$\phi^3$ Analyticity and Finite-Energy Sum Rule For Inclusive Reactions

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ABSTRACT

Using $\phi^3$ theory as a model, the analytic structure of the six-point function is investigated in the kinematical region appropriate to inclusive reactions. With some idea about the analyticity, a finite-energy sum rule is derived. This sum rule can be used to study the concept of generalized duality. The most striking feature of the sum rule is a possibility that the "triple-Regge vertex function" can be calculated by the data on the inclusive reaction with relatively low $M^2$, i.e., the resonance production region.
I. INTRODUCTION

It has been conjectured that the cross section for

\[ a + b \rightarrow c + \text{anything} \]  \hspace{1cm} (1)

is related to the absorptive part of a scattering amplitude for

\[ a + b + \bar{c} \rightarrow a + b + \bar{c} \]  \hspace{1cm} (2)

when later is analytically continued to the proper kinematical region. \(^1\) Then various asymptotic behaviors of (1) can be obtained from that of (2). It is assumed that the asymptotic behaviors of (2) can be obtained by the \( O(2, 1) \) expansion. \(^2\) Subsequently, it has been verified in the context of field theory that the amplitude for the reaction (2), when continued analytically, indeed has the absorptive part which is proportional to the cross section for the reaction (1). \(^3\)

We see the analogy between the four-point function and the six-point function developing. The inclusive cross section and the six-point function satisfy a relationship similar to that between the total cross section and the four-point function. The \( O(2, 1) \) expansion in the six-point function corresponds to the Regge expansion in the four-point function. We therefore see that machinery developed for the four-point function (Forward-dispersion relations, Finite-energy sum rules, etc.) may be perhaps applicable to the six-point function. What follows is the first attempt along this line.

In order to start the program, we must first get some idea about the analyticity. No doubt the problem of analyticity and crossing for
the six-point function will be complicated. At present we can gain
insight only by investigating a reliable model. For this purpose, we
will use $\phi^3$ theory as our guide.

The kinematical variables for our problem are

\[
S = (p_\alpha + p_\epsilon)^2, \quad p_\alpha^2 = m_\alpha^2
\]
\[
t = (p_\alpha - q_\beta)^2, \quad p_\beta^2 = m_\beta^2
\]
\[
M^2 = (p_\alpha + p_\epsilon - q_\sigma)^2, \quad q_\sigma^2 = j^2
\]

where the momenta are defined by Fig. 1.

The result of our analysis indicates that the analyticity on the $M^2$
plane for fixed $t \leq 0$ and large $s$ and $s/M^2$ is directly related to the
analyticity in the mass variable of an ordinary Regge residue function.
Given the possibility that there might be some complex branch point on the
$M^2$ plane in addition to the singularity obtained from unitarity, we
must be cautious in applying analytic-function theory to the scattering
amplitude. We will, however, assume, for now, that such complex
branch points are absent. This assumed analyticity, together with
ideas about triple-Regge dominance, yields a sum rule which corresponds
to the finite-energy sum rule for the four-point function.

In Sec. II. we discuss the optical theorem for the six-point function.
In Sec. III we consider possibilities for complex cuts and state a theorem
on the analyticity of the relevant Feynman diagram. In Sec. IV. we
prove the theorem. This section can be skipped without loss of continuity. In Sec. V we derive a finite-energy sum rule. In Sec. VI we present sum rules which require additional assumption about fixed poles, etc.

II. GENERALIZED OPTICAL THEOREM

The cross section for reaction (1) can be written as

$$\frac{d\sigma}{d\omega} = \frac{\alpha_s^2}{(2\pi)^2} \Delta(S,t,\Lambda^2)$$

$$\Delta(S,t,\omega) = \frac{\mathcal{E}_a \mathcal{E}_b}{(2\pi)^2} \int d^4x \, \epsilon^{\alpha_\mu_1 \cdots \alpha_{\mu_6}} \left[ \gamma_{\alpha} \left( q_{\mu_1} \cdots q_{\mu_6} \right) \right]$$

where $\phi_c(x)$ is the field operator for the particle $c$. Let $T$ be the amplitude for the process shown in Fig. 2.

$$T = \frac{\mathcal{E}_a \mathcal{E}_b}{(2\pi)^2} \int d^4x \, e^{i\mathbf{x} \cdot \mathbf{p}} \left( \gamma^\alpha \right) \left( q_{\mu_1} \cdots q_{\mu_6} \right) \left| p_1 \right| \left| p_2 \right|$$

$T$ is a function of 25 Lorentz scalars that can be constructed out of the six four vectors and thus it has singularities for 25 different channels.
It has been shown that in the forward limit when \( s \) and \( t \) are fixed, \( s > s \)-channel threshold, \( t < 0 \), the absorptive part of \( T \) in \( M^2 \) is proportional to \( A \). We would like to sketch the reasoning behind above statement.

Let us first define what we mean by the forward limit. Since the limit is used to relate the cross section to the absorptive part of \( T \), all the four vectors must approach a real limit. That is \( \lim p_i = \lim p_i' = \) real four vector, \( \lim q = \lim q' = \) real four vector. But it is important to keep in mind that the direction and the rate at which these four vectors approach the limit is not specified. For example, in the special frame in which \( p_a = 0 \), we can have

\[
\begin{align*}
\vec{p}_a &= \left( m, \, 0, \, 0, \, 0 \right), \quad \vec{p}_i' &= \left( m, \, 0, \, 0, \, 0 \right) \\
\vec{p}_b &= \left( \vec{e}_i, \, \vec{e}_b, \, \vec{e}_i', \, \vec{e}_b' \right), \quad \vec{p}_b' &= \left( \vec{e}_i, \, \vec{e}_b, \, \vec{e}_i', \, \vec{e}_b' \right) \\
\vec{q} &= \left( \vec{e}_a, \, \vec{e}_{\perp a}, \, \vec{e}_{\perp a}', \, \vec{e}_{\perp a}' \right), \quad \vec{q}' &= \left( \vec{e}_a, \, \vec{e}_{\perp a}, \, \vec{e}_{\perp a}', \, \vec{e}_{\perp a}' \right)
\end{align*}
\]

In the forward limit, all \( \epsilon \)'s approach zero. But it is our choice as to how they go to zero. For what follows, we make the distinction between primed and unprimed variables only if it is important to keep track of \( i \epsilon \)'s. In the forward limit when \( s \) and \( t \) are fixed and \( s > s \)-channel threshold, \( t < 0 \), only those variables that are linearly related to \( p_b \cdot q \), \( p_b' \cdot q' \), or \( p_b' \cdot q \) can vary. They are
\[ M^2 = (p_a + i s - q)^2 \]

\[ M_a^2 = (p_a - q)^2 = s^2 - 2t - M_0^2 \]

\[ M_b^2 = (p_b + q)^2 = s^2 - 2t - M_0^2 \]

\[ \chi_a = (p_a - q)^2 = 2m^2 + \gamma - s^2 - t \]

\[ \chi_b = (p_b - q)^2 = 2m^2 + \gamma - s^2 - t \]

\[ \chi_7 = (p_a + q)^2 = s^2 - 2t + \gamma - s^2 - t \]

\[ \chi_8 = (p_b + q)^2 = s^2 - 2t + \gamma - s^2 - t \]

where we have set \( p_a^2 = p_b^2 = m^2 \). These channels are shown in Fig. 3.

