident energies, may be obtained by replacing in Eq. (3.72) the energy differences $(w_0 - w_n)$ by an average excitation energy $\bar{w}$. The summation on n may then be performed by closure, so that

$$\langle \mathbf{r} | V^{(2)} | \mathbf{r}' \rangle = \frac{2M}{\hbar^2} G_0^{(+)}(k', \mathbf{r}, \mathbf{r}') A(\mathbf{r}, \mathbf{r}'). \qquad (3.75)$$

Here $G_0^{(+)}(k', \mathbf{r}, \mathbf{r}')$ is the free Green's function (2.6) corresponding to a wave number $k' = (k_i^2 - 2\bar{w})^{1/2}$ and

$$A(\mathbf{r}, \mathbf{r}') = \langle 0 | V(\mathbf{r}, X) V(\mathbf{r}', X) | 0 \rangle - \langle 0 | V(\mathbf{r}, X) | 0 \rangle \langle 0 | V(\mathbf{r}', X) | 0 \rangle \quad (3.76)$$

where the symbol X denotes collectively the target coordinates. We note that the expression (3.75) contains explicitly an imaginary part, so that "absorption" corrections due to the non-elastic processes are now taken into account. A study of the optical potential

$$<\underset{\sim}{r} | V_{opt} | \underset{\sim}{r}'> = V^{(1)}(\underset{\sim}{r}) + <\underset{\sim}{r} | V^{(2)} | \underset{\sim}{r}'>. \tag{3.77}$$

where $V^{(1)}$ is given by Eq. (3.71) and $<r | V^{(2)} | r'>$ by Eq. (3.75) has been made recently in the eikonal approximation for elastic electron-atom scattering at intermediate energies (Joachain and Mittleman, 1971a,b). We shall return to this question in Section IV.

Let us now return to the Lippmann-Schwinger equation (3.67) for the operator F. An alternative way of solving this equation is to express F in terms of two-body scattering matrices. To this end we define the objects

$$t_j = v_j + v_j \, G_d^{(+)} (1 - \Pi_o) \, t_j \tag{3.78}$$

where we recall that v_j is the two-body interaction between the incident particle and the j th target scatterer. The operator F is then given by the Watson equations (Goldberger and Watson, 1964)

$$F = 1 + \sum_{j=1}^{N} G_d^{(+)} (1 - \Pi_o) \, t_j \, F_j \tag{3.79a}$$

with

$$F_j = 1 + G_d^{(+)} \sum_{k(\neq j)=1}^{N} (1-\Pi_o) t_k \, F_k \tag{3.79b}$$

and the optical potential is given by

$$V_{opt} = <0 | \sum_{j=1}^{N} t_j \, F_j | 0> \tag{3.80}$$

This expression is still exact, but the coupled Watson equations (3.79) are in general very difficult to solve since the operators t_j include the effect of the internal target Hamiltonian. However, in the weak binding limit (i.e., when the incident particle has high energy compared to the binding energy of a target particle) one can use the impulse approximation to write $t_j \simeq T_j$, where T_j is a genuine two-body scattering matrix for the collision of the incident particle with a free target scatterer j. In this case the Watson equations (3.79) read

$$F = 1 + G_d^{(+)} (1 - \Pi_0) \sum_{j=1}^{N} T_j F_j \qquad (3.81a)$$

with

$$F_j = 1 + G_d^{(+)} (1 - \Pi_0) \sum_{j(\neq k)=1}^{N} T_k F_k \qquad (3.81b)$$

and the optical potential is given by

$$V_{opt} = \langle 0 | \sum_{j=1}^{N} T_j F_j | 0 \rangle . \qquad (3.82)$$

Solving the Watson equations (3.81) by iteration, we then obtain for V_{opt} the multiple scattering series

$$V_{opt} = \langle 0 | \sum_{j=1}^{N} T_j | 0 \rangle + \langle 0 | \sum_{j(\neq k)=1}^{N} T_j G_d^{(+)} (1 - \Pi_0) T_k | 0 \rangle + \ldots \qquad (3.83)$$

A detailed analysis of these single scattering and double scattering contributions to V_{opt} may be found in Goldberger and Watson (1964) for hadron-nucleus scattering in the weak binding limit. For a "large" nucleus of mass number A such that the concept of nuclear density is meaningful, the first term on the right of Eq. (3.83) yields the optical potential

$$V_{opt}(\underset{\sim}{r}) = -\frac{2\pi c^2}{E} A\, f_0\, \rho(\underset{\sim}{r}) \tag{3.84}$$

where E is the (laboratory) energy of the incident particle, f_0 is the (laboratory) forward hadron-nucleon scattering amplitude averaged over the spins and isospins of the target nucleons, and $\rho(\underset{\sim}{r})$ is the nuclear density normalized to one. The double scattering term in Eq. (3.83) involves correlations between the target nucleons and has been studied by several authors (Lax, 1951; Francis and Watson, 1953, Glauber, 1959; Beg, 1960; Johnston and Watson, 1961; Goldhaber and Joachain, 1968).

Until now we have assumed that the incident particle is distinct from each of the target particles. The scattering of a particle identical with target scatterers has been considered by Takeda and Watson (1955), Bell and Squires (1959), Lippmann, Mittleman and Watson (1959) and Feshbach (1962). The Feshbach method is particularly useful for low-energy scattering, a case which we shall not consider here.

The multiple scattering approach to the determination of the optical potential may also be formulated within the framework of the Glauber approximation (Glauber, 1959). In this case we write the eikonal elastic scattering amplitude as

$$F_{el} = \frac{k}{2\pi i}\int d^2\underset{\sim}{b}\, e^{i\underset{\sim}{\Delta}\cdot\underset{\sim}{b}}\, [e^{i\chi_{opt}(\underset{\sim}{b})} - 1] \tag{3.85}$$

where $\chi_{opt}(\underset{\sim}{b})$ is the optical phase shift function. If we identify this amplitude with the Glauber (many-body) elastic amplitude (3.57), we define the Glauber optical phase shift function χ^G_{opt} such that

$$e^{i\chi^G_{opt}(\underset{\sim}{b})} = \langle 0 | e^{i\chi^G_{tot}(\underset{\sim}{b}, \underset{\sim}{b}_1, \ldots \underset{\sim}{b}_N)} | 0 \rangle . \tag{3.86}$$

From this relation one readily deduces that χ_{opt} is in general complex and has a positive imaginary part as soon as non-elastic scattering can occur. We also note that within the eikonal approximation, we may define an optical potential which corresponds to the phase shift function χ_{opt}. It is a local operator $V_{opt}(\underset{\sim}{r})$ such that

$$\chi_{opt}(\underset{\sim}{b}) = -\frac{1}{\hbar v_i} \int_{-\infty}^{+\infty} V_{opt}(\underset{\sim}{b}, z) dz. \tag{3.87}$$

Glauber (1959) has given a detailed discussion of Eq. (3.86) for high-energy hadron-nucleus scattering. For a large nucleus with uncorrelated nucleons, he finds that

$$\chi^G_{opt}(\underset{\sim}{b}) = \lambda f_0 \int_{-\infty}^{+\infty} \rho(\underset{\sim}{b}, z) dz \tag{3.88}$$

where $\lambda = 2\pi k^{-1}$ is the de Broglie wavelength of the incident particle. We note that this result agrees with that obtained by computing first V_{opt} in the "single scattering" approximation of Watson's multiple scattering theory [Eq. (3.84)] and then "eikonalizing" the resulting potential by means of Eq. (3.87).

If the interaction V between the incident particle and the target system is known, as in atomic collision problems, we may use Eqs. (3.58) and (3.86) to expand the Glauber optical phase shift χ^G_{opt} in powers of V (and inverse powers of v_i). Finally, we note that χ^G_{opt} may also be expressed in terms of the quantity Γ_{tot} defined by Eq. (3.50). That is,

$$e^{i\chi^G_{opt}(\underset{\sim}{b})} - 1 = - \langle 0 | \Gamma_{tot}(\underset{\sim}{b}, \underset{\sim}{b}_1, \ldots \underset{\sim}{b}_N) | 0 \rangle \tag{3.89}$$

and Γ_{tot} may in turn be expanded as the multiple scattering series (3.54). The first term of this series is easily shown to yield the familiar impulse approximation for F_{el}, as we shall illustrate in Section V.

IV. ATOMIC COLLISIONS

1. The Scattering of Fast Charged Particles by Atoms.

Because the basic Coulomb interaction is well known, it should be possible to investigate systematically the validity of some of the theoretical methods discussed above, for "simple" atomic collisions. We shall give here a brief survey of recent work concerning the non-relativistic scattering of a fast, charged, "elementary" particle by an atom.

The simplest high-energy approximation used in atomic collisions is certainly the first Born approximation (3.16) together with some modifications of it such as the unitarized Born approximation (Seaton, 1961) and the Ochkur approximation (Ochkur, 1963). The unitarized Born approximation is just the first Born approximation for the corresponding K-matrix element, while the Ochkur approximation is a simplified version of the Born approximation in which only the leading term of the T-matrix element in powers of k_i^{-1} (the inverse of the incident wave number) is retained. The computation of these first order approximations is generally rather straightforward, at least for collisions involving two fragments in the final state and when simple, uncorrelated wave functions (for example of the Hartree-Fock type) are used to describe the bound atomic systems involved in the collision.

Second Born calculations imply a summation over the intermediate states of the system and are therefore much harder to perform, even approximately (see for example Holt and Moiseiwitsch, 1968; Holt, Hunt and Moiseiwitsch, 1971a, 1971b; Woollings and McDowell, 1973; Byron and Joachain, 1973b). As an illustration of these difficulties, let us consider the elastic scattering of an electron by an atom of atomic number Z, and at first neglect exchange effects between the incident and target electrons. The initial and final momenta of the electron are denoted respectively by $\underset{\sim}{k}_i$ and $\underset{\sim}{k}_f$, with $|\underset{\sim}{k}_i| = |\underset{\sim}{k}_f| = k$. Neglecting recoil effects, we choose the nucleus of the target atom as the origin of our coordinate system and denote the coordinate of the projectile electron by $\underset{\sim}{r}$, while the positions of the atomic electrons are given by $\underset{\sim}{r}_j (j = 1, 2, \ldots Z)$. We use atomic units (a.u.) such that the unit of length is the "first Bohr radius" a_0 while the unit of energy is e^2/a_0 (i.e., twice the Rydberg). The free motion of the two colliding particles is then described by the Hamiltonian

$$H_0 = -\frac{1}{2} \nabla_{\underset{\sim}{r}}^2 + h \tag{4.1}$$

where h is the internal target Hamiltonian, with eigenstates $|n>$ and internal energies w_n. We assume that the target is in the state $|0>$ before and after the collision.

The full Hamiltonian of the system is such that

$$H = H_0 + V \tag{4.2}$$

where V, the interaction potential between the incident electron and the target atom, is given by

$$V = -\frac{Z}{r} + \sum_{j=1}^{Z} \frac{1}{|\underset{\sim}{r}_j - \underset{\sim}{r}|}. \tag{4.3}$$

The second Born scattering amplitude for elastic scattering (neglecting exchange) is then given by

$$F_{el}^{B2} = F_{B1} + \overline{F}_{B2} \tag{4.4}$$

where F_{B1} is the corresponding first Born amplitude and

$$\overline{F}_{B2} = 8\pi^2 \int d\underset{\sim}{\kappa} \sum_n \frac{<\underset{\sim}{k}_f, 0 | V | \underset{\sim}{\kappa}, n> <\underset{\sim}{\kappa}, n | V | \underset{\sim}{k}_i, 0>}{\kappa^2 - k^2 + 2(w_n - w_0) - i\epsilon}. \tag{4.5}$$

Here we have written the asymptotic initial and final free states (which are eigenstates of H_0) respectively as $|\underset{\sim}{k}_i, 0>$ and $|\underset{\sim}{k}_f, 0>$ while a general eigenstate of H_0 is denoted by $|\underset{\sim}{\kappa}, n>$. The normalization adopted is such that

$$<\underset{\sim}{\kappa}', n' | \underset{\sim}{\kappa}, n> = \delta_{nn'} \; \delta(\underset{\sim}{\kappa} - \underset{\sim}{\kappa}'). \tag{4.6}$$

The summation over the index n appearing in Eq. (4.5) evidently implies an integration when states belonging to the continuum are concerned. As in the case of the evaluation of the second order contribution to the optical potential [see Eq. (3.72)], we may obtain a useful approximation for the quantity $\overline{F}_{B2}$ by replacing the energy differences $(w_0 - w_n)$ by an average excitation energy $\bar{w}$. The sum on intermediate states can then be done by closure, and after performing the integration on the plane wave part of the matrix elements one obtains

$$\overline{F}_{B2} = \frac{2}{\pi^2} \int d\underset{\sim}{\kappa} \; \frac{1}{\kappa^2 - k_i'^2 - i\epsilon} \; \frac{1}{K_i^2 K_f^2} < 0 | \left[\sum_{j=1}^{Z} \left(e^{-i\underset{\sim}{K}_f \cdot \underset{\sim}{r}_j} - 1 \right) \right] \left[\sum_{k=1}^{Z} \left(e^{i\underset{\sim}{K}_i \cdot \underset{\sim}{r}_k} - 1 \right) \right] | 0> \tag{4.7}$$

where $\underset{\sim}{K}_i = \underset{\sim}{k}_i - \underset{\sim}{\kappa}$, $\underset{\sim}{K}_f = \underset{\sim}{k}_f - \underset{\sim}{\kappa}$ and $k_i' = (k_i^2 - 2\tilde{w})^{1/2}$. If the state $|0>$ is written as an antisymmetrized product of orbitals [whose radial part is assumed to be the sum of terms of the form $r^\ell \exp(-ar)$] the matrix elements in Eq. (4.7) may be readily evaluated and the remaining integration on $\underset{\sim}{\kappa}$ can be reduced to a single integral by using the Feynman parametization technique (Feynman, 1949). Detailed results of such calculations will be discussed below for electron-hydrogen and electron-helium scattering.

Let us now consider the application of the Faddeev-Watson multiple scattering (FWMS) expansions to intermediate and high-energy atomic collisions. Since a recent discussion of this method for three-body atomic problems has been given by Chen (1972), we shall only emphasize a few important points. First of all, we recall that the FWMS expansions are expressed in terms of off-shell two-body T-matrices. For the Coulomb interaction, several representations of the two-body T-matrix are available (see for example J. Chen and A. Chen, 1972). However, at incident energies larger than the three-body break-up threshold, particular care must be exercised in handling the cuts of the Coulomb T-matrix (Nuttall and Stagat, 1971; Chen, Chen and Kramer, 1971; Chen and Kramer, 1971, 1972).

The application of the FWMS expansion (3.38), limited to first order terms, has been studied for several elastic scattering processes by Chen, Chen, Sinfailam and Hambro (1971), and Sinfailam and Chen (1972). Significant differences between the first order FWMS expansion and the first Born approximation were found at rather high energies, as shown in Fig.6 for the case of electron and positron elastic scattering by hydrogen atoms. This effect does not appear in calculations using the Born series and may be entirely spurious. It illustrates some of the difficulties involved in trying to apply the Faddeev-Watson multiple scattering expansion to atomic collision problems.

Three-body rearrangement collisions have also been analyzed by means of first order FWMS expansions, obtained by using the multiple scattering series (3.39) or (3.41) and keeping only the two first terms on the right. (Shastry, Kumar and Callaway, 1970; Chen and Hambro, 1971; Chen and Kramer, 1972). Of particular interest is the electron-transfer or pick-up reaction $p + H \rightarrow H + p$, in which p is a proton and H an hydrogen atom. The role of the proton-proton interaction in this reaction (at high energies) had already been the subject of numerous investigations (Oppenheimer, 1928; Brinkman and Kramers, 1930; Bates and Dalgarno, 1952; Jackson and Schiff, 1953; Drisko, 1955; Bassel and Gerjuoy, 1960; Bates, 1962; Mapleton, 1967, McCarroll and Salin, 1967, Coleman, 1968). The first order FWMS results of Chen and Kramer (1971, 1972) indicate that at very high laboratory energies ($E > 2$ MeV) the cross sections tend towards the first Born results [Eq. (3.16)] of Jackson and Schiff (1953), thus exhibiting a high-energy dependence of the form E^{-6}. However, since the second order Born terms yield an $E^{-5.5}$ energy dependence in the high-energy limit (Drisko, 1955; Mapleton, 1967), it is desirable to examine higher-order terms of the FWMS expansions. Preliminary results on the second order FWMS terms have already been obtained (Carpenter and Tuan, 1970; Chen, Chen and Kramer, 1971) but further calculations (and experiments) seem desirable before definite conclusions can be drawn.

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We now turn to the application of eikonal approximations to intermediate and high energy collisions of a charged particle by an atom. We only outline here the various methods which have been proposed. A more detailed analysis of electron-hydrogen and electron-helium collisions is given respectively in Sections IV. 2 and IV. 3.

The many-body Glauber amplitude, given by Eq. (3.45), has been evaluated for elastic electron-hydrogen collisions (Franco, 1968; Birman and Rosendorff, 1969; Tai, Teubner and Bassel, 1969) and for the excitation of the lowest levels of hydrogen by electron impact (Ghosh and Sil, 1969, Ghosh, Sinha and Sil, 1970; Tai, Bassel, Gerjuoy and Franco, 1970; Bhadra and Ghosh, 1971; Sheorey, Gerjuoy and Thomas, 1971; Gerjuoy, Thomas and Sheorey, 1972). Since exchange scattering is ignored in these calculations, proton-hydrogen collisions may be treated in a formally identical manner, except for a change in the scale of the momentum transfer. Such computations have been performed by Franco and Thomas (1971), Bhadra and Ghosh (1971) and Ghosh and Sil (1971). All these calculations on atomic hydrogen, using the Glauber formula (3.45) may be reduced to the evaluation of a single dimensional integral or even, as shown by Thomas and Gerjuoy (1971) to a finite sum of hypergeometric functions (see also Gerjuoy, 1972). More recently, the Glauber amplitude has also been evaluated for the ionization of atomic hydrogen (Hidalgo, McGuire and Doolen, 1972, McGuire, et al., 1973).

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For target atoms more complex than atomic hydrogen, the reduction of the Glauber amplitude (3.45) to a tractable form is more difficult to achieve. For electron-helium elastic scattering, Franco (1970) has reduced the Glauber scattering amplitude to a three-dimensional integral by using a

Hartree-Fock ground state wave function. Glauber calculations for elastic electron and proton scattering by helium have also been performed by Johnson and Brolley (1970), while the excitation of the 2^1S state has been studied by Yates and Tenney, (1973). A general reduction procedure of the Glauber amplitude (3.45) for many-electron atoms has been proposed by Franco (1971). Finally, we mention that Glauber-type calculations for elastic scattering and excitation of the 2s-2p transition in lithium have been performed by Mathur, Tripathi and Joshi (1971, 1972). These authors, however, disregard the effect of the 1s core electrons, an approximation which is too crude to permit a detailed comparison of their results with the experimental data. More realistic Glauber-type calculations on electron scattering by Li, Na and K have been performed recently by Walters (1973).

We have already pointed out in Section III. 3 that for inelastic collisions the Glauber scattering amplitude (3.45) can only be derived from the more general expression (3.56) by neglecting the longitudinal momentum transfer. The importance of treating correctly the kinematics for inelastic atomic collisions has been stressed by Byron (1971) and by Chen, Joachain and Watson (1972). Byron (1971) has also given a derivation of the general Glauber expression (3.56) by treating in the eikonal approximation the complete set of close-coupling equations (with exchange neglected). He then used the Monte-Carlo technique to perform the multidimensional integrals appearing in Eq. (3.56) for the excitation of various states of atomic hydrogen and helium by electron impact. Similar calculations using the Monte-Carlo method have also been made by Byron and Joachain (1972) for the excitation of the 2^3S state of helium by electron impact, which is a pure rearrangement (knock-out) process when spin-dependent interactions are neglected.

We now come to eikonal calculations involving the optical model formalism (Joachain and Mittleman, 1971a, b; Chen, Joachain and Watson, 1972, Joachain and Vanderpoorten, 1973a, b). Starting from the optical potential (3.77) and using the eikonal approximation, Joachain and Mittleman have shown that the direct elastic scattering amplitude for the collision of a charged particle by an atom is given by

$$F_{el} = \frac{k}{2\pi i}\int d^2\underset{\sim}{b}\; e^{i\underset{\sim}{\Delta}\cdot\underset{\sim}{b}}\left[e^{i\chi^{(2)}_{opt}(\underset{\sim}{b})}-1\right] \qquad (4.8)$$

where the (second order) optical phase shift function $\chi^{(2)}_{opt}(\underset{\sim}{b})$ is obtained from

$$\chi^{(2)}_{opt}(\underset{\sim}{b}) = -\frac{1}{v_i}\int_{-\infty}^{+\infty} V^{(1)}(\underset{\sim}{b}, z)dz + \frac{i}{v_i v_i'}\int_{-\infty}^{z} dz \int_{-\infty}^{+\infty} dz' \exp[-i(k_i - k_i')(z-z')]$$

$$A(\underset{\sim}{b}, z; \underset{\sim}{b}, z'). \qquad (4.9)$$

Here $V^{(1)}$ is the static (first order) optical potential, as given by Eq. (3.71), the quantity $A(\underset{\sim}{r};\underset{\sim}{r}')$ is defined by Eq. (3.76) and $v_i = k_i/M$ is the initial relative velocity of the two colliding particles (M being their reduced mass). Moreover, an average excitation energy $\bar{w}$ of the target states has been introduced, such that $Mv_i'^2/2 = k_i'^2/(2M) = k_i^2/(2M) - \bar{w}$. We note that within the framework of the eikonal approximation we may use Eq. (3.87) to extract from Eq. (4.9) the equivalent local (second order) optical potential

$$V^{(2)}_{opt}(\underset{\sim}{r}) = V^{(1)}(\underset{\sim}{r}) - \frac{i}{v_i'}\int_{-\infty}^{z} dz' \exp[-i(k_i - k_i')(z-z')] A(\underset{\sim}{b}, z; \underset{\sim}{b}, z'). \quad (4.10)$$

Direct integration of the second term on the right of Eq. (4.9) is still very laborious, even for the simplest targets. However, we note that the quantity

$$\mathrm{Im}\,\chi^{(2)}_{opt} = -\frac{1}{v_i}\int_{-\infty}^{+\infty} \mathrm{Im}\,V^{(2)}_{opt}(\underset{\sim}{b}, z)dz = -\frac{1}{2v_i v_i'}\int_{-\infty}^{+\infty} dz \int_{-\infty}^{+\infty} dz' \qquad (4.11)$$

$$\exp[-i(k_i - k_i')(z-z')] A(\underset{\sim}{b}, z; \underset{\sim}{b}, z')$$

may be evaluated with a reasonable amount of computational effort. Therefore the leading <u>absorption corrections</u>, induced by unitarity from the open channels, can be calculated explicitly in this formalism.

It is interesting to compare the second order optical phase shift $\chi^{(2)}_{opt}$ given by Eq. (4.9) with Glauber optical phase shift χ^{G}_{opt} obtained from Eq. (3.86). Thus, we first write

$$\begin{aligned}\chi^{G}_{opt}(\underset{\sim}{b}) &= -i \log \langle 0 | e^{i\chi^{G}_{tot}} | 0 \rangle \\ &= \langle 0 | \chi^{G}_{tot} | 0 \rangle - i \log \langle 0 | \exp i \, [\chi^{G}_{tot} - \langle 0 | \chi^{G}_{tot} | 0 \rangle] \, | 0 \rangle\end{aligned} \tag{4.12}$$

Then, using Eq. (3.58) and expanding the right-hand side of Eq. (4.12) in powers of v_i^{-1}, we find that

$$\chi^{G}_{opt}(\underset{\sim}{b}) = -\frac{1}{v_i}\int_{-\infty}^{+\infty} V^{(1)}(\underset{\sim}{b}, z)dz + \frac{i}{2v_i^2}\int_{-\infty}^{+\infty} dz \int_{-\infty}^{+\infty} dz' \, A(\underset{\sim}{b}, z; \underset{\sim}{b}, z') + \ldots \tag{4.13}$$

Hence, by comparing this result with Eq. (4.9), we see that the Glauber optical phase shift, or the corresponding optical potential V^{G}_{opt} such that

$$\chi^{G}_{opt}(\underset{\sim}{b}) = -\frac{1}{v_i}\int_{-\infty}^{+\infty} V^{G}_{opt}(\underset{\sim}{b}, z)dz \tag{4.14}$$

contains no real second order terms and corresponds to the choice $\bar{w} = 0$ for the average excitation energy of the target. The fact that $\bar{w} = 0$ in the Glauber approximation has important consequences. Indeed, using Eqs. (3.57) and (3.86), one can readily deduce that the Glauber many-body elastic scattering amplitude F^{G}_{el} diverges logarithmically at zero momentum transfer (Franco, 1968a). This undesirable feature is removed in the eikonal optical model of Joachain and Mittleman (1971a, b).

The optical model formalism may also be used together with the eikonal DWBA method to analyze inelastic or rearrangement atomic processes (Chen, Joachain and Watson, 1972, Joachain and Vanderpoorten, 1973a, b). For example, in the case of a direct transition such that the target, initially in the state $|0\rangle$, is left in the state $|n\rangle$, the eikonal DWBA transition matrix element obtained from Eq. (3.25), is simply

$$T_{ba}^{eik} = (2\pi)^{-3} \int d\mathbf{r}\; e^{i\boldsymbol{\Delta}\cdot\mathbf{r}} \exp\{i[\Lambda_i(\mathbf{b},z) + \Lambda_f(\mathbf{b},z)]\}\, V_{n0}(\mathbf{b},z) \tag{4.15}$$

where

$$\Lambda_i(\mathbf{b},z) = -\frac{1}{v_i}\int_{-\infty}^{z} U_i(\mathbf{b},z')dz',$$
$$\Lambda_f(\mathbf{b},z) = -\frac{1}{v_f}\int_{z}^{\infty} U_f(\mathbf{b},z')dz', \tag{4.16}$$

and

$$V_{n0}(\mathbf{b},z) = \langle n|V|0\rangle$$

Here U_i and U_f are respectively the initial and final distorting potentials, while v_i and v_f are the relative velocities in the initial and final channel. At the expense of treating to first order that part of the interaction which is responsible for the inelastic transition, this method leads to reasonably simple expressions. These take into account explicitly the longitudinal momentum transfer, allow the evaluation of exchange effects, and may be applied to complex target atoms. Applications of this method to electron-hydrogen and electron-helium collisions will be discussed below.

To conclude this section, we would like to mention the very interesting approach recently developed by Bransden et al. (Bransden and Coleman, 1972; Bransden, Coleman and Sullivan, 1972; Sullivan, Coleman and Bransden, 1972; Berrington, Bransden and Coleman, 1973) to analyze the scattering of charged particles by atoms. Starting from the set of close coupling equations, these authors retain explicitly a group of states in a truncated expansion of the full wave function. The remaining states are accounted for by the introduction of suitable second order potentials, similar to those discussed above. This method has already been applied successfully to the scattering of electrons and protons by atomic hydrogen and helium, as we shall illustrate in Sections IV. 2 and IV. 3.

2. Electron Scattering By Atomic Hydrogen

We shall now analyze in more detail the scattering of electrons by atomic hydrogen at intermediate and high energies. We begin by considering elastic collisions, and follow the treatment of Byron and Joachain (1973b) who have carried out a detailed comparison of the Born and the Glauber eikonal series. We write the Born series for the direct elastic scattering amplitude as

However, one can still alter the phases of the fields and this, we will now show, is sufficient to reduce U to dependence on a single parameter.

To see this, consider that the most general 2×2 unitary matrix can be written as

$$U = e^{i\Delta} \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix},$$

where Δ is some real phase and the complex numbers α and β satisfy $|\alpha|^2 + |\beta|^2 = 1$. Suppose we define new (primed) fields, differing from the original ones by a phase

$$\begin{pmatrix} c \\ u \end{pmatrix} = e^{i\Gamma} \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} c \\ u \end{pmatrix}' \qquad \begin{pmatrix} d \\ s \end{pmatrix} = e^{i\Lambda} \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}'.$$

In terms of these new fields, the current assumes the form

$$J = e^{i(\Delta + \Lambda - \Gamma)} \, \overbrace{\bar{c} \; \bar{u}}' \begin{pmatrix} \alpha' & \beta' \\ -\beta'^* & \alpha'^* \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}',$$

where $\alpha' = \alpha e^{i(\phi - \psi)}$ and $\beta' = \beta e^{i(\phi + \psi)}$. Clearly, we now have the freedom to remove all relative phases and reduce J to a function of a single parameter. To bring this expression above into conventional form, choose ϕ and ψ to make α' and β' purely imaginary and then choose $\Lambda - \Gamma$ to make the overall matrix real. Since $|\alpha|^2 + |\beta|^2 = 1$, we may define $\cos\theta_C = |\beta|$, $\sin\theta_C = |\alpha|$. Then the hadronic current becomes

$$J_H = \overbrace{\bar{c} \; \bar{u}}' \begin{pmatrix} -\sin\theta_C & \cos\theta_C \\ \cos\theta_C & \sin\theta_C \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}'.$$

The sole remaining parameter θ_C is called the Cabibbo angle. Hereafter,

$$F^G_{el} = \frac{k}{i}\int_0^\infty dbb\ J_0\ (\Delta b) < 0 \left| \left[e^{i\chi^G_{tot}} - 1 \right] \right| 0> \qquad (4.21)$$

and similarly we find from Eq. (3.60) that

$$\overline{F}_{Gn} = \frac{k}{i}\frac{i^n}{n!}\int_0^\infty dbb\ J_0(\Delta b) < 0 \left| \left[\chi^G_{tot} \right]^n \right| 0> . \qquad (4.22)$$

It is apparent from Eq. (4.22) that, as in the case of potential scattering [cf. the discussion following Eq. (2.43)] the terms of the Glauber multiple scattering series (3.59) are alternatively purely real and purely imaginary. This, again, is in contrast with the Born series (4.17), where already the term $\overline{F}_{B2}$ contains a real as well as an imaginary part.

We have already pointed out in Section IV. 1 that by using an average excitation energy $\bar{w}$ it is possible to reduce the quantity $\overline{F}_{B2}$ to the expression (4.7) which then can be evaluated in a straightforward manner. In fact, for simple target atoms like hydrogen and helium one may even include exactly a few states in the summation on n appearing in Eq. (4.5), and then evaluate the sum on the remaining states by closure methods (see for example Holt, Hunt and Moiseiwitsch, 1971b, Woollings and McDowell, 1972, Byron and Joachain, 1973b). Of particular interest is the limit of large values of k for which, at small scattering angles ($\theta < \bar{w}/k^2$), Byron and Joachain find that Re $\overline{F}_{B2}$ varies like k^{-1}, while Im $\overline{F}_{B2}$ behaves like k^{-1} log k. We note that this behavior of $\overline{F}_{B2}$ is different from that found in Eq. (2.50) for the case of potential scattering. In particular, we emphasize that Re $\overline{F}_{B2}$ now gives the dominant correction to the first Born differential cross section at small angles. At larger angles $\theta > \bar{w}/k^2$ one retrieves the "potential scattering" behavior such that Re $\overline{F}_{B2}$ varies like k^{-2} and Im $\overline{F}_{B2}$ like k^{-1} for larger values of k.

Let us now examine the Glauber multiple scattering series (3.59). The second order term $\overline{F}_{G2}$, which is purely imaginary, may easily be shown to diverge logarithmically at $\Delta = 0$. Indeed, the corresponding quantity Im $\overline{F}_{B2}$ also diverges logarithmically as $\bar{w}$, the average excitation energy, is set equal to zero. As shown explicitly by Byron (1971), the many-body Glauber result (3.57) precisely assumes that $\bar{w} = 0$. Although the quantities Im $\overline{F}_{B2}$ and Im $\overline{F}_{G2}$ differ substantially at very small momentum transfers because of the divergence of Im $\overline{F}_{G2}$, a detailed study of these two quantities shows that otherwise they agree very well, even in the backward direction and for rather low values of k. This is reminiscent of the relationship (2.46) proved in potential scattering for Yukawa-type potentials.

For $n \geq 3$, the terms $\overline{F}_{GN}$ of the Glauber multiple scattering series (3.59) are finite, even at $\Delta = 0$. It is therefore very likely that these terms will agree with the corresponding terms of the Born series (i.e., $\overline{F}_{G3}$ with Re $\overline{F}_{B3}$, $\overline{F}_{G4}$ with i Im $\overline{F}_{B4}$, etc.) at least for small scattering angles. Since the direct evaluation of the quantity Re $\overline{F}_{B3}$ (which yields contributions of order k^{-2} to the differential cross section) is an extremely difficult task, it seems therefore reasonable to use

$\overline{F}_{G3}$ in place of Re $\overline{F}_{B3}$. Thus we write the direct elastic scattering amplitude (through terms of order k^{-2}) as

$$F_{el} = \overline{F}_{B1} + \text{Re}\,\overline{F}_{B2} + \overline{F}_{G3} + i\,\text{Im}\,\overline{F}_{B2} + \ldots \quad (4.23)$$

and shall refer to this treatment as the eikonal-Born series method (Byron and Joachain, 1973b).

