Perpendicular Diffusion of Charged Test Particles in Magnetostatic Slab Turbulence

John W. Bieber, Richard L. Mace, and W. H. Matthaeus,
Bartol Research Institute, University of Delaware, Newark, DE 19716, USA

Abstract

The perpendicular diffusion of charged particles in random magnetostatic slab fields is investigated numerically, primarily as a means to check the validity of a recent theory (Bieber and Matthaeus, 1997) for the perpendicular diffusion coefficient $\kappa_\perp$. The numerical results suggest the following: (1) the theory agrees very well with the numerical experiments for large $\Omega\tau \gg 1$, where Ω is the particle gyrofrequency and τ is a rigidity dependent decorrelation time; (2) for intermediate values, i.e., for $2 \leq \Omega\tau \leq 4$ the theory provides a better fit to the numerical data than does the standard quasilinear result $\kappa_\perp = vP_{xx}(0)/4B_0$; (3) for $\Omega\tau < 1$ the theory does not fit the experimental data quantitatively or qualitatively. It is suggested that the breakdown for $\Omega\tau < 1$ is related to the form of the initial ansatz for the velocity correlation function in this region of parameter space.

1 Introduction:

Theoretical understanding of perpendicular diffusion of charged particles is crucial in a variety of astrophysical settings, as it, along with drifts, provides a primary mechanism for transporting charged particles across magnetic field lines. These processes are especially relevant in the outer heliosphere, because the large–scale magnetic field there is nearly perpendicular to the radial direction. In spite of its intrinsic importance, a fundamental theory of perpendicular transport is not as yet widely agreed upon, and observations continue to pose challenges (Dwyer et al, 1997) for theoretical explanations. As such, investigations of cosmic ray modulation often adopt empirically determined perpendicular diffusion coefficients, represented by simple *ad hoc* functional forms (e.g., Cummings et al., 1994; Burger, 1990). While this is a useful approach, a more complete theory of cosmic ray transport would employ transport coefficients based purely upon deductive physical principles. Such a theory is well established for hard–sphere scattering in a magnetized plasma (Gleeson, 1969), and for related extensions based upon the Boltzmann equation (Jones, 1990). However, cosmic rays in the solar wind are presumably scattered by magnetic turbulence, and here our understanding is severely limited in a number of different ways.

Most existing theories of charged particle diffusion (Forman et al. 1974; Jokipii & Parker 1969) are constrained by the following points: (i) they apply strictly only to slab turbulence; (ii) they are limited to weakly turbulent plasmas with $\langle \delta B^2\rangle^{1/2}/B_0 \ll 1$; and (iii) they are valid only if the particle gyroradius is well in excess of the correlation length of the turbulence (high energy/rigidity assumption). Bieber & Matthaeus (1997) attempted to address these limitations using the Taylor-Green-Kubo expression (Taylor, 1922; Green, 1951; Kubo, 1957) for the diffusion coefficient in terms of the velocity autocorrelation,

$$D_{ij}(t) = \int_0^\infty \mathrm{d}\tau_0 \, \langle v_i(t+\tau_0)v_j(t)\rangle \tag{1}$$

to obtain $\kappa_\perp$, the diffusion coefficient of charged particles relative to a mean magnetic field (see also Forman, 1977). In the above $v_i(t)$ denotes the ith Cartesian component of the particle velocity and $\langle \ldots \rangle$ denotes an ensemble average (see, e.g., Montgomery & Tidman, 1964). As a consequence of an *Ansatz* concerning the particle velocity autocorrelation, they obtained the result

$$\kappa_\perp = \tfrac{1}{3}\rho_L^2 \nu \frac{\Omega^2}{\nu^2+\Omega^2} = \tfrac{1}{3} v\rho_L \frac{\Omega\tau}{1+\Omega^2\tau^2}, \tag{2}$$

where v is the particle speed, $\rho_L = v/\Omega$ is the gyroradius, ν is the rate at which a particle trajectory decorrelates from a helical orbit, $\tau = 1/\nu$, and Ω is the relativistic particle gyrofrequency. The correspondence $\kappa_\perp \leftrightarrow D_{xx}$ was used. The perpendicular diffusion coefficient $\kappa_\perp$ enters through the cosmic ray diffusion tensor $\kappa_{ij} = (\kappa_\parallel - \kappa_\perp) b_i b_j + \kappa_\perp \delta_{ij} + \kappa_A \varepsilon_{ijk} b_k$, where $\mathbf{b}$ is a unit vector in the mean magnetic field direction, $\kappa_\parallel$ is the parallel diffusion coefficient, and κ_A describes the diffusive effects associated with gradient and curvature drift (Jokipii et al. 1977).