The absorptive part in \( M^2 \), when \( s = s_0 + i \epsilon_1 \), \( s' = s_0 + i \epsilon_2 \), \( t \) real is

\[
2 \text{Im} \left[ \text{Im} \left( \frac{1}{s - s'_+ i + i t} \right) \right] \Rightarrow s'_+ = s_0 + i \epsilon_2, \quad s'_0 = s_0 + \epsilon_3, \quad s'_0 = s_0 + i \epsilon_2 \]

\[
M_a^2 = 2t + 2m^2 - M_b^2 - i \epsilon_3 \quad M_b^2 = 2t + 2m^2 + i \epsilon_3 \quad M_0^2 = 2t + 2m^2 + i \epsilon_3 \quad M_3^2 = 2t + 2m^2 + i \epsilon_3 \]

\[
\chi_7 = s_0 + p^2 + i (-\epsilon_2 + \epsilon_3) \quad \chi_8 = s_0 + p^2 + i (-\epsilon_2 - \epsilon_3) \]

\[
\chi_9 = s_0 + M_0^2 + 2m^2 + i (\epsilon_2 + \epsilon_3) \quad \chi_{10}^s = s_0 + M_0^2 + 2m^2 + i (\epsilon_2 - \epsilon_3) \]

\[
\chi_7 = s_0 + M_0^2 + 2m^2 + i (\epsilon_2 - \epsilon_3) \quad \chi_8 = s_0 + M_0^2 + 2m^2 + i (\epsilon_2 + \epsilon_3) \]

\[
\chi_9 = s_0 + M_0^2 + 2m^2 + i (\epsilon_2 + \epsilon_3) \quad \chi_{10} = s_0 + M_0^2 + 2m^2 + i (\epsilon_2 - \epsilon_3) \]

\[
(5)
\]
Note that if we choose \( \epsilon_1, \epsilon_2, \epsilon_3 \) such that \( |\epsilon_1| > |\epsilon_3|, |\epsilon_2| > |\epsilon_3| \)
only the discontinuity in \( M_2 \) and \( M_4 \) contributes to the difference. All
other channel variables are evaluated in the same side of their respective
cuts. (That is, small imaginary part for all variables, except \( M_2 \) and \( M_4 \) does
not change sign between two terms on the right-hand side of Eq. 5, the unitarity
equation. In other words \( T \) has singularities corresponding to each channel
associated with variables listed in footnote 4, but it is possible to
isolate a sheet on the \( M_2 \) plane which contains only the singularities
due to the \( M_2 \) and \( M_4 \) channels. From now on "\( M_2 \) plane" refers to
this sheet. The absorptive part of \( T \) in \( M_2 \) can be evaluated from
Eq. 3.

\[
\int_{\mu^2} \mu \to \mu^* \left( \sum_{\epsilon_1, \epsilon_2, \epsilon_3} \frac{E_1 E_2}{ \mu^2 + m_1^2} (q^2 - \mu^2)^2 (2\pi)^4 \right)
\]

\[
\times \left[ \delta^4 \left( p_1 - p_2 - q - p_3 \right) \langle f_{\mu} \mid \Phi_{\mu}^+ \mid \phi_{\mu} \rangle \langle \Phi_{\mu}^+ \mid \phi_{\mu}^+ \mid f_{\mu} \rangle \right]
\]

\[
+ \tilde{\epsilon}_3 \left( \delta^4 \left( p_3 - p_1 - q - p_2 \right) \langle f_{\mu} \mid \Phi_{\mu}^+ \mid \phi_{\mu} \rangle \langle \Phi_{\mu}^+ \mid \phi_{\mu}^+ \mid f_{\mu} \rangle \right)
\]
Note that the first term on the right hand side is non-zero only if
\( p_a + p_b - q = p_n \) and the second term is non-zero only if \( p_a - p_b - q = p_n \).
These two regions do not overlap. Consider the region where
\( p_a + p_b - q = p_n \). We want to show that

\[
\mathcal{A}_{s, t, \mu^c} \left( s = s_c + i \epsilon_c, t = t_c - i \epsilon_c, \mu^c \right) = \mathcal{A}
\]

where

\[
\mathcal{A} = \lim_{q^2 \to 0} \frac{1}{2\pi i} \sum_{\mu^c} \left[ \frac{m_e}{m_n} \right]^{1/2} \left( \frac{q^2}{\mu^c} \right) \frac{1}{2\pi} \frac{1}{(2\pi)^4} \left\{ \frac{1}{c_{\mu^c}} \left| \frac{1}{c_{\mu^c}} \right| \right\}
\]

Of course the distinction between \( \text{Abs T} \) and \( \mathcal{A} \) are "in" and "out" states.

Let us define an analytic function \( F(s, M^2, t) \) such that

\[
\lim_{\epsilon_1, \epsilon_2 \to 0} F_n \left( s = s_0 + i \epsilon, M^2, t, \epsilon_1, \epsilon_2 \right) = \sqrt{\frac{E_a E_b}{m_n^2}} \left\langle \frac{d^2}{d^2} \phi^+_n \left( \phi^+_n \right) \right\rangle
\]

Then Eq. 6 is proven if we can show that

\[
\lim_{\epsilon_1, \epsilon_2 \to 0} F_n \left( s = s_0 + i \epsilon, M^2, t, \epsilon_1, \epsilon_2 \right) = \sqrt{\frac{E_a E_b}{m_n^2}} \left\langle \frac{d^2}{d^2} \phi^+_n \left( \phi^+_n \right) \right\rangle
\]

since the continuation of \( T \) from \( s' = s_0 + i \epsilon_1 \) to \( s' = s_0 - i \epsilon_1 \) is given by the continuation of \( F_n \) from \( s' = s_0 + i \epsilon_1 \) to \( s' = s_0 - i \epsilon_1 \). Let \( t < 0 \) be below the t-channel threshold. By reducing \( b \) we obtain
where we have performed the $x_0$ integration by using a integral representation for the theta function. If we evaluate Eq. 8 in the rest frame of $\mathbf{a}$, 

$$s = 2(m^2 + mE + i\epsilon_b + iE\epsilon_b),$$

the continuation to the opposite side of the cut in $s$ is equivalent to continuing $p_0^a$ to the other side of its cut. On the other side of the cut in $p_0^a$, the sign of $i\epsilon$ changes.

$$\Gamma(s_0 - i\epsilon_1, M^2 + \epsilon_2, t) =$$

$$\left[ \langle p_a | \phi_b(x_0, s) \frac{1}{p_0^a - p_0^b - H + i\epsilon} \phi_c^\dagger(x_0, s) | \nu \rangle + \langle p_a | \phi_c(x_0, s) \frac{1}{p_0^a - p_0^b - H + i\epsilon} \phi_b(x_0, s) | \nu \rangle \right]$$

$$\left[ \langle p_a | \phi_b(x_0, s) \frac{1}{p_0^a - p_0^b - H + i\epsilon} \phi_c^\dagger(x_0, s) | \nu \rangle + \langle p_a | \phi_c(x_0, s) \frac{1}{p_0^a - p_0^b - H + i\epsilon} \phi_b(x_0, s) | \nu \rangle \right]$$
If $b$ is reduced in $\sqrt{\frac{E_a E_b}{m_a m_b}} < n |\phi(0)| p_a p_b^{\text{in}}$, it is quite easily seen that indeed Eq. 7 holds.