Before we compute the elastic differential cross section we recall that the leading (Ochkur) term of the first order exchange amplitude is of order k^{-2}. A consistent calculation of the elastic differential cross section through order k^{-2} therefore requires the inclusion of this term, which we call G_1. The elastic differential cross section (for unpolarized beam and target, and if no attempt is made to distinguish the various final spin states) is then given by

$$\frac{d\sigma_{el}}{d\Omega} = \frac{1}{4}\left| F_{el} + G_1 \right|^2 + \frac{3}{4}\left| F_{el} - G_1 \right|^2 . \quad (4.24)$$

As an example, we display in Fig. 7 the result of such a calculation for the elastic scattering of electrons by atomic hydrogen at an energy of 100 eV. Also shown on Fig. 7 are the first Born approximation results and the Glauber approximation cross section $d\sigma^G_{el}/d\Omega = \left| F^G_{el} \right|^2$. We note that $d\sigma^G_{el}/d\Omega$ is quite different from $d\sigma_{el}/d\Omega$ as given by Eq. (4.23). Indeed the Glauber differential cross section diverges at $\theta = 0^o$ (because of the term Im $\overline{F}_{G2}$) and lacks the exchange term G_1 together with the important term Re $\overline{F}_{B2}$. The remaining curve on Fig. 7 corresponds to positron-hydrogen scattering, calculated from the eikonal-Born series amplitude (4.23). Whereas the eikonal-Born series method predicts significant differences between electron and positron scattering, the Born and Glauber approximations do not distinguish the two cases. The (relative) measurements of Teubner, Williams and Carver (quoted in Tai, Teubner and Bassel, 1969) have been normalized to the eikonal-Born series curve at $\theta = 30^{0}$. They could, however, be equally well normalized to the Glauber or the first Born curve and therefore do not provide a good test of the various theories. As we shall see below, the situation is quite different in elastic electron-helium scattering.

We now consider briefly some inelastic transitions induced in atomic hydrogen by the impact of fast electrons. Calculations using the Glauber approximation (3.45) have been performed by several authors (Ghosh and Sil, 1969; Ghosh, Sinha and Sil, 1970, Tai, Bassel, Gerjuoy and Franco, 1970; Bhadra and Ghosh, 1971; Sheorey, Gerjuoy and Thomas, 1971; Gerjuoy, Thomas, and Sheorey, 1972) and reviewed by Gerjuoy (1972). As an example, we show in Fig. 8 the results of the calculations of Tai, Bassel, Gerjuoy and Franco (1970) for the differential cross section corresponding to the excitation of the 2s states of hydrogen by incident electrons of 100 eV. We see from this figure that at small angles the predictions of Tai et al. differ substantially from the first Born approximation and from the eikonal DWBA calculations of Chen, Joachain and Watson (1972) (using static distorting potentials) and of Joachain and

Vanderpoorten (1973a) [using Glauber (complex) distorting potentials].

As another example, we show in Fig. 9 the total cross section for excitation of the 2p state of hydrogen by electron impact. Here, in addition to the first Born approximation and the Glauber results of Tai et al. (1970), we have also displayed the eikonal calculations of Byron (1971), the four-channel approximation results of Sullivan, Coleman and Bransden (1972) and the eikonal DWBA calculations of Joachain and Vanderpoorten (1973a). Also shown for comparison are the close-coupling results of Burke, Schey and Smith (1963). The experimental data are those of Long, Cox and Smith (1968). They are normalized at high energies to the first Born values.

Another interesting quantity is the polarization P of the radiation emitted from the final state of the excitation process $e^- + H(1s) \rightarrow e^- + H(2p)$. This polarization results from the relative population of the magnetic sublevels of the 2p states. The corresponding $2p \rightarrow 1s$ line occurs at 1216 Å and has been studied experimentally by Ott, Kauppila and Fite (1967). Using the Glauber expression (3.45) which neglects the longitudinal momentum transfer, Tai et al. (1970) found a selection rule $\Delta m = \pm 1$ for $s \rightarrow p$ transitions which leads to a constant polarization $P = -3/11$. This result is in strong disagreement with the experimental data of Ott, Kauppila and Fite, who find that the polarization P is positive from threshold to about 250 eV. By using the more general and kinematically correct expression (3.56), Byron (1971) obtained theoretical values of P in much better agreement with the experimental data. Gerjuoy, Thomas and Sheorey (1972) have also obtained good agreement with experiment by using the Glauber expression (3.45), with the axis of quantization chosen perpendicular to the momentum transfer, and then transforming the calculated cross sections to refer them to a quantization axis in the direction of $\underset{\sim}{k}_i$. The results of the four-channel approximation of Sullivan, Coleman and Bransden (1972) reproduce the experimental shape of P as a function of the energy but lie somewhat below the experimental values. It is worth noting that in this case the first Born approximation agrees surprisingly well with measurements.

Returning to the evaluation of excitation cross sections, we note that, as in the case of elastic scattering, the Glauber expression (3.45) predicts identical results for the excitation by electron or positron impact. Using the more general Eq. (3. 56), which properly accounts for the longitudinal momentum transfer, Byron (1971) has found significant differences between electron and positron excitation of the 2s states of hydrogen. Since positron scattering is presently not feasible, experimental data on proton scattering by hydrogen (in the energy range 50 - 150 keV, i.e., at velocities corresponding to the electron case) would be very useful to settle this question. It is worth noting that the eikonal DWBA method of Chen, Joachain and Waston (1972) and the approach of Bransden and Coleman (1972) also predict differences between electron and positron (proton) scattering.

3. Electron-Helium Collisions

We now turn to the scattering of electrons by helium at intermediate and high(atomic) energies. In this case the theoretical calculations are obviously harder to perform than for atomic hydrogen, but on the other hand accurate, absolute experimental data for various processes have recently become available. It is therefore with helium targets that the various theories examined above can presently be tested in the most reliable way.

We begin by analyzing elastic electron-helium scattering, following the eikonal-Born series method of Byron and Joachain (1973b). By using an analytical fit (see for example Byron and Joachain, 1966) to the Hartree-Fock ground state helium wave function (Roothaan, Sachs and Weiss, 1960), the reduction of the second Born expression (4.7) proceeds as in the case of hydrogen. Similarly, the Glauber expressions (4.21) and (4.22) can also be evaluated in this case. The eikonal-Born series direct elastic scattering amplitude is still given by Eq. (4.23), and the elastic differential cross section now reads

$$\frac{d\sigma_{el}}{d\Omega} = \left| F_{el} - G_1 \right|^2 \qquad (4.25)$$

where G_1 again refers to the leading Ochkur) term of the exchange amplitude.

As an illustration of these calculations, we display in Figs. 10 and 11 the eikonal-Born series results, the Glauber approximation (Franco, 1970) and the first Born approximations for the elastic scattering of 500 eV electrons by the helium ground state. The experimental points refer to the absolute measurements of Bromberg (1969). Fig. 10, which shows the differential cross section, clearly exhibits the small angle behavior of the various theoretical predictions, while Fig. 11, which displays the quantity $(d\sigma/d\Omega) \times \sin\theta$ is more appropriate to analyze the larger angle behavior. It is apparent from the examination of Figs. 10 and 11 that the eikonal-Born series results [using Eq. (4.23)] are consistently better than the first Born or the Glauber approximation predictions.

As in the case of elastic electron-hydrogen scattering, the Glauber differential cross section diverges in the forward direction and misses the important contribution arising from the term Re $\overline{F}_{B2}$. It is

worth stressing that the difficulties encountered here with the many-body Glauber approximation clearly result from the long range nature of the atomic interactions; they do not appear in problems involving short-range interactions, as we shall see in Section V.

It is also interesting to include in the comparison with experiment the methods involving second order optical potentials. Thus in Fig. 12 we compare (for elastic electron-helium scattering at 300 eV) the experimental results of Vriens, Kuyatt and Mielczarek (1968) [as renormalized recently by Chamberlain, Mielczarek and Kuyatt, 1970] with the optical model calculations of Joachain and Mittleman (1971a, b) and the recent calculations of Berrington, Bransden and Coleman (1973). Also shown on Fig. 12 are the first Born values, the eikonal-Born series results of Byron and Joachain (1973b) and the Glauber values (Franco,1970). Since the calculations of Joachain and Mittleman (1971a, b) use a phenomenological, one parameter polarization potential to fit the data at $\theta = 5^{o}$, it appears from Fig. 12 that the most satisfactory results obtained at present are those of Berrington, Bransden and Coleman (1973) and of Byron and Joachain (1973b). It is also worth noting that these two approaches yield forward scattering amplitudes which are in good agreement with the analysis of Bransden and McDowell (1970), based on forward dispersion relations.

We now describe briefly a few inelastic electron-helium processes. We show in Fig. 13 the differential cross section for the process $e^- +$ He $(1^1S) \rightarrow e^- +$ He (2^1S) at an incident electron energy of 200 eV. The experimental data are those of Vriens, Simpson and Mielczarek (1968), as renormalized by Chamberlain, Mielczarek and Kuyatt (1970). The various theoretical predictions shown are those of the first Born approximation, of the second Born calculation performed by Woollings and McDowell (1972), of the eikonal DWBA method (Joachain and Vanderpoorten, 1973b) and of the four-channel calculations of Berrington, Bransden, and Coleman (1973). In particular, Berrington et al. show that the $2^1S - 2^1P$

coupling, which they take into account explicitly, strongly influences the angular distribution in the forward direction and brings it into agreement with the experimental data. The first Born and eikonal DWBA results, on the contrary, are too low at small scattering angles.

Let us now consider the excitation process $e^- + He(1^1S) \rightarrow e^- + He(2^1P)$. In this case the strong 1^1S - 2^1P coupling completely dominates and there is no appreciable effect arising from the neglect of the 2^1S - 2^1P coupling. We should therefore expect in this case good results from the eikonal DWBA method (Joachain and Vanderpoorten, 1973b). This is confirmed by the examination of the total cross sections shown in Fig. 14. We also note that the eikonal calculations of Byron (1971) are in good agreement with the experimental results of de Jongh and van Eck (1971) and of Donaldson, Hender and McConkey (1972).

To conclude this section, let us examine the excitation of triplet states of helium by electron impact, taking as a particular example the reaction

$$e^- + He(1^1S) \rightarrow e^- + He(2^3S) \tag{4.26}$$

As we already mentioned in Section IV. 1, this process is a pure rearrangement ("knock-out" or exchange) collision provided that very small spin-dependent interactions are neglected. Although the reaction (4.26) received a large amount of attention (Joachain and Mittleman, 1965; Ochkur and Brattsev, 1965; Bell, Eissa and Moiseiwitsch, 1966, Miller and Krauss, 1968, Kang and Choi, 1968, Joachain and Van den Eynde, 1970), no satisfactory explanation was found for the forward peaking observed (Vriens, Simpson and Mielczarek, 1968; Chamberlain, Mielczarek and Kuyatt, 1970) in the differential cross section at small angles and for incident electron energies ranging from 100 eV to 225 eV. In particular, the first Born and the Ochkur approximations badly fail in this case, as can be seen from the examination of Fig. 15. The reasons for this failure have been given by Byron and Joachain (1972) who have also performed many-body eikonal calculations (using the Monte-Carlo integration method) for the reaction (4. 26). Their results, shown in Fig. 15 are seen to be in good agreement with experiment. Given the interest concerning the theoretical treatment of rearrangement collisions, more experiments on the reaction (4. 26) would be very desirable.

V. HIGH-ENERGY HADRON-DEUTERON COLLISIONS

The topic at hand is a vast one which we shall discuss not in general terms but with the intent of illustrating the applicability of multiple scattering expansions to a practical problem. We must needs be selective in our coverage, so while we will describe some of the complexities of high-energy scattering in detail, we shall have to ignore others. For the reader whose primary concern is hadron-deuteron scattering we therefore list a few of the issues we have not treated, together with one or two modern references which provide access to the literature.

(i) scattering in the resonance region (Landau, 1971).

(ii) neutron cross sections (Musgrave, 1971; Julius, 1972).

(iii) high-momentum spectators (Musgrave, 1971).

(iv) Fermi motion (Atwood and West, 1972; West, 1972).

(v) Presence of isobars in the deuteron wave function (Kerman and Kislinger, 1969; Nath, Weber, and Kabir, 1971).

1. High-Energy Hadron-Nucleus Scattering.

We consider a hadron X of initial laboratory energy E and momentum $\underline{k}$ incident on a nucleus of mass number A. We use units such that $\hbar = c = 1$. We assume that the incident particle travels much faster than the characteristic nuclear velocities, and that it interacts with the target nucleons via two-body spin-independent interactions. (The generalization to spin-dependent interactions will be discussed briefly below.) Furthermore, we shall only consider for the moment small angle elastic or "mildly" inelastic collisions. The transition amplitude from an initial nuclear state $|0\rangle$ to a final nuclear state $|m\rangle$ is then given (in the laboratory system) by the Glauber expression (3.51), namely (Glauber, 1959)

$$F^G_{m0} = \frac{ik}{2\pi}\int d^2b\, e^{i\underset{\sim}{q}\cdot b} <m | \Gamma_{tot}(b, b_1, \ldots b_N) | 0> \qquad (5.1)$$

where $\underset{\sim}{q}$ is the laboratory momentum transfer and Γ_{tot} may be written as [see Eq. (3.54)]

$$\Gamma_{tot} = \sum_{j=1}^{A} \Gamma_j - \sum_{j \neq \ell} \Gamma_j \Gamma_\ell + \ldots (-1)^{A-1} \prod_{j=1}^{A} \Gamma_j \qquad (5.2)$$

The multiple scattering series (5.2) which contains A terms, has been particularly useful to analyze the scattering of high-energy hadrons by light nuclei. We shall return shortly to this point in connection with hadron-deuteron scattering. We note here that according to Eq. (2.38), generalized to a high-energy two-body collision, the quantity

$$f_j(\underset{\sim}{q}) = \frac{ik}{2\pi}\int d^2b\, e^{i\underset{\sim}{q}\cdot b}\, \Gamma_j(b) \qquad (5.3)$$

is just the eikonal (laboratory) two-body scattering amplitude of the incident particle X by the j th nucleon. Hence, using Eqs. (5.1) and (5.2), we immediately deduce that the "single scattering" or "impulse" approximation, obtained by retaining only the terms linear in Γ_j on the right of Eq. (5.2), leads to the hadron-nucleus scattering amplitude

$$F_{m0} \simeq \sum_{j=1}^{A} f_j(\underset{\sim}{q}) <m | e^{i\underset{\sim}{q}\cdot b_j} | 0>. \qquad (5.4)$$

In particular, for elastic scattering, and assuming that all the f_j's are identical ($f_1 = f_2 = \ldots f$) we recover the familiar result of the "impulse" approximation, namely

$$\frac{d\sigma_{el}}{d\Omega} \simeq \left(\frac{d\sigma}{d\Omega}\right)_f | S(\underset{\sim}{q}) |^2 \qquad (5.5)$$

where

$$\left(\frac{d\sigma}{d\Omega}\right)_f = |f|^2 \tag{5.6}$$

is the elastic differential cross section for the scattering of the incident particle by a <u>free</u> nucleon, and

$$S(\underset{\sim}{q}) = \sum_{j=1}^{A} <0|e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}_j}|0> \tag{5.7}$$

is the elastic form factor of the target bound state. Since $S(0) = A$, Eq. (5.6) predicts that in the impulse approximation the coherent (elastic) differential cross section for hadron-nucleus scattering is enhanced by a factor A^2 in the forward direction by respect to the corresponding hadron-nucleon cross section. In fact, because hadrons interact strongly with nucleons, multiple collision effects are important in hadron-nucleus collisions. They lead to an A-dependence of the forward differential cross section which increases less rapidly than A^2, although the angular distribution still remains heavily concentrated in the forward direction. This strong forward peaking is the major characteristic of high-energy coherent hadron-nucleus scattering.

The elastic scattering of hadrons by "large" nuclei is conveniently studied by means of the eikonal optical model summarized at the end of Section III. For example, using Eqs. (3.85) and (3.88), together with additional corrections for Coulomb and target correlation effects, Goldhaber and Joachain (1968) have analyzed the experimental data of Belletini et al. (1966) on high-energy proton scattering by a variety of nuclei. Their analysis includes a study of inelastic collisions which dominate at larger angles. Goldhaber and Joachain have also proposed a simple eikonal DWBA method to deal with coherent production reactions such as

$$\pi + \text{nucleus} \rightarrow A_1 + \text{nucleus} \tag{5.8}$$

or

$$p + \text{nucleus} \rightarrow N^* + \text{nucleus, etc.} \tag{5.9}$$

This formalism has been applied to extract the A_1-nucleon cross section from the analysis of coherent A_1 production in freon (Goldhaber, Joachain, Lubatti and Veillet, 1969).

We shall not pursue further hadron scattering by nuclei other than deuterium. The interested reader will find additional information and references in recent work (Stodolsky, 1966; Drell and Trefil, 1966; Formanek and Trefil, 1967, Bassel and Wilkin, 1967, 1968; Czyz and Lesniak, 1967; Goldhaber and Joachain, 1968; Ross, 1968; Margolis, 1968, Kolbig and Margolis, 1968, Trefil, 1969, Kofoed-Hansen, 1969, Feshbach and Hüfner, 1970, Feshbach, Gal and Hüfner, 1971; Moniz and Nixon, 1971; Bassichis, Feshbach and Reading, 1972) as well as in the review articles of Glauber (1967, 1968), Wilkin (1968) and Czyz (1971).

2. Hadron-Deuteron Scattering in the Glauber Formalism

Let us now concentrate on hadron-deuteron collisions, which have been studied extensively by using the Glauber generalization of the eikonal approximation. We follow here the analysis of Franco and Glauber (1966). The basic formula for elastic and mildly inelastic collisions is still Eq. (5.1), where

$$\Gamma_{tot} = 1 - \exp\{i[\chi_n(\underset{\sim}{b}-\tfrac{1}{2}\underset{\sim}{s}) + \chi_p(\underset{\sim}{b}+\tfrac{1}{2}\underset{\sim}{s})]\}. \quad (5.10)$$

The quantities χ_n and χ_p are phase shift functions contributed respectively by the neutron and the proton, while the vector $\underset{\sim}{s}$ is the projection of the internal relative vector $\underset{\sim}{r}_d$ of the deuteron in the plane of impact parameters. If we define the quantities

$$\Gamma_n(\underset{\sim}{b}) = 1 - \exp[i\chi_n(\underset{\sim}{b})] \quad (5.11)$$

and

$$\Gamma_p(\underset{\sim}{b}) = 1 - \exp[i\chi_p(\underset{\sim}{b})] \quad (5.12)$$

we may write Eq. (5.10) as

$$\Gamma_{tot} = \Gamma_n(\underset{\sim}{b} - \tfrac{1}{2}\underset{\sim}{s}) + \Gamma_p(\underset{\sim}{b} + \tfrac{1}{2}\underset{\sim}{s}) - \Gamma_n(\underset{\sim}{b} - \tfrac{1}{2}\underset{\sim}{s})\,\Gamma_p(\underset{\sim}{b} + \tfrac{1}{2}\underset{\sim}{s}) \quad (5.13)$$

leading to the physical interpretation in terms of single and double scattering, as we expect from the discussion following Eq. (3.54). To analyze this situation in more detail, we note that the functions Γ_n and Γ_p can be expressed in terms of the hadron-neutron and hadron-proton scattering amplitudes f_n and f_p by an approximate two-dimensional Fourier inversion. [See Eq. (5.3).] Thus

$$\Gamma_n(\underset{\sim}{b}) \simeq \frac{1}{2\pi ik}\int d^2\underset{\sim}{q}\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{b}}\, f_n(\underset{\sim}{q}) . \quad (5.14)$$

A similar formula holds for Γ_p. Returning to Eq. (5.1), we now have

$$F^G_{m0} = \langle m \Big| \Big\{ e^{i\frac{1}{2}\underset{\sim}{q}\cdot\underset{\sim}{s}} f_n(\underset{\sim}{q}) + e^{-i\frac{1}{2}\underset{\sim}{q}\cdot\underset{\sim}{s}} f_p(\underset{\sim}{q}) + \frac{i}{2\pi k}\int d^2\underset{\sim}{q}' \; e^{i\underset{\sim}{q}'\cdot\underset{\sim}{s}} f_n(\underset{\sim}{q}' + \tfrac{1}{2}\underset{\sim}{q})\, f_p(-\underset{\sim}{q}' + \tfrac{1}{2}\underset{\sim}{q}) \Big\} \Big| 0 \rangle \tag{5.15}$$

and for elastic scattering

$$F^G_{el} = f_n(\underset{\sim}{q})\, S(\tfrac{1}{2}\underset{\sim}{q}) + f_p(\underset{\sim}{q})\, S(-\tfrac{1}{2}\underset{\sim}{q}) + \frac{i}{2\pi k}\int d^2\underset{\sim}{q}' \; S(\underset{\sim}{q}')\, f_n(\underset{\sim}{q}' + \tfrac{1}{2}\underset{\sim}{q})\, f_p(-\underset{\sim}{q}' + \tfrac{1}{2}\underset{\sim}{q}) \tag{5.16}$$

where $S(\underset{\sim}{q})$ is the form factor of the deuteron ground state, namely

$$S(\underset{\sim}{q}) = \int e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}_d} \,|\psi_o(\underset{\sim}{r}_d)|^2\, d\underset{\sim}{r}_d \tag{5.17}$$

Here $\psi_o(\underset{\sim}{r}_d)$ is the ground state deuteron wave function. The formulae (5.15) and (5.16) clearly justify the interpretation of the collision in terms of single and double scattering processes. The two types of diagrams which contribute to the scattering are shown in Fig. 16. Evidently, these diagrams do not, at this point, have any more content than the formulae (5.15) or (5.16). We shall return to the analysis of diagrams in Section V.3 when dealing with analytic properties of scattering amplitudes.

We may immediately obtain the total hadron-deuteron cross section from Eq. (5.16) by using the optical theorem. Thus, writing $\sigma^d_{tot} = 4\pi \text{ Im } F^G_{el} / k$, one finds that (Franco and Glauber, 1966)

$$\sigma^{d}_{tot} = \sigma^{n}_{tot} + \sigma^{p}_{tot} - \delta\sigma \tag{5.18}$$

where σ^{n}_{tot} and σ^{p}_{tot} are respectively the total hadron-neutron and hadron-proton total cross sections and $\delta\sigma$, the "cross section defect", is given by

$$\delta\sigma = -\frac{2}{k^2}\int S(\underset{\sim}{q})\, R_e\,[f_n(\underset{\sim}{q})\, f_p(-\underset{\sim}{q})]\; d^2\underset{\sim}{q} \tag{5.19}$$

If the average neutron-proton interaction has much larger range than the hadron-nucleon interaction, one can readily derive from Eq. (5.19) the approximate formula

$$\delta\sigma \simeq -\frac{4\pi}{k^2}\, \mathrm{Re}[f_n(0)\, f_p(0)]\;\; \langle \mathbf{r}_d^{-2}\rangle \tag{5.20}$$

where $\langle r_d^{-2}\rangle$ is the inverse square of the neutron-proton distance averaged over the deuteron ground state. Further, if the amplitudes $f_n(0)$ and $f_p(0)$ are purely imaginary ("black nucleons"), one obtains the very simple result (Glauber, 1959)

$$\delta\sigma \simeq \frac{1}{4\pi}\; \sigma^{n}_{tot}\; \sigma^{p}_{tot}\; \langle r_d^{-2}\rangle \tag{5.21}$$

A variety of angular distributions can be derived from Eqs. (5.15) and (5.16). The elastic differential cross section is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{el} = |F^{G}_{el}|^{2}. \tag{5.22}$$

The total scattered intensity is obtained from

$$\left(\frac{d\sigma}{d\Omega}\right)_{sc} = \sum_{m} |F^{G}_{m0}|^{2} \tag{5.23}$$

and can be evaluated by using the closure relation on the deuteron final states $|m\rangle$. Inelastic processes in which the deuteron is dissociated into two free nucleons are calculated from

$$\left(\frac{d\sigma}{d\Omega}\right)_{in} = \left(\frac{d\sigma}{d\Omega}\right)_{sc} - \left(\frac{d\sigma}{d\Omega}\right)_{el} \quad . \tag{5.24}$$

The corresponding total cross sections σ_{el}, σ_{sc}, and $\sigma_{in} = \sigma_{sc} - \sigma_{el}$ are directly obtained by integrating Eqs. (5.22) - (5.24) over the angles, while the "absorption" cross section

$$\sigma_{abs} = \sigma_{tot}^{d} - \sigma_{sc} \tag{5.25}$$

corresponds to all processes where the incident hadron disappears during the collision or reappears with one or several produced particles.

The generalization of these considerations to include the spin and isospin degrees of freedom of the incident particle and the target nucleons has been carried out by several authors (Franco and Glauber, 1966; Wilkin, 1966; Glauber and Franco, 1967; Alberi and Bertocchi, 1968, 1969b). For example, collision processes contributing to charge-exchange scattering by the deuteron in the case of an incident hadron of isotopic spin 1/2 are represented in Fig. 17, whereas in Fig. 18 the double charge-exchange process leading to no net transfer of charge is shown. This last effect, first pointed out by Wilkin (1966), is small relative to the other cross-section corrections. Indeed, if $f_c(\underset{\sim}{q})$ is the charge-exchange amplitude, one obtains now for the cross-section defect, instead of Eq. (5.19) (Glauber and Franco, 1967),

$$\delta\sigma = -\frac{2}{k^2}\ \mathrm{Re}\left\{\int S(\underset{\sim}{q})\,\frac{1}{2}\left[f_p(\underset{\sim}{q})\,f_n(-\underset{\sim}{q}) + f_n(\underset{\sim}{q})\,f_p(-\underset{\sim}{q}) - f_c(\underset{\sim}{q})\,f_c(-\underset{\sim}{q})\right]\,d^2\underset{\sim}{q}\right\}, \tag{5.26}$$

or

$$\delta\sigma = -\frac{2}{k^2}\ \mathrm{Re}\left\{\int S(\underset{\sim}{q})\left[2f_n(\underset{\sim}{q})\,f_p(\underset{\sim}{q}) - \frac{1}{2}f_p^{\,2}(\underset{\sim}{q}) - \frac{1}{2}f_n^{\,2}(\underset{\sim}{q})\right]\,d^2\underset{\sim}{q}\right\}. \tag{5.27}$$

If the hadron-nucleon force range is small compared with the average neutron-proton interaction, one may again approximate

$$\delta\sigma \simeq -\frac{4\pi}{k^2}\ \mathrm{Re}\left[f_n(0)\,f_p(0) - \frac{1}{2}\left[f_n(0) - f_p(0)\right]^2\right]\langle r_d^{-2}\rangle \tag{5.28}$$

which under the assumption of purely imaginary amplitudes $f_n(0)$ and $f_p(0)$ reduces to [compare with Eq. (5.21)]

$$\delta\sigma \simeq \frac{1}{4\pi}\left[\sigma_{tot}^{n}\,\sigma_{tot}^{p} - \frac{1}{2}(\sigma_{tot}^{n} - \sigma_{tot}^{p})^2\right]\ \langle r_d^{-2}\rangle. \tag{5.29}$$

Franco and Glauber (1966) have applied the theory outlined above to a detailed investigation of antiproton-deuteron collisions in the (lab) energy range 0.13 to 17.1 GeV, using various ground-state deuteron wave functions. They assume that at high energies the antiproton-nucleon amplitudes are such that

$$f_{\bar{p}n}(\underset{\sim}{q}) = f_{\bar{p}p}(\underset{\sim}{q}) \equiv f_{\bar{p}N}(\underset{\sim}{q}), \tag{5.30}$$

and can be parameterized as

$$f_{\bar{p}N} = i(k_i\,\sigma_{\bar{p}N}/4\pi)\,e^{-\frac{1}{2}\alpha^2\underset{\sim}{q}^2}. \tag{5.31}$$

Using as input the measured experimental data (Elioff et al., 1962; Galbraith et al., 1965; Czyzewski et al., 1965; Coombes et al., 1958; Armenteros et al., 1960; Foley et al., 1963b; Ferbel et al., 1965) on antiproton-proton collisions, they obtained total and absorption antiproton-deuteron cross sections in good agreement with experiment (Elioff et al., 1962; Galbraith et al., 1965; Chamberlain et al., 1957) and showing an appreciable double scattering effect (see Fig. 19). They also investigated spin-dependent effects and concluded that their influence on the cross-section defect should be small. Franco (1966) has also analysed the antiproton-deuteron elastic angular distribution for small momentum transfers in the region of incident momenta between 2.78 and 10.9 GeV/c. In a subsequent work, Glauber and Franco (1967) studied the reaction

$$K^+ + d \to K^0 + p + p, \tag{5.32}$$

which, together with K^+p collisions, is used to extract information about the K^+n charge-exchange reaction (Butterworth et al., 1965)

$$K^+ + n \to K^0 + p. \tag{5.33}$$

They show that the effect of the charge-exchange correction on the values of the (pn), ($\bar{p}$n), and (K^+n) total cross sections which are obtained indirectly through deuteron measurements is very small for incident hadron momenta above 2 GeV/c.

We now turn to a more detailed analysis of the angular distribution of elastic hadron-deuteron scattering. We start with proton-deuteron elastic scattering, which has been studied in the GeV range by various authors (Harrington, 1964; 1968 a, b; Franco, 1966, 1968b; Franco and Coleman, 1966; Kujawski, Sachs, and Trefil, 1968; Franco and Glauber, 1969). To understand qualitatively the main features of the angular distribution, let us return to Eq. (5.16). We first note from the alternation of sign in Eq. (5.13) that the double scattering term has opposite sign to the single scattering term. In fact, if the amplitudes f_n and f_p were purely imaginary, the double scattering term would completely cancel the contribution of the single scattering amplitude at $-t \simeq 0.5\ (\mathrm{GeV}/c)^2$. The contribution of the single and double scattering terms for such a parametrization of the amplitudes is displayed in Fig. 20, which also shows that the single scattering term dominates near the forward direction. At larger momentum transfers the double scattering term, which decreases much more slowly with increasing q, becomes the dominant contribution to the scattering amplitude.

Let us now analyze more closely the intermediate region of momentum transfers where the single and double scattering terms interfere destructively. Since the proton-neutron and proton-proton scattering amplitudes both have small real parts we do not expect the differential cross section to exhibit a zero, but instead to show a sharp dip in the interference region. This region is therefore of special interest, since it depends delicately upon the phases of the hadron-nucleon amplitudes.

The first experimental data on p-d elastic scattering (Kirillova et al., 1964; Belletini et al., 1965; Zolin et al., 1966; Coleman et al., 1966, 1967) gave encouraging agreement with Glauber's theory. For example, the large-angle measurements at 2.0 GeV (Coleman et al., 1966) confirmed the importance of the double scattering term in the region of four-momentum transfers

$$0.5\ (\mathrm{GeV}/c)^2 \leq -t \leq 1.5\ (\mathrm{GeV}/c)^2 \tag{5.34}$$

and were in good agreement with the theoretical calculations of Franco and Coleman (1966). However, these larger-angle data did not fully cover the important intermediate region. It remained for Bennett et al., (1967) to perform a crucial p d experiment at 1 GeV, which showed agreement with the theory in the small and larger momentum transfer ranges, but displayed only a shoulder (no dip) in the interference region (see Fig. 21). This result was confirmed by measurements at 582 MeV (Boschitz, quoted in Glauber 1969). A similar feature was observed in $\pi^- d$ elastic scattering experiments (Bradamante et al., 1968).

Several suggestions were proposed to understand this apparent paradox: momentum-transfer dependence of the phases of the proton-neutron and proton-proton amplitudes (Bennett et al., 1967), spin effects (Kujawski, Sachs, and Trefil, 1968; Franco, 1968), influence of three-body forces (Harrington, 1968a) or of inelastic intermediate states (Pumplin and Ross, 1968; Alberi and Bertocchi, 1969a; Harrington, 1970). There is one crucial fact though, which leads to the resolution of the puzzle, namely that interference minima are observed in the elastic scattering of protons by the spin-zero nuclei He^4, C^{12}, and 0^{16} (Palevsky, et al., 1967; Boschitz, et al., 1968). It is therefore tempting to associate the absence of the dip with the quadrupole deformation of the spin-one deuteron (Harrington, 1968a).

