On the transport of anomalous cosmic rays: the Parker propagator for spherical solar modulation

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Abstract

An analytical solution for the Green's function of the fundamental transport equation of cosmic rays, i.e. of the Parker equation, is presented. Among the new features of the approach are a simultaneous dependence of the coefficient of spatial diffusion on the configuration as well as momentum space coordinates and an incorporation of wave-particle interactions due to the effect of transit-time damping. After the determination of the transport parameters for three turbulence models the analytical solutions are applied to the transport of anomalous cosmic rays.

1 Introduction:

Anomalous cosmic rays (hereafter abbreviated ACR) have been first detected in the early seventies (e.g. Garcia-Munoz et al. 1973). This mainly low energetic particles are thought to originate from neutral atoms which are swept into the heliosphere from the local interstellar medium because of the motion of the solar system relative to this medium and that have become subsequently ionized in the heliospheric space by the solar ultraviolet radiation or by charge exchange with solar wind ions (Fisk et al. 1974). These mostly singly ionized atoms, the so-called pickup ions, are picked up by the solar wind electromagnetic fields and are convected outward with this solar plasma flow, while undergoing adiabatic cooling and momentum diffusion in the ambient wave fields. Once they reach the solar wind termination shock, preaccelerated pickup ions undergo diffusive shock acceleration and create the population of ACR (Pesses et al. 1981). After this acceleration an as yet unknown fraction diffuses backwards into the inner heliosphere against the solar wind, i.e. the anomalous component is subjected to the effects of solar modulation. This causes, with decreasing heliocentric distance r, a rapid decrease in ACR-density due to the outward flow of the solar plasma and a shift of the particle flux maximum to lower momenta resulting from the adiabatic deceleration by this radially magnetized diverging solar wind plasma. In order to take these effects into account, one has to use the appropriate equation describing the transport of cosmic rays, i.e. the Parker equation, which was first derived 1965 by E. Parker (Parker 1965). He has given a variety of solutions for simplified cases which show the effects of spatial diffusion, convection and energy loss.

In the past 34 years many analytical solutions were presented in the literature (e.g. Fisk and Axford 1969; Gleeson and Webb 1974; Cowsik and Lee 1977). Although these solutions describe the effects of solar modulation and its two major features, they still were not really exact in a mathematical sense because of using approximations and asymptotic forms or making assumptions about source functions in their derivations.

It is the purpose of this paper to present, to the best of our knowledge, for the first time one of several exact solutions of Parker's equation, which are valid for arbitrary source functions. All these analytical solutions are determined by the coefficient of spatial diffusion and hence they depend characteristically on the composition and topology of heliospheric turbulence, which will be considered in the next section from the plasma wave viewpoint. We will do this for three different models of turbulence.

2 Spatial diffusion coefficients

Energetic charged particles like cosmic rays in general and, in particular, the mostly singly ionized anomalous component are embedded in the solar wind plasma and therefore they can interact resonantly with plasma waves, which are in the low-frequency range and in a low-β plasma mainly determined by their magnetic field component. Hence, the phase space distribution function of the particles adjust to a quasi-isotropic state due to pitch-angle diffusion. This quasi-isotropic distribution function obeys the equation of transport.

Within the framework of quasilinear theory the coefficient of spatial diffusion can be defined as a pitch-angle averaged Fokker-Planck coefficient $D_{\mu\mu}$, which is determined by the composition and the geometry of the plasma wave turbulence,

$$\kappa_{\|} = \frac{v^2}{8} \int_{-1}^{+1} d\mu \frac{(1-\mu^2)^2}{D_{\mu\mu}}, \tag{1}$$

where $\mu = p_{\|}/p$ and v are the pitch-angle and the particel velocity, respectively.

In this paper we consider three different models of turbulence, i.e. on the one hand shear Alfvén waves in a slab turbulence (A) and on the other hand fast magnetosonic waves in an isotropic turbulence model (F). The third model consists of a mixture of slab Alfvén waves and isotropic fast magnetosonic waves (M). Schlickeiser (1989) and then Schlickeiser and Miller (1998) have calculated the coefficients of spatial diffusion for these three cases. In their calculations they assumed for the plasma wave spectrum a Kolmogorov-like power-law dependence above some minimum wavenumber k_{min} with index $q > 1$.