We stress again the most important point that there is a sheet in $M_2$ plane which contains only the $M_2$ and $M_1$ channel singularities. If the singularities from the other channels cannot be separated, there is no simple relation between the cross section for the inclusive reaction and the absorptive part of the six-point function. Let us take a particular example $a = \pi^-$, $b = \text{proton}$, and $c = K^-$. The process which gives the right hand cut, shown in Fig. 3a, is non zero when

$$M^2 \geq (\mu_\pi + m^2).$$

(For the left hand cut see below). The break in the cut is due to the requirement that $q^2 = \mu_K^2$. The cut in the region $(\mu_K + m^2)^2 \leq M^2 \leq (\sqrt{s} - \mu_K)^2$ corresponds to the emission of $K$ as it can easily be verified that $q_0 \geq \mu_K$. The cut in the region $M^2 \geq (\sqrt{s} + \mu_K)^2$ corresponds to the three-particle scattering process since $q_0 \leq -\mu_K$ provided $s$ and $s'$ are analytically continued to the proper side of the cut.

The second term in Eq. 7 is shown in Fig. 3b. We are interested in the case where $s$ and $s'$ are held fixed and large. In particular $(p_a - q)^2 = t < 0$, $(p_a - p_b')^2 \approx -s < 0$, $(q + p_b')^2 = s + \mu^2 - M^2 + t$. Then this diagram corresponds to the cross section for the inclusive process $K^- + p \to \pi^+ + \text{anything}$ and the scattering process $K^- + \pi^+ + p \to \text{anything}$. 
The incident energy for these two reactions are $(p_b + q^-) = s + \mu_{-}^2 - M^2 + t$ and $M^2$ respectively. The momentum transfer between $\pi$ and $K$ is $t$. The cut on the $M^2$ plane corresponding to this process is located at the position

$$M_{1,2}^2 = 2t + 2m^2 - m_i^2 \geq (\mu_{++} + \mu_{--})^2 \quad M_{1,2}^2 = m_i^2.$$ 

This cut corresponds to the left hand cut shown in Fig. 4.

**III. COMPLEX CUTS?**

So far we have been discussing the singularities whose existence is guaranteed by unitarity. Are there any other singularities? It is our task to investigate the additional singularity structure of the amplitude $T$, besides the cuts shown in Fig. 4, in the region

$$|M^2| \leq (\sqrt{s} - \mu_{-})^2.$$ 

We do not have to look far to find such singularities. In fact a box diagram shown in Fig. 5 will give a complex branch point on the physical sheet in the region $|s/M^2| = 0(1)^5$. Another region of interest is where $|s/M^2| >> 1$. In this region such a trivial example cannot be found. Therefore, restricting oneself to region $|s/M^2| >> 1$, we will now investigate the singularity structure implied by certain class of Feynman diagrams in the $\phi^3$ theory. We will notice some very important simplifications.

In the region $|s/M^2| >> 1$, we expect that the dominating process in the inclusive reaction is the Regge exchange shown in Fig. 6. In
In particular in the case of $\pi^+ p \rightarrow \pi^+ + X$ at $M^2 = m_p^2$, Fig. 6 represents an elastic $\pi^+ p \rightarrow \pi^+ p$ process which is dominated by the Pomeron exchange. Experimentally in the reaction $p + p \rightarrow p + X$, it is seen that $I = 1/2$ baryon resonances are produced and the cross section is constant in energy. Furthermore, the $\Delta_{33}$ resonance production cross section goes down rapidly with energy. This indicates that in the reaction $pp \rightarrow p + X$, Pomeron exchange gives the cross section which is constant in $s$ and the lower lying trajectories for $p$ gives the contribution which decrease in $s$. In fact these experiments tell us that Fig. 6 is the dominant contribution. We use this experimental result to say that in the limit of large $|s/M^2|$, only certain class of diagrams is important in the $\phi^3$ theory. Consider the diagram shown in Fig. 7.

(i) The four point function associated with the lower black blob corresponds to the arbitrary sum of diagrams in the $\phi^3$ theory such that it behaves as $\left[-(p_2 + k)^2\right] \alpha(t) \beta(k_1^2, (k + p_2 - q)^2, t)$ in the limit of large $(p_2 + k)^2$. Similarly for the upper black blob. Furthermore, we assume that the asymptotic behavior of $\beta(m_1^2, m_2^2, t)$ on the complex $m_1^2, m_2^2$ plane is such that a double dispersion relation can be written. The ladder diagrams satisfy these criteria.

(ii) The checked blob is a six-point function which represents arbitrary Feynman diagrams with $n$ number of propergators and
number of loops.

From the experimental evidence presented above, in the limit of large $s/M^2$ the set of diagrams belonging to Fig. 7 gives the dominating contribution to the amplitude. We therefore restrict ourselves to these diagrams. A crucial question is whether the class of diagrams contained in Fig. 7 possesses singularities other than those required by analyticity. To answer this, we will prove a following theorem in the next section.

**Theorem:** In the limit of large $s$, the necessary condition for the diagram (Fig. 7 satisfying (i) and (ii) above) to possess complex branch points on the physical $M^2$ plane is that $\beta(m_1^2, m_2^2, t)$ possesses complex branch point on the $m_1^2$ or $m_2^2$ plane or a branch point at $m_1^2$ or $m_2^2 = \mu_0^2$.

If $\beta(m_1^2, m_2^2, t)$ possesses only a cut on the real axis at $\mu_0^2 < m_1^2, m_2^2$, the analyticity of the diagram Fig. 7 can be deduced from that of Fig. 8 and only the cut due to the unitarity shown in Fig. 3 is present in the amplitude $T$. This theorem reduces the study of the six-point function analyticity to that of four-point Regge-residue function in this particular kinematical limit. For example, if we sum over only the leading logarithms in the ladder diagram, $\beta(m_1^2, m_2^2, t) = \text{constant}$. Thus to this order, Fig. 7 contains cuts only on the real axis corresponding to the unitarity cut shown in Fig. 4. We feel however uneasy to restrict ourselves to the leading log since the nonleading log is also important.
in obtaining the asymptotic behavior of the residue function

\[ \beta(m_1^2, m_2^2, t) \rightarrow \sqrt{2m_1^2} \lambda(t) \]

as \( m_1 \to \infty \) \hspace{1cm} (9)

Incidentally, the asymptotic Eq. 9 does not give any complex cut on the \( m_1^2 \) plane.

IV. THEOREM

This section contains a proof of the theorem. A reader who is not interested in the detail may skip this section without losing continuity. The Feynman amplitude of Fig. 7 with \( f \) loops and \( n \) propagators in the checked blob may be written as

\[
F < \prod_{i=1}^{f} \frac{\beta(k_i^2, (k_i^2 - q^2)^2, t)}{\left(k_i^2 - m_i^2\right)^2 - \mu_r^2} \frac{\lambda^x}{\left(\sum_{j=1}^{n} (k_j^2 - m_j^2)^2\right)^n} \]

where we have labeled the momenta flowing through the loops by \( k_i \), momenta associated with the internal lines \( q_r \) and the mass of the internal particles was taken to be \( \mu_0^2 \). By the asymptotic behavior assumed above (i), we can write an integral representation
\[
\frac{\beta(k_1, q, k_2, \tau)}{(k_1^2 - \mu_1^2)(k_1^2 - \mu_2^2)} = \int \frac{\delta\left(\mu_2, \mu_2, t, \mu_2\right)}{(k_1^2 - \mu_1^2)(k_1^2 - \mu_2^2)} d\mu_1^2 d\mu_2^2
\]

\[+ \int \frac{\delta\left(\mu_1^2, \mu_2^2, t, \mu_2^2\right)}{(k_2^2 - \mu_2^2)(k_2^2 - \mu_1^2)} d\mu_2^2 + \int \frac{\delta\left(\mu_1^2, \mu_2^2, t, \mu_1^2\right)}{(k_1^2 - \mu_1^2)(k_1^2 - \mu_2^2)} d\mu_1^2
\]

\[= \int \frac{\delta\left(\mu_1^2, \mu_2^2, t, \mu_2^2\right)}{(k_2^2 - \mu_2^2)(k_2^2 - \mu_1^2)} d\mu_2^2 d\mu_1^2
\]