The wavefunction for a deuteron of spin projection M can be written as

$$\Phi_M(r) = \frac{u(r)}{r} Y_{oo}(\hat{r}) \langle \tfrac{1}{2}\, \tfrac{1}{2}\, m_1\, m_2 \,|\, 1\, M\rangle \chi^p_{m_1} \chi^n_{m_2}$$
$$+ \frac{w(r)}{r} Y_{2,M-m_1-m_2}(\hat{r}) \langle 2\; 1\; M-m_1-m_2\;\; m_1+m_2 \,|\, 1\, M\rangle$$
$$\times \langle \tfrac{1}{2}\, \tfrac{1}{2}\, m_1\, m_2 \,|\, 1\; m_1+m_2\rangle \chi^p_{m_1} \chi^n_{m_2}, \qquad (5.35)$$

where $\chi^p_{m_1}$ and $\chi^n_{m_2}$ are proton and neutron Pauli spinors of projection m_1 and m_2, and a summation over m_1 and m_2 is implicit. The S-wave and D-wave radial wave functions are chosen real and normalized by

$$\int_0^\infty dr\, (u^2+w^2) = 1 \qquad (5.36)$$

In the case of πd scattering, four scattering amplitudes must be considered, when spin is not ignored. In them, one may recognize (Michael and Wilkin, 1968; Sidhu and Quigg, 1973), the contributions from spherical, quadrupole, and magnetic form factors

$$\phi_S(\tfrac{1}{2}q) = \int_0^\infty dr\, j_o(\tfrac{1}{2}qr)\, [\,|u(r)|^2 + |w(r)|^2],$$

$$\phi_Q(\tfrac{1}{2}q) = \int_0^\infty dr\, j_2\,(\tfrac{1}{2}qr)\,[\,2u(r)\,w(r) - |w(r)|^2/\sqrt{2}\,]\,,$$

$$\phi_M(\tfrac{1}{2}q) = \int_0^\infty dr\,\{j_0\,(\tfrac{1}{2}qr)\,[\,|u(r)|^2 - \tfrac{1}{2}|w(r)|^2] + j_2\,(\tfrac{1}{2}qr)\,[\,u(r)w(r)/\sqrt{2} + \tfrac{1}{2}|w(r)|^2\,]\,\}, \tag{5.37}$$

the squares of which are plotted in Fig. 22, for the hard-core model of Reid (1968). In a simple model (Michael and Wilkin, 1968) in which the πN spin-flip amplitude is neglected and the non-flip amplitude is positive imaginary, the contribution of the quadrupole form factor to the differential cross section for πd scattering remains finite at the position of the diffraction zero in the contribution of the spherical form factor, and fills in the dip. This cooperation is displayed in Fig. 23.

Because the pion-nucleon scattering amplitudes are so well known, more detailed calculations have been possible. Alberi and Bertocchi (1969b) reanalyzed the data of Bradamante, et al., (1968) by taking into account the deuteron D-state and using πN amplitudes given by phase shift analyses. Some of their results are shown in Fig. 24, which exhibits impressive agreement between theory and experiment. [For a detailed account of this work, see Bertocchi (1969).] At higher energies the Regge pole fits of Barger and Phillips (1968, 1969) have been exploited by Alberi and Bertocchi (1969b), Michael and Wilkin (1969), and Sidhu and Quigg (1973). These calculations agree very well with the $\pi^- d$ elastic differential cross sections of Fellinger, et al., (1969)

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and Bradamante et al. (1968, 1969, 1970a) for incident pion momenta between 2 and 15.2 GeV/c, in the single-scattering regime and in the region of the break. Typical calculations are shown in Fig. 25 - 27. At larger angles (in the double-scattering regime), the theoretical curves lie systematically above the data. The number of detailed computations which exhibit these features supports the inference of the CERN-Trieste group (Bradamante, et al., 1971) that the disagreement in the double-scattering regime (which is also observed in pd scattering) cannot be ascribed to the uncertainty in our present knowledge of the hadron-nucleon scattering amplitudes.

Similar considerations apply to proton-deuteron elastic scattering. The calculations of Franco and Glauber (1969) are compared with the experimental data at 1 and 2 GeV in Fig. 28. Recent measurements of pd elastic scattering at 9.7, 12.8 and 15.8 GeV/c (Bradamante, et al. 1970b) and at higher energies (Allaby, et al., 1969a, b; Amaldi, et al., 1972) are also in excellent agreement with the theory, except in the double-scattering regime.

A number of authors (Fäldt, 1971; Gunion and Blankenbecler, 1971; Cheng and Wu, 1972; Namyslowski, 1972b) have suggested that the Glauber theory without consideration of deuteron recoil overestimates the overlap integral and hence the cross section in the double-scattering regime. While it is appealing to think that the relative motion of initial- and final-state deuterons should diminish the wavefunction overlap, the magnitude of the correction (as estimated, for example, in a covariant formalism by Namyslowski,1972b) is approximately 20%, whereas the data appear to demand a factor of two suppression. Additional precision experiments seem required before it is worthwhile to take the leap of formulating the entire problem covariantly, with the complications of spin fully included.

Since the scattering amplitude for elastic hadron-deuteron scattering in the intermediate momentum transfer region is dominated by quadrupole transitions between the deuteron S and D states, it is strongly dependent on the relative orientations of the momentum transfer and the deuteron spin. Thus, as Franco and Glauber (1969) remarked and Alberi and Bertocchi (1969b) demonstrated by explicit calculations, interesting effects could appear in experiments involving polarized deuteron targets. Indeed, with such a target, the interference dip can appear or not depending on the particular experimental arrangement, namely on the orientation of the polarization axis. Another interesting experiment using the spin-dependence arising from the D-wave component of the deuteron to produce high-energy aligned deuterons has been proposed by Harrington (1969a). He pointed out that this spin dependence could be studied in a double-scattering experiment in which a high energy deuteron beam is scattered from two hydrogen targets in succession. The experiment was carried out by Bunce, et al. (1972) using the external deuteron beam of the Princeton-Pennsylvania Accelerator at 3.6 GeV/c, a momentum corresponding to p-d scattering at 1.0 GeV/c where the differential cross section has been measured by Bennett, et al. (1967). In a double-scattering experiment in which a deuteron beam is polarized by the first scattering and analyzed by the second scattering, the differential cross section of the second scattering has the azimuthal dependence

$$N(\phi) = N_o \, (1 + A\cos 2\phi + B\cos\phi). \qquad (5.38)$$

The momentum transfer of the second scattering was fixed at $-t_b = (0.23\pm0.016)\ (\mathrm{GeV}/c)^2$, and the azimuthal asymmetry $N(\phi)$ was measured over a range of momentum transfers of the first scattering. The measured values of the parameters A and B are shown in Fig. 29, together with fits based on Glauber theory, in which the D-state probability and real part of the NN scattering amplitude enter as parameters. Thus the experiment allows a Glauber model-dependent method for measuring the real parts of NN amplitudes at high energies.

We also mention the experiments of Carter et al. (1968), who measured π-d cross sections, and of Chase et al. (1969) on inelastic pion-deuteron scattering at 5.53 GeV/c, leading to an outgoing pion plus anything in the final state (missing-mass experiment). The inelastic intensity, calculated from Eq. (5.24), was found to be in good agreement with the data. [See also Hsiung et al. (1968).]

We now turn to a comparison of the Glauber method with the Faddeev-Watson multiple scattering equation. Bhasin (1967) has studied the first four terms of the expansion (3.38) for elastic hadron-deuteron scattering,

$$T_{11}^{(-)} = T_2 + T_3 + T_2 G_0^{(+)} T_3 + T_3 G_0^{(+)} T_2 + \ldots \tag{5.39}$$

As expected, the two first terms on the right reduce to Glauber's single scattering terms if one ignores the dependence of T_2 or T_3 on the energy of the third particle and also assumes the two-body off-the-energy-shell amplitudes to be functions only of the momentum transfer. With these assumptions and additional requirement that $\underset{\sim}{k}_i \simeq \underset{\sim}{k}_f$, the double scattering terms $T_2G_0T_3$ and $T_3G_0 T_2$ also reduce to the Glauber "eclipse" correction. Pumplin (1968) and Bhasin and Varma (1969) have investigated the importance of the off-shell corrections on the double scattering terms. They find that the corresponding effect for proton-deuteron scattering is largest in the interference region between single and double scattering. However, Harrington (1969b) has recently shown that in a potential model the off-energy-shell effects in the double scattering term must cancel the contribution of the remaining part of the multiple scattering series in the high-energy limit. (See also Section V. 3.) It should be noted here that only in high-energy diffraction theory does the multiple scattering series terminate after A terms. In the deuteron case considered here the triple, quadruple, ... terms are small, since they contain at least one (unlikely) backward scattering. Their sum could well annihilate the off-energy-shell contribution to the double scattering term, if the mechanism described by Harrington also works for interactions which cannot be described by potentials.

While we are still discussing the multiple scattering series, it is worth mentioning a recent paper by Kofoed-Hansen (1969), who has pointed out that truncated versions of the Glauber series (5.2) could produce misleading results, since the series is slowly converging in terms of multiplicity. This remark evidently does not apply to the deuteron case--where the multiplicity is two--but it is relevant in cases such as nucleus-nucleus collisions (Franco, 1967, 1970) as well as in quark model or multiple scattering theory of hadron-hadron scattering (Harrington and Pagnamenta, 1967, 1968, 1969; Deloff, 1967; Barnhill, 1967; Schrauner, Benofy and Cho, 1967; Chou and Yang, 1968; Frautschi and Margolis, 1968, Durand and Lipes, 1968).

We now consider briefly the effect of three-body forces in hadron-deuteron collisions. Harrington (1968b) has studied corrections to the Glauber expression due to the scattering of the incident hadron from a pion being exchanged by the two target nucleons. Numerical estimates indicate that such an effect on the total cross section is quite small($<$ 1% at very high energies), but could possibly influence the differential cross section at large momentum transfers.

As we have emphasized above, the Glauber method is at its best for collisions in which the inelasticity is small, in particular for elastic scattering, for which the results in the high-energy small-angle limits are in excellent agreement with the data. Even in that case, however, one should keep in mind that several correction terms, typified by the contribution of inelastic intermediate states (see Fig. 30) should be included in the scattering amplitude. There is no simple way to take into account the contribution of such inelastic intermediate states within the framework of Glauber's method. Fortunately, because of the mass difference, the reaction

$$p + p \rightarrow p + N^{*} \tag{5.40}$$

has a minimum momentum transfer greater than zero, so that two relatively violent scatterings of this type, leaving the deuteron in its bound state, are not likely to occur with high probability compared with the single and double scattering terms discussed before. Such "truly inelastic" corrections have been considered for proton-nucleus scattering by Pumplin and Ross (1968) and for pion-deuteron scattering by Alberi and Bertocchi (1969a) and Harrington (1970). We discuss them further in connection with Gribov's Reggeon calculus approach in Section V. 3. The excellent agreement between conventional Glauber theory and the 19.1 GeV/c pd data of Allaby, et al. (1969) indicates that inelastic corrections are negligible at that momentum.

More serious problems arise when one wants to study coherent production reactions such as

$$\pi + \text{Nucleus} \rightarrow A_1 + \text{Nucleus} \quad (5.41a)$$

$$\pi + d \rightarrow A_1 + d \quad (5.41b)$$

or

$$p + He^4 \rightarrow N^* + He^4, \text{ etc.} \quad (5.41c)$$

We shall return to this question in Section V. 3. We note, however, that existing discussions of unstable hadron-nucleon cross sections ignore the issue of whether an unstable hadron has time to materialize as such before rescattering. Suppose the A_1 to be a normal resonance, and consider reaction (5.41a). Of particular interest is the term which describes the pion interacting with one nucleon and being excited into an A_1 which subsequently scatters from a second nucleon. Does enough time elapse between the excitation and rescattering for the excited pion to pull itself together as an A_1? The simplest estimates (Goldhaber, 1972), stimulated by the recent experiment of Bemporad, et al. (1971) on pion + nucleus → (three or five pions) + nucleus which indicates (after a Goldhaber-Joachain analysis) rather small cross sections for non-resonant three pion and five pion systems on nucleons, suggest that the answer is no.

Coherent production of vector mesons would seem to be a special, and favorable, case since according to the ideas of vector dominance the incoming photon actually exists part of the time as an off-mass-shell vector meson. Some experimental results on the reaction $\gamma d \rightarrow \rho^0 d$ are discussed in Section V.3.

3. Hadron-Deuteron Scattering and Regge Theory

How to calculated Regge cuts (branch cuts in the angular momentum plane) is one of the challenging theoretical problems of the present day, for which no solution seems close at hand. We therefore choose an historical approach to the relation between the Glauber formalism and Regge theory. In this way we shall encounter some of the false steps which have been taken in the past, and try to convey the theoretical atmosphere of the present. Some insight is gained into the connection between diffraction and Regge poles if, following Udgaonkar and Gell-Mann (1962), we understand the shrinkage of the diffraction peak by an optical analogue.

At high energies hadron-hadron scattering is apparently dominated by Pomeranchuk exchange. The X-Y elastic scattering invariant amplitude, which we represent in Fig. 31, has the form for small angles

$$A_{XY}(s,t) = \{i - \cot[\pi\alpha_P(t)/2]\}\ \gamma_X(t)\ \gamma_Y(t)\ s_0 \left(\frac{s}{s_0}\right)^{\alpha_P(t)}, \qquad (5.42)$$

where $s = -(p_X + p_Y)^2$ is the square of the total c.m. energy, $t = -(p_X - p'_X)^2$ is the square of the four-momentum transfer, s_0 is the Regge scale energy-squared, and $\alpha_P(t)$ is the Pomeranchuk trajectory function: $\alpha_P(0) = 1$. The total cross section is given in terms of this amplitude by the optical theorem;

$$\sigma_{total}(s) \approx \frac{1}{s} \mathrm{Im}[A_{XY}(s,0)] = \gamma_X(0)\, \gamma_Y(0). \tag{5.43}$$

Here we have explicitly exposed the factorization property of the pole residues. Let us rewrite (5.42) as

$$A(s,t) = \{i - \cot[\pi\alpha_P(t)/2]\}\, s_0 \left(\frac{s}{s_0}\right)^{\alpha_P(t)} \sigma_{total}(s) \cdot [\gamma_X(t)\, \gamma_Y(t)/\gamma_X(0)\, \gamma_Y(0)] . \tag{5.44}$$

Now assume that the Pomeranchuk trajectory is linear, $\alpha_P(t) = 1 + \epsilon t$, and that the residue functions are slowly varying, so we may set the factor in square brackets equal to 1. Then for small t, we have

$$A(s,t) \approx i\, s\, \sigma_{total}(s)\, e^{\epsilon t \log \left(\frac{s}{s_0}\right)}, \tag{5.45}$$

which exhibits, for $\epsilon > 0$, the shrinkage of the diffraction peak.

We write the partial-wave series for A(s,t)

$$A(s,t) = 8\pi i \sum_{\ell} (2\ell + 1)\, P_\ell\, (\cos\theta_s)\, (1 - e^{2i\delta_\ell}). \tag{5.46}$$

We turn the sum over ℓ into an integral, introduce the impact parameter $b = 2\ell\, s^{-\frac{1}{2}}$, and use $P_\ell\, (\cos\theta_s) \approx P_\ell\, (1 + 2t/2) \approx J_0[b(-t)^{\frac{1}{2}}]$. We then calculate f(s,t), an amplitude such that $\frac{d\sigma}{dt} = |f(s,t)|^2$, which for NN scattering at high energies is $f(s,t) \approx [4s\,(\pi)^{\frac{1}{2}}]^{-1} A(s,t)$, so that

$$f(s,t) = \frac{i}{2(\pi)^{\frac{1}{2}}} \int_0^\infty 2\pi b db \, [1 - S(b,s)] \; J_0[b(-t)^{\frac{1}{2}}]$$

$$= \frac{i}{2(\pi)^{\frac{1}{2}}} \int d^2\underset{\sim}{b} [1 - S(\underset{\sim}{b},s)] \; e^{i\underset{\sim}{b}\cdot\underset{\sim}{q}} \qquad , \qquad (5.47)$$

where the transmission coefficient $S(\underset{\sim}{b},s) = e^{2i\delta_\ell}$ and $q^2 = -t$. In the exponential approximation(5.45) Fourier inversion gives the absorption coefficient [now $\sigma \equiv \sigma_{total}(s)$]

$$1 - S(\underset{\sim}{b},s) \approx \frac{\sigma}{8\pi} [\epsilon \log(\frac{s}{s_0})]^{-1} \exp\{-b^2/[4\epsilon \log(\frac{s}{s_0})]\}. \qquad (5.48)$$

Evidently the effective radius-squared (the value of b^2 for which the absorption coefficient is $1/e$ times its value at $b = 0$) is $4\epsilon \log(\frac{s}{s_0})$, which increases logarithmically with s. Likewise the transparency, which we define as $[1 - S(b = 0, s)]^{-1}$, is logarithmically increasing with s because of the factor $\epsilon \log(\frac{s}{s_0})$. Finally we find that the elastic cross section

$$\sigma_{el} = \int d^2\underset{\sim}{b} \, |1 - S(\underset{\sim}{b},s)|^2 \approx \frac{\sigma^2}{32\pi \, \epsilon \log(\frac{s}{s_0})}, \qquad (5.49)$$

tends to zero as $s \to \infty$.

Let us describe a nucleus approximately as a composite system specified by a wave function referring to the individual coordinates of the constituent nucleons. Assuming that high-energy NN scattering is controlled by Regge poles, we compute the amplitude for high-energy

N-d scattering. The probability distribution $|\psi|^2$ of the nucleon positions is integrated over the beam direction (z coordinates) to give a probability distribution $P(\underset{\sim}{b}_1, \underset{\sim}{b}_2)$ of two-dimensional vectors $\underset{\sim}{b}_i$. Then the transmission coefficient for the deuteron, $S_d(\underset{\sim}{b}, s)$, is just the averaged product of the transmission coefficients for the constituent nucleons:

$$S_d(\underset{\sim}{b}, s) = \int d^2 b_1 \int d^2 b_2 \; P(\underset{\sim}{b}_1, \underset{\sim}{b}_2)\, S(\underset{\sim}{b} - \underset{\sim}{b}_1, s)\, S(\underset{\sim}{b} - \underset{\sim}{b}_2, s). \quad (5.50)$$

Now we take the deuteron c.m. as the origin so that $\underset{\sim}{b}_1 = \frac{1}{2}\underset{\sim}{\rho}$, where $\underset{\sim}{\rho}$ is the two-dimensional relative coordinate. Let the wave function--ignoring spin--be $\psi(\underset{\sim}{\rho}, z)$ and define

$$G(p^2) = \int_{-\infty}^{\infty} dz \int d^2\rho \, |\psi(\underset{\sim}{\rho}, z)|^2 \, e^{i\underset{\sim}{p}\cdot\underset{\sim}{\rho}}. \quad (5.51)$$

Then we get for the scattering amplitude and total cross section

$$f_{Xd}(s,t) = \frac{i}{4(\pi)^{\frac{1}{2}}} \Bigg\{ 2\sigma \; G(-t/4)\, B(t) \left(\frac{s}{s_0}\right)^{\alpha(t)-1} \quad (5.52)$$

$$- \frac{\sigma^2}{8\pi^2} \int d^2 p \; G(p^2)\, B\left(-(\tfrac{\underset{\sim}{q}}{2} - \underset{\sim}{p})^2\right) B\left(-(\tfrac{\underset{\sim}{q}}{2} + \underset{\sim}{p})^2\right)$$

$$\times \left(\frac{s}{s_0}\right)^{\alpha\left(-(\frac{\underset{\sim}{q}}{2}+\underset{\sim}{p})^2\right) + \alpha\left(-(\frac{\underset{\sim}{q}}{2}-\underset{\sim}{p})^2\right) - 2} \Bigg\}$$

and

$$\sigma^{tot}_{Xd} = 2\sigma - \frac{\sigma^2}{8\pi^2} \, \mathrm{Re} \int d^2 \underset{\sim}{p} \; G(p^2)\, [B(-p^2)]^2 \left(\frac{s}{s_0}\right)^{2\alpha(-p^2)-2}, \quad (5.53)$$

where $B(t) = \{1 + i \cot[\pi\alpha(t)/2]\} s_0 \, \gamma_X(t)\, \gamma_Y(t)/\gamma_X(0)\, \gamma_Y(0)$ is the Regge

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residue function.

In addition to the Pomeranchuk pole term, with a coefficient twice as large in the forward direction as in the NN case, there is an eclipse term which corresponds to a continuous "smear" of Regge poles, i.e. to a Regge cut with branch point at

$$\alpha_c = 2\alpha(t/4) - 1. \qquad (5.54)$$

This is the result of Udgaonkar and Gell-Mann (1962). At very high energies, the eclipse term at t = 0 vanishes like $1/\log(s/s_0)$ and $\sigma^{tot}_{Xd} \rightarrow 2\sigma$. [See also Gribov, Ioffe, Pomeranchuk, and Rudik (1962).] This is sensible because, as we saw above, the nucleons become very transparent at high energies. For intermediate energies, the eclipse term can be identified with Glauber's.

Abers et al. (1966) observed that from the point of view of Feynman graphs the double scattering term contains no Regge cut, so the validity of the result of Udgaonkar and Gell-Mann and, by extension, of Glauber theory at high energies is questionable. To compress this discussion somewhat we draw from a recent lecture by Wilkin (1969). We may represent the Glauber terms graphically as the impulse (or single scattering) terms of Figs. 32a, b and the eclipse (or double scattering) term of Fig. 32c. Regarded as a Feynman diagram, the double scattering graph has no Regge cut, because the off-mass-shell part of the loop integral cancels the Regge cut from the on-mass-shell contribution which is obtained by replacing the propagator by a delta function. Thus it contributes asymptotically only as s^{-3}, not as $s/\log s$, which is given by the Glauber formula. A general Feynman diagram as in Fig. 33 has a j-plane branch cut on the physical sheet only if both the left hand and the right hand blobs have nonzero third double spectral functions [$\rho_{su}(s,t)$] in the t-channel sense. In other words, crossed lines are required on both sides of the graph; the simplest diagram with a Regge cut appears in Fig. 34 [cf. Mandelstam (1963), Wilkin (1964)]. Such a result must be a source of embarrassment either for Glauber theory as embodied in the calculation of Udgaonkar and Gell-Mann or for Feynman graphs, if not for both. On the one hand Feynman diagrams are "fundamental" and therefore to be believed. On the other, Glauber theory has been checked experimentally for energies up to a few GeV.

One may try to circumvent the difficulty by imputing to the projectile hadron an internal structure which includes a cross (e.g., Fig. 34b) and claiming that the compositeness of hadrons restores the Regge cut. Such a calculation was performed by Abers et al. (1966), who thereby proposed to replace the Glauber eclipse with a complicated expression dependent upon the internal structure of the projectile. Assigning a particular internal structure to the projectile seems artificial, especially when the imputed structure may be absent. As Quigg (1970) emphasizes, the statement $\rho_{su} \neq 0$ is equivalent to the statement that the projectile has definite (s-channel) signature. To the extent that exchange degeneracy is exact, hadrons do not have definite signature and the cross, artificial or not, simply does not correspond to physics. Under the assumption that duality diagrams are meaningful for Reggeon-hadron scattering, Finkelstein (1971) derived a selection rule for Regge cuts which makes more precise the conflict between arbitrary imputed structure and exchange degeneracy. This phenomenological argument provides strong circumstantial evidence against the imputed structure Feynman graph approach.

Landshoff (1969) has estimated the energy at which the Glauber theory result (the Regge cut of Udgaonkar and Gell-Mann) ceases to be valid numerically under the assumption that the relevant amplitude is given by the Feynman graph of Fig. 32c, without assigning any structure to the projectile. As the deuteron is very lightly bound, the critical laboratory energy at which the Glauber theory should break down is very large. On the basis of heuristic arguments about the off-mass-shell behavior of scattering amplitudes, Landshoff estimates

$$E_{\text{critical}} \approx m_{\text{projectile}} (M_{\text{nucleon}} / \text{Deuteron Binding Energy})^{\frac{1}{2}} . \tag{5.55}$$

For incident nucleons this is about 20 GeV. Thus while the Feynman diagram considered has no cut in the j plane its numerical properties are quite similar over a wide range of energy to those of the Glauber eclipse term.

Further doubt has been cast upon the simple diagram approach by a potential theory calculation of Harrington (1969b). In Glauber theory the amplitude for scattering from a potential V is given by [see Section II. 3]

$$f(q) = \frac{k}{2\pi i}\int d^2\underset{\sim}{b}\, e^{i\underset{\sim}{q}\cdot\underset{\sim}{b}}\, [e^{i\chi(\underset{\sim}{b})} - 1], \tag{5.56}$$

where

$$\chi(\underset{\sim}{b}) = -\frac{1}{v}\int_{-\infty}^{\infty} dz\, V(\underset{\sim}{b}, z). \tag{5.57}$$

We let $\underset{\sim}{q} \equiv \underset{\sim}{k} - \underset{\sim}{k}'$ and invert the Fourier integral (5.56). Thus

$$\tilde{f}(\underset{\sim}{b}) = \frac{1}{2\pi}\int d^2\underset{\sim}{q}\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{b}}\, f(\underset{\sim}{q}). \tag{5.58}$$

In momentum space we have

$$\tilde{V}(\underset{\sim}{p}) = \frac{1}{(2\pi)^3}\int d^3\underset{\sim}{X}\, e^{i\underset{\sim}{p}\cdot\underset{\sim}{X}}\, V(\underset{\sim}{X}), \tag{5.59}$$

and the phase shift expressed in terms of $\tilde{V}$ is

$$\chi(\underset{\sim}{b}) = -\frac{2\pi}{v_i}\int d^2\underset{\sim}{q}\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{b}}\, \tilde{V}(\underset{\sim}{q}). \tag{5.60}$$

We expand the integrand of (5.56) in powers of $i\chi(\underset{\sim}{b})$,

$$f = \frac{k}{2\pi i}\int d^2\underset{\sim}{b}\, e^{i\underset{\sim}{q}\cdot\underset{\sim}{b}} \sum_{n=1}^{\infty} \frac{[i\chi(\underset{\sim}{b})]^n}{n!}, \tag{5.61}$$

and substitute (5.60) into (5.61) to obtain

$$f = -2\pi i\, k \sum_{n=1}^{\infty} \frac{1}{n!}\int d^2\underset{\sim}{q}_1 \ldots \int d^2\underset{\sim}{q}_{n-1} \left(\frac{-2\pi i}{v_i}\right)^n \times \tilde{V}(\underset{\sim}{q}_1)\, \tilde{V}(\underset{\sim}{q}_2) \ldots \tilde{V}(\underset{\sim}{q}_{n-1})\, \tilde{V}(\underset{\sim}{q} - \sum_{i=1}^{n-1} \underset{\sim}{q}_i). \tag{5.62}$$

This represents an infinite sum of ladder graphs in which the Feynman loop integrals are integrated only over transverse momentum components. We can reexpress (5.56) in terms of the Born amplitude

$$f_B(\underset{\sim}{q}) = -2\pi^2 \tilde{V}(\underset{\sim}{q}), \tag{5.63}$$

$$f = -2\pi i\, k \sum_{n=1}^{\infty} \frac{1}{n!} \int d^2 \underset{\sim}{q}_1 \ldots \int d^2 \underset{\sim}{q}_{n-1} \times \left(\frac{i f_B(\underset{\sim}{q}_1)}{v_i \pi} \right) \cdots \left(\frac{i f_B(q - \sum_{i=1}^{n-1} \underset{\sim}{q}_i)}{v_i \pi} \right) \tag{5.64}$$

Thus we have a prescription for calculating the "absorptive corrections" to any Born term f_B. Wilkin next applies these rules to πd scattering to give some intuitive background to Harrington's result. First notice that the vertex d $\rightarrow$ np is merely a deuteron wave function which we write in momentum space as $\phi(\underset{\sim}{p})$. If the πp amplitude of Fig. 32b is the Born term $f_B^p(\underset{\sim}{q})$ we get

$$\int d^3 q' \;\; \phi(q/4 - q')\, \phi(q/4 + q')\, f_B(q), \tag{5.65}$$

which is the expected result. It is straightforward to verify that the right answer is obtained for Fig. 32c.

Now consider the graphs in Fig. 35. Remarkably, both of these give the same answer,

$$\int d^3 \underset{\sim}{p} \int d^2 \underset{\sim}{q}_1 \;\; \phi(\underset{\sim}{p})\, \phi(\underset{\sim}{p} + \underset{\sim}{q}_1)\, f_B{}^n\, (\underset{\sim}{q}_1 - \underset{\sim}{q}/2) \times \int d^2 \underset{\sim}{q}_2\, f_B{}^p(\underset{\sim}{q}_2)\, f_B{}^p(\underset{\sim}{q}/2 + \underset{\sim}{q}_1 - \underset{\sim}{q}_2), \tag{5.66}$$

which is recognizable as part of the Glauber multiple scattering term expanded in a Born series. Thus the Glauber theory includes triple scattering terms such as those in Fig. 35. Notice that the ordering of the πp and πn potential interactions does not affect the contribution of the graph. This is true for any complicated graph, as can be proved from the rules obtained above. It is then a basic property of Glauber theory that the order in which the interactions take place does not matter. A picturesque explanation of this fact [Wilkin (1969)] is that in deriving Glauber theory it is always assumed that the incident energy is large and any changes are very small. Complementary to this certainty in energy is an uncertainty in time: it is impossible to tell which interaction takes place first and hence there is a commutativity among the several scatterings. Glauber theory exploits this independence of time order by lumping all the πp interactions together at one end of the (Glauber, not Feynman!) diagram and pushing all the πn interactions to the other end.

Harrington's calculation goes further. Employing the Faddeev multiple scattering series (cf. Sec. III. 2 of this review) he proves that in the high energy limit and in the Glauber approximation the off-shell contribution to the double scattering term is canceled by the higher order terms in the series. The proof consists in observing that in the high-energy limit the scattering amplitude is given by the Glauber approximation

$$T \sim T_{\text{Glauber}} = \sum_n T^{(n)}_{\text{Glauber}}, \qquad (5.67)$$

where $T^{(n)}_{\text{Glauber}}$ is $T^{(n)}$ after the Glauber approximations have been made. If we break the linearized propagator into its δ-function [δ] and principal value [P] (off-mass-shell) parts and correspondingly separate $T^{(2)}_{\text{Glauber}}$ as

$$T^{(2)}_{\text{Glauber}} = T^{(2)}_{\text{Glauber},\,\delta} + T^{(2)}_{\text{Glauber},P}, \qquad (5.68)$$

then

$$T_{\text{Glauber}} = T^{(1)}_{\text{Glauber}} + T^{(2)}_{\text{Glauber},\,\delta}. \qquad (5.69)$$

Thereby it follows that in the high-energy limit the off-shell contribution to $T^{(2)}$ must be canceled by the higher-order terms in the multiple scattering series

$$T_P^{(2)} + \sum_{n=3}^{\infty} T^{(n)} \sim T^{(2)}_{\text{Glauber},\,P} + \sum_{n=3}^{\infty} T^{(n)}_{\text{Glauber}} = 0. \qquad (5.70)$$

It is not known whether this exact cancellation carries over to the relativistic domain, but the likelihood that more complicated diagrams will continue to be important means that the use of a few Feynman graphs to debunk (or derive!) Glauber is a very dubious procedure. There is a lesson here for Regge cut calculations in nonnuclear hadron-hadron scattering as well. [We do not pursue nondeuteron scattering any further here, but for the connection between multiple scattering and Regge cuts see the discussion by Jackson (1970).]

We now turn to the question of singularities in the Mandelstam variables. We shall not dwell on the analytic structure of the hadron-deuteron scattering amplitude in the momentum variables, for we are able to refer the reader to the elegant review by Ericson and Locher (1969) on hadron-nucleus forward dispersion relations. In the language of S-matrix theory, the lightly bound structure of the deuteron is evidenced through the existence of anamalous threshold singularities (so called because they cannot be discerned in straightforward fashion from unitarity) in $d \rightarrow ab$ Regge residue functions [Karplus et al. (1958)]. A rather complete discussion of the singularities of the dpn Regge residue function has recently been given by Lee (1968). Here we content ourselves with recalling for the reader what anomalous singularities are, by giving an intuitive discussion due to Bohr (1960).

Consider the virtual process $d \rightleftarrows np$. The deuteron is stable in the usual sense because $M_d < M_p + M_n$. For states below threshold, with energies $|\omega_i| < M_i$, a virtual decay can take place if all the particles have positive imaginary momenta ($+i\kappa$) in the z direction, say. The four-momentum vector of a particle with imaginary three-momentum is Euclidean: $M^2 = \omega^2 + \kappa^2$. The energy momentum conservation equation can be represented geometrically by a triangle in the ω-κ plane as in Fig. 36. For the virtual decay to occur all the energies ω and pseudomomenta κ must be positive, which means the triangle will close if $M_d^2 > M_p^2 + M_n^2$. Hence an anomalous singularity will occur for the deuteron because the deuteron mass satisfies

$$(M_p + M_n)^2 > M_d^2 > M_p^2 + M_n^2. \tag{5.71}$$

For the deuteron this anomalous threshold lies very near the physical region, at

$$t_o = 4M_p^2 - \frac{(M_d^2 - M_p^2 - M_n^2)^2}{M_d^2} \approx 0.03(\mathrm{GeV}/c)^2. \tag{5.72}$$

In most phenomenological studies the full complications of kinematics (in particular, of the anomalous threshold) have been ignored. As an example we cite the analysis of coherent $K^*(890)$ production $Kd \to K^*d$ at 4.5 GeV/c of Eisner et al. (1968), in which the deuteron is treated as a structureless spin-1 object. Typically statistics have been so low that more sophisticated analysis would be unwarranted. For example, see Buchner et al. (1969) for coherent K^* production at 3 GeV/c. Alberi and Bertocchi (1969a) estimated the contribution of inelastic intermediate meson states in $\pi d \to \pi d$. Again the subtleties of kinematics were ignored as the Regge pole parametrization was used to give the Phragmén-Lindelöff theorem connection between asymptotic energy dependence and the phase of an amplitude. Given the success of theories for $\pi d \to \pi d$ which take proper account of spin [cf. V.2] , the corrections due to inelastic intermediate states are likely to be small. An exception to the general rule is the paper by Barger and Michael (1969) in which the full gore of Lee's kinematics is applied to $pp \to \pi^+ d$, despite the relative absence of data.

