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Velocity Correlation Functions and Cosmic-Ray Transport

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Abstract

The method based on the velocity correlation function, developed by Kubo in 1957, can be used, under very general conditions, to evaluate spatial diffusion coefficients, if conditions are statistically homogeneous over several coherence times. Here we address aspects of interest in cosmic-ray transport and where the application of Kubo's formalism is not obvious. We consider an idealized model of perpendicular transport in a homogeneous magnetic field, called compound diffusion, in which the particles scatter back and forth along field lines and do not move normal to them. Transport normal to the average field occurs solely from the random walk of the field lines. Compound diffusion is non-Markovian, which leads to a situation in which the perpendicular displacement increases as the fourth root of time, in contrast to the square root of time, which is the dependence of standard diffusion. Among other things, we show that, the non-Markovian nature of the motion gives rise to a long-term anticorrelation in velocity, which causes the spatial diffusion coefficient to vanish. This behavior of compound diffusion can also be recovered from the Laplace transform of the velocity correlation function. Further implications of the long-term anticorrelation are discussed.

1 Introduction:

The transport of energetic charged particles in a turbulent magnetic field is often diffusive, where the time evolution of the omnidirectional particle density, $f_0(x_i, t)$ is described by a diffusion equation with diffusion tensor κ_{ij}. We consider here some consequences of a particular way of looking at the diffusion. For a random, diffusive motion, the spatial diffusion tensor, κ_{ij}, can be related, under very broad conditions, to the velocity correlation function Kubo (1957)

$$\kappa_{ij} = \int_0^{\tau} < v_j(t) v_i(t + t') > \, dt' \qquad (1)$$

in the limit that $\tau \to \infty$. Here $\Delta x_i = x_i(t + t') - x_i(t)$ is the spatial displacement of particle positions between times t and $t + t'$; the brackets $< >$ denote averages over an ensemble. We assume that the fluctuating velocities are statistically homogeneous over the time and length scales of interest, so the velocity correlation function, $< v_j(t) v_i(t + t') >$, depends only on the time difference t'. For any physical random velocity, $< v_j(0) v_i(t') >$ must go to go to zero at large t', and the integral in (2) approaches a constant value for for $t' \to \infty$. The virtue of this method is that only the particle velocity needs to be considered.

Forman (1977) was the first to apply Kubo's formalism to the transport of cosmic rays. It has recently been invoked by Bieber and Matthaeus (1997), who postulated a simple exponential form for the correlation tensor $< v_i(0) v_j(t) >$ to infer the corresponding perpendicular diffusion coefficient, $\kappa_\perp$, and effective drift, which is related to the antisymmetric component of κ_{ij}.

In this paper we address the special case of the perpendicular diffusion of particles tied to the turbulent magnetic field lines, which is important in understanding the transport of energetic charged particles in the heliosphere, and for which the application of Kubo's formalism is not obvious.

2 Compound Diffusion

The most poorly-understood area of cosmic-ray transport at present is the transport of particles perpendicular to the direction of the average magnetic field. This motion is due to at least two distinct effects. Particles may scatter across field lines and the field lines may depart from the mean field due to the random walk and mixing of field lines (Jokipii and Parker, 1969). This random walk of the field lines plays an important role in the perpendicular diffusion [see, e.g. Jokipii, (1966); Forman, Jokipii, and Owens, (1974), and Giacalone and Jokipii (1999)].

Low-rigidity particles in certain cases may be effectively tied to magnetic field lines, so it is useful to consider an idealized, but physically consistent, approximation, in which particles are assumed to be strictly tied to the field lines. The particles are assumed scatter back and forth along the field lines, in which case the particle perpendicular transports arises solely from the random walk of field lines. This then can serve as a starting point for understanding the more general problem of particle transport. This approximation has been termed compound diffusion, and has been used to discuss transport of cosmic rays in the galaxy (e.g. Getmantsev (1963); Lingenfelter et al (1971); Allan (1972).

Compund diffusion may be written as the convolution of two diffusive processes. Particles scatter back and forth and spread strictly along the field lines with a diffusion coefficient $\kappa_{\|}$, and the field lines, in turn, diffuse perpendicular to the mean field's (z) direction with a diffusion coefficient D_L. The mean square displacement in a perpendicular direction, say x, is then proportional to the length travelled along the field line, which, from simple scaling properties, is proportional to $\Delta t^{1/2}$. A quantitative calculation evaluating the convolution of the x and t motions yields

$$< \Delta x^2 >= 2D_L <| \xi |>= 4D_L \sqrt{\frac{\kappa_{\|} \Delta t}{\pi}} \tag{2}$$

which is slower than the standard diffusion, where $< \Delta x^2 >= 2\kappa_{\perp} t$, and so is fundamentally non-Markovian.