(11)

Where the path of integration \(\mu_1^2, \mu_2^2\) may be complex depending on the singularity structure of \(\beta(m_1^2, m_2^2, t)\). Finite number of subtraction constants will not affect our argument below. Using the representation

\[
\frac{(-s)^x}{\pi \alpha} = \frac{1}{\pi} \int \frac{(m^2)^x}{s - m^2 + i \epsilon} dm^2
\]

(12)

which is valid for \(\alpha < 0\), we can rewrite Eq. 10.

\[
\mathcal{F} \times \int_{\mu_1^2}^{\infty} \int_{\mu_2^2}^{\infty} \delta_1(\mu_1^2, \mu_2^2, t, \mu_1^2) \delta_2(\mu_2^2, \mu_2^2, t, \mu_2^2) \int_{\mu_3^2}^{\infty} \int_{\mu_4^2}^{\infty} \left( m_1^2 \right) \left( m_2^2 \right) \left( m_3^2 \right) \left( m_4^2 \right) \left( \frac{s_{m_1, m_2}}{\pi} \right)^2 G
\]

(13)

where
\[ G = \int \frac{d^4k_4}{(2\pi)^4} \prod_{j=1}^{n+6} \frac{d^4k_j}{(2\pi)^4} \delta \left( \sum_{j=1}^{n+6} x_j - 1 \right) \left[ D + \sum_{i \in \mathbb{C}} C_i \right]^{n+6 - 2\ell} \]

where \( C \) is a function of \( x_j \)'s only and \( D \)

\[ D = \sum_{\ell=1}^{n+6} \frac{1}{\ell!} \frac{\delta^{(\ell)}(x_1, \ldots, x_{n+6})}{(n+6)!} \frac{m^2}{\ell} + \sum_{j=1}^{25} f_j(x_1, \ldots, x_{n+6}) x_j \]

\[ + \left( \sum_{i=6}^{n+6} x_i \mu_i^2 + m_i^2 x_i + \mu_i^0 x_i + \mu_i^2 x_i + \mu_i^0 x_i + \mu_i^2 x_i \right) C \]

\( x_j \) are all possible invariants that can be constructed out of six four...
vectors. They are given in footnote 4. $\overline{m}_k^2$ are external masses.

With Eq. 16, $m_1^2, m_2^2$ integration in Eq. 13 can be performed explicitly.

Note that in order to perform this integration, it is necessary to keep $\alpha < 0$, $D' = D + m_1^2 x_1 + m_2^2 x_2 \neq 0$. Such a region exists (e.g., where $\overline{m}_k^2 \approx 0, x_j \approx 0$) and analytical continuation to their physical values can be performed after the integration. The result is

$$F \propto \int_{D'} \frac{\Gamma^2(\alpha + 1)}{\Gamma(\nu - 2\alpha + \nu - 2\alpha)} \int \frac{d\mu_i}{\mu_i^2} \int \frac{d\mu_j}{\mu_j^2} \frac{\Gamma(\nu + 2 - \nu - 2\alpha)}{\Gamma(\nu - 2\alpha)} \int \frac{d\mu_k}{\mu_k^2} \int \frac{d\mu_l}{\mu_l^2}$$

(18)

where

$$I = e^{-2\pi i \alpha} \prod_{j=1}^{n_i} d\mu_j$$

$$D' = D + m_1^2 x_1 + m_2^2 x_2$$

(19)
In the forward limit and large $s$, we have

$$s' = -s_2 = -s_3', \quad t = t', \quad \chi_1 = \chi_2 = \chi_3 = 0$$

$$\chi_4 = \chi_5 = (p_0 - \hat{g})^2, \quad \chi_6 = \chi_7 = -t + 2(m_s^2 + s'^2);$$

$$\chi_8 = \chi_9 = (p_0 + g)^2, \quad \chi_{10}, \ldots, \chi_{18} = \text{masses}^2.$$

Writing $D'$ explicitly

$$D' = \left( \sum_{f_4} \sum_{f_{11}} + \sum_{f_{15}} + \sum_{f_{2}} + \sum_{f_{5}} + \sum_{f_{10}} + \sum_{f_{12}} \right) \omega^2$$

$$+ \left( \sum_{f_{13}} \sum_{f_{15}} + \sum_{f_{14}} \right) \mu^2 + \sum_{f_{4}} (p_{0} - \hat{g})^2 + \sum_{f_{5}} (p_{0} - \hat{g})^2$$

$$+ \left( \sum_{f_{9}} \sum_{f_{10}} \right) [-t + 2(m_s^2 + s'^2)] + \sum_{f_{7}} (p_{0} + \hat{g})^2 + \sum_{f_{8}} (p_{0} + \hat{g})^2$$

$$+ \left( \sum_{f_{10}} \sum_{f_{12}} \right) s + \left( \sum_{f_{17}} \sum_{f_{14}} \right) s' + \left( \sum_{f_{18}} \sum_{f_{19}} \right) t$$

$$+ \sum_{f_{12}} M^2 + \sum_{f_{13}} M^2 + \sum_{f_{14}} M^2 + \sum_{f_{15}} M^2$$

$$- \left( \sum_{r=6}^{m_6} \chi_r \right) \sum_{r=6}^{m_6} \mu_r^2 + \mu_i^2 + \mu_k^2 + \mu_{l'}^2 X_4 + \mu_{l''}^2 X_5 + \mu_{m'}^2 X_6 + \mu_{m''}^2 X_6$$

The equality among the invariants in the forward limit is true only for the real part. At this stage it will be seen below that it is important to distinguish $s$ and $s'$. We can simplify Eq. 23 by relation Eq. 4, the result is
\[ D' = \sum_{i=0}^{n+6} g_i M_i^2 + g_{12} s + g_{13} s' + g_{14} t + g_{15} \mu_{1}^2 + g_{16} \mu_{12}^2 \]

\[ - \left( \sum_{i=0}^{n+6} x_i \mu_{i}^2 + \mu_{1}^2 x_{3} + \mu_{1}^2 x_{4} + \mu_{2}^2 x_{5} + \mu_{2}^2 x_{6} \right) C. \]

where

\[
\begin{align*}
\hat{g}_1 &= (f_4 + f_5 - f_7 - f_8 + f_{12} - f_{13} - f_{24} + f_{25}) \\
\hat{h}_2 &= (-f_4 + f_5 - f_{10} - f_{20} + f_{24} - f_{25}) \\
\hat{g}_3 &= (-f_5 + f_7 + f_{17} - f_{21} + f_{24} - f_{25}) \\
\hat{g}_4 &= (-f_4 - f_5 - f_6 + f_7 + f_8 - f_9 + f_{10} + f_{11} - 2 f_{25}) \\
\hat{g}_5 &= (f_3 + f_5 + f_{15} + f_{25} + f_{3} + f_{5}) \\
\hat{g}_6 &= (f_2 + f_4 + f_6 + 2 f_4 + 2 f_7 + 2 f_8 + 2 f_9 + 2 f_10 + 2 f_11 + f_{12} + f_{13} + 2 f_{15} + e f_{25}).
\end{align*}
\]

Eq. 18 with \(D'\) function given by Eq. 24 is a general form of the amplitude.