Having analyzed Glauber theory in the J-plane and the singularities in the Mandelstam variables, we now consider a modification of the theory recently proposed by Gribov (1969b). This author has argued that for incident momenta $\gtrsim$ 10 GeV/c the screening effect changes markedly as inelastic rescattering becomes competitive with the elastic rescattering responsible for the conventional Glauber screening. Similar proposals have been advanced on intuitive or phenomenological grounds by Pumplin and Ross (1968), by Alberi and Bertocchi (1969a), and by Harrington (1970). Gribov's proposed modification of Glauber theory has its roots in his earlier work on a Reggeon calculus (Gribov, 1967) and on the question of the vanishing of the eclipse term at very high energies (Gribov, 1969a). The essence of the suggestion is that inelastic scattering leading not only to discrete resonances, but also to continuum excitation, be taken into account in the computation of rescattering corrections. These inelastic intermediate states are indicated in Fig. 37; their contributions are to be evaluated as usual, by putting the intermediate states on the mass shell.

It is straightforward to apply these ideas to an evaluation of the πd total cross section defect at high energies. Neglecting spin and assuming all production amplitudes to be purely positive imaginary, Gribov (1969b) writes the inelastic screening correction as

$$\delta\sigma_{\text{inelastic}} \approx 2\sum_{\ell}\int dt\ \rho(4t)\, d\sigma_{\ell}(t)/dt, \qquad (5.73)$$

where $\rho(t)$ is the deuteron form factor and $d\sigma_{\ell}/dt$ is the differential cross section for production of the ℓ^{th} intermediate state. As data becomes available on the inclusive reaction

$$\pi p \to p + \text{anything} \qquad (5.74)$$

[See, for example, Antipov, et al., 1972] it may prove useful to recast (5.73) as

$$\delta\sigma_{\text{inelastic}} \approx 2\int d(\mathcal{M}^2/s)\int_{-\infty}^{t_{\min}(\mathcal{M}^2)} dt\ \rho(4t)\,\frac{d\sigma}{d(\mathcal{M}^2/s)dt} \qquad (5.75)$$

where $d\sigma/d(\mathcal{M}^2/s)dt$ is the inclusive cross section to produce a proton recoiling against missing mass $\mathcal{M}$.

Several attempts have been made to estimate inelastic screening effects on the basis of (5.73). Gurvits and Marinov (1970) predicted that inelastic effects should diminish above 20 GeV/c. However, their conclusion was based on the identification of a decreasing experimental cross section as "diffractive," and hence with a purely imaginary amplitude, in conflict with analyticity, and should be disregarded. Kancheli and Matinyan (1970), employing the triple-Regge techniques introduced by Kancheli (1970), traced qualitatively the energy dependence of the eclipse term. They found that, with the onset of inelastic rescattering the eclipse term increases until the inelastic screening reaches its asymptotic limit, then decreases as the conventional Glauber term diminishes à la Udgaonkar, Gell-Mann, Gribov, Ioffe, Pomeranchuk, and Rudik, and approaches a constant limit given solely by inelastic screening. This is in accord with the expectations of Gribov (1969b).

More recently, Sidhu and Quigg (1973) have given a quantitative estimate of the inelastic screening to be expected at high energies. They included as intermediate states all those multipion states which may be reached from the incident pion by diffractive excitation. For simplicity the differential cross sections are parametrized as exponentials

$$d\sigma_\ell / dt = A_\ell \sigma_\ell \exp\left[A_\ell \sigma_\ell (t - t_\ell(p_{lab}))\right], \qquad (5.76)$$

where t_ℓ is the minimum squared momentum transfer required to produce the state ℓ from an incident beam of momentum p_{lab}, and A_ℓ is the slope of the differential cross section. If the deuteron form factor is approximated by an exponential as well, $\rho(t) = e^{\frac{1}{4}at}$, one may simplify (5.73) to

$$\delta\sigma_{inelastic} \approx 2 \sum_\ell A_\ell \sigma_\ell (a + A_\ell)^{-1} \exp\left[a t_\ell(p_{lab})\right]. \qquad (5.77)$$

They took as intermediate states all channels containing an odd number of pions (> 1) and assigned them the cross section suggested by the Nova model for inclusive distributions (Jacob and Slansky, 1972). Choosing $A_\ell = 2.5(\mathrm{GeV}/c)^{-2}$ for every ℓ, they computed the inelastic screening contributions shown in Fig. 38, which they estimate reliable within a factor of two in magnitude. The energy dependence is in agreement with the qualitative description given by Kancheli and Matinyan (1970).

Recent measurement of the $\pi^{\pm}p$ total cross sections reveal several interesting features. Unlike the $\pi^{\pm}p$ total cross sections [Fig. 39], which remain constant above 30 GeV/c, the πd cross sections [Fig. 40] continue to fall. To the extent that the πp cross sections are constant, the decrease of the πd cross sections must be laid to inelastic screening corrections. Gorin, et al. (1972) determined the amount of screening directly from the high-energy data as

$$\delta\sigma = \sigma_t(\pi^+p) + \sigma_t(\pi^-p) - \tfrac{1}{2}[\sigma_t(\pi^+d) + \sigma_t(\pi^-d)]. \quad (5.78)$$

Their results are shown in Fig. 41 together with the corresponding results of Galbraith, et al. (1965). The newly-measured screening corrections are clearly increasing with energy; the Serpukhov points are well-fitted by the form $\delta\sigma = (1.39 + 0.004[p_{lab}/(1 \text{ GeV/c})]^{1.05})$mb. Taken literally, they seem to indicate the presence of additional screening corrections, above and beyond those predicted by Glauber theory, of roughly the magnitude that Sidhu and Quigg (1973) found plausible in Gribov's (1969b) formulation. Less convincing evidence for a similar effect in pd scattering is shown in Fig. 42, compiled by Kreisler, et al. (1968).

Following the lead of Kancheli and Matinyan (1970), Quigg and Wang (1973) have calculated the πd total cross section defect using as input a triple-Regge analysis of the reaction $\pi^- p \rightarrow p$ + anything published by Paige and Wang (1972). In this way the phases of amplitudes are prescribed by the Regge pole signature factors, and need not be assumed. The results of their calculation (cf. Fig. 41) are in remarkable agreement with the trend of the Serpukhov data, and differ markedly from the conventional Glauber theory prediction at high energies. Indeed it does not seem too much to hope that deuteron corrections can provide an important consistency check on the Reggeon calculus program in which Gribov vertices are extracted from data on inclusive reactions.

To conclude this section, we shall now present a few remarks concerning the experimental situation. As we have indicated in the introduction to this review of high-energy hadron-deuteron scattering, Glauber theory has been tested and refined extensively for elastic hadron-deuteron collisions. Such detailed comparison of theory with experiment has not yet been made in inelastic reactions, and we therefore wish to close by making some simple remarks about inelastic scattering. Little is known about the catastrophic case in which the deuteron is broken up and one of the constituent nucleons is transformed into a nucleon resonance or a hyperon. A purely experimental investigation of great value is the comparison of N^* production cross sections off deuterons with the corresponding cross sections off protons. For example, examination of

$$K^+ d \to K^o \Delta^{++} n_s \text{ vs } K^+ p \to K^o \Delta^{++} \tag{5.79}$$

will reveal whether the neutron is truly a spectator or not. This kind of information is needed for one critically to assess the evidence for "exotic" I = 2 exchange reported in a comparison of $\gamma p \to \pi^\pm \Delta$ with $\gamma d \to \pi^\pm \Delta\, n_s$. [See the discussion by Diebold (1969).] One such comparison has been published by Buchner, et al. (1971) who claim that at 2.97 GeV/c the differential cross section for $K^+ d \to K^{*o} \Delta^{++} n_s$ is not distinguishible from the differential cross section for $K^+ p \to K^{*o} \Delta^{++}$. In their data the impulse approximation seems completely adequate.

Backward hadron-deuteron scattering is a case in which the Glauber approximation would presumably break down. The most straightforward reaction is pd → dp, for which Bertocchi and Capella (1967) proposed a double scattering mechanism with nucleon exchange which was in satisfactory agreement with the data of Coleman, et al. (1966). No single (known) particle exchange is allowed in $\pi d \to d\pi$, so any explanation of this reaction will suffer all the ambiguities of exotic Regge cut box graphs for hadron-hadron scattering.

Coherent excitation of the projectile seems a more tractable problem theoretically, and several experiments have been proposed (Bertocchi and Caneschi, 1967; Formanek and Trefil, 1967) as means to unstable hadron-nucleon cross sections. Of these we mention in particular

$$\begin{aligned}
&\text{(i)} \quad \pi d \to A_1 d, \\
&\text{(ii)} \quad pd \to N^*(1688)d, \\
&\text{(iii)} \quad Kd \to Qd, \\
&\text{(iv)} \quad \gamma d \to \rho^o d.
\end{aligned} \tag{5.80}$$

All of these final states may be obtained by vacuum exchange from the initial states. Using the multiple scattering formalism, we can formulate the problem to show explicitly what is to be learned from this class of experiments.

For a general coherent production

$$X + d \to X^* + d \tag{5.81}$$

we generalize the multiple scattering expansion (3.38) in an obvious way to write

$$\begin{aligned}
T = T_p + T_n + E_{X^*n}\, G_{X^*} T_p + E_{X^*p}\, G_{X^*}\, T_n + T_n + T_n\, G_X\, E_{Xp} + \\
+ T_p\, G_X\, E_{Xn} + \ldots ,
\end{aligned} \tag{5.82}$$

where E_{ij} describes the elastic scattering of particules i and j and T_k is the amplitude corresponding to the process $Xk \to X^* k$. For applications one assumes in the spirit of Harrington (1969b) that the infinite series implied by (5.82) can be replaced by the on-shell contributions to the terms we have displayed explicitly. Then for the reactions (5.80) above everything is known (or otherwise measurable) except the X^*-nucleon elastic scattering amplitude. Thus diffractive excitation of hadron resonances off deuterons becomes a technique for studying unstable hadron-nucleon scattering. Here we have committed the usual sin, discussed in Section V. 2, of assuming that the excited object indeed corresponds to X^* when it interacts with the second nucleon.

We have already remarked that this implicit assumption is more plausible for reaction (iv) than for the others. In a recent experiment Anderson, et al. (1971) have studied $\gamma d \rightarrow \rho^0 d$ at 6, 12 and 18 GeV over a wide range of momentum transfer. Their data, which are shown in Fig. 43, greatly extend the older results of Hilpert, et al.(1970). The shape of the differential cross section is the one characteristic of elastic hadron-deuteron scattering that we have seen already in Figs. 21, 24 - 29. Using the spin formalism of Michael and Wilkin (1969) and assuming equality of the $\rho^0 p$ and $\rho^0 n$ elastic scattering amplitudes, Anderson et al. extracted from their data the differential cross section for ρ^0-nucleon scattering over a limited range of momentum transfer. Their results are compared with the differential cross sections for $\pi^- p \rightarrow \pi^- p$ in Fig. 44. The rough agreement exhibited there is in accord with simple quark model ideas. Because of the ambiguities in the details of the theory in the region of the break in $d\sigma/dt$ for $\gamma d \rightarrow \rho^0 d$, the analysis cannot reliably be extended to larger values of -t.

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FIGURE CAPTIONS

Fig. 1. The real part of the scattering amplitude for a superposition of two Yukawa potentials of the form given in Eq. (2.52), with $U_0 = -20$, $a = 1$, $\rho = 1.125$, and $ka = 5$. The solid curve shows the exact result, the dashed curve gives the eikonal result, and the dashed-dotted curve is the second Born approximation. (From Byron, Joachain and Mund, 1973.)

Fig. 2. Same as Fig. 1 except that the imaginary part of the amplitude is shown. (From Byron, Joachain and Mund, 1973.)

Fig. 3. The real part of the scattering amplitude for a superposition of two Yukawa potentials of the form given in Eq. (2.52), with $U_0 = -20$, $a = 1$, $\rho = 1.125$, and $ka = 2$. The solid curve shows the exact result and the dashed curve gives the eikonal result. (From Byron, Joachain and Mund, 1973.)

Fig. 4. Same as Fig. 3 except that the imaginary part of the amplitude is shown. (From Byron, Joachain and Mund, 1973).

Fig. 5. Illustration of the distorted-wave Born approximation for a process $A + B \rightarrow C + D$.

Fig. 6. Comparison of the energy dependence of the differential cross

section for electron and positron elastic scattering by hydrogen atoms, as given by the first order FWMS expansion and by the first Born approximation. [Taken from Sinfailam and Chen, 1972.]

Fig. 7. Differential cross section for elastic scattering of electrons and positrons by atomic hydrogen at an energy of 100 eV. The solid curve is obtained for electrons by using the eikonal-Born series method (Byron and Joachain, 1973b). The dash-double dot curve is the corresponding one for positrons. The dashed curve represents the first Born approximation, and the dash-dot curve corresponds to the Glauber approximation. The experimental points refer to the work of Teubner, Williams and Carver, quoted in Tai, Teubner and Bassel (1969). [From Byron and Joachain, 1973b.]

Fig. 8. Differential cross section for the excitation of the 2s state of atomic hydrogen by electrons at an incident energy of 100 eV. Curve 1: Glauber approximation (Tai, Bassel, Gerjuoy and Franco, 1970); Curve 2: First Born approximation; Curve 3: Eikonal DWBA method with static distorting potentials (Chen, Joachain and Watson, 1972); Curve 4: Eikonal DWBA method with Glauber distorting potentials (Joachain and Vanderpoorten, 1973a). [Taken from Joachain and Vanderpoorten, 1973a.]

Fig. 9. Total cross section for the excitation of the 2p state of atomic hydrogen by electron impact as a function of the incident energy. Curve 1: First Born approximation; Curve 2: four-channel approximation of Sullivan, Coleman and Bransden (1972); Curve 3: Glauber approximation (Tai, Bassel, Gerjuoy and Franco, 1970); Curve 4: Eikonal DWBA method with Glauber distorting potentials (Joachain and Vanderpoorten, 1973a); Curve 4': same as curve 4, except that the quantity $Q_{\perp}$ defined by Long, Cox and Smith (1968) is shown; Curve 5: Eikonal calculation of Byron (1971), using Eq. (3.56); (x): four-state close-coupling calculation for σ_{2p} (Burke, Schey and Smith, 1963); (+): four-state close coupling calculation for $Q_{\perp}$ (Burke et al., 1963). The dots are the experimental data of Long, Cox and Smith (1968). [Taken from Joachain and Vanderpoorten, 1973a.]

Fig. 10. Differential cross section for elastic scattering of electrons by helium at an incident electron energy of 500 eV. The solid curve is obtained from the eikonal-Born series method of Byron and Joachain (1973b). The dash-dot curve represents the Glauber approximation; the dashed curve is the first Born approximation. The dots correspond to the absolute measurements of Bromberg (1969). [Taken from Byron and Joachain, 1973b.]

Fig. 11. Same as Fig. 10, except that the quantity $d\sigma/d\Omega \times \sin\theta$ is shown in order to exhibit more clearly the larger angle behavior of the various theoretical curves. [Taken from Byron and Joachain, 1973b.]

Fig. 12. Differential cross section for elastic electron-helium scattering at 300 eV. Curve 1: eikonal-Born series calculations of Byron and Joachain (1973b); Curve 2: calculations of Berrington, Bransden and Coleman (1973); Curve 3: eikonal optical model results of Joachain and Mittleman (1971a, b). The dash-dot curve represents the Glauber approximation and the dashed curve the first Born approximation. The experimental points refer to the measurements of Vriens, Kuyatt and Mielczarek (1968) as renormalized by the absolute measurements and of Chamberlain, Mielczarek and Kuyatt (1970).

Fig. 13. Differential cross section for the process $e^- + He(1\,^1S) \rightarrow e^- + He(2\,^1S)$ at an incident energy of 200 eV. Curve 1: First Born approximation; Curve 2: Eikonal DWBA method with static distorting potentials (Joachain and Vanderpoorten, 1973b); Curve 3: Eikonal DWBA method with Glauber distorting potentials (Joachain and Vanderpoorten, 1973b); Curve 4: four-channel calculations of Berrington, Bransden and Coleman (1973); (x): Second Born results of Woollings and

McDowell (1972). The dots refer to the measurements of Vriens, Simpson and Mielczarek (1968) renormalzed by Chamberlain, Mielczarek and Kuyatt (1970). [Taken from Joachain and Vanderpoorten, 1973b.]

Fig. 14. Total cross section for the process $e^- + He(1\,^1S) \rightarrow e^- + He(2\,^1P)$ as a function of the incident electron energy. Curve 1: four-channel calculation of Berrington, Bransden and Coleman (1973); Curve 2: First Born approximation; Curve 3: Eikonal DWBA method with Glauber distorting potentials (Joachain and Vanderpoorten, 1973b); Curve 4: Eikonal calculations of Byron (1971). The triangles (Δ) are the experimental points of de Jongh and Van Eck (1971); the circles refer to the measurements of Donaldson, Hender and McConkey (1972). [Taken from Joachain and Vanderpoorten, 1973b.]

Fig. 15. The differential cross section for the process $e^- + He(1\,^1S) \rightarrow e^- + He(2\,^3S)$ at an incident electron energy of 225 eV. The solid curve refers to the first Born approximation. The dash-dot curve corresponds to the Ochkur approximation (Ochkur and Brattsev, 1965). The squares are the results of the many-body eikonal approximation (Byron and Joachain, 1972). The dots refer to the measurements of Vriens, Simpson, and Mielczarek (1968), renormalized by Chamberlain, Mielczarek, and Kuyatt (1970). [From Byron and Joachain, 1972.]

Fig. 16 The two types of diagrams which contribute to elastic hadron-deuteron scattering in the high-energy diffraction theory. [See Eq. (5.16).] (a) Single scattering diagram; (b) double scattering diagram. Another single scattering diagram with proton and neutron interchanged also contributes to the scattering.

Fig. 17. The various processes which contribute to charge-exchange scattering by the deuteron in the case of a positively charged incident hadron of isotopic spin 1/2.

Fig. 18. The double charge-exchange process.

Fig. 19. The total and absorption cross sections for antiproton-deuteron scattering. From Franco and Glauber (1966).

Fig. 20. The contributions to proton-deuteron elastic scattering from the single and double scattering terms in the region 15 to 20 GeV/c. From Glauber (1969).

Fig. 21. The proton-deuteron elastic scattering data of Bennett, et al. (1967), showing the absence of a dip in the "intermediate" region of momentum transfers.

Fig. 22. Squares of the deuteron form factors given by the hard core model of Reid (1968).

Fig. 23. A simple model calculation [Michael and Wilkin, 1968] showing how the contribution from quadrupole transitions fills in the expected dip in the πd differential cross section.

Fig. 24. Comparison of the theoretical calculations by Alberi and Bertocchi (1969b) with the πd elastic scattering data of Bradamante, et al. (1968) at 895 MeV/c. The dashed curve corresponds to a pure S-wave deuteron wave function. The solid curve includes the effect of the D-wave.

Fig. 25. The differential cross section for $\pi^- d$ elastic scattering at 9 GeV/c calculated by Sidhu and Quigg (1973) is compared with the data of the CERN-Trieste Group (Bradamante, et al., 1971). In addition to the statistical errors shown, the data carry an absolute normalization error of 20%.

Fig. 26. Same as Fig. 25 at 13.0 GeV/c.

Fig. 27. Same as Fig. 25 at 15.2 GeV/c.

Fig. 28. Comparison of the theoretical predictions of Franco and Glauber (1968) with the proton-deuteron elastic scattering experiments (a) at 1 GeV by Bennett, et al. (1967); (b) at 2 GeV by Coleman, et al. (1966).

Fig. 29. Fits to the data of Bunce, et al. (1972) using the Glauber model. (a) The coefficient A in $N(\varphi) = N_o(1 + A\cos 2\varphi + B\cos\varphi)$; (b) The coefficient B; (c) Differential cross section from Bennett, et al. (1967).

Fig. 30. Diagram corresponding to the contribution of an inelastic intermediate state for elastic scattering.

Fig. 31. Reggeon exchange diagram for X-Y elastic scattering, which is governed by Pomeranchuk (P) exchange.

Fig. 32. Graphical representation of the Glauber series for hadron (dashed line) - deuteron scattering: (a) and (b) impulse terms; (c) eclipse term. The wavy lines are Regge poles, the solid line the proton and the dotted line the neutron.

Fig. 33. General Feynman graph for two-Reggeon exchange in (quasi) two-body scattering. The blobs may have complicated structure.

Fig. 34. (a) The simplest Feynman graph which has a Regge cut; (b) redrawn for hadron-deuteron scattering.

Fig. 35. Triple scattering Feynman graphs which appear in the Born series for the Glauber eclipse term.

Fig. 36. The virtual dissociation $d \rightleftarrows np$ for imaginary momenta of the three particles. The length of a vector is proportional to the mass of the corresponding particle.

Fig. 37. Gribov's (1969b) proposed double scattering diagram which contains the full spectrum of physical states into which the projectile may be excited. In conventional Glauber theory, only the projectile itself is retained in the intermediate state.

Fig. 38. Inelastic screening corrections to the pion-deuteron total cross section calculated in a nova model by Sidhu and Quigg (1973). Compare the experimental results shown in Fig. 41.

Fig. 39. Pion-nucleon total cross sections at high energies. The π^-p points are from Foley, et al. (1967) [◇] and from Gorin, et al. (1971) [○]; the π^+p points are from Foley, et al. (1967) [◆] and from Denisov, et al. (1971) [●].

Fig. 40. Pion-deuteron total cross sections at high energies. Notice that whereas the pion-nucleon total cross sections shown in Fig. 39 are essentially constant between 30 and 60 GeV/c, the pion deuteron cross sections continue to decrease.

Fig. 41. Experimental results for the screening correction $\delta\sigma$ are

shown together with the expectations of Glauber theory (solid curve). Data are from Galbraith, et al. (1965) [◇] and from Gorin, et al. (1971) [○]. The dotted line is a best fit of the form $A + Bp^C$ to the Serpukhov data. The dashed line is the calculation of Quigg and Wang (1973) which combines Gribov's scheme with a triple-Reggeon fit to inclusive spectra.

Fig. 42. Compilation of data on the pd total cross section defect (from Kreisler, et al., 1968).

Fig. 43. Differential cross sections for the reaction $\gamma d \rightarrow \rho^o d$ at 6, 12, and 18 GeV/c from the experiment of Anderson, et al. (1971). The solid lines are Glauber theory fits made to extract information on ρ^o-nucleon scattering.

Fig. 44. Differential ρ^o-nucleon scattering cross sections as derived from the experiment of Anderson, et al. (1971). Only experimental errors are indicated; the authors estimate a theoretical uncertainty of about 10%. For comparison, the solid lines represent $\pi^- p$ elastic scattering results of Foley, et al. (1963a) at 7, 13, and 17 GeV/c.