Considering, for simplicity, in wavenumber-space forward and backward propagating plasma waves with same intensities and assuming equal spectral shapes, scales and equal intensities of Alfvén and fast mode waves, that means $q_A = q_F$, $k_{min,A} = k_{min,F}$ and $(\delta B_A)^2 = (\delta B_F)^2$, and, furthermore, using the empirical relationship $\kappa_{\perp}^{(i)} = a\kappa_{\|}^{(i)}$ $(a = const. \ll 1)$ and confining the considerations to large heliocentric distances, i.e. $\kappa_{rr}^{(i)} \simeq \kappa_{\perp}^{(i)}$, one can find for nonrelativistic particles the following unified representation of the three different diffusion coefficients,

$$\kappa_{rr}^{(i)} = \kappa_{rr,0}^{(i)} \left(\frac{r}{r_E}\right)^{\alpha} \left(\frac{p}{p_A}\right)^{\beta^{(i)}}, \tag{2}$$

where i refer to the different models A, F and M. The reference values $\kappa_{rr,0}^{(i)}$ are different but of the same order of magnitude for all three models. Here r and p are the heliocentric distance respectively the momentum of the particle and the reference values $r_E = 1AU$ and p_A denote the Earth‘s orbit respectively the particle momentum for particles propagating with Alfvén speed v_A. It has to be pointed out that especially the exponent $\beta^{(i)}$ is determined by the composition and geometry of the heliospheric turbulence, that means $\beta_A = 3 - q$ for slab Alfvén waves, $\beta_F = 2$ in the case of isotropic fast mode waves, and $\beta_M = 1$ for the mixed turbulence.

3 The Parker propagator

After having established the relevant parameter of transport in three turbulence models, we have to enter into the considerations with regard to the spherically symmetric, steady state transport equation for the quasi-isotropic phase space distribution function $F(r,p)$,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \kappa_{rr}^{(i)} \frac{\partial F}{\partial r}\right) - V\frac{\partial F}{\partial r} + \frac{p}{3r^2}\frac{\partial}{\partial r}\left(r^2 V\right)\frac{\partial F}{\partial p} = -S(r,p), \tag{3}$$

which describes on the left hand side the effects of spatial diffusion and spatial convection as well as convection in momentum space. $S(r,p)$ denotes the source function and V the solar wind speed.

Using equation (2) and, for mathematical generality, $V(r) = V_0 r^{\delta}$ equation (3) may be manipulated such that the following form results:

$$y\frac{\partial^2 F}{\partial y^2} + \left[b - y\right]\frac{\partial F}{\partial y} + \frac{2+\delta}{3\nu}\tau\frac{\partial F}{\partial \tau} = -\frac{r[y,\tau]}{V(r[y,\tau])\nu} S(r[y,\tau],\tau) \qquad (\alpha - \delta < 1). \tag{4}$$

Here we have introduced the new variables (see also Jokipii 1967) $\tau = p$ and

$$y(r,p) := \frac{\nu}{(1+\delta-\alpha)^2}\frac{rV}{\kappa_{rr}^{(i)}}.$$

Furthermore, we have used the abbreviations $\nu = (1+\delta-\alpha+\frac{2+\delta}{3}\beta)$ and $b = (2+\delta)/(1+\delta-\alpha)$.

The general solution for the distribution function can be expressed by the Green's function $G(y, y_0, \tau, \tau_0)$, i.e Parker's propagator,

$$F(y,\tau) = \int dy_0 \int d\tau_0 G(y, y_0, \tau, \tau_0)\left[\frac{r[y_0,\tau_0]}{V(r[y_0,\tau_0])\nu} S(r[y_0,\tau_0],\tau_0)\right], \tag{5}$$

which has to satisfy, after appling the Laplace transform technique, the following ordinary and inhomogeneous differential equation, which results from equation (4):

$$y\frac{d^2 g}{dy^2} + [b-y]\frac{dg}{dy} - sg = -\frac{3\nu}{(2+\delta)\tau_0}\delta(y-y_0). \tag{6}$$