Central to the approach of Bieber & Matthaeus (1997) are (1) an *Ansatz* for the functional form of the correlation function $\langle v_x(t) v_x(t+\tau_0)\rangle$, and (2) an additional *Ansatz* concerning the physical origin of the decorrelation rate ν that appears in the first *Ansatz*. In particular, the random walk of magnetic field lines is presumed to govern the gyrophase decorrelation, and the latter is assumed to be the dominant effect in causing the falloff of the particle velocity autocorrelation. In the particular case of magnetostatic slab turbulence one has (Jokipii & Parker 1969; Bieber & Matthaeus 1997)

$$\tau = \frac{1}{\nu} = \frac{2}{3\pi} \frac{\rho_L}{\lambda_c} \frac{B_0^2}{\langle \delta B^2 \rangle} \tau_g = \frac{4}{3} \frac{\rho_L}{\lambda_c} \frac{B_0^2}{\langle \delta B^2 \rangle} \frac{1}{\Omega}, \tag{3}$$

where $\tau_g = 2\pi/\Omega$ is the particle gyroperiod, and λ_c is the correlation length of the magnetic field in the mean field direction. Equation (3) implies, for $\langle \delta B^2 \rangle^{1/2}/B_0 \ll 1$, that the mean time for decorrelation exceeds the particle gyroperiod unless $\rho_L/\lambda_c < (3\pi/2)\langle \delta B^2 \rangle / B_0^2$. As pointed out by Bieber & Matthaeus (1997), non-slab analogs of (3) can be derived using more complex models of field line transport (e.g., Gray et al.(1996); Pommois et al.(1998)). The advantage of the Ansatz approach is that it derives a functional form for $\kappa_\perp$ without requiring at the onset a theory for the rate of decorrelation, ν. We are not aware of any reason that Eq. (2) should not remain a reasonable approximation for $\nu \ll \Omega$, $\nu \sim \Omega$ and $\nu \gg \Omega$. Previous theories (Forman et al. 1974) were limited by the constraint that $\nu \ll \Omega$. For magnetostatic slab geometry one has, from (3), $\Omega\tau = \frac{4}{3}(\rho_L/\lambda_c) B_0^2 / \langle \delta B^2 \rangle$. This implies, for solar wind fluctuations where $\langle \delta B^2 \rangle / B_0^2 \sim 1$, that $\rho_L \gg \lambda_c$, which puts a lower limit on the particle energy and rigidity, $B_0 \rho_L = |\mathbf{p}| c/|q|$. This rules out most of the cosmic rays measured by spacecraft for typical conditions at 1AU, which have, in this sense, intermediate energies.

In this paper we test the theory presented by Bieber & Matthaeus (1997) by evaluating the perpendicular diffusion coefficient numerically, employing a test particle code. Our primary aim will be to explore the intermediate to strong scattering regimes denoted by $\nu \simeq \Omega$ and $\nu > \Omega$, respectively, for which there is currently no other well established theory. We examine only the simplest case of slab geometry here, partly for the simplicity it offers, but more importantly, to have a simple basis for comparison with well-established analytical theories. Numerical tests of the theory for different magnetic field models and, indeed, for various $\langle \delta B^2 \rangle^{1/2}/B_0$, await further study [however, see Giacalone (1998) and Giacalone & Jokipii (1999)]. Further details of this study will be presented elsewhere (Mace & Matthaeus 1999).

2 Equations and Numerical Tests

For magnetostatic fields the magnetic field $\mathbf{B}(\mathbf{x}, t)$ is prescribed and time independent, and the electric field $\mathbf{E}(\mathbf{x}, t)$ is identically zero. The equation of motion of test particles with charge q is

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{v}(t) = \frac{q}{|q|} \Omega \tau_A \mathbf{v}(t) \times \mathbf{B}(\mathbf{x}, t).$$

The quantity τ_A is the Alfvén crossing time, $\tau_A = \lambda / v_A$, where λ is the turbulence correlation length, and v_A is the characteristic Alfvén speed. $\Omega = \Omega_0/\gamma$ is the relativistic gyrofrequency, and $\Omega_0 \equiv |q| B_0 / mc$ is the non-relativistic gyrofrequency. Note that since $\mathbf{E} = 0$, particle energy is conserved and hence γ is constant. The parameter $\Omega\tau_A$ arises as a consequence of our choice of normalizations (Ambrosiano et al., 1988).