We are interested in a particular kinematical region, namely large \( s \) and \( s' \), and \( s \) is evaluated on upper side of the cut in \( s \), \( s \) is evaluated on the lower side of the cut in \( s' \), and \( t \leq 0 \). We can not simply take the large \( s \) limit of Eq. 18 along the real axis since the integral representation Eq. 18 is not defined there. In order to get around this point, we define function \( h_2 \) and \( h_3 \)

\[
\begin{align*}
\hat{g}_2 (x_1, \ldots, x_{n+6}) &= x_1 h_2 (x_1, \ldots, x_{n+6}) \label{eq:18} \\
\hat{g}_3 (x_1, \ldots, x_{n+6}) &= x_2 h_3 (x_1, x_2, \ldots, x_{n+6}).
\end{align*}
\]
We note that $f_j \propto x_1$ for $j = 1, 3, 4, 6, 8, 9, 10, 11, 12, 13, 16, 21, 24, 25$ and $f_j \propto x_2$ for $j = 1, 3, 5, 6, 7, 9, 10, 12, 14, 15, 17, 20, 24, 25$. These follow since we must cut the line associated with $x_1$ or $x_2$ to form the invariants $x_j$ listed in the footnote 4. We divide the integration region of Eq. 16 into four parts by inserting

$$\left[ \Theta(h_1) + \Theta(-h_1) \right] \left[ \Theta(h_2) + \Theta(-h_2) \right] = 1 \quad (25)$$

into the Eq. 16. Later we will be looking for a term proportional to $s^{2\alpha}$ which comes from outside the region where $h_1, h_2 >> \left| \frac{4}{s} \right|$. Therefore, we can write $I$ as sum of four integrals. (If the integration region where $h_2$ or $h_3 \approx 0$ is important, then it requires extra care.) Calling $I_1, \ldots, I_4$ terms with $\Theta(h_2) \Theta(h_3), \Theta(h_2) \Theta(-h_3), \Theta(-h_2) \Theta(h_3)$, and $\Theta(-h_2) \Theta(-h_3)$ respectively. We see that $I_1$ has cuts when $s, s' > 0$, $I_2$ has cuts when $s > 0, s' < 0$, $I_3$ and $I_4$ have cuts when $s < 0, s > 0$ and $s < 0, s' < 0$, respectively. Therefore large $s, s'$ limit can be taken in the direction where it is regular in $s$ and $s'$, that is $s, s' \to -\infty$ for $I_1, s \to -\infty, s' \to \infty$ for $I_2$, etc. We will demonstrate the technique for $I_4$. The technique can be applied for $I_2, \ldots, I_4$ also. Writing

$$I_1 = e^{-2\pi i \chi} \int \prod_{j=1}^{16} dx_j \delta(S x_j - 1) C^{\nu+2-2 \delta-2 \chi} \Theta(h_2) \Theta(h_3) x_1^{-\chi} x_2^{\delta-1} \quad (26)$$
We take the large $s, s'$ limit of Eq. 26. Note, however, that Eq. 26 converges only for $\alpha < 0$. Therefore, what we must do is to single out the region of integration where $I_4$ behaves like $s^\alpha s'^{-\alpha}$ and analytically continue to $\alpha < 0$ after doing the integration explicitly.

Note that for $\alpha < 0$, such a term is not the leading term. Furthermore, when $s, s' \to \infty$, the integral representation ceases to be valid since $I_4$ will diverge when $s, s'$ reaches the threshold value for their respective channels. When all other invariants are kept below threshold, in particular negative, integral is well defined when $s, s' \to \infty$. $I_4$ is well defined on the upper half $s$ and $s'$ planes as well as on the negative real axis, and therefore using Schwartz reflection principle, it is analytic on the physical sheet of $s$ and $s'$ plane except for the cut on the real positive axis. Therefore, we can continue $s, s' \to -\infty$ limit to obtain $s \to \infty + i\epsilon$ and $s' \to \infty - i\epsilon$.

(Assumption about the real integration range for $\mu_1, \ldots, \mu_4$ is important here.) Note the presence of $x_4^{-\alpha-1}, x_2^{-\alpha-1}$ in the numerator of Eq. 26.

When $s$ and $s'$ are large, the integration region $x_1^{-1}, x_2^{-1}$ gives the dominant contribution proportional to $s^\alpha s'^{-\alpha}$. When $h_1$ or $h_2 \sim |1/s|$, the contribution proportional to $s^\alpha s'^{\alpha}$ does not arise. Therefore, we can restrict ourselves to the region $h_1, h_2 \gg |1/s|$. This justifies the splitting of $I$ into $I_1, \ldots, I_4$. First we fix $s'<0$ and take $s \to -\infty$. Setting $y = x_4 sh_4 / R$. 


where $R = D - s g_2$. In taking the larger $s$ limit, the $x_i$ appearing in $R$ as well as in the $s$ function and $c$ can be set to zero. i.e., $f_j$, $j = 1, 3, 4, 6, 8, 9, 10, 11, 12, 13, 21, 24, 25$ drops out of the problem. In particular, we note that $f_4, f_8, f_{24}, f_25$ corresponding to $M_2', M_3', \chi_4', \chi_8'$ drop out. So Eq. 27, for large $s$, does not contain singularities from channels shown in Fig. 3c, d, e, h. (Later we will see that when large $s$ limit is taken, the limiting expression does not contain the singularities from channels shown in Fig. 3f, g and contains only those in Fig. 3a, b.) Note that the path of integration depends on what we take for the phase of $sh_1/R$. The singularity of the integrand at $y = -1$ never makes the integral diverge. In fact we take the path of integration shown in Fig. 10. Since the value of integral is zero everywhere except along the positive real axis,
The large $s'$ limit can be taken in the same way. Continuing to the region $\alpha > 0$, and continuing $s$ and $s'$ from the negative axis to $s_0 + i\epsilon$ and $s_0 - i\epsilon$ respectively along the positive real axis, we have

$$I_1 \sim (-s)^{2\alpha} \pi^{-\alpha} \Gamma(-\alpha) \Gamma(\alpha - 2\alpha + 4\alpha) \int_{\text{s} = 3}^{\text{s} = 6} \frac{d\nu}{\nu} \frac{C^{n-2\lambda - 2\alpha + 2}}{\nu^n} \Theta(h_1) \Theta(h_2) \frac{C^{\alpha - 2\alpha - 1}}{(\nu')^{n-2\lambda - 4\alpha - 4}}$$

$$R' = \left[ R \right]_{\nu = 0}$$

(28)

Finally,

$$I_1 \approx \left[ S \right]^{2\alpha} \pi^{-\alpha} \Gamma(-\alpha) \Gamma(\alpha - 2\alpha + 4\alpha) \int_{\text{s} = 3}^{\text{s} = 6} \frac{d\nu}{\nu} \frac{C^{n-2\lambda - 2\alpha + 2}}{\nu^n} \Theta(h_1) \Theta(h_2) \frac{C^{\alpha - 2\alpha - 1}}{(\nu')^{n-2\lambda - 4\alpha - 4}}$$

(29)

where

$$\kappa = \left( f_{22} - f_{23} \right) \lambda_1 + \left( f_{18} - f_{19} \right) t$$

$$+ \left( \tilde{f}_2 + \tilde{f}_4 \right) x_3^2 + \tilde{f}_5 + 2 \tilde{f}_{23} x_4^2$$

(30)

$$- \left( \sum_{i=6}^{\mu} x_i / \mu - \mu x_3 + \mu x_4 + \mu x_5 x_6 \right) C$$

All invariants which were multiplied by $x_4$ and $x_5$ were eliminated.
where the subscript 1 corresponds to the contribution of \( I_1 \) to \( F \).