Here $g(y,s)$ denotes the Laplace transformed Green's function and s is the Laplace variable. The homogeneous part is the confluent hypergeometric differential equation, also called Kummer's equation. Following the standard method of solving such equations and constructing the Green's functions one can derive, having executed the inverse Laplace transformation, the exact solution for the differential particle flux (Stawicki 1999):

$$\begin{aligned} j(r,p) = p^2 F(r,p) = \frac{3}{b}\int dr_0 \int dp_0 \frac{S(r_0,p_0)}{V(r_0)}\frac{p_0 y_0}{f(p,p_0)}\left(\frac{r_0}{r}\right)^{\frac{1+\alpha}{2}}\left(\frac{p_0}{p}\right)^{\frac{3\alpha-4\delta-5}{2(2+\delta)}} \\ \times\ \exp\left\{-\frac{y_0}{f(p,p_0)}\left(1+h^2(r,r_0,p,p_0)\right)\right\} I_{\frac{1+\alpha}{1+\delta-\alpha}}\left[\frac{2y_0}{f(p,p_0)}h(r,r_0,p,p_0)\right]. \end{aligned} \tag{7}$$

To simplify the notation we have introduced the functions

$$f(p,p_0) = 1-\left(\frac{p}{p_0}\right)^{\frac{3\nu}{2+\delta}} \quad \text{and} \quad h(r,r_0,p,p_0) = \left(\frac{r}{r_0}\right)^{\frac{1+\delta-\alpha}{2}}\left(\frac{p}{p_0}\right)^{\frac{3}{2b}}$$

as well as the modulation parameter $y_0 = y(r_0,p_0)$. $I_n(z)$ is a modified Bessel function of the first kind. This solution is valid for arbitrary source functions in which super-alfvénic charged particles of momentum p_0 are injected continuously from a spherical source surface at radius r_0 into the inner heliosphere. Notice that, in contrast to the solutions of the last 34 years, no assumptions were made with regard to Kummer's functions or the source function. Consequently, equation (7) is an exact solution of Parker's equation for the condition $\alpha - \delta < 1$.

Considering small heliocentric distances, i.e. going into the inner heliosphere, one can obtain, with the aid of the asymptotic form of the Bessel function at small radii, the expression

$$j(r\to 0,p) = \frac{3p^2}{b\Gamma(b)}\int dr_0 \int dp_0 \frac{S(r_0,p_0)}{p_0 V(r_0)} y_0^b f(p,p_0)^{-b}\exp\left\{-\frac{y_0}{f(p,p_0)}\right\}, \tag{8}$$

where $\Gamma(z)$ is the gamma function. This approximation is finite and shows, that the ACR flux depends not only at large but also at small distances, e.g. the Earth's orbit, characteristically on the source functions and their structure. In the same way we can approximate equation (7) for low momenta, i.e. $p \to 0$, through the form

$$j(r,p\to 0) = \frac{3p^2}{b\Gamma(b)}\int dr_0 \int dp_0 \frac{S(r_0,p_0)}{p_0 V(r_0)} y_0^b \exp\{-y_0\}. \tag{9}$$

Notice, that for the case of p sufficiently small, we get $j \propto p^2$. That means the phase space distribution function is constant as p approaches zero.

4 Conclusions

In this paper we presented one of several exact solutions of the solar modulation transport equation of cosmic rays. All these solutions, which are determind characteristically and sensitively by the composition and topology of the turbulent wave fields are valid for arbitrary source functions, which contain radial dependence as well as a dependence of momentum and can be freely choosen. In the simplest case, one can use a monoenergetic injection at a fixed radius to the sun, e.g. the heliospheric distance of the solar wind termination shock. A more realistic dependence in momentum is the power-law or modified power-law injection of pre- and diffusively shock-accelerated pickup ions at the so-called heliospheric shock.

Comparisons of the corresponding solutions derived with the analytical Parker propagator for this most interesting case with ACR observations (e.g. Christian, Cummings and Stone 1995) should provide a very useful tool to address the general problem of the radial variation of the spatial diffusion of energetic particles within the heliosphere.

Besides, these solutions should render a check on computer codes which treat the solar modulation problem more general.

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