Test particle simulations were carried out in a one-dimensional box of length $L = 1000\lambda$. The magnetic field, $\mathbf{B}(z)$, was stored on a grid of spacing $\Delta z = L/N$ where N was an even integer which we fixed at $N = 2^{22} = 4194304$. Each simulation used 1000 particles. The magnetic field configuration was generated through a spectrum $S(\mathbf{k})$ in $\mathbf{k}$-space. The field grids in real space were then produced via inverse Fast Fourier Transform (FFT). The large number of grid points ensures that both the turbulence correlation scale λ and the maximal gyroradius $\rho_L = v/\Omega$ were well resolved. The large box size, $L = 1000\lambda$, was chosen to reduce the subtle effects of box periodicity. In addition, to accurately depict the physics of perpendicular diffusion, we need to approximate the effect of $L \to \infty$. This provides a stable estimate of the power at the lowest wavenumber, The continuum limit is associated with wavenumber spacing $\Delta k \sim 1/L \to 0$. With our choice of box length we achieve $\Delta k\lambda = \pm 2\pi/1000 \simeq \pm 6.28 \times 10^{-3}$. For further detail, see Mace & Matthaeus (1999).

For our one dimensional field configuration the turbulent magnetic field component satisfied $\delta\mathbf{B}(z) = \delta B_x(z)\mathbf{e}_x + \delta B_y(z)\mathbf{e}_y$ with the full magnetic field given by $\mathbf{B}(z) = B_0\mathbf{e}_z + \delta\mathbf{B}(z)$. The fluctuations $\delta B_x(z_m)$ and $\delta B_y(z_m)$ were generated from a discrete Fourier transform. A single realization was used throughout each simulation run. In all the runs the spectral density of the magnetic field fluctuations was proportional to $S(k_n) = (1 + k_n^2\lambda^2)^{-\nu}$ where λ is a measure of the parallel correlation length of the turbulence and $\nu = 5/6$, providing a $\propto k^{-5/3}$ inertial range, $|k| \gg 1/\lambda$. The parameter λ is related to the true correlation length, $\lambda_c = \pi^{1/2}\Gamma(\nu)\lambda/\Gamma(\nu - \frac{1}{2})$. The particle positions were loaded randomly in the simulation box. Initial velocities were in a cold spherical shell distribution – the velocity magnitude v was held constant and a random $\cos\theta$ and ϕ were chosen for each particle. The time integration scheme is a $4th$ order Runge-Kutta-Cash-Karp method with adaptive timestepping (Press et al.,1992). During these runs we monitor accuracy in several ways, including computation of the ratio $(\langle E\rangle - E_0)/E_0$, where E is the energy of a particle at the end of the run and E_0 is the initial energy of the particle. The high degree to which particle energy is conserved (worst case is two parts in 10^5) gives us significant confidence in the code's accuracy.

3 Numerical Results

Figure 1 summarizes a number of runs investigating perpendicular spatial diffusion. The figure illustrates the behavior of $\kappa_\perp$ as a function of ρ_L/λ and $\Omega\tau$ (annotated at the top of the plot frame). Each data point (filled circle) corresponds to a numerically computed value of the normalized diffusion coefficient $\kappa_\perp/(\rho_L^2\Omega)$. On the same figure the solid line,

$$\frac{\kappa_\perp}{\rho_L^2\Omega} = \frac{1}{3}\frac{\Omega\tau}{1 + \Omega^2\tau^2}, \tag{4}$$

corresponds to the theoretical result of Bieber and Matthaeus (1997), for the special case of magnetostatic slab turbulence that we have adopted. The field line random walk limit (FLRW), in which particles always follow field lines, is given by (4) after ignoring the one in the denominator, and is illustrated in Fig. 1 with a dot-dashed line. For large values of $\Omega\tau \gg 1$, the FLRW line corresponds to the theory of Forman et al. (1974), but it should be emphasized that their theory breaks down when this inequality is not satisfied.

Strictly speaking, the dashed curve should not be extended below $\Omega\tau = 1$ as there is currently no well-established theory for perpendicular transport in this “strong scattering” regime. One observes in Fig. 1 the very good agreement of the Bieber & Matthaeus (1997) theory (solid line) with the numerically determined points, provided that $\Omega\tau > 2$, or, equivalently, $\rho_L/\lambda > 0.011$. In fact between $\Omega\tau = 4$ and $\Omega\tau = 1$, this theory is slightly better than the FLRW prediction.