Assuming that \( \mu_1 \) integrations are on the real axis \( \mu_1^2 > \mu_0^2 / 2 \), we can deduce the analyticity of \( F \) on the \( \mathbb{M}^2 \) plane from Eqs. 30 and 31.

Note that \( K \) is exactly the denominator function for the four point function when two of the external particle has mass \( t \). Since the region \( x_1 \sim 1/s, x_2 \sim 1/s' \) gives the contribution, it can be represented by Fig. 11. The four-point function to arbitrary order in the coupling constant has been discussed in many places. \(^{12,13}\) The only possible additional complication in our problem is that two of the masses are \( t < 0 \), and that some of the internal masses \( \mu_1, \ldots, \mu_4 \) are integrated from \( \mu_0^2 \) to \( \infty \). But we note that Ref. \(^{13}\) shows that the propagator is negative definite below threshold when all the external particles are on their mass shell. The continuation from their mass shell to \( t < 0 \) will make the denominator more negative. Same is true for any \( \mu_1^2 > \mu_0^2 \).

Since the integral over \( \mu_1^2 \) are convergent, we see that \( F \) is analytic on the upper half \( \mathbb{M}^2 \) plane as well as \( \mathbb{M}^2 < 4 \mu_0^2 \) on the real axis. Then the Schwartz reflection principle can be used to see that \( F \) is analytic everywhere on the physical sheet \( \mathbb{M}^2 \geq 4 \mu_0^2 \) on the real axis. Note that

\[
\begin{align*}
F & \propto \left| \frac{2^{2n}}{n!} \prod_{j=1}^{n} \frac{1}{(n+4j-2\delta)} \frac{\pi^{2n}}{2\mu_0^2 n! \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{d^2 \mu_1 \cdots d^2 \mu_n}{\Delta_1 \cdots \Delta_n} \right|^2 \mu_1^2 \cdots \mu_n^2 \mu_1^2 \cdots \mu_n^2 \mu_1^2 \cdots \mu_n^2
\end{align*}
\]
these arguments will be false if \( \mu^2 \) is complex, that is if the residue function \( \beta(m_1^2, m_2^2, t) \) has a singularity on the complex \( m_1^2, m_2^2 \) plane.

IV. SUM RULE

The theorem states that if there is any complex branch point, the source of such a branch point is in the Regge residue function of the ordinary two-to-two scattering amplitude. We are not prepared, here, to make any statement about the Regge-residue function. We would rather take the point of view that if the results of assuming no complex branch point on the \( M^2 \) plane does not agree with experiment then we know a possible source of the problem.

In this section, we assume that no other branch point except those coming from unitarity exists. The singularity structure for \( I, \) is shown in Fig. 12. (For those who skipped Section IV, \( I, \) is the part of \( T \) which contains all the leading singularities in the limit of large \( s \).) The discontinuity across the cut, according to Eq. 6 is proportional to the inclusive cross section. Therefore, if we know the \( M^2 \) dependence of the amplitude around the circle of radius \( M_0^2 \), we can use the formula

\[
\oint (M^2)^n \left( \sum \right) dM^+ = 0
\]
to obtain the relationship between experimentally measurable quantities.\textsuperscript{14}

The triple Regge expansion supplies the $M^2$ dependance around the circular contour. According to the triple Regge expansion we have

$$
T \rightarrow \frac{\pi}{4 \pi^2} \sum_{ijk} \left( \gamma_j \gamma_k \frac{\alpha_j(t) + \alpha_k(t)}{M^2} \right) \left( \frac{\beta_{ijk}(t)}{M^2} \right) \prod \frac{\delta M}{\pi \left( \alpha_j(t) - \alpha_j(t) - \alpha_k(t) \right)} + \sum_r F_{ijk} \gamma_j
$$

where

$$
\gamma_j = \delta_{\alpha_{ijk}} \left( \frac{\alpha_j(t)}{M^2} \right) \frac{\alpha_j(t) + 1}{\alpha_j(t) - 1} \gamma_i = \epsilon \left( \alpha_j(t) - \alpha_j(t) - \alpha_k(t) \right) + 1
$$

$\beta_{\text{abj}}(t)$ is a Regge-residue function associated with particle $a$, $b$ and Regge trajectory $j$ coupling, $g_{ijk}(t)$ is the triple-Regge residue function. They are normalized in the same way as Ref. 15. These notations are defined by Fig. 13. In particular our $g_{ijk}(t)$ where $P$ stands for the Pomeron, corresponds to $g_P(t)$ in Ref. 15. $\alpha(0)$ and $\alpha_j(t)$ are the Regge trajectory functions. When $\alpha_j(0) - \alpha_j(t) - \alpha_k(t) = \gamma = \text{integer}$, the first term of Eq. 33 seems to have spurious poles. They are cancelled by either (i) zero in $g_{ijk}(t)$ or (ii) by $F_{ijk}^r$. The spurious poles have been studied in Ref. 10 by computing a particular Feynman diagram in the $\phi^3$ theory. It was found that for $\gamma \leq 0$, $F_{ijk}^n = 0$ and $g_{ijk}(t)$ has a zero, for $\gamma \geq 1$, $F_{ijk}^n$ is present to cancel the poles. It is therefore, quite reasonable to assume that $F_{ijk}^n = 0$ for $n \leq 0$. For $n \geq 1$, $F_{ijk}^n$ is a
polynomial and even if it is present,

\[ \oint F^{r}_{ij\kappa} (m^{2})^{\eta} dM^{2} = 0 \]

and gives no contribution to Eq. 32. Using Eq. 6 and argument on the left hand cut presented in Sec. II, we obtain

\[ \left( \int_{m^{2}}^{M^{2}} (m^{2})^{\eta} \frac{d\sigma}{dt dM^{2}} \right)_{a + b \rightarrow c + \lambda} = (-1)^{\eta} \int_{m^{2}}^{M^{2}} (m^{2})^{\eta} \frac{d\sigma}{dt dM^{2}} \right)_{c + b \rightarrow a + \lambda} \]

\[ = \sum \left( (-1)^{\eta} \right) \frac{\eta_{j} \eta_{k}}{M_{0}^{2} S_{2}} \left( \frac{S_{p}}{M_{0}^{2}} \right)^{i} \left( \frac{S_{p}}{M_{0}^{2}} \right)^{i} \frac{\alpha_{j}(t) - \alpha_{k}(t)}{\beta_{b} \beta_{c} \gamma_{d} \gamma_{e} \gamma_{j} \gamma_{k} (t)} \right) \]

\[ \alpha_{e}(t) \rightarrow \alpha_{j}(t) - \alpha_{k}(t) \gamma_{1} \]

(34)

Note that only even signatured Regge poles contribute in "i". This is because the inclusive cross section is always symmetric in \( j \) and \( k \).