3 The Kubo Formulation Applied to Compound Diffusion

Kubo's formalism states essentially that the mean square displacement, $< \Delta x^2 >$, in a time, Δt, can be obtained from very general principles as

$$< \Delta x^2 >= \left\langle \left[\int_0^{\Delta t} v_x(t') dt' \right]^2 \right\rangle = 2 \int_0^{\Delta t} (\Delta t - t') < v_x(0) v_x(t') > \, dt' \tag{3}$$

which, for Δt large compared with the coherence time of $v_x(t)$, yields diffusive motion with a diffusion coefficient given by

$$\kappa_{xx} = \frac{\langle \Delta x^2 \rangle}{\Delta t} = \int_0^{\infty} < v_x(0) v_x(t') > \, dt' \tag{4}$$

The only requirement, in addition to statistically homogeneous conditions, is that the velocity correlation function $< v_x(0) v_x(t) >$ should vanish sufficiently fast as the time lag, τ increases. Under these conditions, the Kubo model would always give a diffusion $\propto \Delta t$, and could not yield compound diffusion, which results in a slower, $\propto \Delta t^{1/2}$ transport. There is clearly a problem with the application ot Kubo's formulation to this problem.

To explore this more deeply, to see where the problem lies, we have considered a simple, transparent model in which the particles propagate either forward or backward along a magnetic field line, which executes random walk about the main field direction in z. The z axis points in the direction of the mean background field; ξ denotes the position, measured along the field line, and $x(\xi)$ stands for the departure of the field line from the mean field. We consider particles released, in random directions, at $\xi = 0$ $(x = 0)$ at time $t = 0$. The variation of the number of forward (n_+) and backward (n_-) moving particles as a function of time, t, and position along the field line, ξ, is governed by the pair of equations:

$$\frac{\partial n_+}{\partial t} + v \frac{\partial n_+}{\partial \xi} = -\frac{n_+ - n_-}{2\tau} + q_+ \delta(t) \delta(\xi) \tag{5}$$

$$\frac{\partial n_-}{\partial t} - v \frac{\partial n_-}{\partial \xi} = -\frac{n_- - n_+}{2\tau} + q_- \delta(t) \delta(\xi) \tag{6}$$

where τ represents the average time of scattering. q_+ and q_- are the number of particles released in positive and negative directions, respectively.

Obviously, the velocity in the x direction is $\pm v(dx/d\xi)$ depending on whether the particle happens to move forward or backward along the field line. Thus to obtain the velocity correlation, $< v_x(0)v_x(t) >$, the mean velocity $(vn_+ - vn_-)$ along the field is to be averaged over position, with the inclusion of the actual orientation of the field line ξ

$$< v_x(0)v_x(t) >= v^2 \int_{-\infty}^{\infty} \left\langle \left(\frac{dx}{d\xi}\right)_{|0} \left(\frac{dx}{d\xi}\right)_{|\xi} \right\rangle (n_+ - n_-)\, d\xi \tag{7}$$

where subscripts imply the position in ξ. Since the source q_+ accounts for positive initial speed, while q_- corresponds to negative intial speed, n_+ and n_- can be taken as the solutions for sources $q_+ = 1/2$ and $q_- = -1/2$. This ensures that the initial speed $v_x(0)$ is properly taken into account.

Instead of considering the directly the velocity correlation $< v_x(0)v_x(t) >$ we consider its Laplace transform, $L_{xx}(s) = \int_0^\infty < v_x(0)v_x(t) > e^{-st}\, dt$. We note that $L_{xx}(s=0)$ yields exactly the corresponding perpendicular diffusion coefficient, κ_{xx}, while the behavior of L_{xx} at small s values brings information on the behaviour of $< v_x(0)v_x(t) >$ for large times ($t \gg \tau$).