For $\Omega\tau < 2$ the simulations differ substantially from either of the theories, but agreement with the FLRW limit is somewhat better. Further details of the numerical method and the simulation results for perpendicular diffusion are given in Mace and Matthaeus (1999). In closing we note that recent computations using composite models of turbulence (Giacalone and Jokipii, 1999) appear to be consistent with the present results. Evidently, and in spite of some recent progress, a full explanation of perpendicular transport will await further theoretical insight.

This research is supported in part by NSF grants ATM-9713595 and ATM-9616610, and NASA grants NAG5-8134 (SECTP) and NAG5-7142.

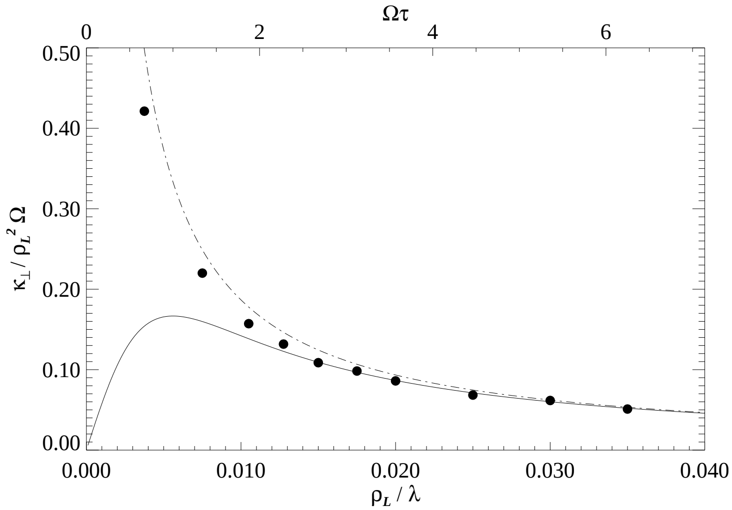

Figure 1: Normalized $\kappa_\perp$ vs. ρ_L/λ. Circles are values from the numerical simulations. Solid curve is Bieber and Matthaeus (1997) theory. Dot-dashed curve is FLRW limit. For these simulations $\langle \delta B^2 \rangle^{1/2}/B_0 = 0.1$ is constant.

References

Ambrosiano, J., Matthaeus, W. H., Goldstein, L., & Plante, D. 1988, *J. Geophys. Res.*, 93, 14383
Bieber, J. W., & Matthaeus, W. H. 1997, *Astrophys. J.*, 485, 655
Burger, R. A. 1990, Physics of the Outer Heliosphere, Oxford: Pergamon, 179
Cummings, A. C., Stone, E. C., and Webber, W. R. 1994, *J. Geophys. Res.*, 99, 11,547
Dwyer, J.R, et al. 1997 *Astrophys. J.*, 490, L115
Forman, M. A. 1977, Astrophys. Space Sci. 49, 83
Forman, M. A., Jokipii, J. R., & Owens, A. J. 1974, *Astrophys. J.*, 192, 535
Giacalone, J. 1998, Space Sci. Rev., 83, 351
Giacalone, J., & Jokipii, J. R. 1999, *Astrophys. J.*, in press
Gleeson, L. 1969, Plan. Space Sci., 17, 31
Gray, P. C., Pontius, D. H., & Matthaeus W. H. 1996, Geophys. Res. Lett., 23, 965
Green, M. S. 1951, J. Chem. Phys., 19, 1036
Jokipii, J. R., Levy, E. H., & Hubbard, W. B. 1977, *Astrophys. J.*, 213, 861
Jokipii, J. R., & Parker, E. N. 1969, *Astrophys. J.*, 155, 777
Jones, F. C. 1990, *Astrophys. J.*, 361, 162
Kubo, R. 1957, J. Phys. Soc. Japan, 12, 570
Mace, R. L., & Matthaeus 1999, Ap. J.,submitted
Montgomery, D. C., & Tidman, D. A. 1964, Plasma kinetic theory (New York: McGraw-Hill)
Pommois, P., Zimbardo, G., & Veltri, P. 1998, Phys. Plasmas, 5, 1288
Press, W. H., et al. 1992 Numerical Recipes, 2nd edition (New York: Cambridge University Press)
Taylor, G. I. 1922, Proc. London Math. Soc., Ser. 2, 20, 196