It is important to point out that for

\[ \frac{d\sigma}{dt dM^{2}} \right)_{c + b \rightarrow a + \lambda} \]

The center of mass squared of \( b \) and \( c \) is

\[ (p_{3}^{2} + q_{3}^{2}) = S + \mu^{2} - \lambda M^{2} + t \]

and it is not fixed along the integration path. When major contribution to the integral comes from lower end of the integral, however, modification due to the energy shift should be small. Note also that if \( a = c \), Eq. 34 reduces to a trivial equation for even \( n \).
V. EXTENSIONS

Eq. 34 in general requires measurements of two inclusive cross section $a+b \rightarrow c+X$ and $c+b \rightarrow a+X$. In this section we discuss sum rules which stem from Eq. 34 but require less experimental data. We see immediately that for $a = c$ and odd $n$ we have

$$\int \left( \frac{d\sigma}{dt} \right)_{a+b \rightarrow c+X} \left( M^2 \right) \left( \frac{d\sigma}{dt} \right)_{c+b \rightarrow a+X} = \sum \frac{1}{16 \pi s \alpha_1 \alpha_2} \frac{1}{\left( \frac{s}{M^2} \right)^{\alpha_1 \alpha_2} \left( M^2 \right)^{\alpha_1 \alpha_2} }$$

where the sum "i" runs over only even signatured trajectories.

The contribution from the cross channel $c + b \rightarrow a + X$ in Eq. 34 comes from the fact that $I$ contains both right and left hand cuts. Suppose now that we can make the separation

\[ (35) \]
where $I_R$ ($I_L$) is an analytic function of $s$, $t$, and $M^2$ which contains only the right (left) hand cut on the $M^2$ plane in the limit of large $s$ and fixed $t$. Let us further assume that they both have a triple-Regge behavior with appropriate phase factors. (i.e. no fixed poles) sum rule can be written for both $I_L$ and $I_R$ separately and we obtain

$$
\int_{t_{\text{cut}}}^{s_{\text{cut}}} \left( \frac{d\sigma}{dt} \right)^n \frac{dM^2}{\partial x} \Omega_{1,2} \sim 1 + \cdots
$$

where $\Omega_{1,2}$ is the slope of the Pomeranchuk trajectory. If $\alpha_P(0) = 1$, $g_{\text{PPP}}(t)$ must have a zero at $t = 0$. The presence of this zero is well known. (b) Note that the left hand side of the sum rule (37) contains the integral over the low missing mass region and thus it contains the integral over the resonances. We might, therefore, expect that the

Let us now discuss the content of this sum rule. (a) Consider a reaction $a + b \rightarrow a + x$. Then the leading Regge trajectory is $i = j = k = \text{Pomeranchuk}$. For $n = 0$, and small $t$, we can write

$$
\int_{t_{\text{cut}}}^{s_{\text{cut}}} \left( \frac{d\sigma}{dt} \right)^n \frac{dM^2}{\partial x} \sim 1 + \cdots
$$

where $\alpha^*$ is the slope of the Pomeranchuk trajectory. If $\alpha_P(0) = 1$, $g_{\text{PPP}}(t)$ must have a zero at $t = 0$. The presence of this zero is well known. (b) Note that the left hand side of the sum rule (37) contains the integral over the low missing mass region and thus it contains the integral over the resonances. We might, therefore, expect that the
concept of duality from two-to-two scattering amplitude to appear here
in its generalized form. This will be true if the sum rule holds for
unusually low $M_0^2$ with only the leading Regge trajectory in the sum over
i. Since the generalized form of duality is widely accepted without any
experimental bases, this is a good opportunity to check it. There is also
a related question concerning how the Pomeron and the ordinary Regge
contributions should be related to the contributions from the resonance
and the background. If we take the analogy with the two-particle
scattering, we associate the contribution of the background in the $M^2$
channel with the Pomeron contribution in i and the contribution of the
resonance with the ordinary Regge contribution in i. All these can be
checked when the data in various reactions become available.

(c) For now, we associate the Pomeron contribution to the left hand
side of Eq. 35. Then we obtain

$$g_{PPP}(t) = \left| \frac{(1 - \alpha_0(\sigma) + 2x' \cdot t)}{\sigma_a \sqrt{\sigma_b}} \right| \int_{M_0^2}^{M_0^2} \frac{dM}{dM} \left[ \frac{d\sigma}{dt_{+\pi^+}} \left( \alpha_+ b - t + x \right) \right]$$

where the right side is to include only that background contribution
which has $s^{-}$ behavior. This equation is useful for obtaining the
value for the triple-Pomeron vertex function. Note that Eq. 39 is the most
reliable way to obtain $g_{PPP}(t)$. The only other way known at present
is to measure the differential cross section in the triple-Regge region. But the cross section is bound to be small due to the zero in $g_{ppp}(t)$ at $t = 0$ discussed above, and away from $t = 0$, the contribution from cuts may play a role. Another advantage of Eq. 39 is that if the background can be properly separated from the resonance, the knowledge of the low energy cross section will put a lower bound on $g_{ppp}(t)$.

Furthermore, note that factorization implies that the right hand side of Eq. 39 is a universal function of $t$ for any $a$ and $b$. A test of the universality can be made in, for example,

\begin{align*}
   p + p &\to p + x, \pi^\pm + He \to He + x, \pi^\pm + p \to p + x, \pi^\pm + p \to \pi^\pm + x, \\
   K^\pm + p &\to p + x, K^\pm + p \to K^\pm + x, p + \bar{p} \to p + x, \bar{p} + p \to p + x, \text{etc.}
\end{align*}

The Regge behavior for the unsigned amplitudes $I_R, I_L$ were assumed in order to obtain the above results. The verification of this assumption is, in itself, extremely interesting. We will illustrate a possibility that the fix pole may exist by a heuristic argument. Consider a Regge + particle $\to$ Regge + particle scattering where initial Regge trajectory has spin $a_k$ and the final Regge trajectory has spin $a_j$. The particle is taken to be spinless. Let the square of the direct channel energy be $M^2$. Then at large $M^2$, the maximum spin flip amplitude behaves as $(M^2)^ {a_i - a_j - a_k}$ where $a_i$ is the Regge trajectory exchanged in the $t$ channel. For example if $a_j = a_k = 1$, the kinematics is same as that of Compton scattering and $a_i$ is a Pomeron. In fact, at $a_i = a_j = a_k = 1$, the spin flip amplitude chooses wrong-signature
nonsense. In the Compton scattering one needs a fixed pole at this point in order to prevent the Pomeron from decoupling. The triple-Pomeron contribution resembles this possibility. Since the triple Pomeron decouples at \( t = 0 \), it may be an indication that the fixed pole corresponding to the Pomeron in the Compton scattering is absent. But it is quite possible that a fixed pole associated with other trajectories may exist.
VI. CONCLUSION

The analyticity of a scattering amplitude has been proven to be a powerful tool in understanding two-to-two reactions. The possibility of using such a tool to three-to-three amplitude becomes exceedingly complicated. We have demonstrated that in the region $|s/M^2| > 1$, there is a good chance that the analyticity of the three-to-three amplitude on the $M^2$ plane becomes very simple.

Using this analyticity, we have written a sum rule Eq. 34. This sum rule enables us to evaluate the triple-Regge residue function from a low missing mass inclusive cross section data. Such information will be very useful for future experiments at NAL.