We adopt the technique of Fourier and Laplace transforms (Fedorov and Shakov, 1993; Kóta 1994). First, taking the Fourier transform of equations (5) and (6) and Laplace transforming the resulting pair of equations yields a solution for $L_{xx}(s)$:

$$L_{xx}(s) = \frac{v}{2}\sqrt{\frac{s\tau}{s\tau+1}} \int_{-\infty}^{\infty} \left\langle \left(\frac{dx}{d\xi}\right)_{|0} \left(\frac{dx}{d\xi}\right)_{|\xi} \right\rangle e^{-k_0|\xi|}\, d\xi \tag{8}$$

Inspection of (8) shows that $L_{xx}(0) = 0$, unless the integral over ξ, which is related to the random walk of field lines, is infinite (this would be the case only if the field had a nonzero regular component in the x direction). Since the Laplace transform at $s = 0$ is identical to κ_{xx}, the derivation above demonstrates that Kubo's theorem yields precisely zero perpendicular diffusion coefficient, κ_{xx}, for compound diffusion. This result could intuitively be anticipated, since compound diffusion produces slower than $\propto t$ diffusion.

A further study of (8) reveals the character of $< v_x(0)v_x(t) >$ in more detail. First we notice that, for small values of s, $k_0 \approx 0$, and the integral over ξ in equation (8) gives the power of field fluctuations at zero wavenumber, which is equivalent to $2D_L$, where D_L is the diffusion coefficient of field line random walk (Jokipii, 1966). For small values of s, $L_{xx} \approx vD_L(st)^{1/2}$, and (8) implies that, for large values of t

$$< v_x(0)v_x(t) >\approx -\frac{vD_L\sqrt{\tau}}{2\sqrt{\pi}}\, t^{-3/2}. \tag{9}$$

The velocity correlation function has a long negative tail to balance the positive values at smaller t, and to give an exactly vanishing integral in (4). This long term behaviour could be obtained directly from considering the solutions for n_+ and n_- Fisk and Axford, (1969) in the $t \gg \tau$ limit, when the exact solutions can be approximated by diffusive time profiles.

Thus we find that, in a broader sense, compound diffusion fits into Kubo's theory. The velocity correlation function $< v_x(0)v_x(t) >$ exhibits a long-term anticorrelation, causing the diffusion coefficient, κ_{xx}, (i.e. the integral (4)) to vanish. This is connected with the non-Markovian nature of the compound diffusion.

At this point it is of interest to establish the connection between the present discussion and some current ideas in time-series analysis. The fact that the mean square displacement, $< \Delta x^2 >$ increases as $\Delta t^{.5}$ means that the motion is non-Markovian. In fact, the case in which $< \Delta x^2 > \propto \Delta t^{2H}$ $(0 < H < 1)$ has been given the name fractional Brownian motion, where H is the Hurst exponent (Mandelbrot and Van Ness, 1968). The case of compound diffusion corresponds to a Hurst exponent $H = .25$.It may be shown (Feder,1988), that if $H > .5$ the process exhibits long-term positive correlation and conversely, if $H < .5$ corresponds to long-term anticorrelation of the process. Clearly, then, the case of no correlation requires $H = .5$. This agrees with the determination using Laplace transforms, described previously. Now, physically, we expect particles in a turbulent magnetic field will loose correlation and we retrieve the standard form $< \Delta x^2 > \propto \Delta t$. But this cannot occur if particles are strictly tied to field lines.

3.1 Summary We considered compound diffusion, which is a non-Markovian diffusion leading to $\Delta x^2 \propto \Delta t^{1/2}$. This idealized but valid motion is seemingly in contradiction with Kubo's theory, which yields $\Delta x^2 \propto \Delta t$. We have shown that compound diffusion fits into the general theory in a broader sense. We determined the Laplace transform of the velocity correlation, and showed that the diffusion coefficient, as defined by (1) turns out to vanish. A study of the Laplace transform revealed, furthermore, that the velocity correlation has a negative non-exponential tail, $< v_x(0)v_x(t) > \propto -t^{-3/2}$ indicating a long-term anticorrelation. The $\Delta x^2 \propto \Delta t^{1/2}$ behavior of the compond diffusion could also be recovered from Kubo's formalism.

The compound diffusion may serve as a starting point for understanding the perpendicular transport of low-rigidity particles. The question is how to proceed from this picture to a model including some scattering across field lines. A small amount of cross-field scattering can be amlified by the subsequent mixing of field lines; originally nearby field lines may separate to great distances. The time scales of these processes may be large for low rigidity particles. In this case, the long non-exponential tail of the velocity correlation, which is a result of the long-term anticorrelation, may be of importance; the velocity correlation function may considerably differ from the simple exponential decay postulated by Bieber and Matthaeus, (1997). These questions need further exploration. We also point out that consideration of temporally-varying magnetic fields suggests that the conclusions derived here apply also to this situation.

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