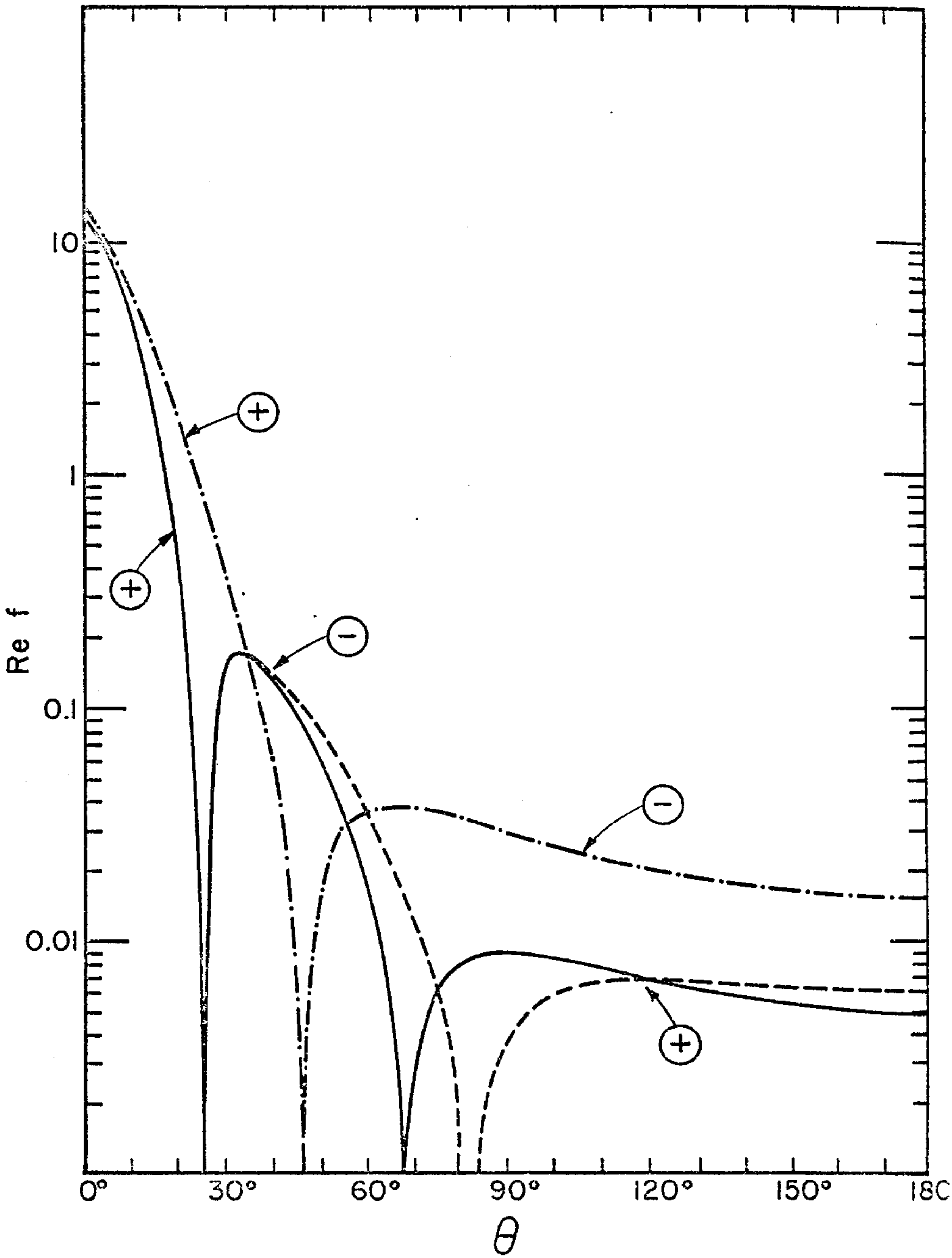


Fig. 1

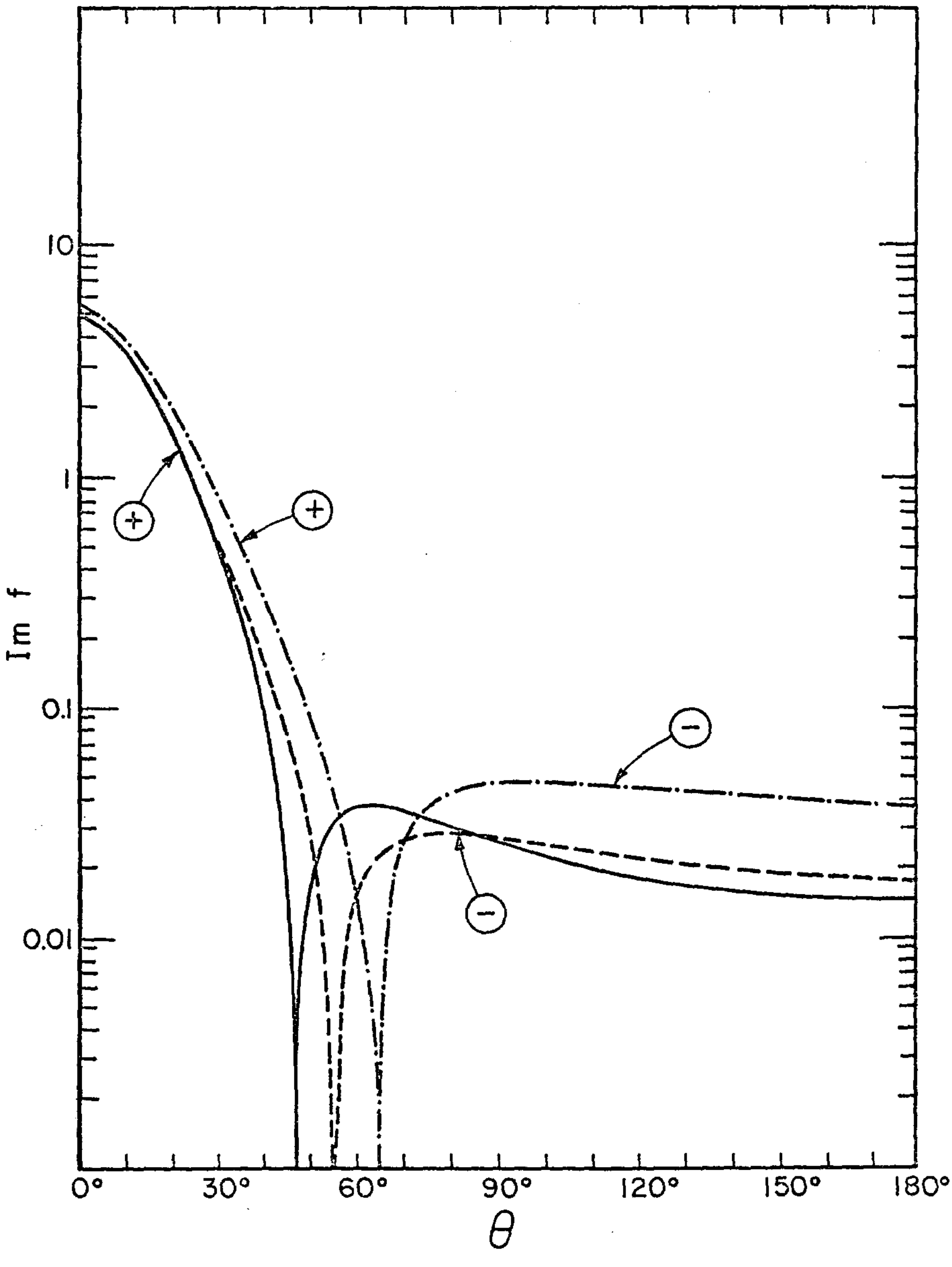


Fig. 2

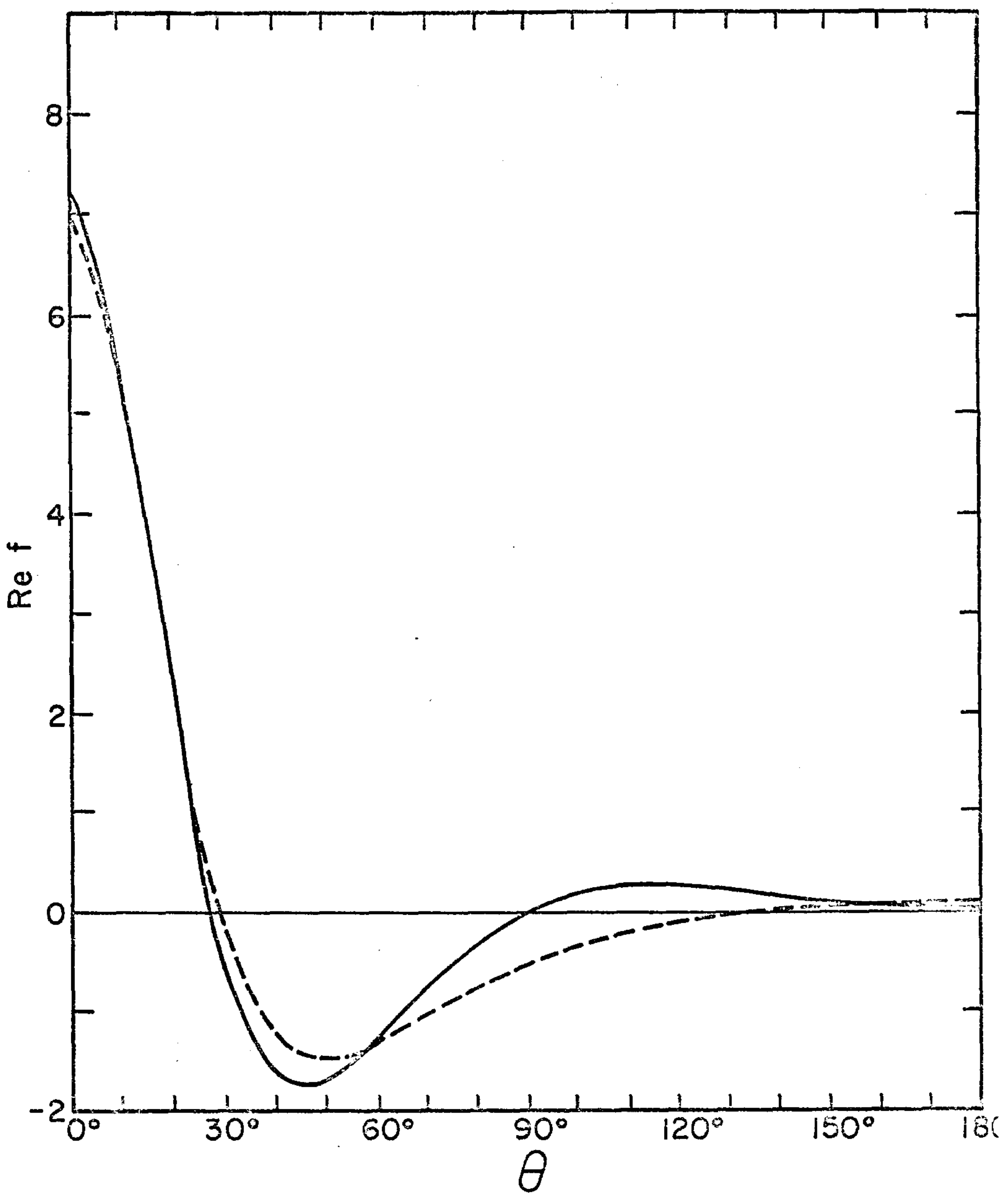


Fig. 3

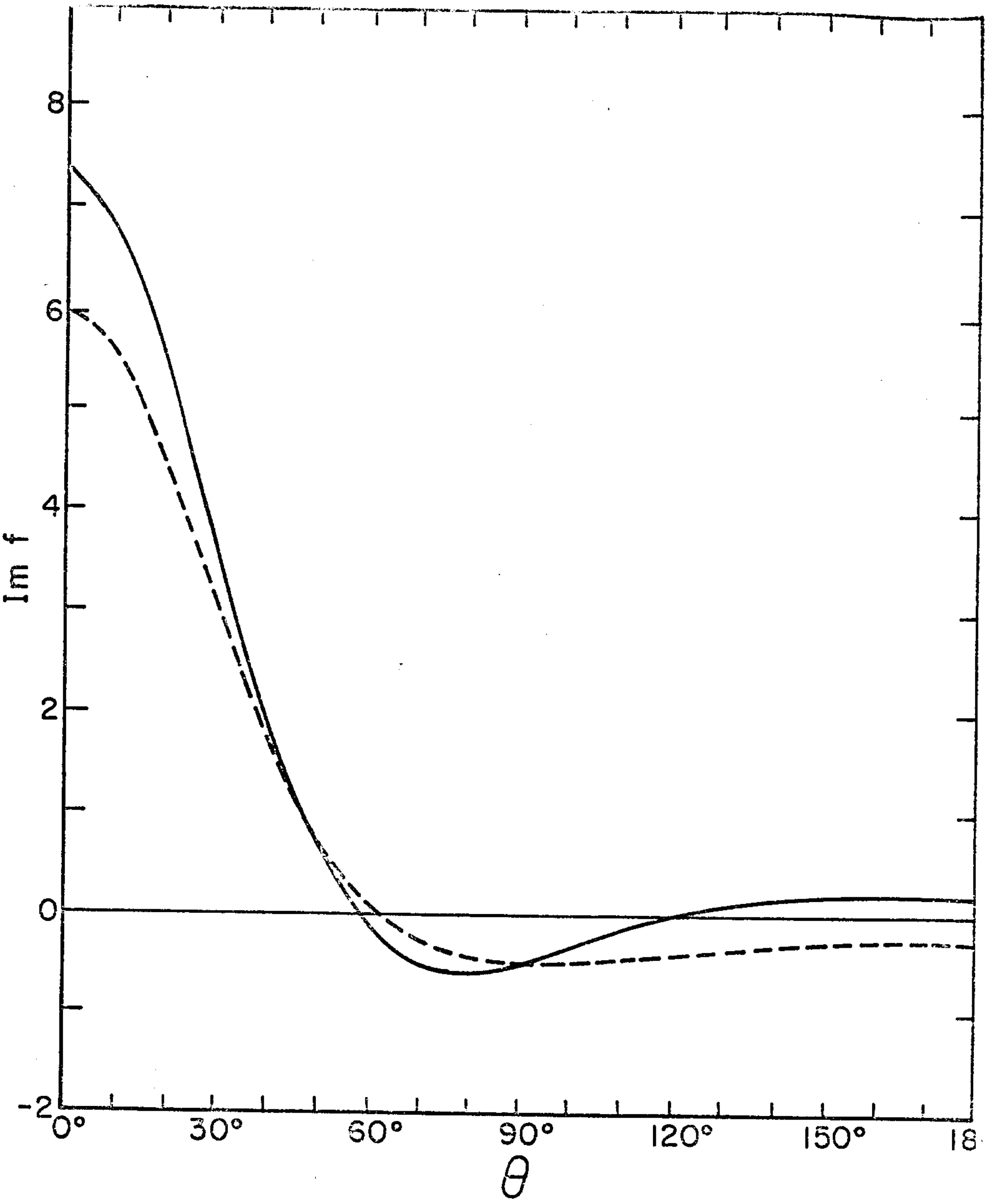


Fig. 4

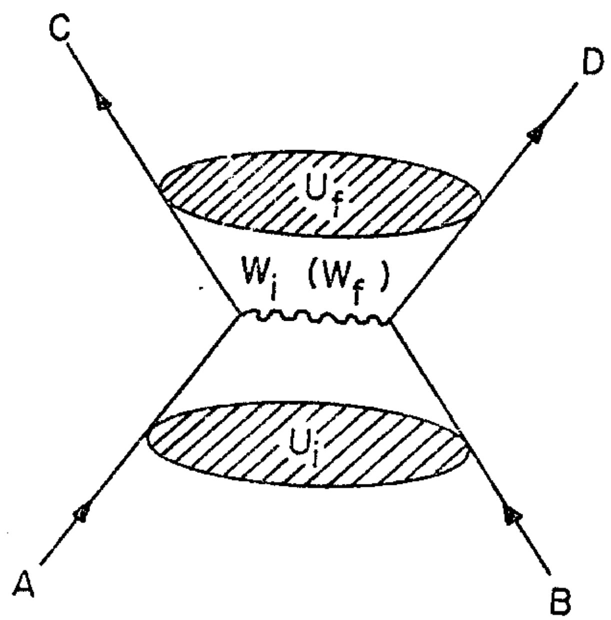
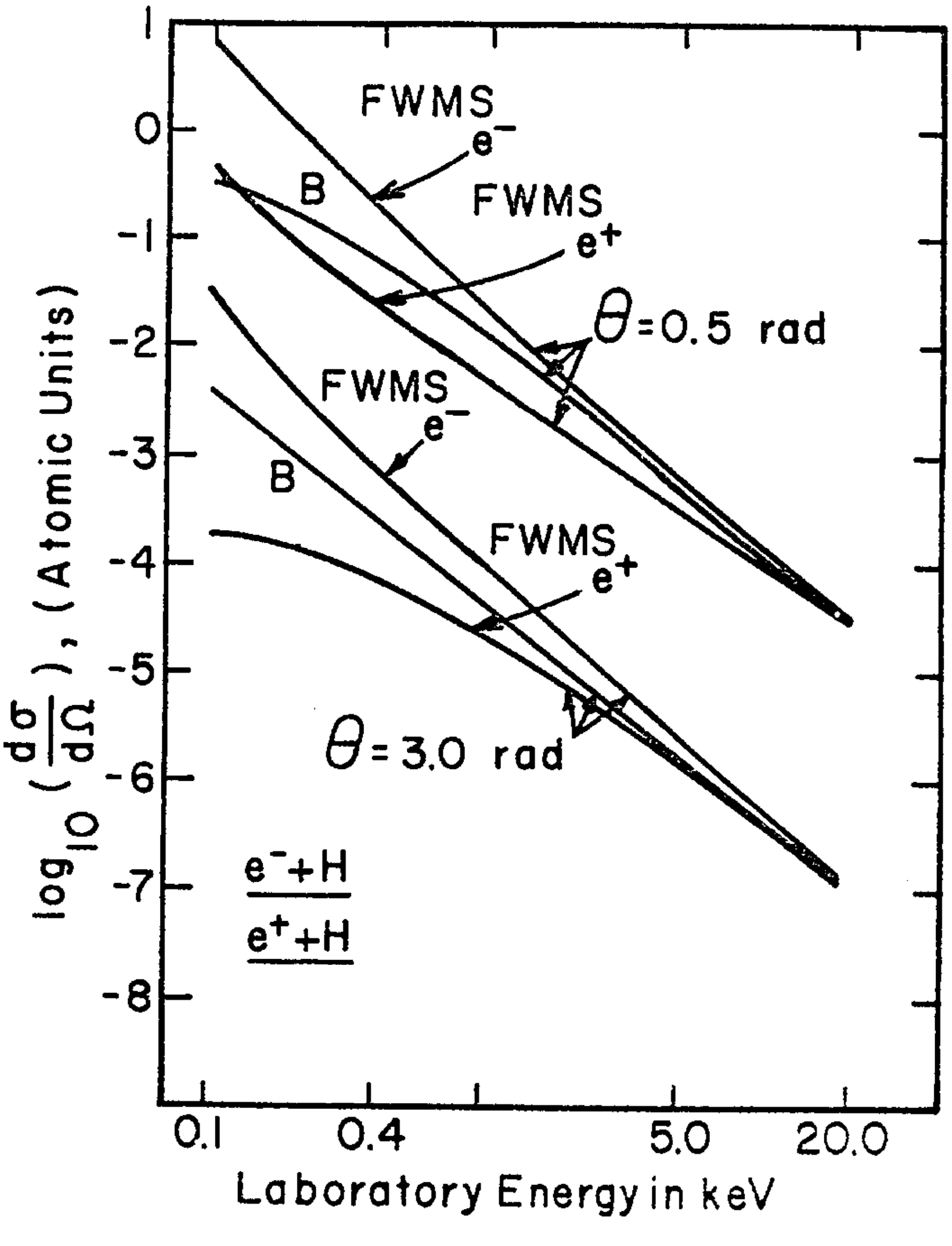


Fig. 5



FWMS e⁻
B
FWMS e⁺
θ=0.5 rad
FWMS e⁻
B
FWMS e⁺
θ=3.0 rad
e⁻+H
e⁺+H
log10 (dσ/dΩ), (Atomic Units)
Laboratory Energy in keV
0.1
0.4
5.0
20.0

Fig. 6

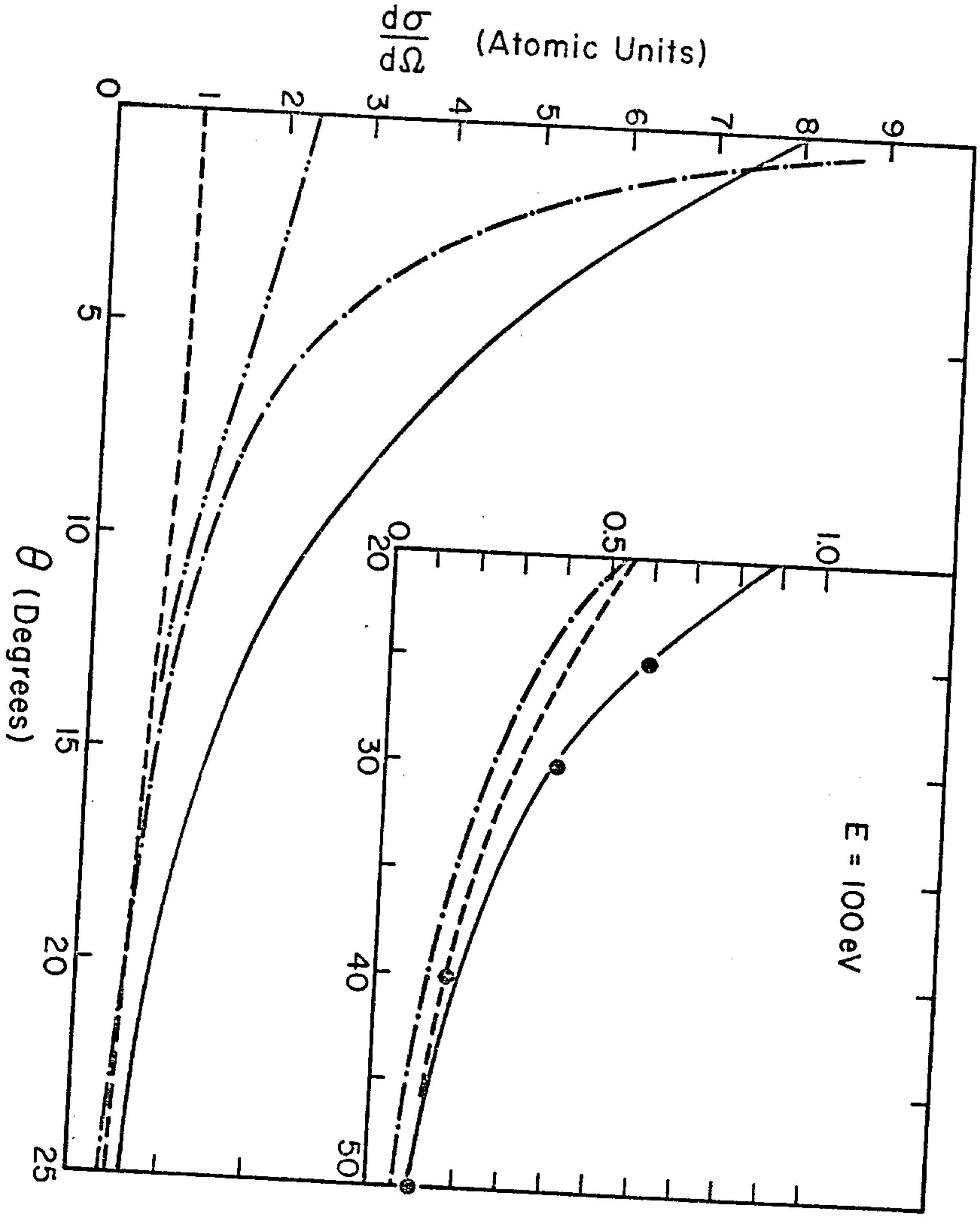
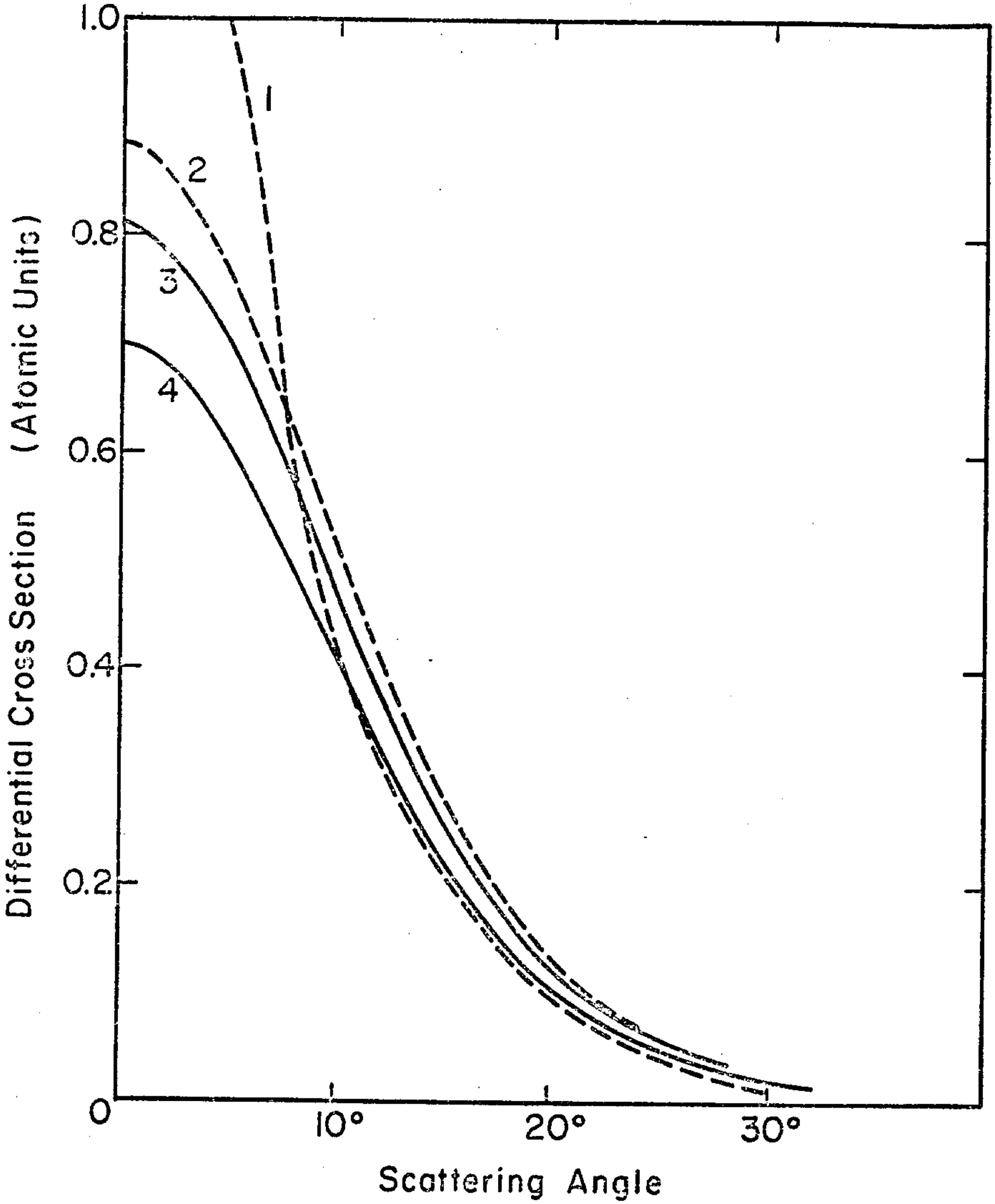


Fig. 7



1.0
0.8
0.6
0.4
0.2
0
10°
20°
30°
1
2
3
4
Differential Cross Section (Atomic Units)
Scattering Angle

Fig. 8

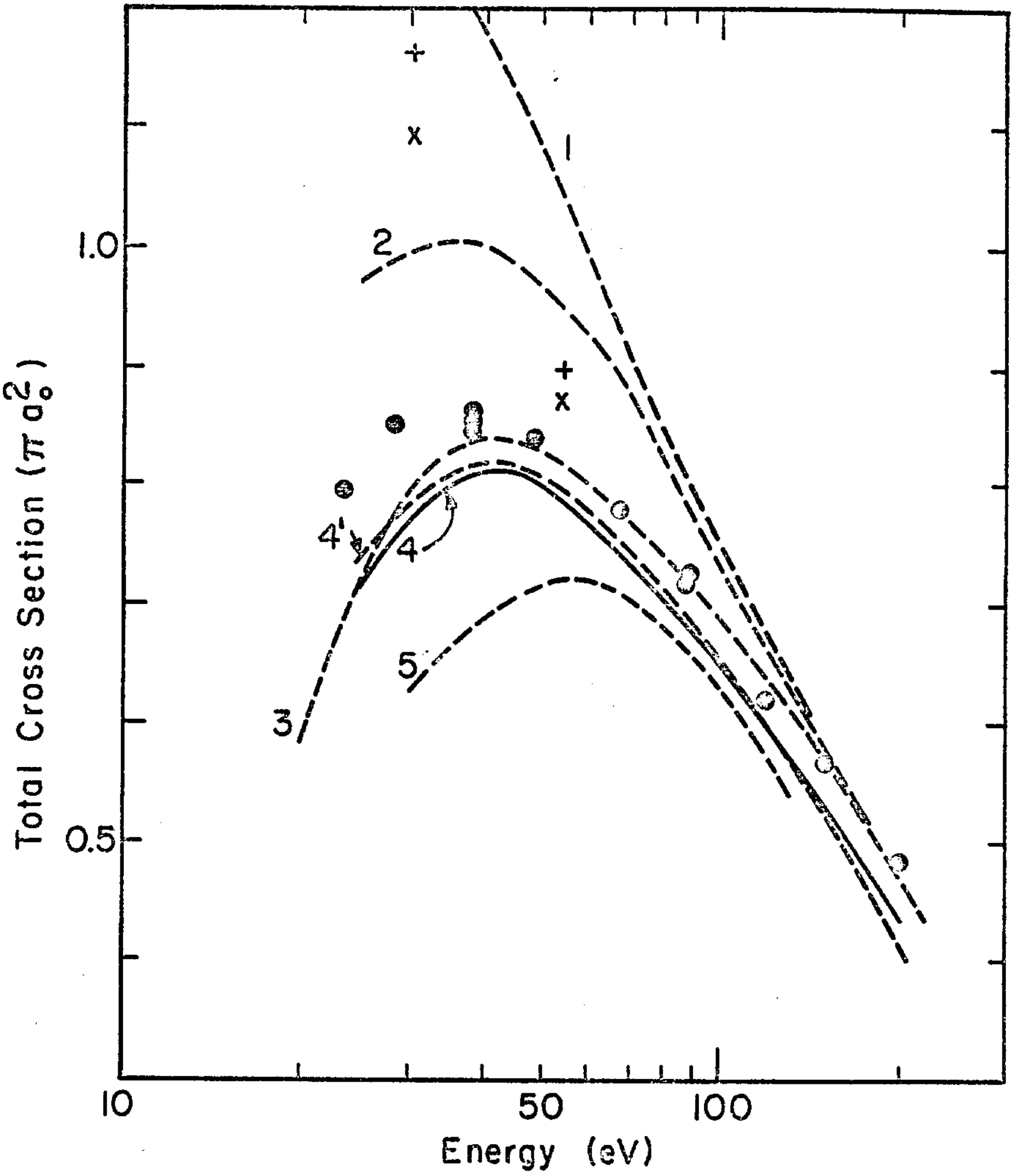


Fig. 9

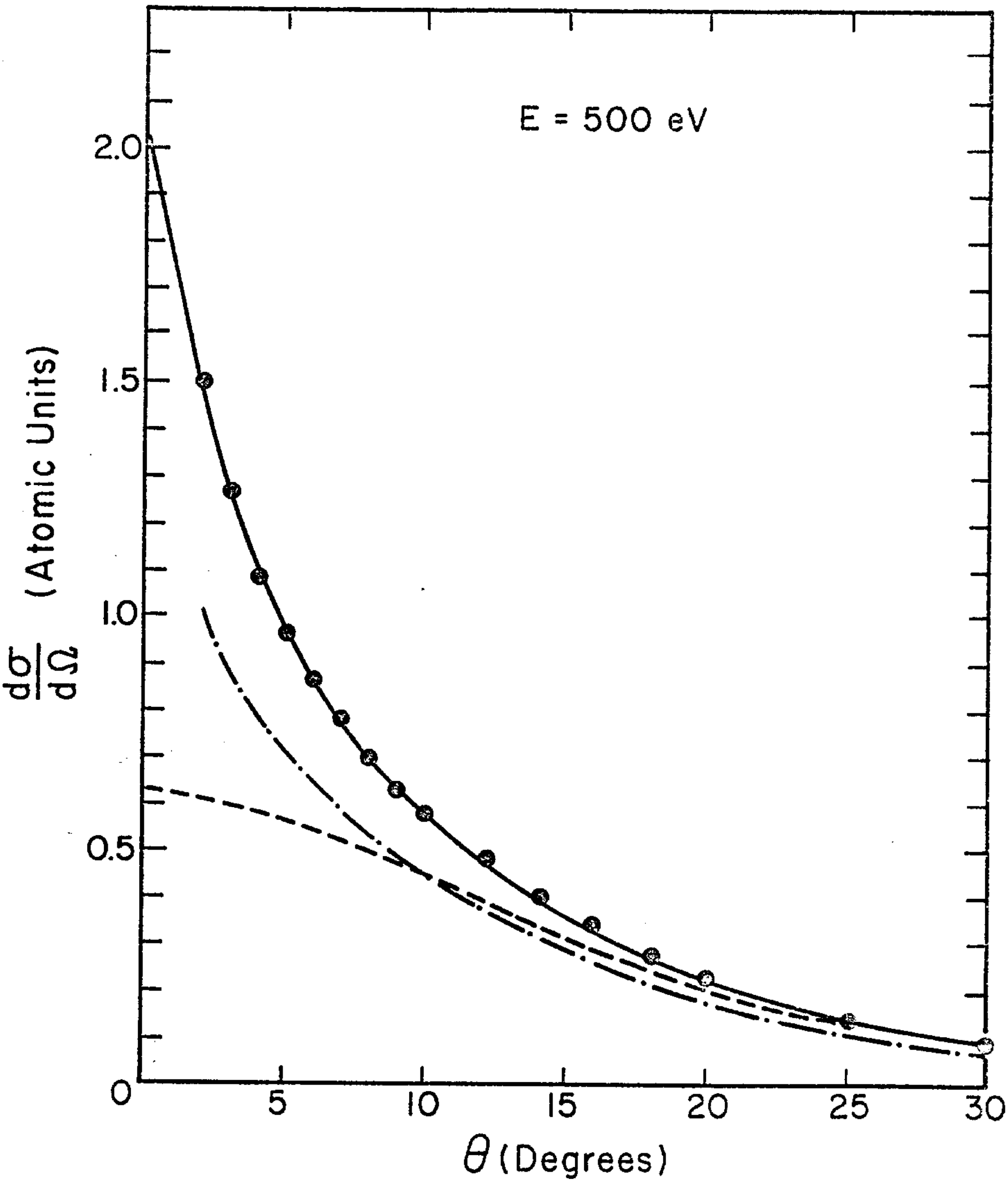
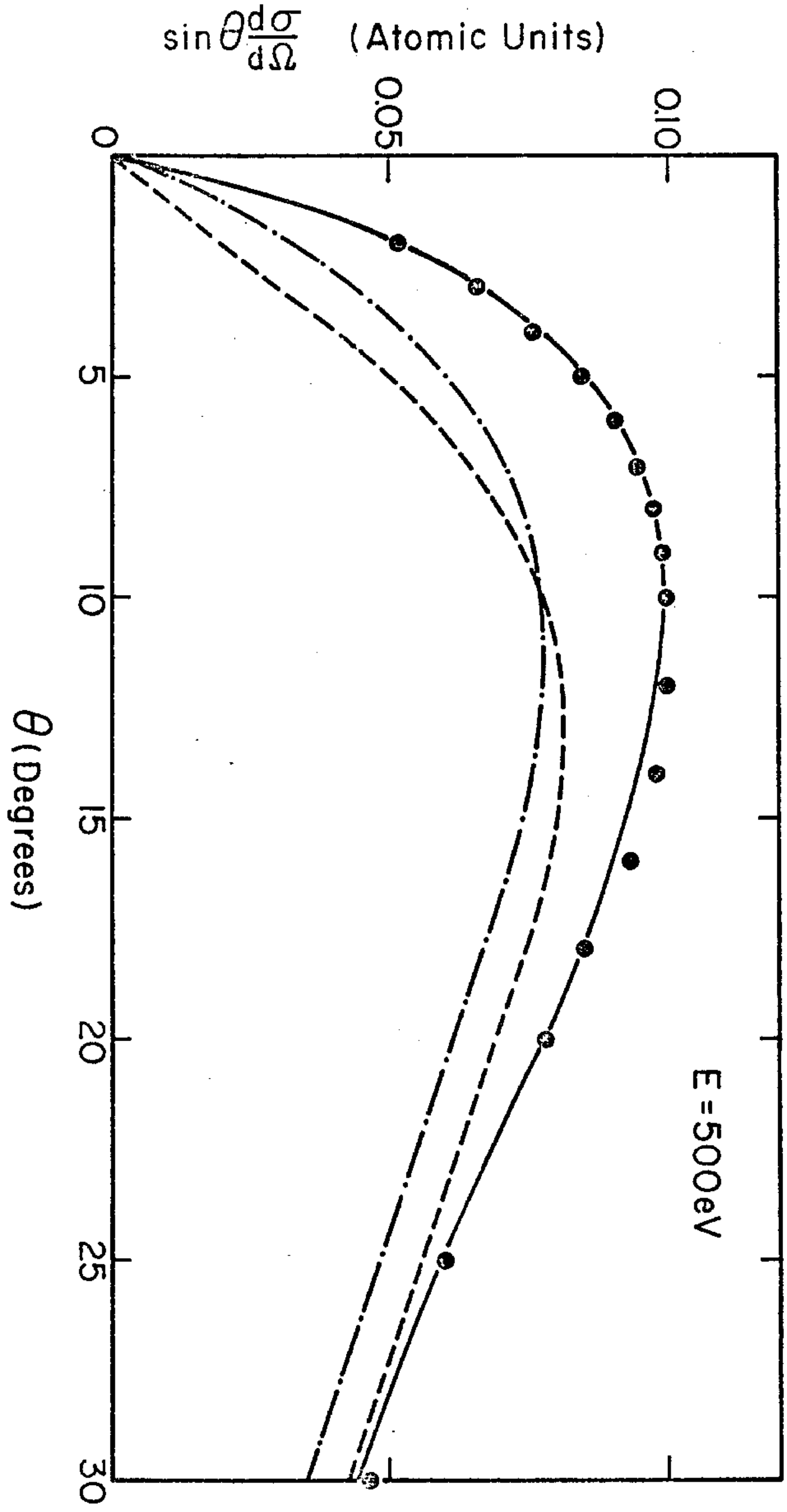
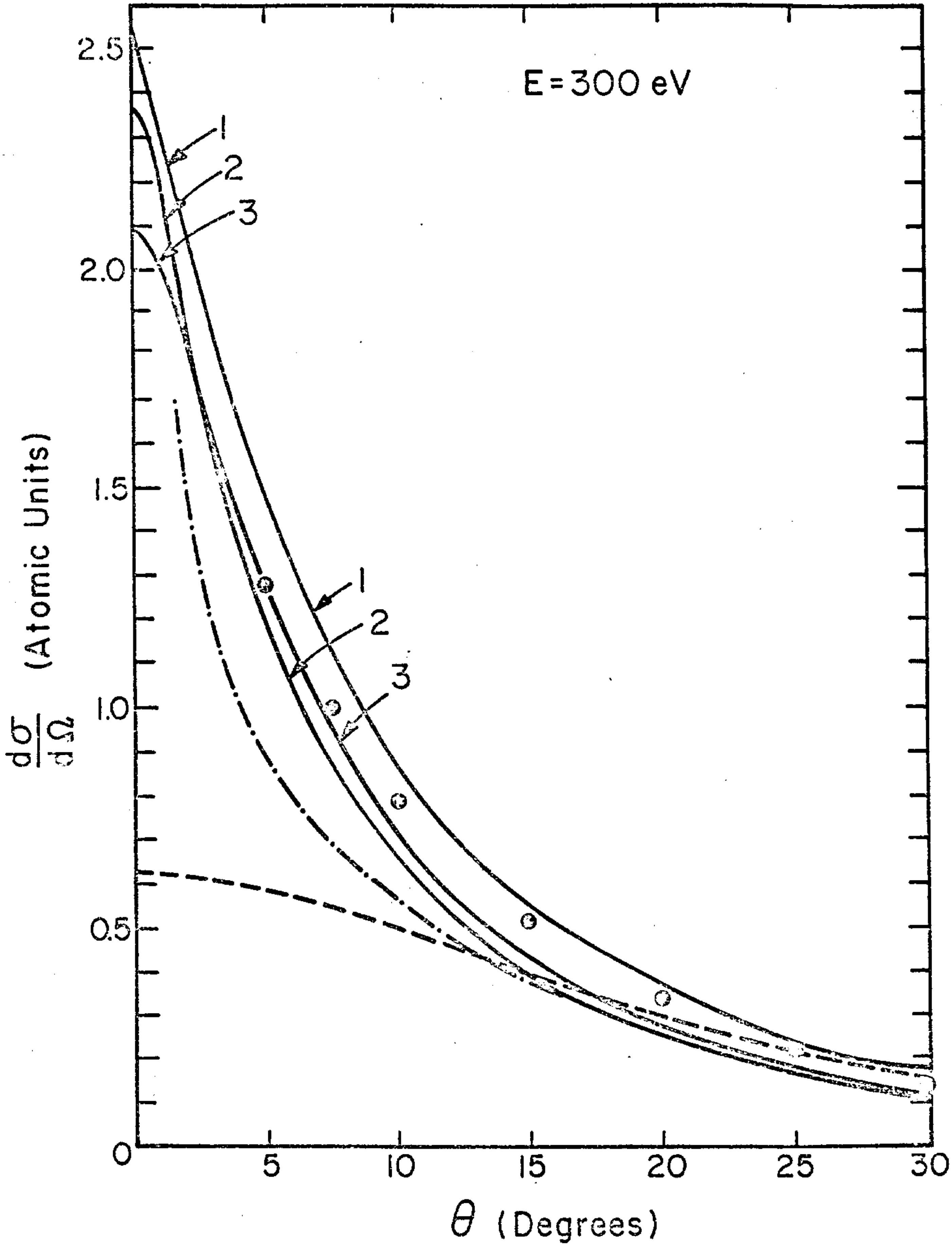

E = 500 eV
2.0
1.5
1.0
0.5
0
5
10
15
20
25
30
$\frac{d\sigma}{d\Omega}$ (Atomic Units)
θ (Degrees)