The successes of the sum rules written here, when they are compared with experiment, will be quite significant. It means that we can apply the techniques used in the two particle scattering to the analysis of inclusive reactions. If the idea of duality in the generalized form is verified through these sum rules, we should gain confidence in the significance of dual models. In order to compare the sum rules with experiments, we need to separate the resonance and background. This is very difficult in the existing experimental data (for example Ref. 5). It is clear that future experiments should be designed such that the separation can be easily achieved. Furthermore, when $a + b \rightarrow a + X$ is being
measured, $\tilde{a} + b \rightarrow \tilde{a} + X$ or $a + \tilde{b} \rightarrow a + X$ should be measured simultaneously.

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REFERENCES AND FOOTNOTES


4. For future reference, we list them.

\[ x_1 = (p_a - p_a')^2, \quad x_2 = (p_b - p_b')^2, \quad x_3 = (q - q')^2, \quad x_4 = (p_b - q)^2, \]

\[ x_5 = (p_b - q')^2, \quad x_6 = (p_a' - q)^2, \quad x_7 = (p_b' - q)^2, \quad x_8 = (p_b' + q)^2, \]

\[ x_9 = (p_a + p_b - p_a')^2, \quad x_{10} = (p_a + p_b - p_b')^2, \]

\[ x_{11} = (p_a + p_b - p_b' + p_a' - q')^2, \quad x_{12} = (p_a + p_b - p_b' + p_a' - q')^2, \]

\[ x_{13} = (p_a + p_b - p_a' - q) - (p_a + p_b - p_a' + q) - (p_a + p_b - p_b' - q) + (p_a + p_b - p_b' + q), \]

\[ x_{14} = (p_a + p_b - p_a' - q) - (p_a + p_b - p_a' + q) - (p_a + p_b - p_b' - q) + (p_a + p_b - p_b' + q), \]

\[ x_{15} = (p_b - p_a - p_b')^2, \quad s = (p_a + p_b)^2 \equiv x_{16}, \quad s' = (p_a' + p_a')^2 \equiv x_{17}, \]

\[ t = (p_a - q)^2 \equiv x_{18}, \quad t' = (p_a' - q')^2 \equiv x_{19}, \quad s_2 = (p_b - p_a')^2 \equiv x_{20}, \]

\[ s_2 = (p_a' - p_a)^2 \equiv x_{21}, \quad M^2 = (p_a + p_b - q)^2 \equiv x_{22}, \]

\[ M_1^2 = (p_a - p_b - q)^2 \equiv x_{23}, \quad M_2^2 = (p_a + p_b - q)^2 \equiv x_{24}, \]

\[ M_3^2 = (-p_a - p_b - q)^2 \equiv x_{25}. \]

5. This part of the argument is due to Stapp Ref. 3.


9. We have put in the form \( \beta [-(k_1 + p_2)^2]^x \) as the contribution from the black blob. Later, we will see that the largest contribution comes
from the region where \((k_1 + p_2)^2\) is large and thus the short cut will be justified.

10 The method used here is same as those used in S. -J. Chang et al. NAL preprint THY-16 (1971).

11 Taking Eq. 15 and performing the \(m_1^2, m_2^2\) integrations one can show that the term \(s^2\alpha\) comes from the region of \(k\) integration where \((k_1 + p_2)^2 = (k' - q')^2 \cong s\). Thus the discussion in footnote 7 is justified.


14 While this work was in progress, the author has received M. Einhorn Berkeley Preprint UCRL-20688 (1971), Olssen CERN Preprint Th 1376 (1971), both of which discuss finite-energy sum rule to some extent.


16 Numerical evaluation of these sum rules is in progress. S.D. Ellis and A.I. Sanda to be published. We would like to acknowledge S. Ellis for discussions which no doubt greatly influenced the development of the material presented in this section.
FIGURE CAPTIONS

Fig. 1 Diagram for an inclusive process.

Fig. 2 Diagram for a six-point function.

Fig. 3 When $s$ and $t$ are fixed, any of these channels have singularities on the $M^2$ plane.

Fig. 4 The singularities of channels $M^2$ and $M^2_4$ on the $M^2$ plane.

Fig. 5 The box diagram which gives complex singularity in the physical sheet.

Fig. 6 The dominant diagram in the inclusive reaction at small $t$ and $|s/M^2| >> 1$.

Fig. 7 The class of diagrams in $\phi^3$ theory that were studied. It has a following property:

(i) The four point function associated with the lower black blob corresponds to the arbitrary sum of diagrams in
the $\phi^3$ theory such that it behaves as
\[ \alpha(t) \beta(k_1^2, (k+p_2-q)^2, t) \] in the limit of large $(p_2+k)^2$. Similarly for the upper black blob.

(ii) The checked blob is a six-point function which represents an arbitrary Feynman diagram with $n$ number of propagators and $l$ number of loops.

Fig. 8 The diagram whose Feynman denominator function is same as that of Fig. 7 in the limit of large $s$.

Fig. 9 Diagram for $G$ defined by Eq. 14.

Fig. 10 Path of integration for Eq. 27.

Fig. 11 The diagram which gives the $s^\alpha, s'^\alpha$ limit when $s, s'$ are large.

Fig. 12 Analyticity of $I, M^2$ plane and the path of integration to obtain the finite-energy sum rule.

Fig. 13 The triple Regge diagram.
Fig. 1
Fig. 2
Fig. 3

(a) $M^2$ with $q$, $p_a$, $p_b$, $q'$, $p'_a$, $p'_b$

(b) $M_1^2$ with $q$, $p_a$, $p_b$, $q'$, $p'_a$, $p'_b$

(c) $M_2^2$ with $q$, $p_a$, $p_b$, $q'$, $p'_a$, $p'_b$

(d) $M_3^2$ with $q$, $p_a$, $p_b$, $q'$, $p'_a$, $p'_b$

(e) $X_4$ with $p_a$, $p'_a$, $p_b$, $q'$, $q'$

(f) $X_5$ with $p_a$, $p'_a$, $p_b$, $q'$, $q'$

(g) $X_7$ with $p_a$, $p'_a$, $p_b$, $q'$, $q'$

(h) $X_8$ with $p_a$, $p'_a$, $p_b$, $q'$, $q'$
Figs. 6, 7, 8
\[ \beta_{ack} \quad \beta_{acj} \quad \beta_{bki} \]
Note added: We have examined $pp \rightarrow p + x$ (Ref. 7 and Alaby et. al. CERN 70-16 (1970) and $\pi - p \rightarrow p + x$ (CERN-IHEP collaboration). Following conclusions were reached: (a) the cross sections are consistent with two term triple-Regge expansion

$$\frac{d\sigma}{dt dM^2} = \frac{m^2}{(16\pi)^2 s} \left[ G_{PPf} \left( \frac{t}{M^2} \right)^{2\alpha(t)} \left( \frac{1}{M^2} \right)^{\alpha_f(0)} + G_{ffP} \left( \frac{s}{M^2} \right)^{2\alpha_f(t)} \left( \frac{1}{M^2} \right)^{\alpha_p(0)} \right]$$

$G_{PPf}$ and $G_{ffP}$ are products of $g$, $\beta$, $\eta$, see S. D. Ellis and A. I. Sanda, NAL-THY-30 submitted to Phys. Rev. Letters; (b) the finite-energy sum rule for inclusive reaction Eq. (32) is indeed satisfied. See S. D. Ellis and A. I. Sanda NAL-THY-47.