Fig. 10



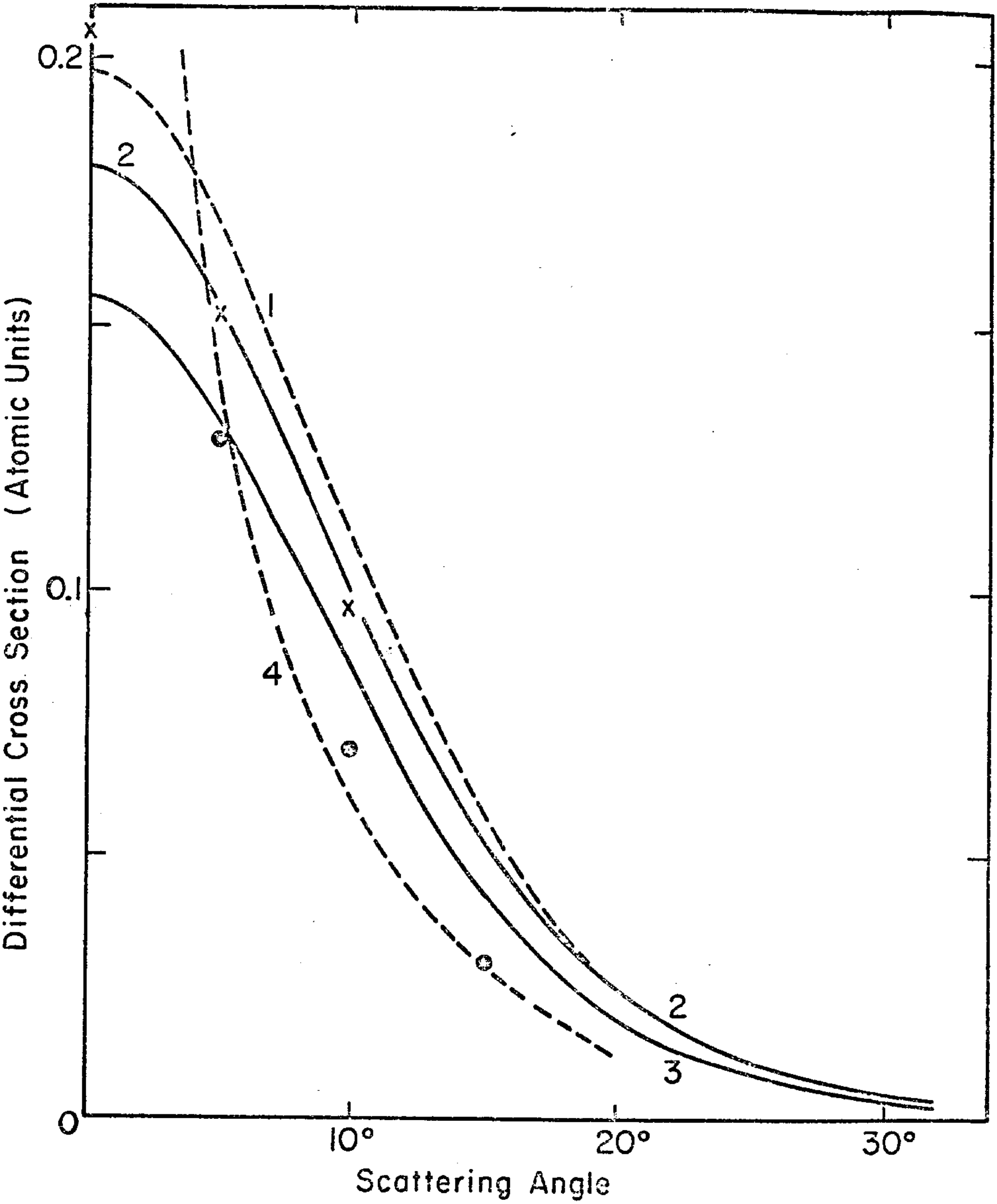
sin θ dσ/dΩ (Atomic Units)
0.05
0.10
0
5
10
15
20
25
30
θ (Degrees)
E = 500eV

Fig. 11



E=300 eV
1
2
3
1
2
3
2.5
2.0
1.5
1.0
0.5
0
5
10
15
20
25
30
dσ/dΩ (Atomic Units)
θ (Degrees)

Fig. 12



0.2
0.1
0
10°
20°
30°
Scattering Angle
Differential Cross Section (Atomic Units)
1
2
2
3
4

Fig. 13

422

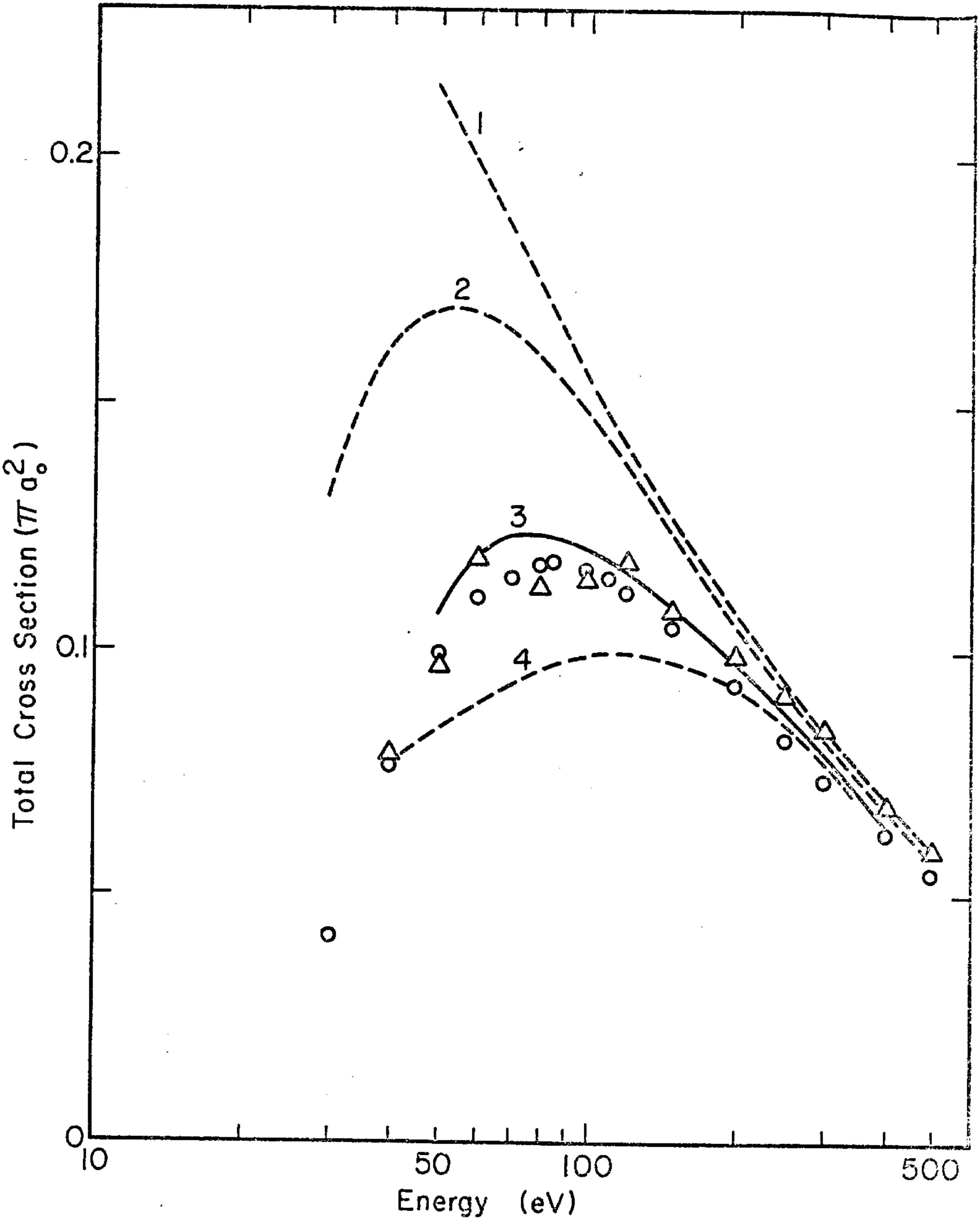
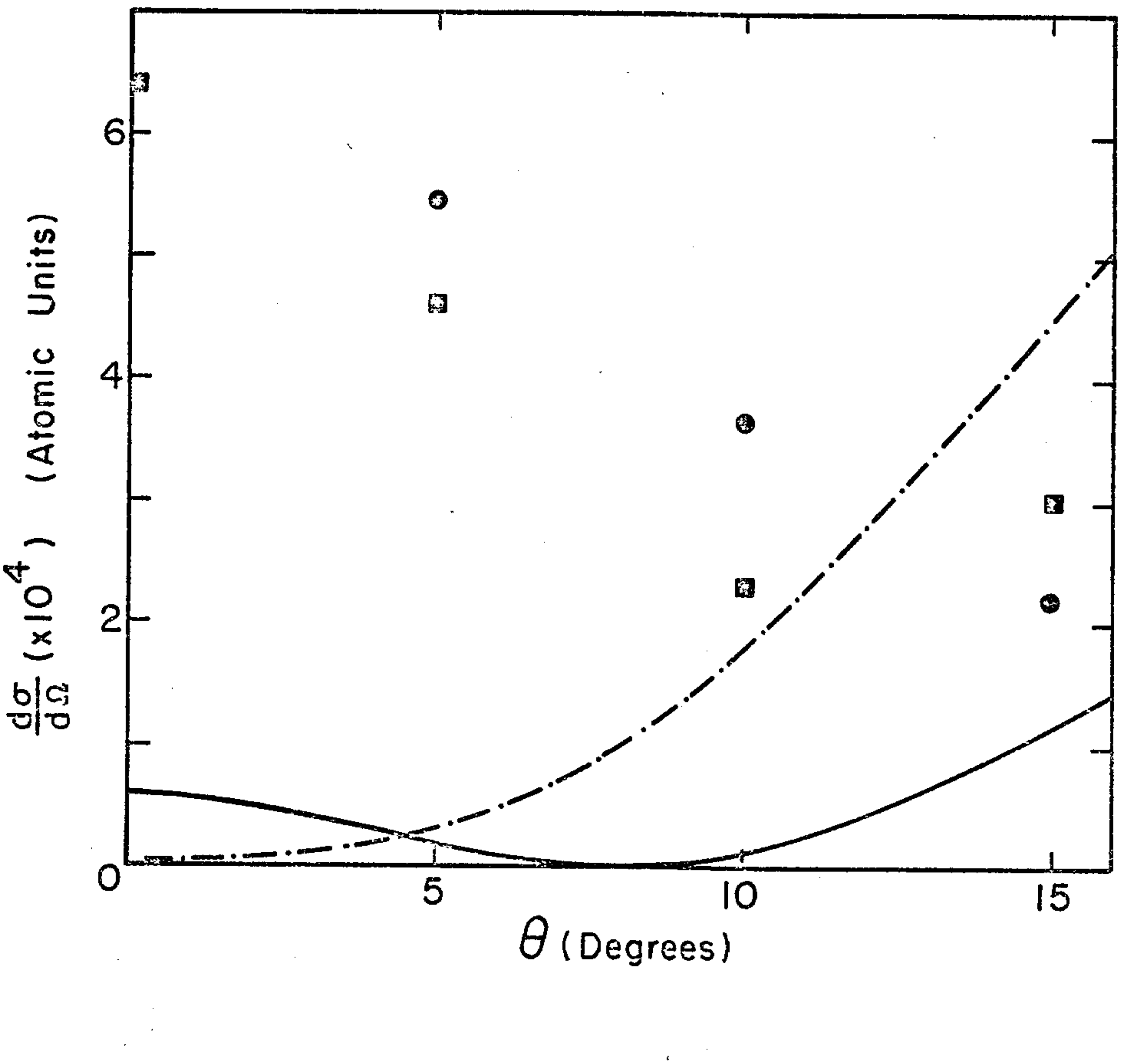


Fig. 14



$\frac{d\sigma}{d\Omega}$ (x10^4) (Atomic Units)
0
2
4
6
5
10
15
θ (Degrees)

Fig. 15

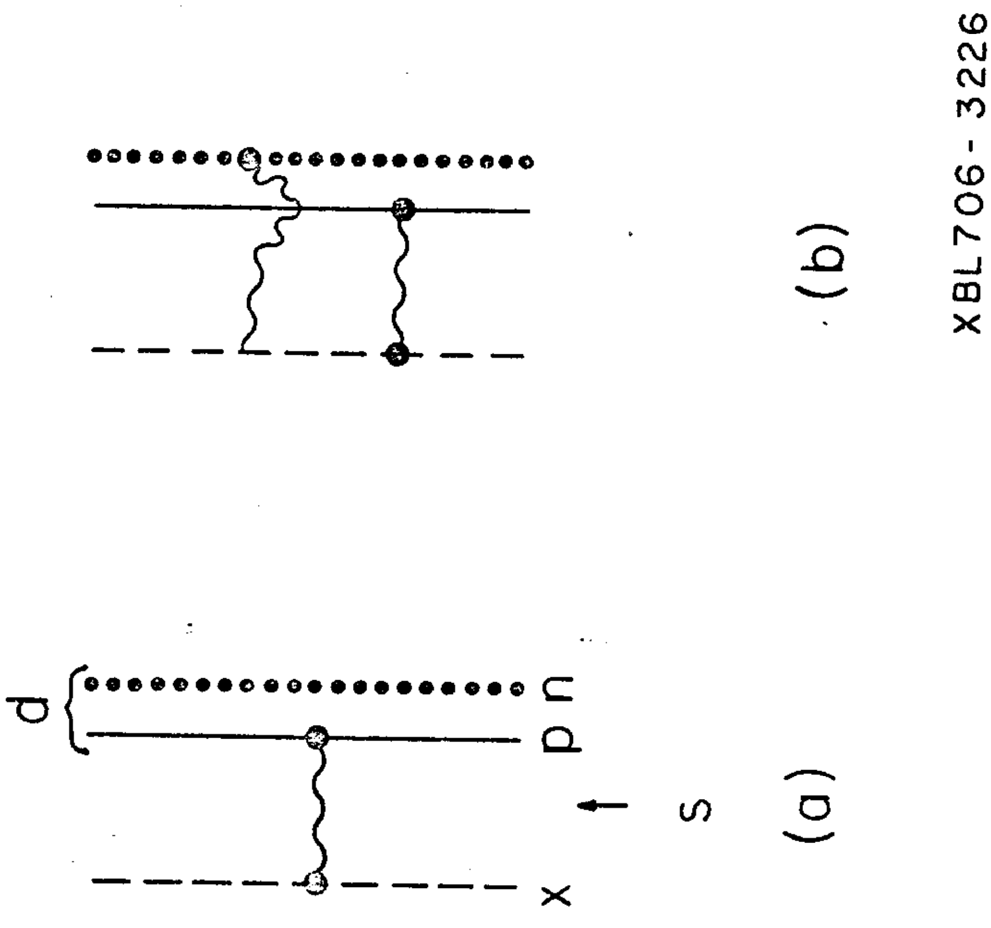


Fig. 16

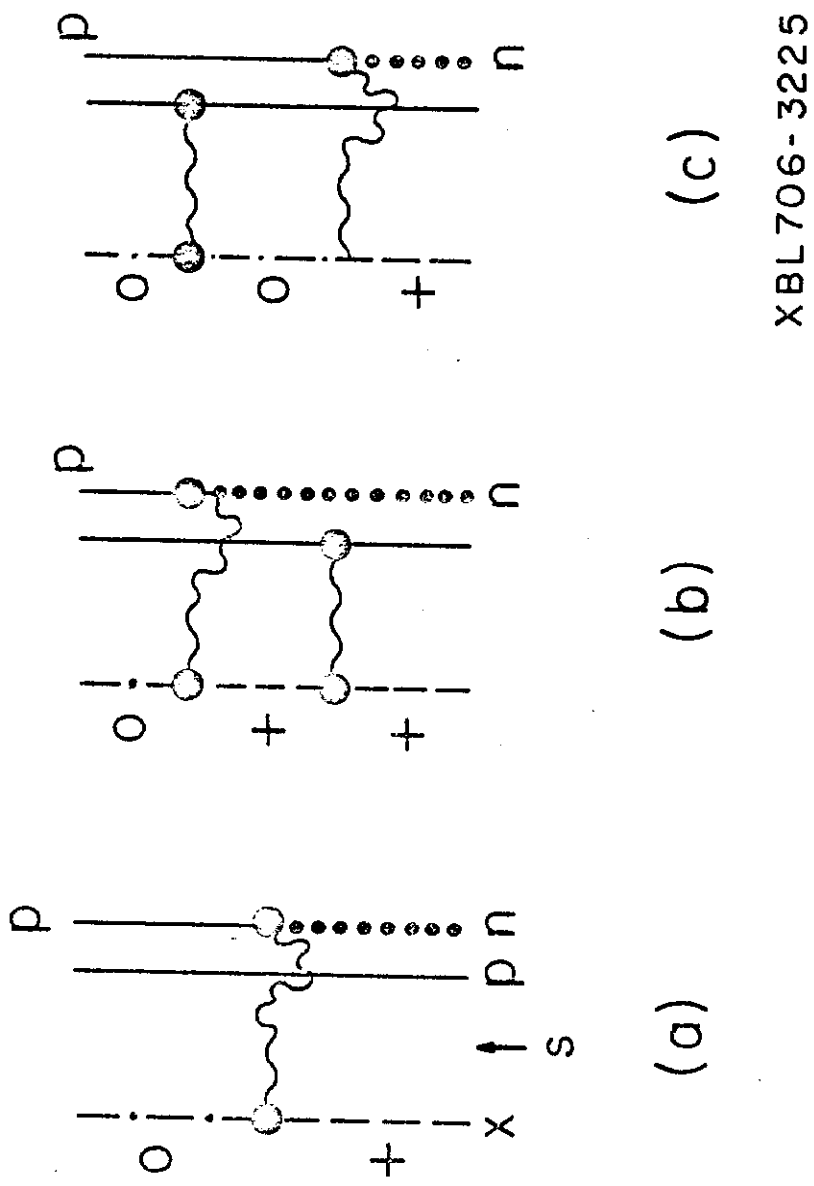
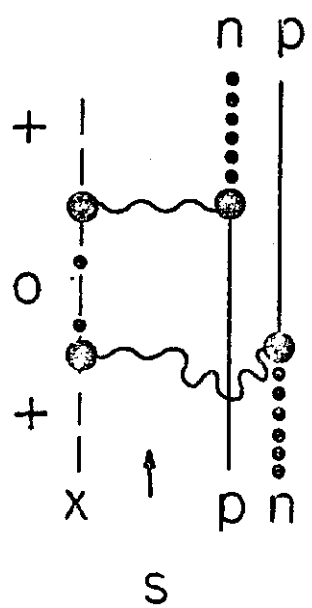
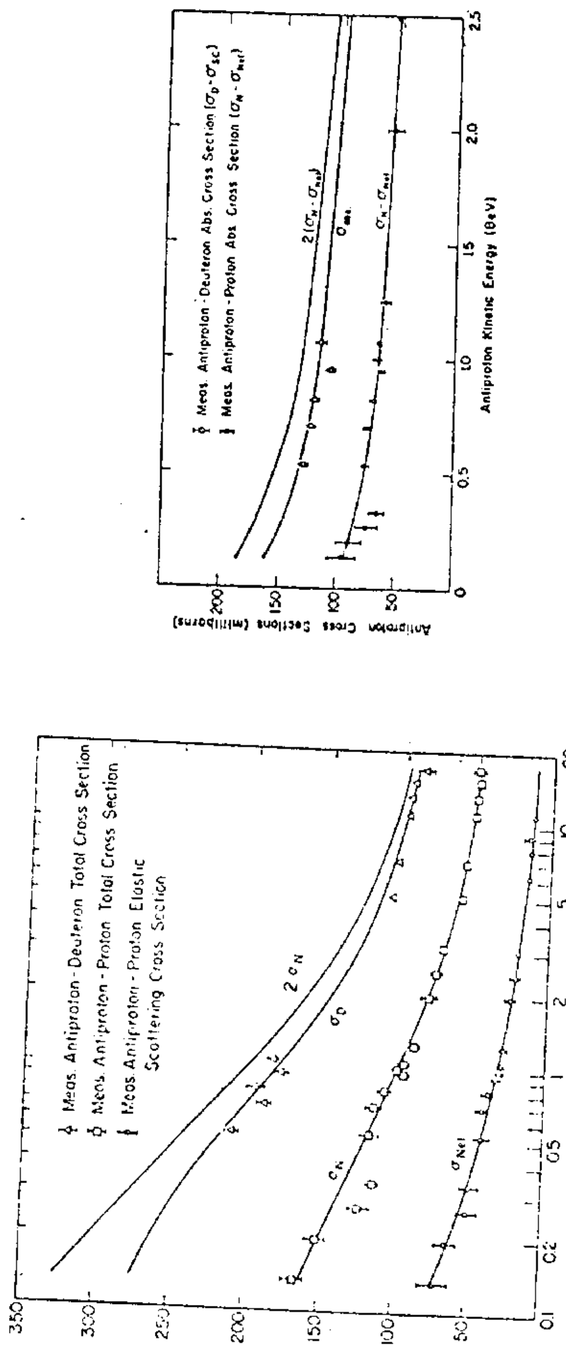


Fig. 17



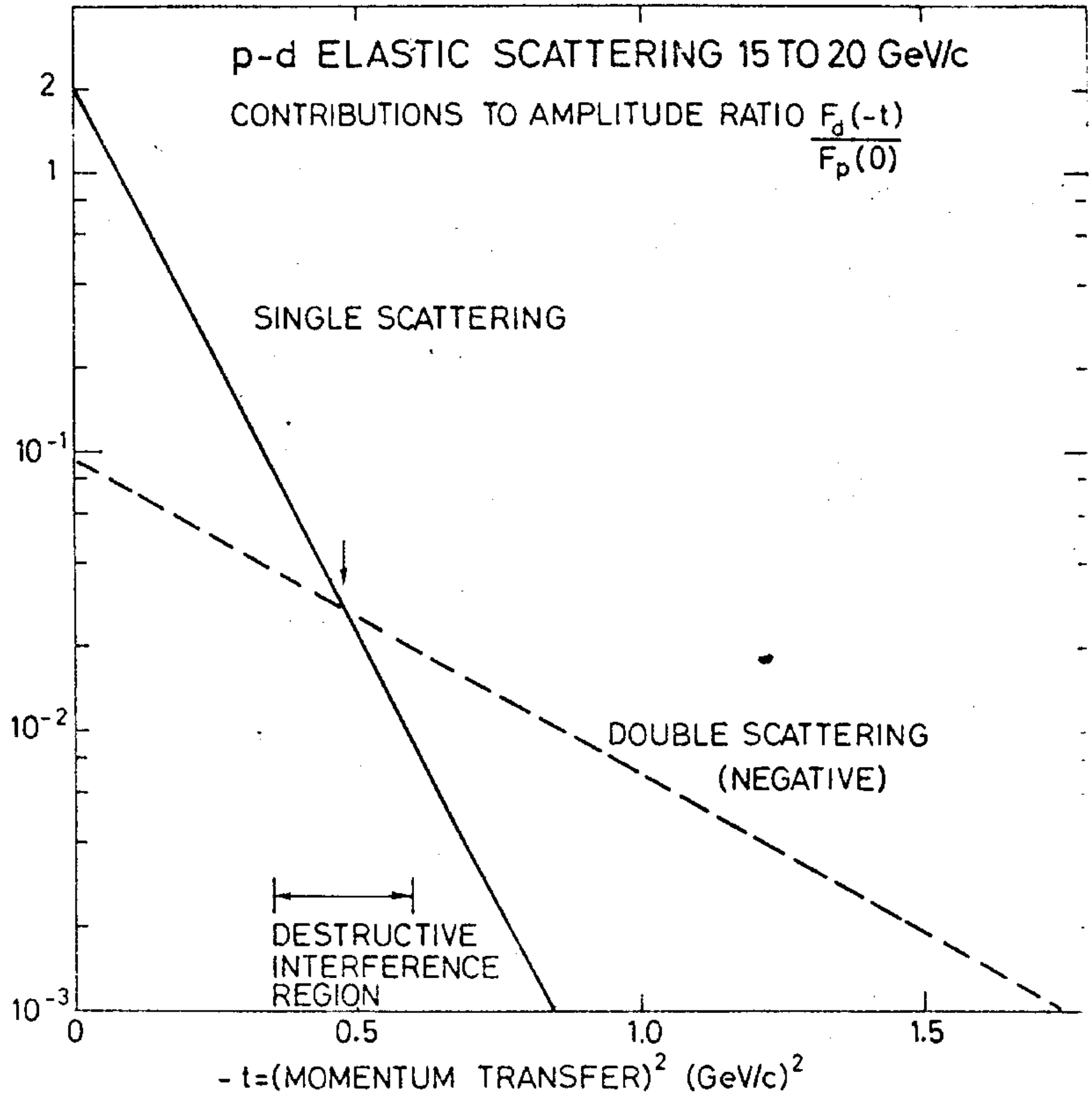
XBL706-3224

Fig. 18



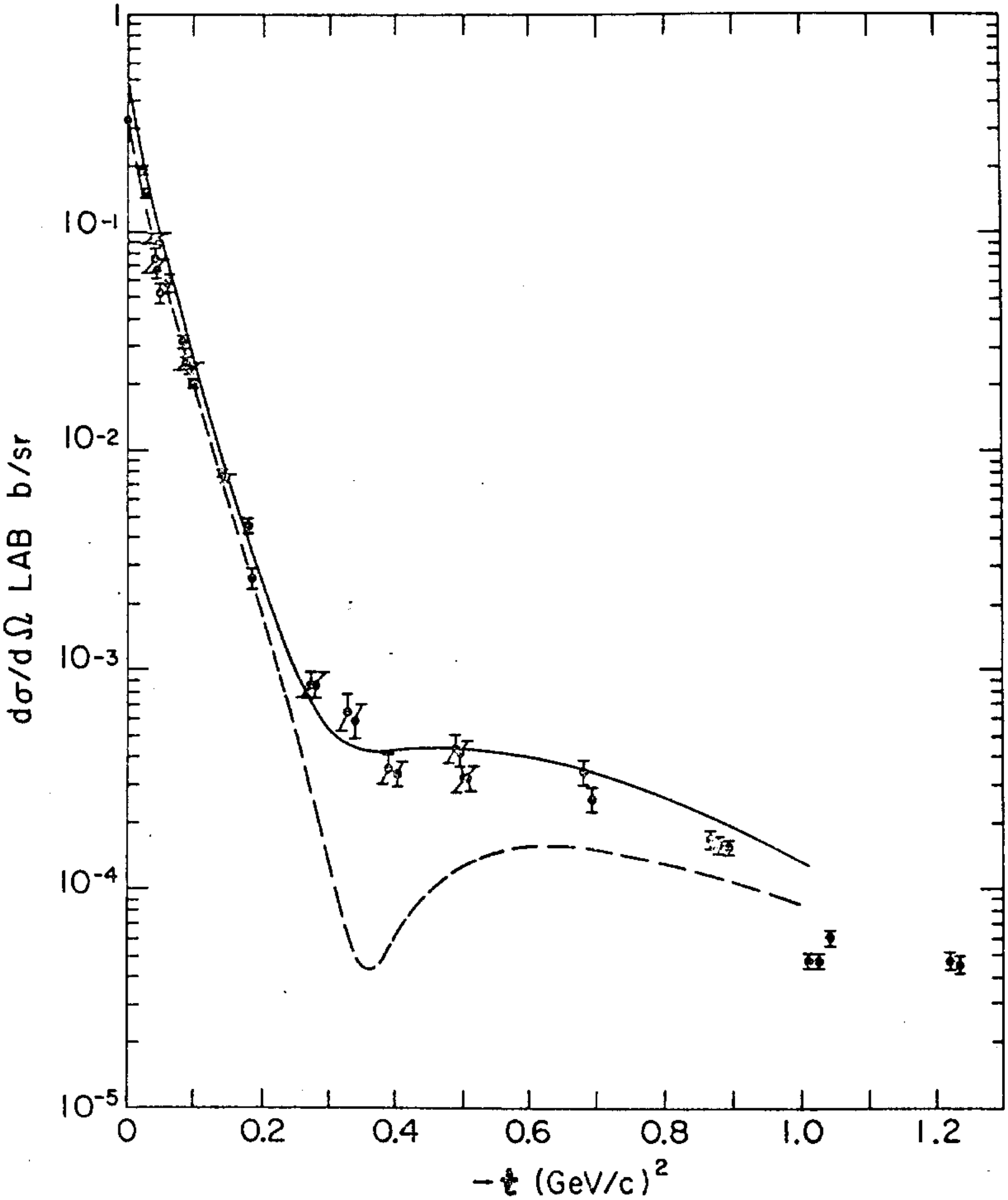
XBL 706-1187

Fig. 19



XBL 706-118

Fig. 20



XBL 706-1189

Fig. 21

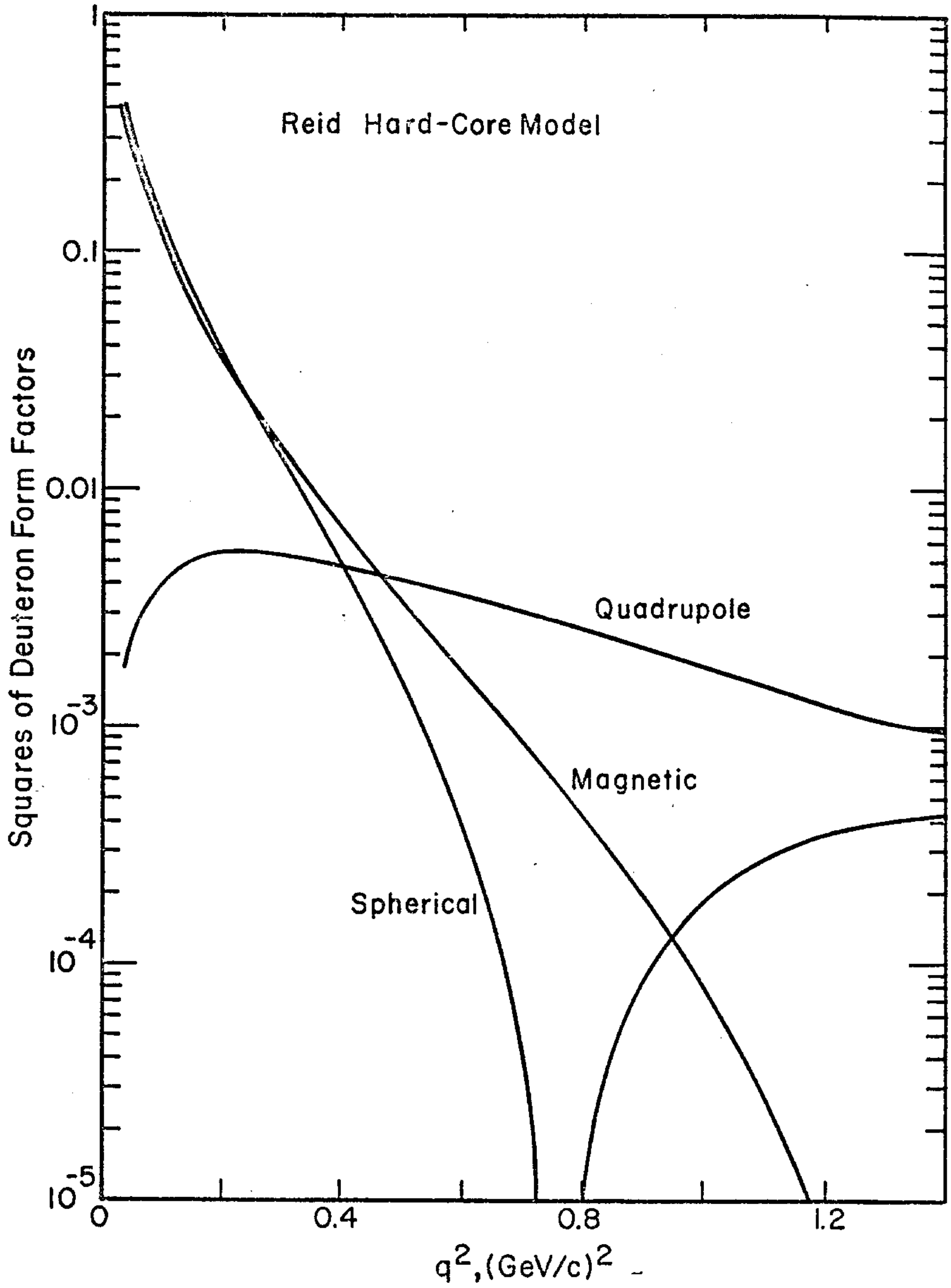
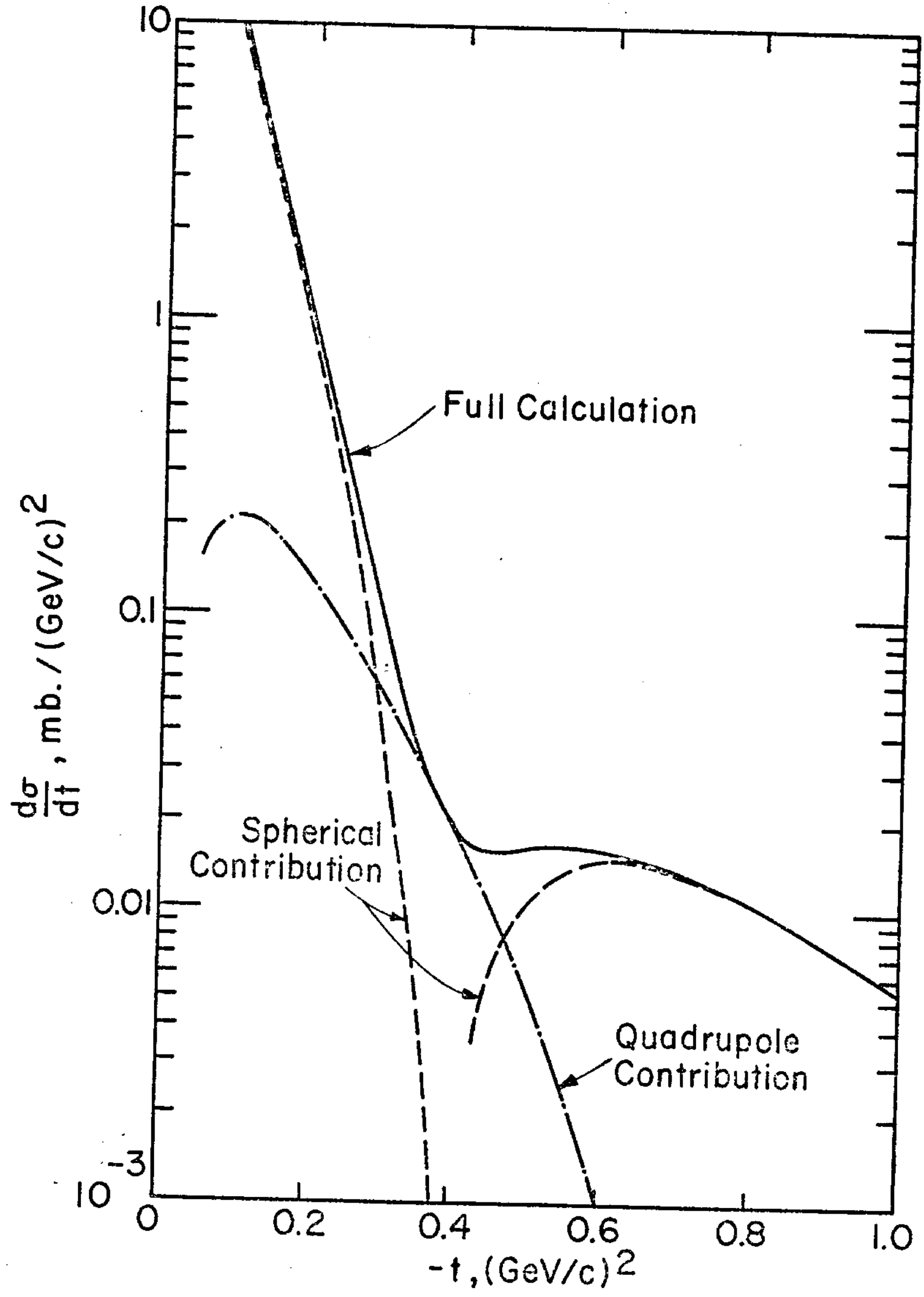
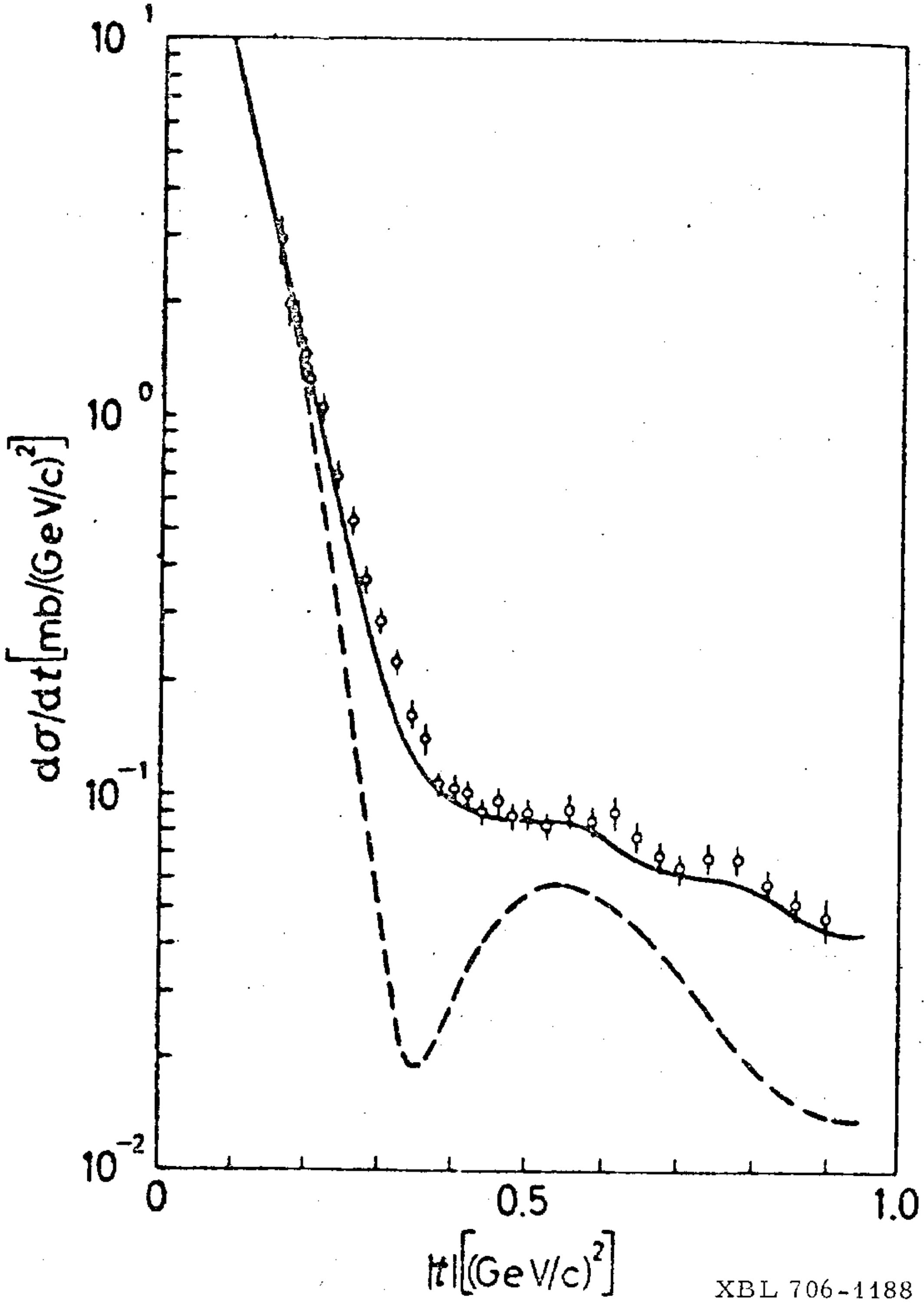

Reid Hard-Core Model
Squares of Deuteron Form Factors
Quadrupole
Magnetic
Spherical
1
0.1
0.01
10^{-3}
10^{-4}
10^{-5}
0
0.4
0.8
1.2
q^2, $(GeV/c)^2$

Fig. 22



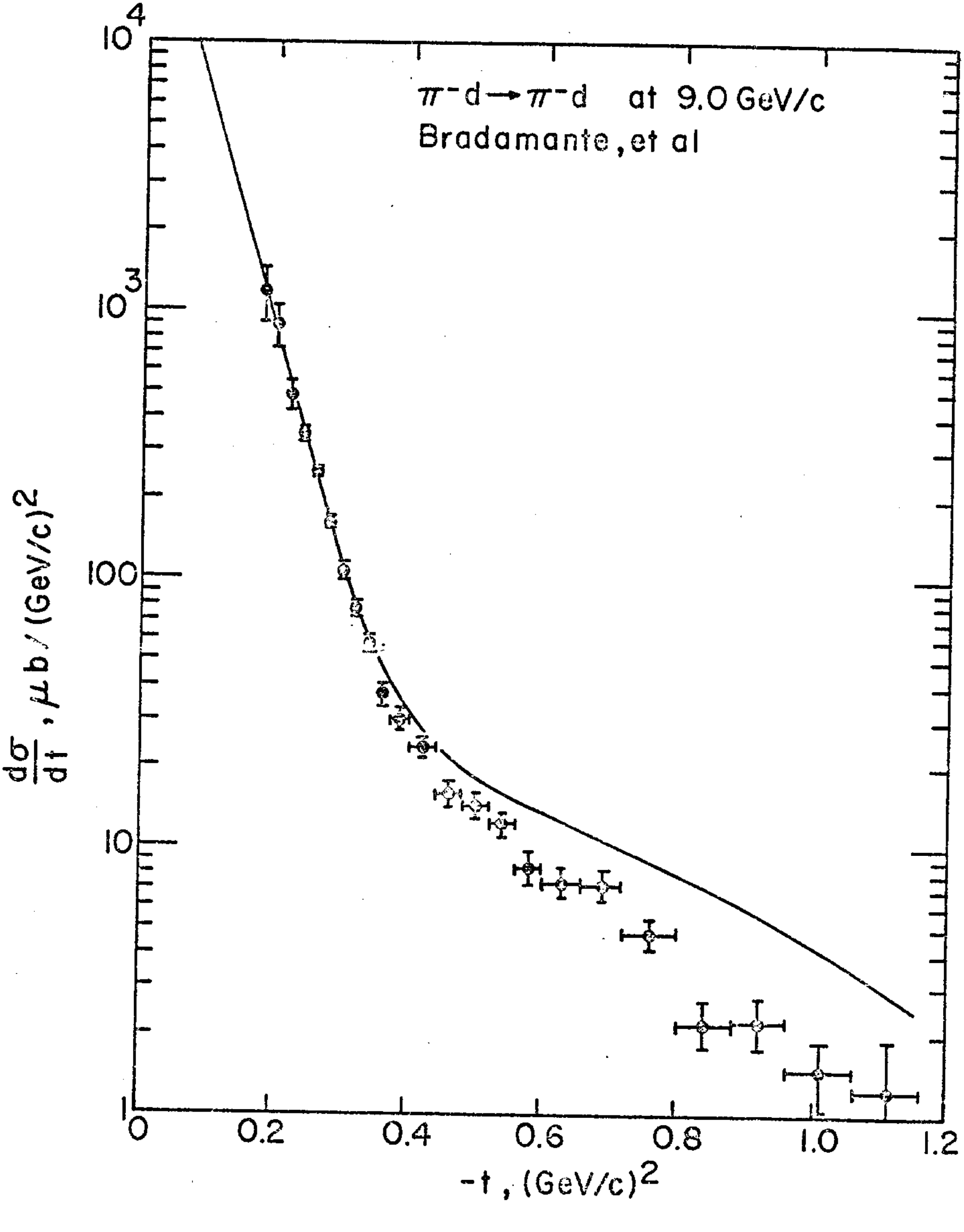
10
1
0.1
0.01
10^{-3}
0
0.2
0.4
0.6
0.8
1.0
$-t, (GeV/c)^2$
$\frac{d\sigma}{dt}$, mb./$(GeV/c)^2$
Full Calculation
Spherical Contribution
Quadrupole Contribution

Fig. 23



XBL 706-1188

Fig. 24



$\pi^- d \rightarrow \pi^- d$ at 9.0 GeV/c
Bradamante, et al
$\frac{d\sigma}{dt}$, μb / (GeV/c)^2
-t, (GeV/c)^2
10^4
10^3
100
10
1
0
0.2
0.4
0.6
0.8
1.0
1.2

Fig. 25

45A

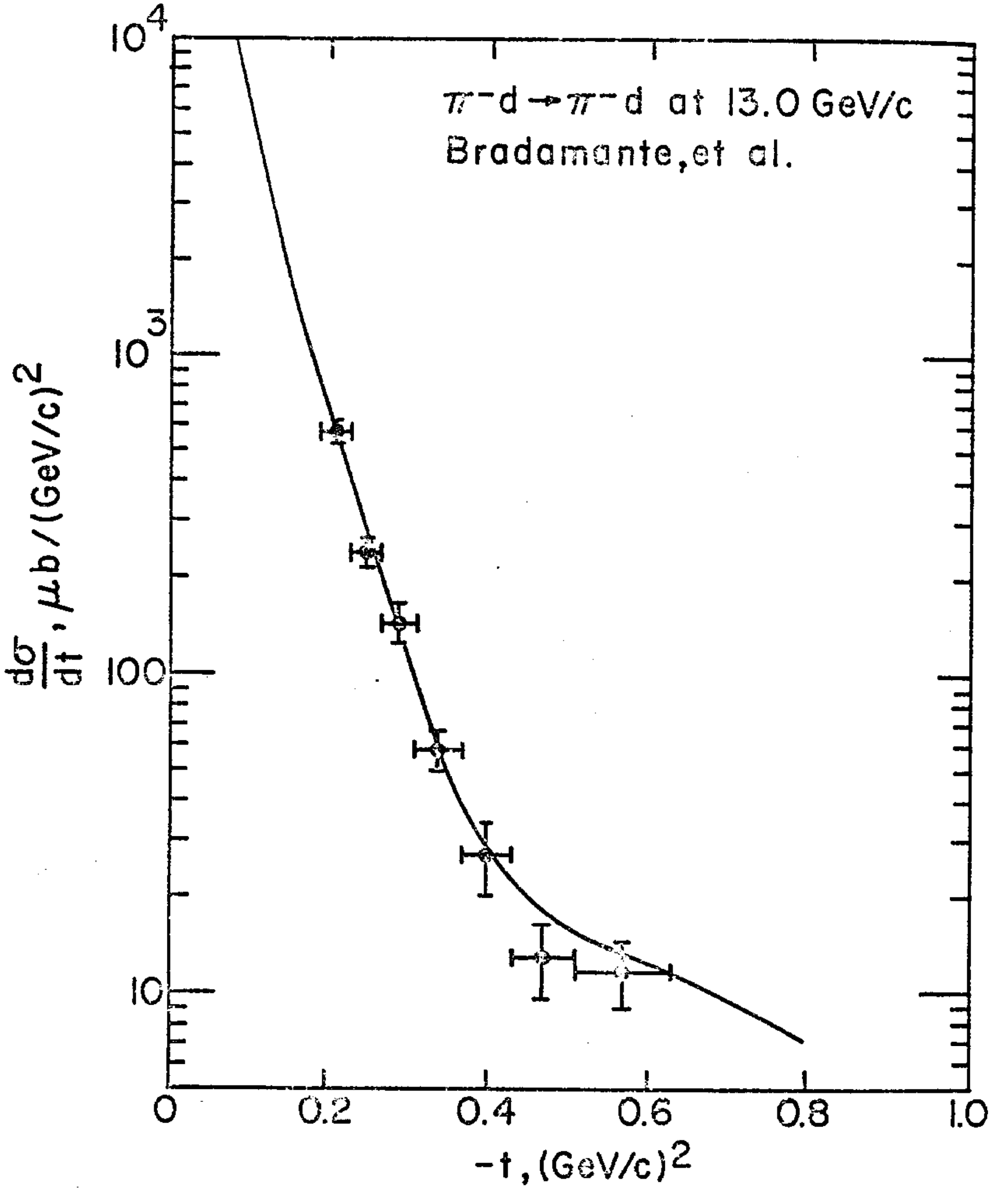
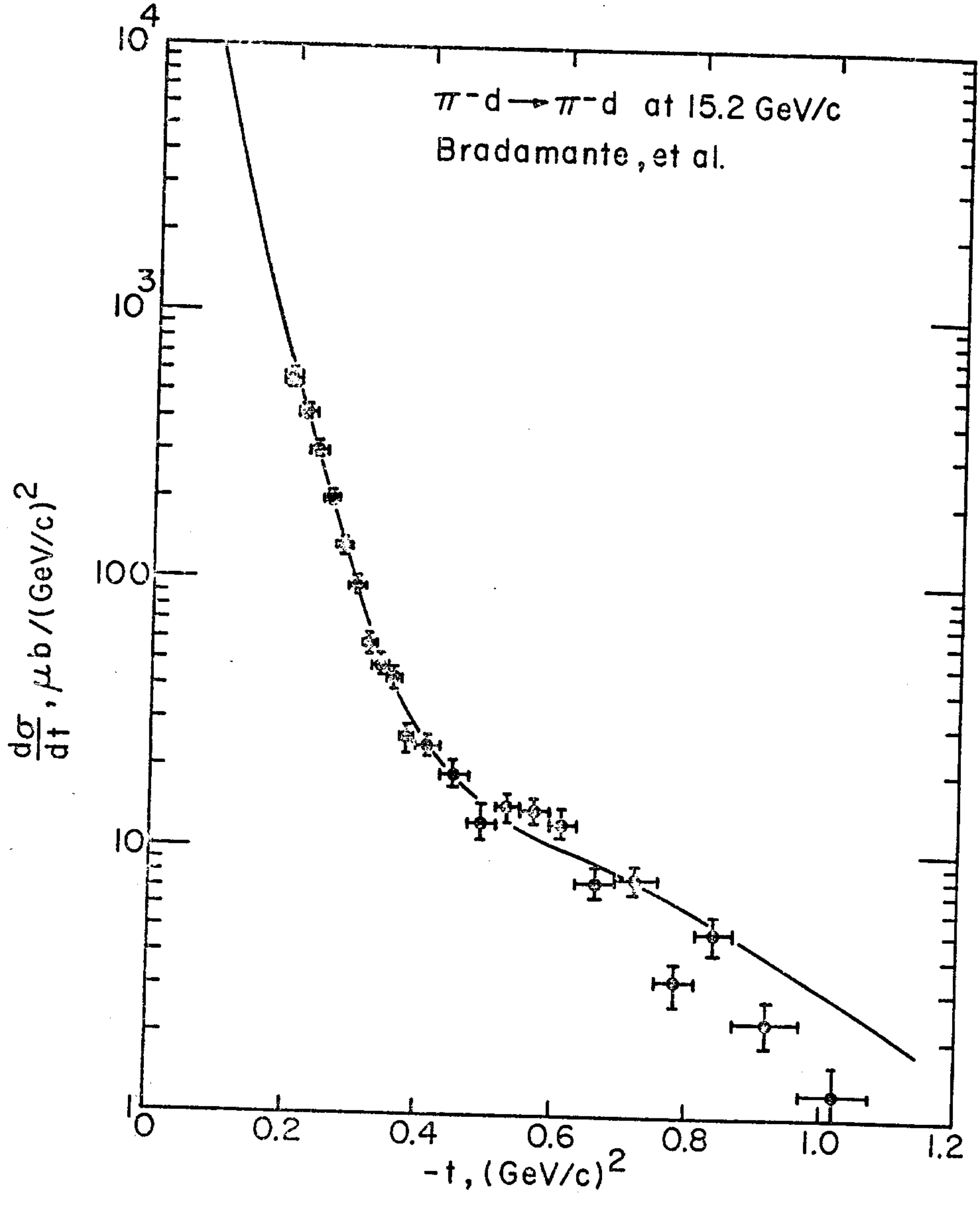

$\pi^- d \rightarrow \pi^- d$ at 13.0 GeV/c
Bradamante, et al.
10^4
10^3
100
10
$\frac{d\sigma}{dt}$, μb/(GeV/c)^2
0
0.2
0.4
0.6
0.8
1.0
$-t$, (GeV/c)^2

Fig. 26



π⁻d → π⁻d at 15.2 GeV/c
Bradamante, et al.
$\frac{d\sigma}{dt}$, μb /(GeV/c)^2
10^4
10^3
100
10
1
0
0.2
0.4
0.6
0.8
1.0
1.2
-t, (GeV/c)^2

Fig. 27

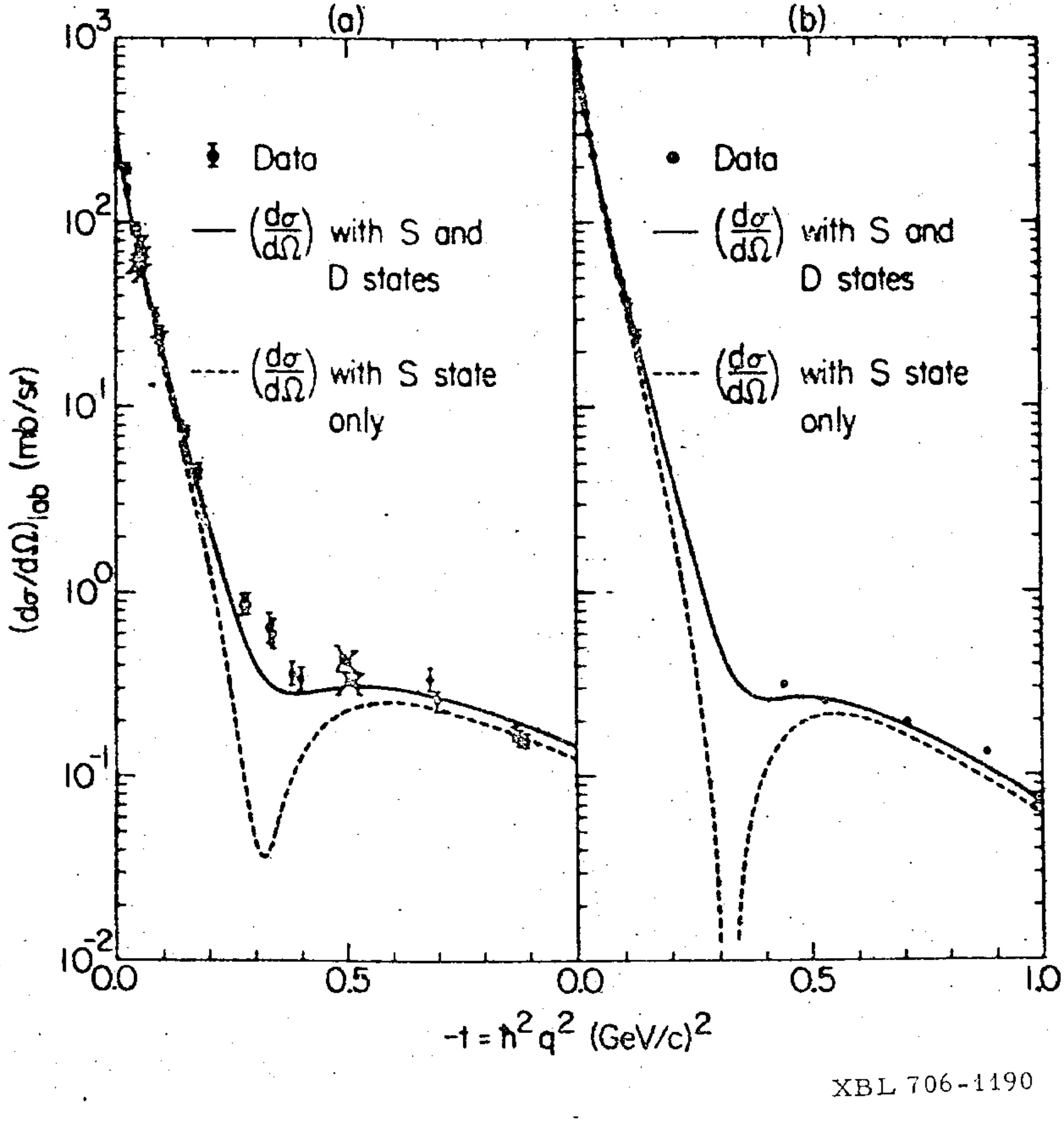
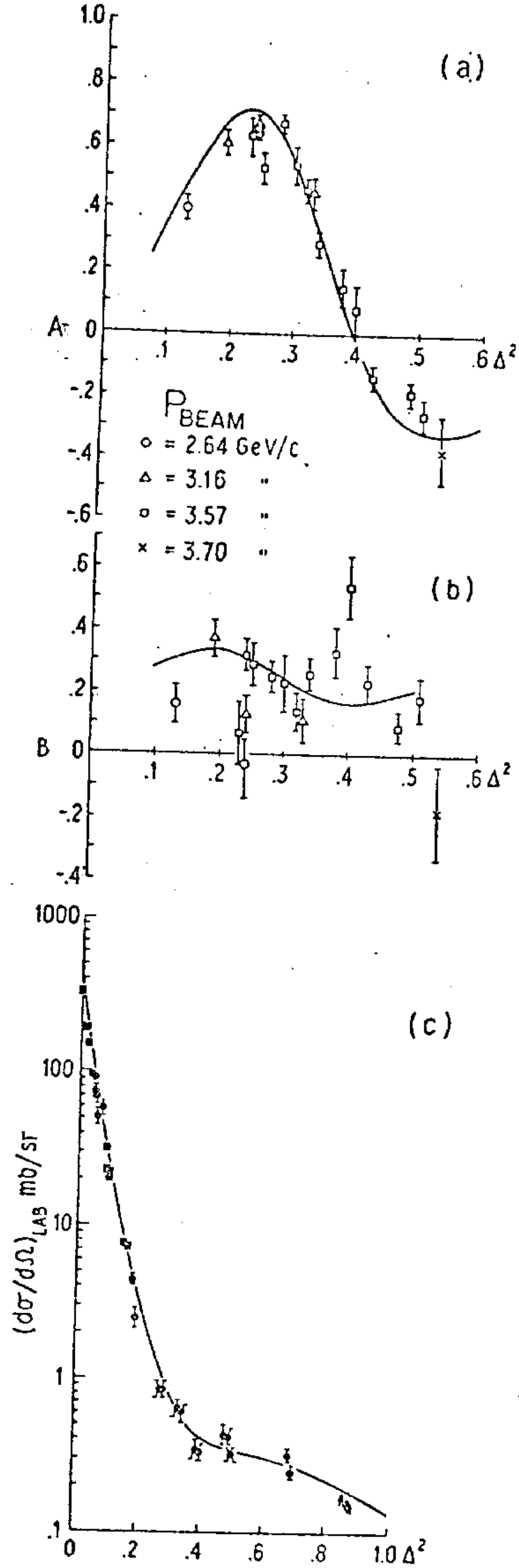
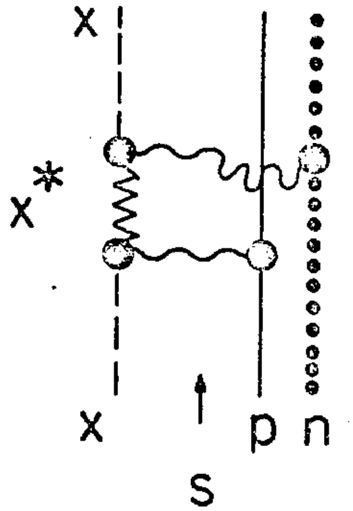

(a)
(b)
Data
$\left(\frac{d\sigma}{d\Omega}\right)$ with S and D states
$\left(\frac{d\sigma}{d\Omega}\right)$ with S state only
$(d\sigma/d\Omega)_{lab}$ (mb/sr)
$-t = \hbar^2 q^2$ (GeV/c)^2
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
0.0
0.5
1.0
XBL 706-1190

Fig. 28



(a)
(b)
(c)
A_T
B
$(d\sigma/d\Omega)_{LAB}$ mb/ST
Δ^2
P_{BEAM}
○ = 2.64 GeV/c
△ = 3.16 "
□ = 3.57 "
× = 3.70 "

Fig. 29



XBL706-3223

Fig. 30

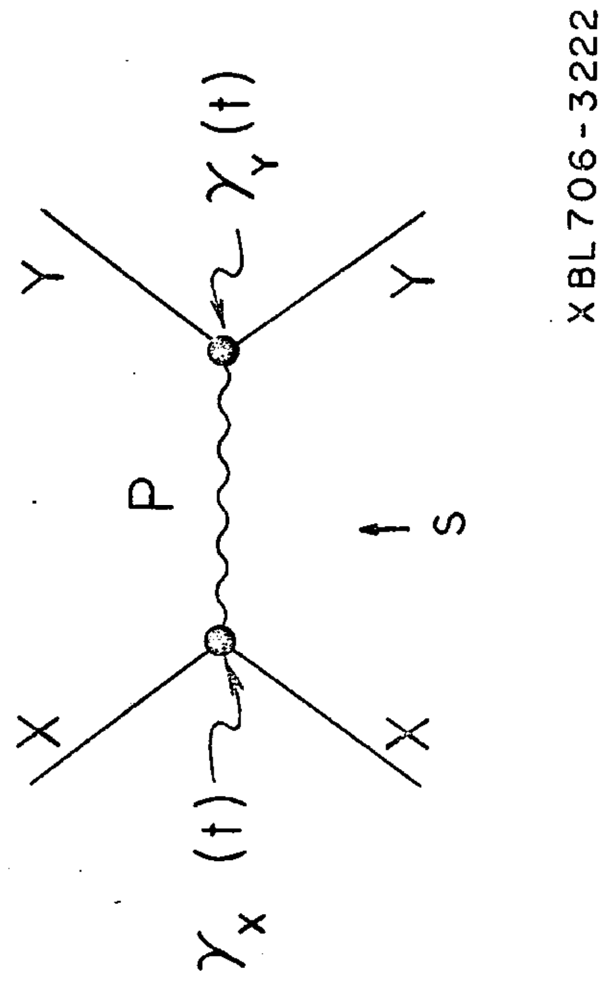


Fig. 31

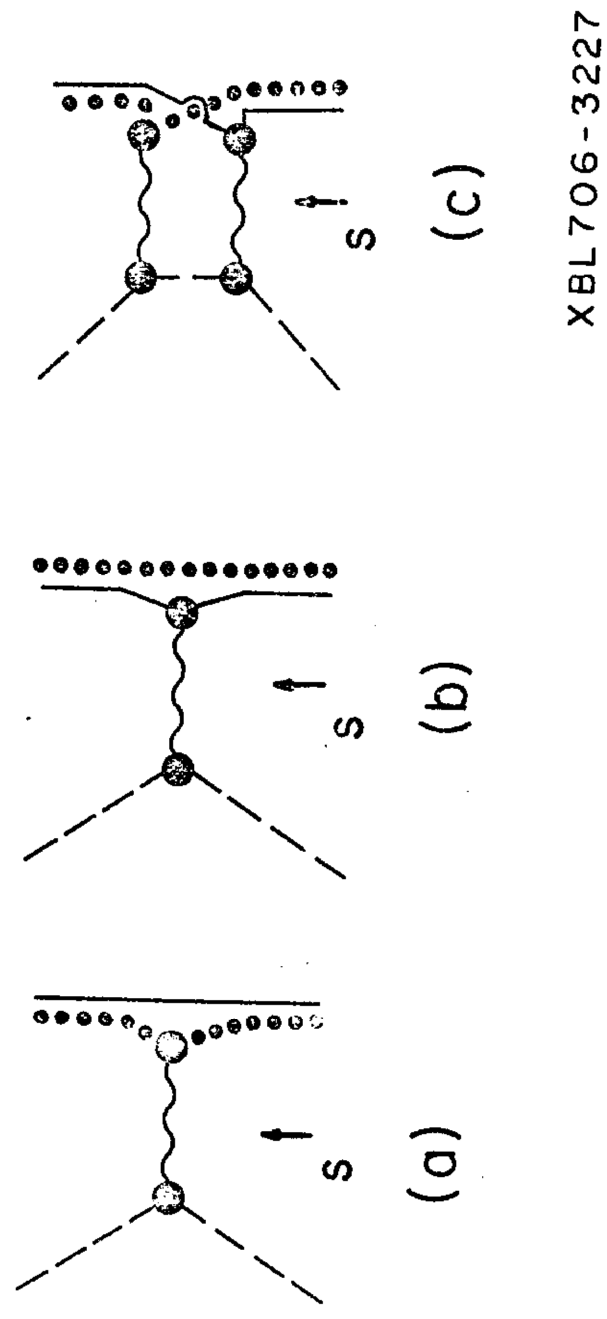


Fig. 32

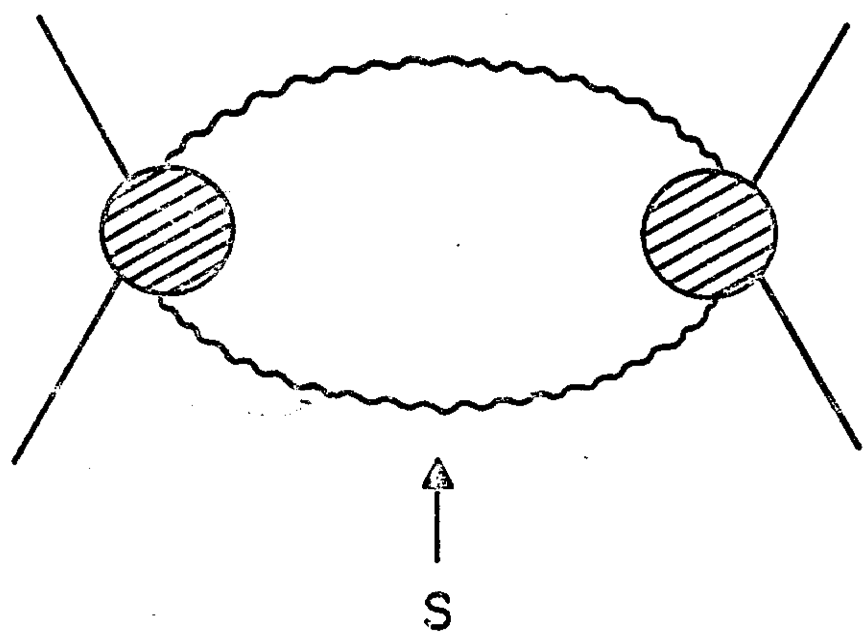


Fig. 33

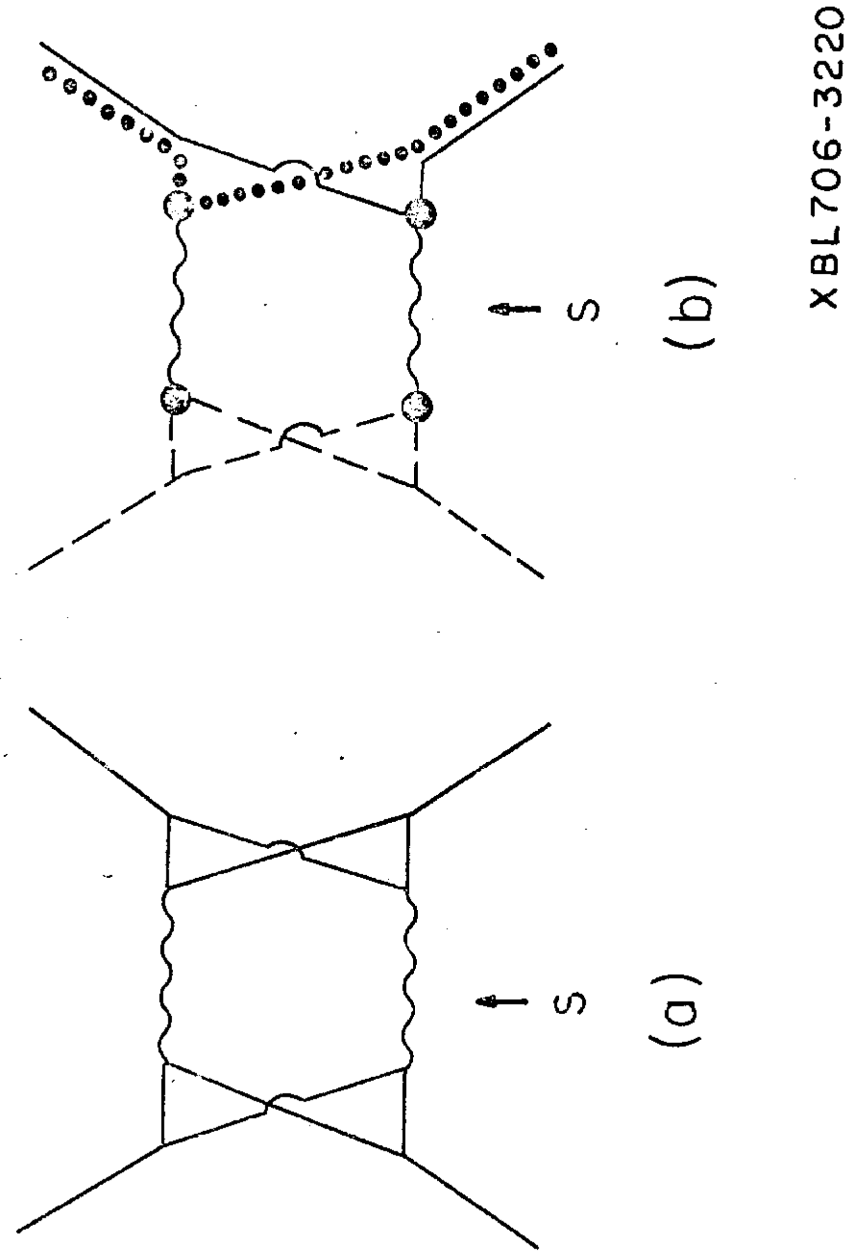


Fig. 34

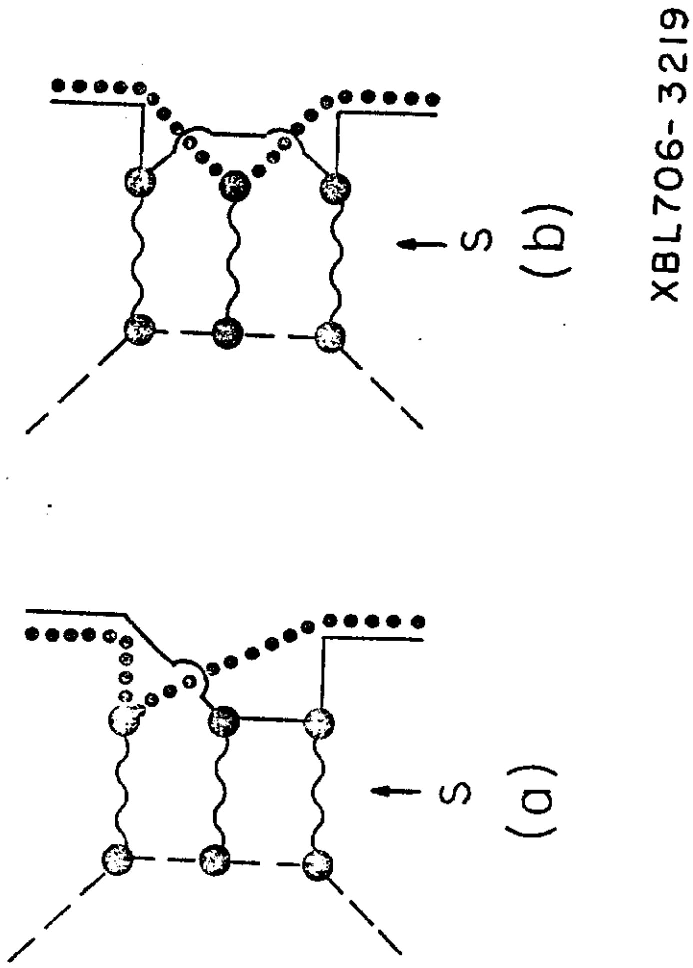


Fig. 35

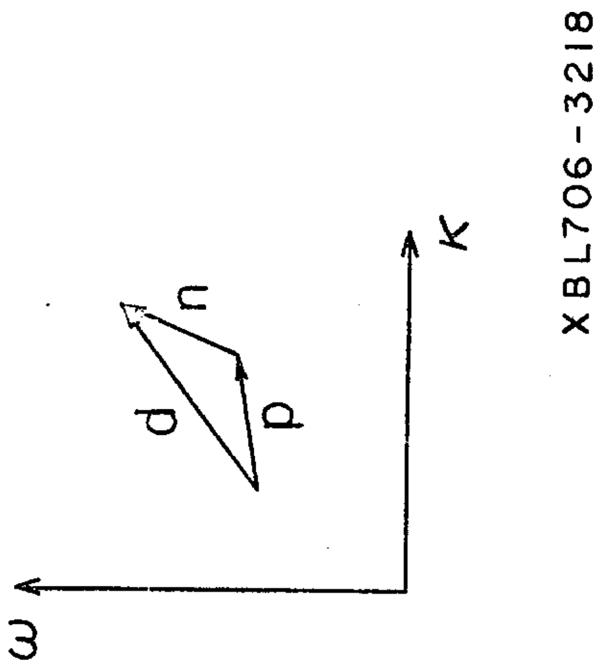


Fig. 36

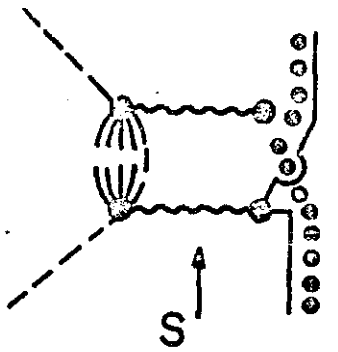


Fig. 37

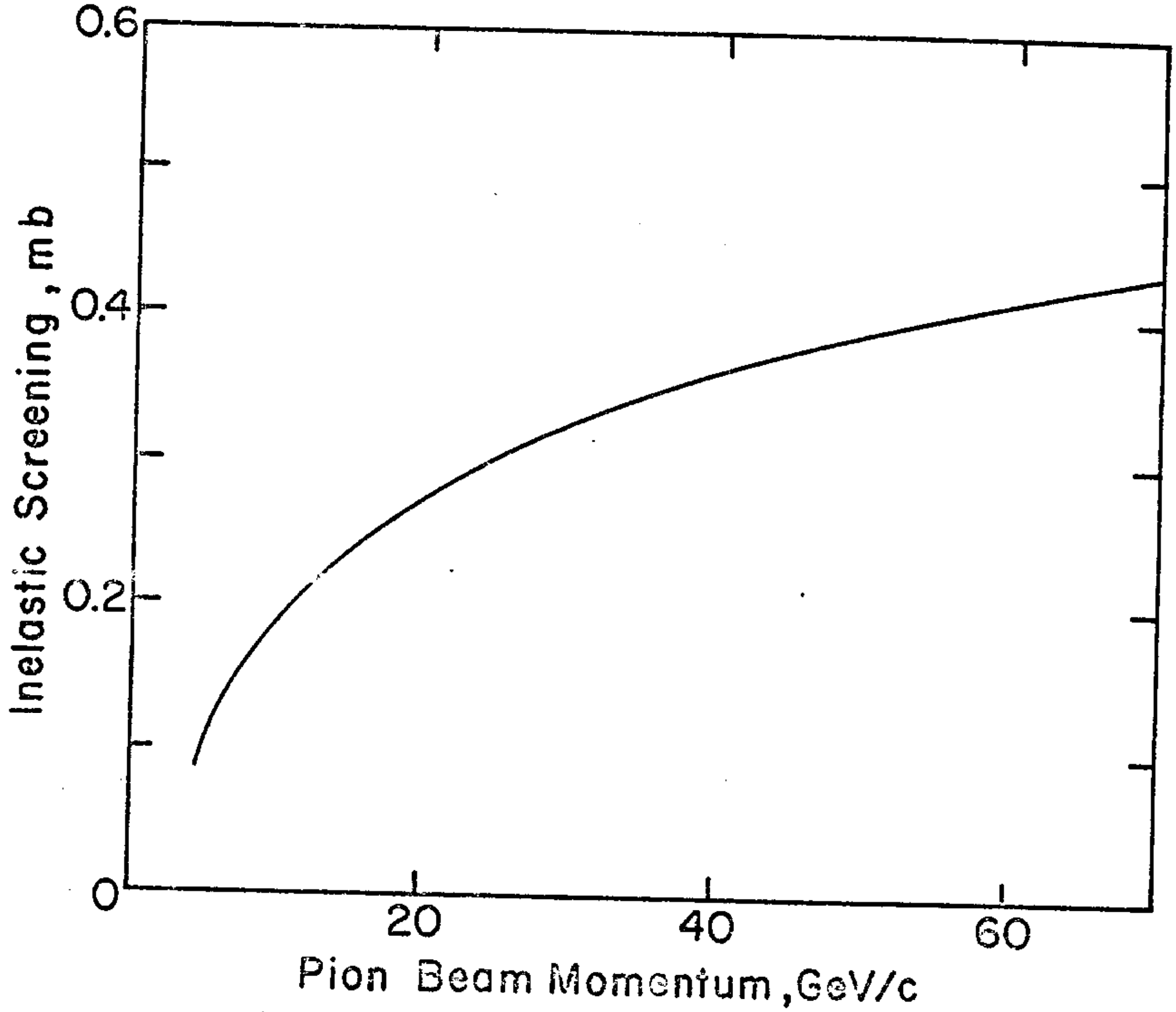
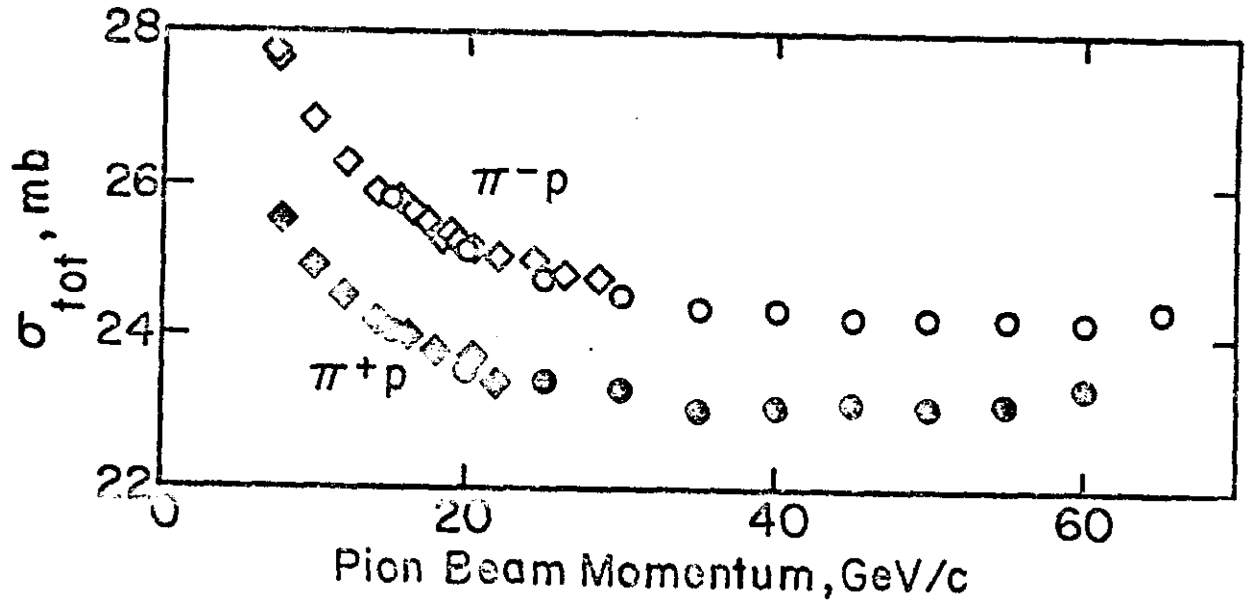

0.6
0.4
0.2
0
Inelastic Screening, mb
20
40
60
Pion Beam Momentum, GeV/c

Fig. 38



28
26
24
22
0
20
40
60
σ_{tot}, mb
π^-p
π^+p
Pion Beam Momentum, GeV/c

Fig. 39

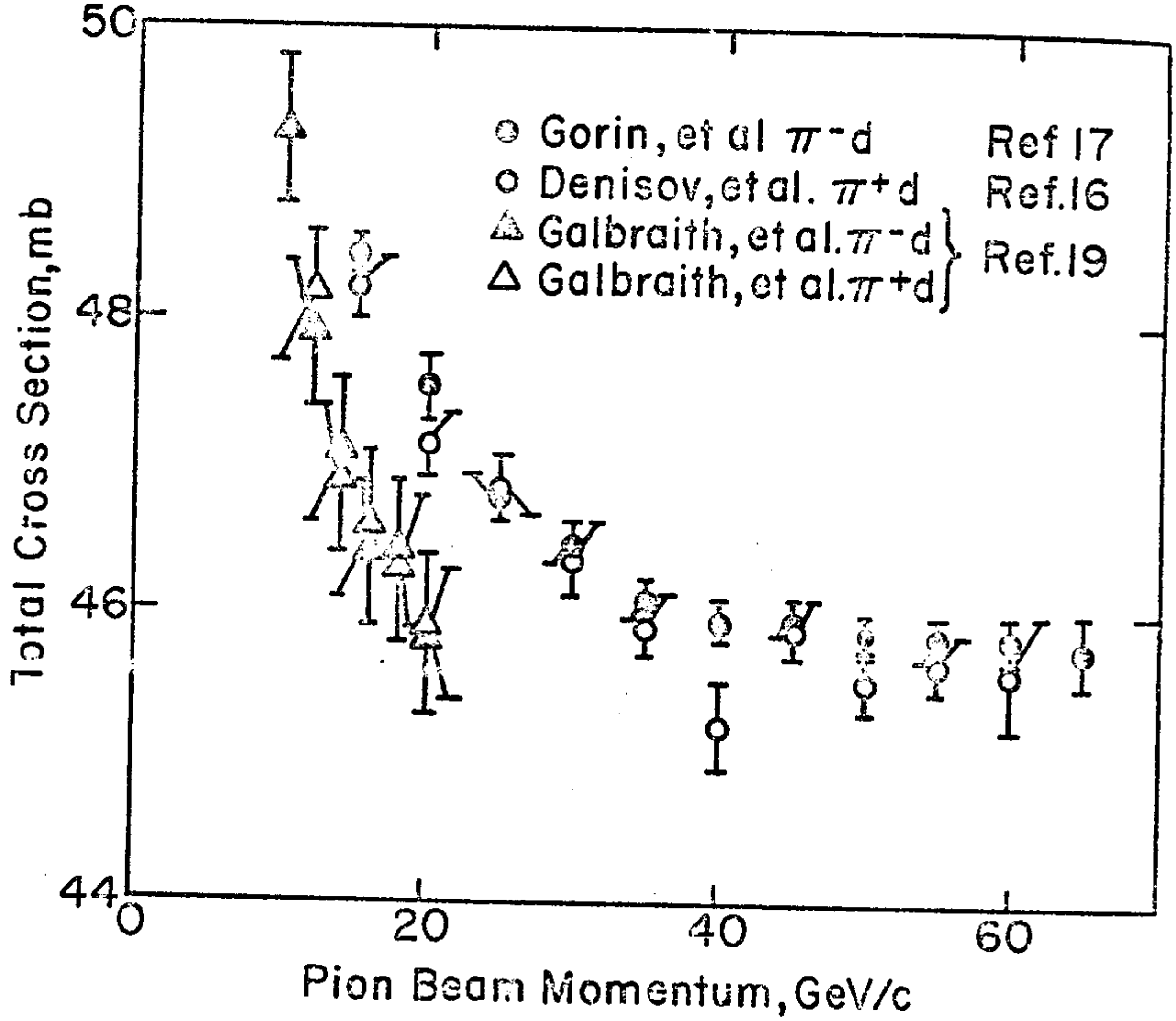

Gorin, et al π⁻d Ref 17
Denisov, et al. π⁺d Ref.16
Galbraith, et al. π⁻d
Galbraith, et al. π⁺d
Ref.19
Total Cross Section, mb
50
48
46
44
0
20
40
60
Pion Beam Momentum, GeV/c

Fig. 40

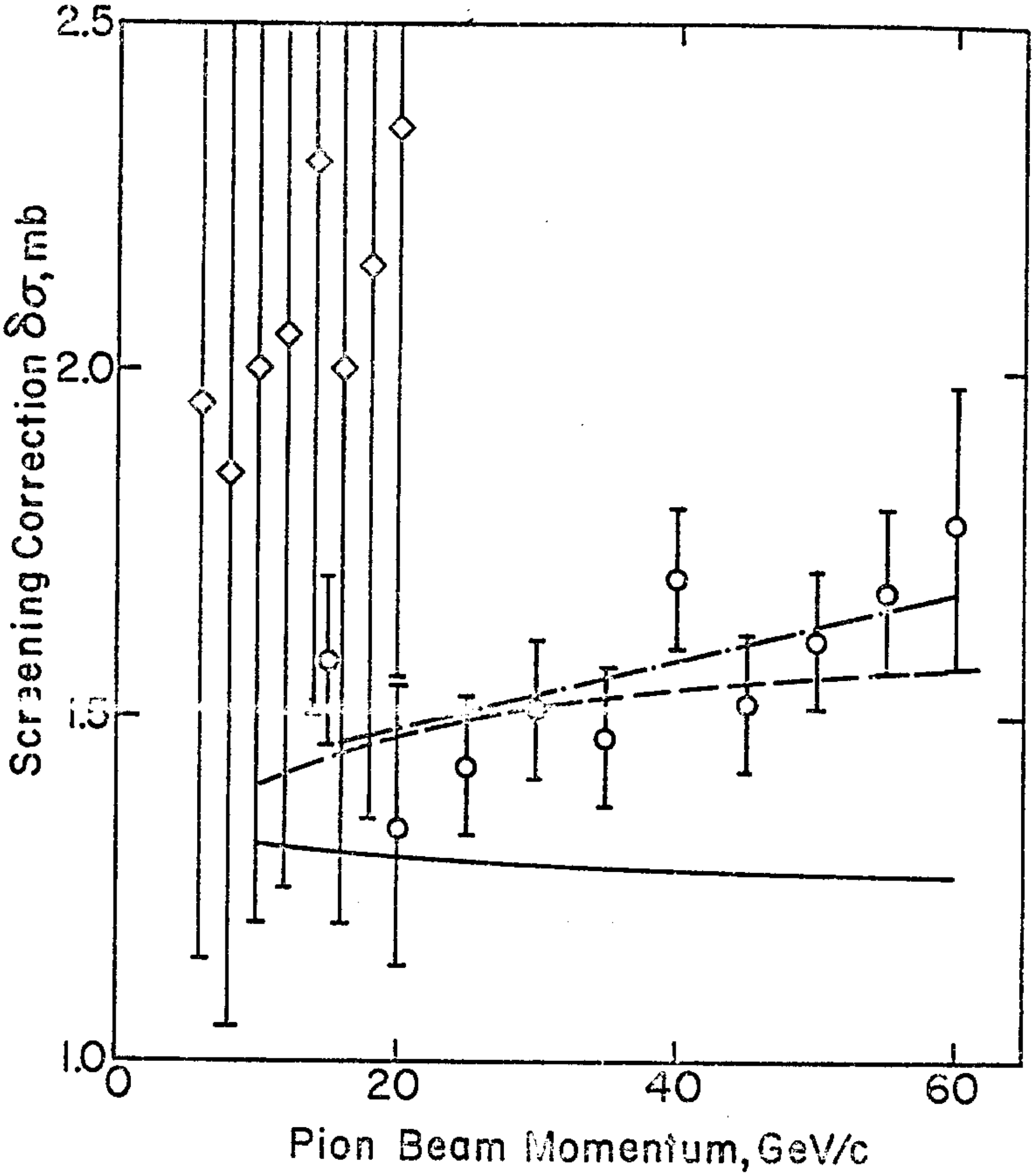
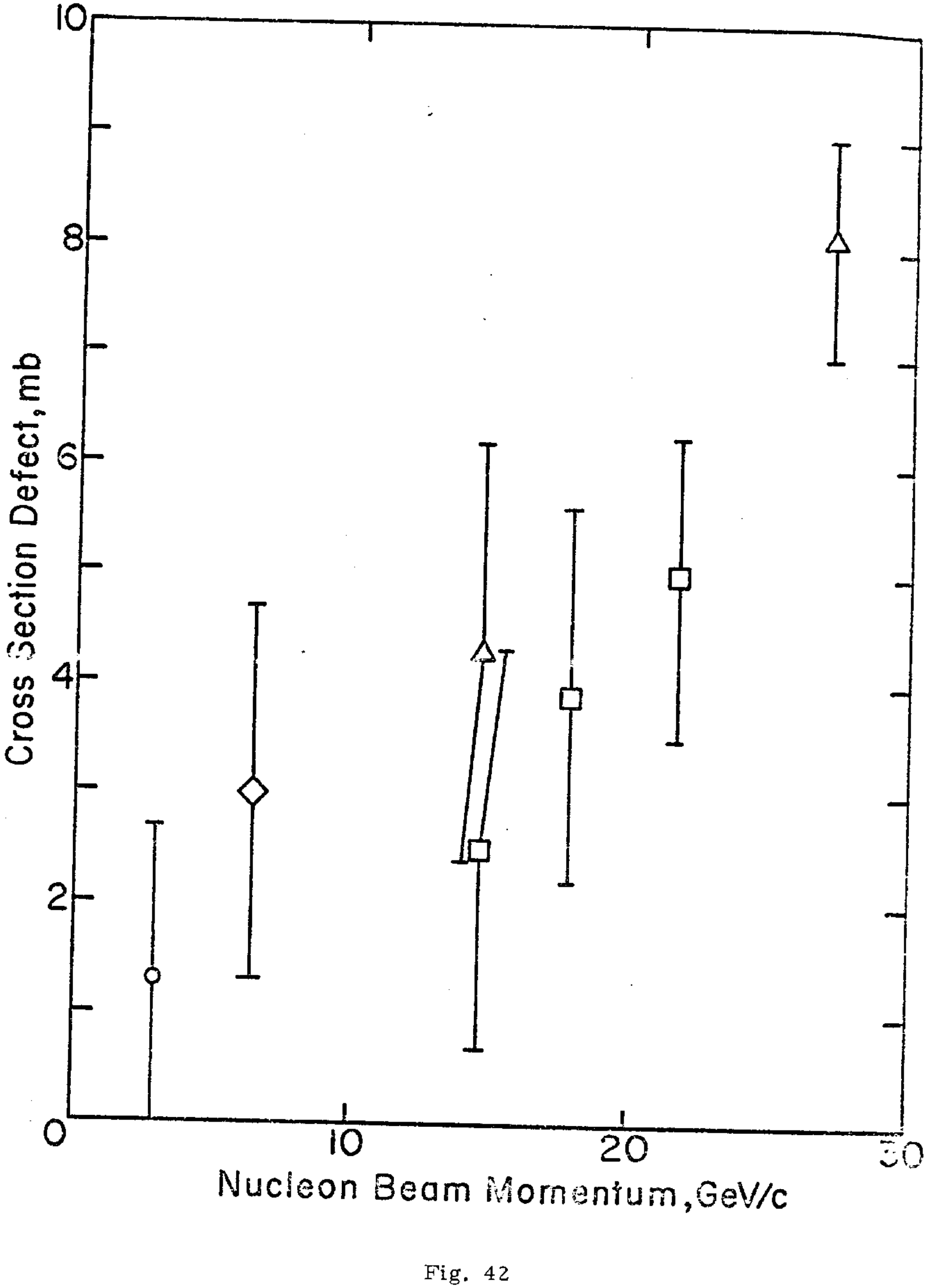
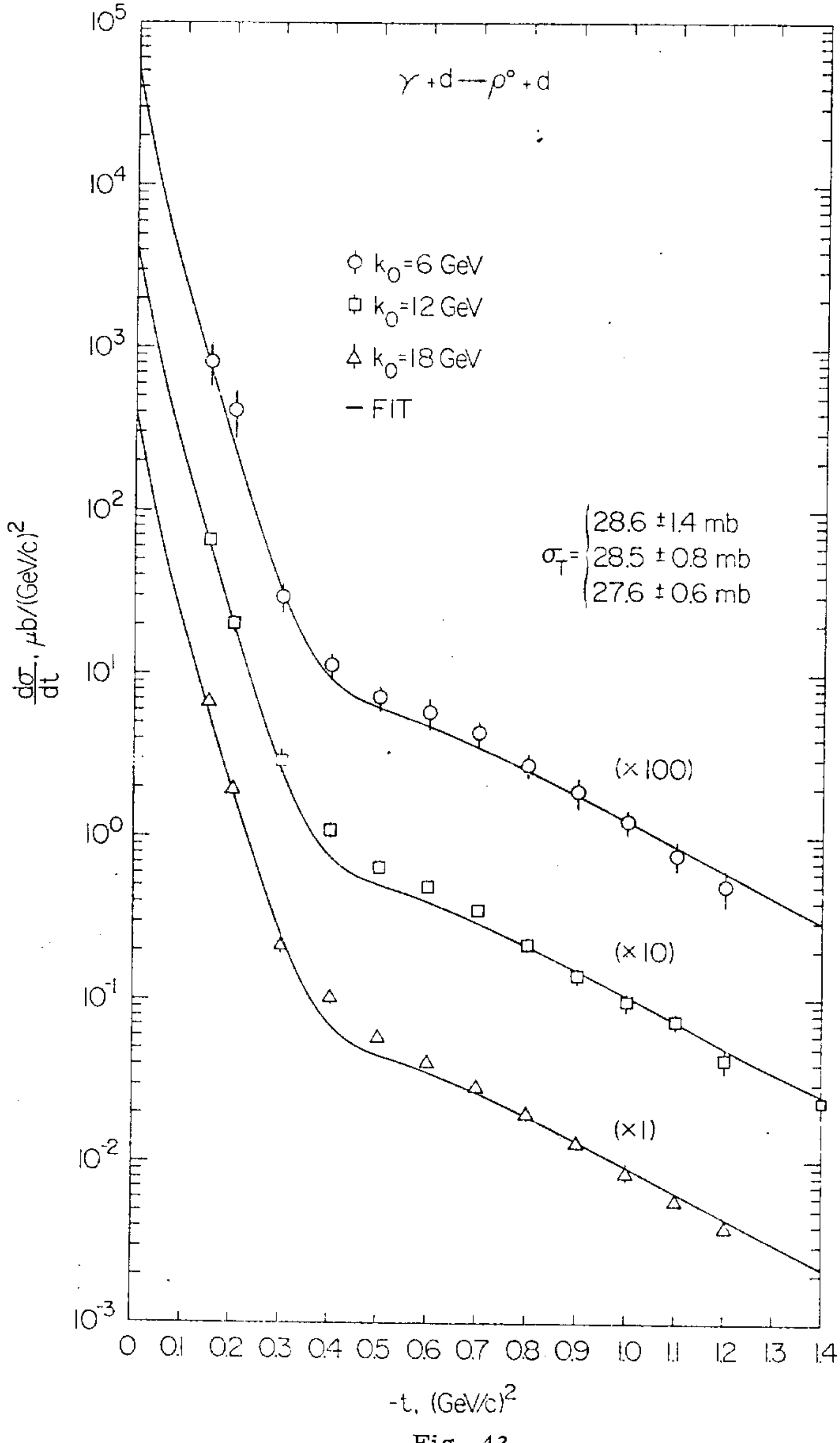

Screening Correction δσ, mb
2.5
2.0
1.5
1.0
0
20
40
60
Pion Beam Momentum, GeV/c

Fig. 41



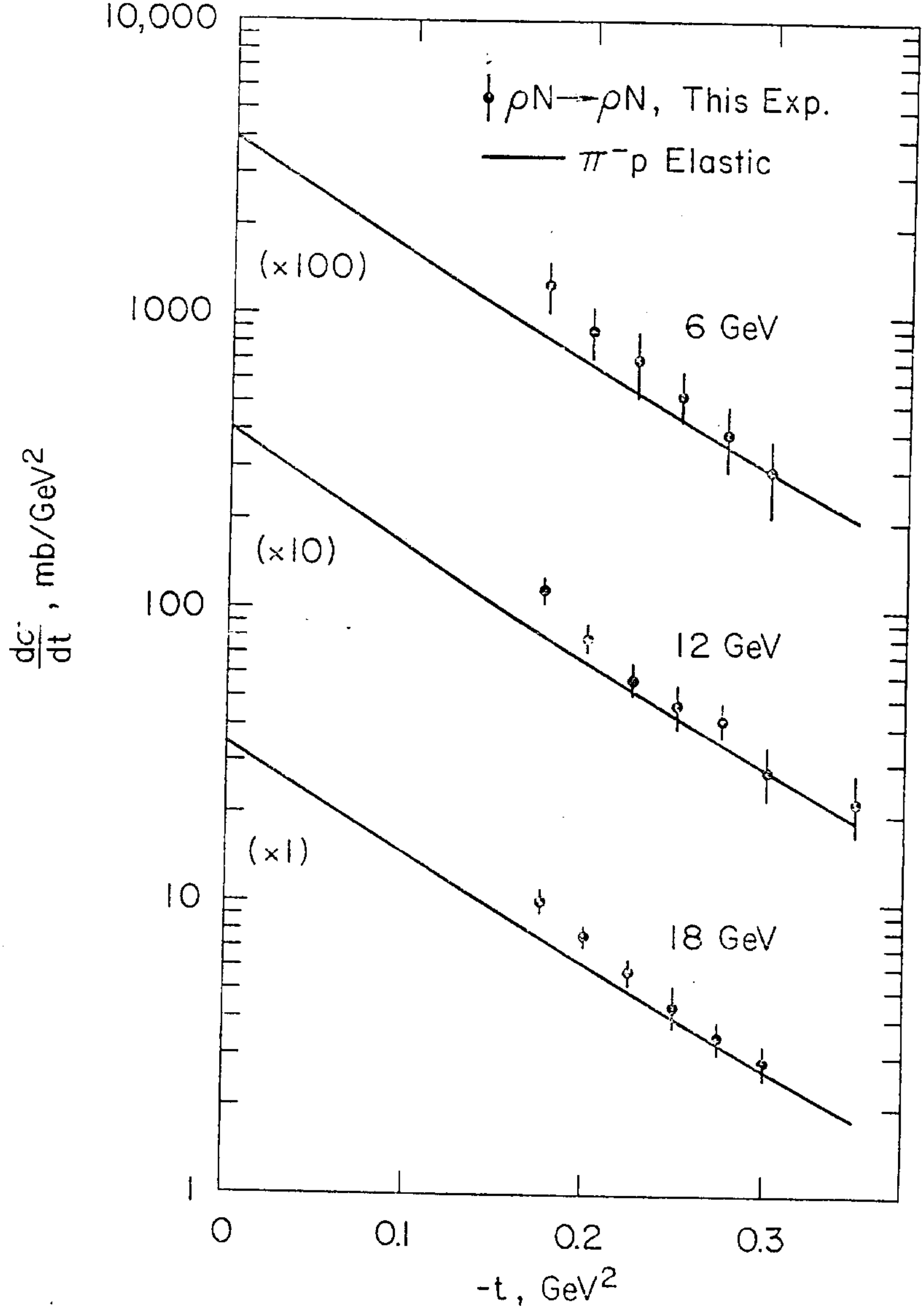
Cross Section Defect, mb
10
8
6
4
2
0
10
20
30
Nucleon Beam Momentum, GeV/c

Fig. 42



$\gamma + d \rightarrow \rho^{\circ} + d$
k_0=6 GeV
k_0=12 GeV
k_0=18 GeV
— FIT
σ_T = 28.6 ± 1.4 mb
28.5 ± 0.8 mb
27.6 ± 0.6 mb
(×100)
(×10)
(×1)
$\frac{d\sigma}{dt}$, μb/(GeV/c)^2
-t, (GeV/c)^2

Fig. 43



ρN→ρN, This Exp.
π⁻p Elastic
(×100)
6 GeV
(×10)
12 GeV
(×1)
18 GeV
10,000
1000
100
10
1
dσ/dt, mb/GeV²
0
0.1
0.2
0.3
-t, GeV²

Fig. 44