The Cosmic Ray Riemann Problem

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Abstract

The Riemann problem is an initial value problem in which one often considers the evolution of a system which has discontinuous initial conditions. In ideal gasdynamics, the problem is well understood. Here we consider the Riemann problem in the presence of cosmic rays. Although the effect of cosmic rays on shock waves is reasonably well understood in the hydrodynamic approximation, the cosmic ray Riemann problem has not been studied in any detail. This problem is particularly important when considering the effect of cosmic rays on the termination shock and heliopause, and the interaction of supernovae shocks with progenitor stellar wind termination shocks.

1 Introduction:

It is, of course, possible for a system to begin with discontinuous initial conditions. Such an initial value problem, the Riemann problem, is well studied for ideal gasdynamics and the solution can be determined analytically (Landau & Lifshitz, 1959). If the initial conditions happen to satisfy the Rankine Hugoniot conditions for a shock wave, the solution is trivial. If, on the other hand, the initial conditions do not satisfy the Rankine Hugoniot relations, the gasdynamic system will generally break up into three waves which allow the two initial states to be connected in such a way that mass, momentum, and energy are conserved. These three waves will generally be a combination of shock waves, rarefactions, and contact discontinuities. The exact combination of waves depends on the initial state.

The standard gasdynamic Riemann solution consists of a contact discontinuity bounded by either two shock waves, two rarefactions, or a shock wave and a rarefaction. The two outer waves (shock or rarefaction) move away from the initial discontinuity into the two initial, constant states. Between these two waves the pressure and velocity are constant while the density consists of two constant states on either side of the contact discontinuity (Landau & Lifshitz, 1959). The actual solution is determined by solving the Hugoniot relation across each of the outer two waves for the common central pressure (see Chorin & Marsden, 1990). One can then determine the two density values and the common velocity. The contact discontinuity convects with this flow velocity. Whether the outer two waves are shocks, rarefactions or a combination is determined by comparing the calculated central pressure with the initial pressures (see Landau & Lifshitz, 1959).

What we consider here is how the addition of cosmic rays effects the standard Riemann problem solution. A considerable body of work has investigated the effect of cosmic rays on shock waves (Axford, Leer, & McKenzie, 1982; Drury &Volk, 1981; Donohue, Zank, & Webb, 1994) but the Riemann problem has not been studied in any detail. Webb et al. (1995) did consider a collision between a gasdynamic shock and a cosmic ray modified shock but did not, however, address the Riemann problem itself.

We use a one-dimensional, Cartesian, two fluid model (Axford, Leer, & McKenzie, 1982). The cosmic rays are assumed to have a negligible mass and contribute only a pressure. The equations are

$$\partial_t \rho + \partial_x(\rho u) = 0 \quad (1)$$

$$\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P_g + \partial_x P_c = 0 \quad (2)$$

$$\partial_t\left(\frac{1}{2}\rho u^2 + \frac{P_g}{\gamma_g - 1} + \frac{P_c}{\gamma_c - 1}\right) + \partial_x\left[\left(\frac{1}{2}\rho u^2 + \frac{\gamma_g P_g}{\gamma_g - 1}\right)u + \frac{\gamma_c P_c}{\gamma_c - 1}u - \frac{\kappa}{\gamma_c - 1}\partial_x P_c\right] = 0 \quad (3)$$

$$\partial_t P_c + u\partial_x P_c - \kappa\partial_x^2 P_c + \gamma_c P_c \partial_x u = 0 \quad (4)$$

where ρ, u, and P_g are the thermal gas density, velocity, and pressure. P_c is the cosmic ray pressure which has a diffusion coefficient κ and evolves according to equation (4), determined by taking a moment of the cosmic ray transport equation. The terms γ_g and γ_c are the gas and cosmic ray spectral indices taken to be 5/3 and 4/3 respectively. The pure gasdynamic equations are obtained by setting P_c to zero. The above equations are solved using a time explicit Eulerian hydrodynamic code (Pauls, Zank, & Williams, 1995).

2 The Riemann Problem

In the cases considered, the velocity is always positive and hence there is a net motion from left to right. To avoid waves propagating out of the right hand side of the box, we choose our discontinuity to be 1/5 of the way along the grid. The grid is initialized to two different constant states, one on either side of this discontinuity. We consider both a purely gasdynamic situation and one in which cosmic rays are included. In both cases we consider three initial value problems. For each problem the gas variables are the same in both the gasdynamic and cosmic ray cases. The initial gas conditions are shown in Table 1. The subscripts 1 and 2 refer to the left hand and right hand states respectively. In all three problems the gas pressures and densities are the same and are chosen such that the the sound speed is greatest to the right of the discontinuity. When cosmic rays are included they are constant across the entire box and have a value equal to the gas pressure on the left of the discontinuity. All the values that are presented are normalized.

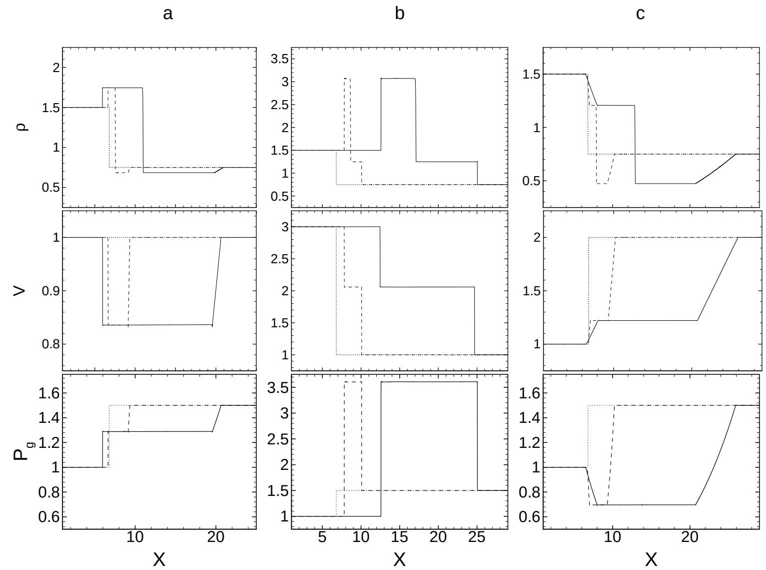

Figure 1: Gasdynamic Riemann problem solutions for the three cases discussed in the text. The gas density, flow velocity, and pressure are plotted.

2.1 Gasdynamics:

Figure 1 shows the solution to the three abovementioned gasdynamic Riemann problems. For each case, we plot the gas density, flow velocity, and pressure. In each plot the dotted line illustrates the discontinuous initial conditions, the dashed line shows the solution at time $t = t_o$ while the solid line gives the solution at time $t = 5t_o$. It is clear that in all three cases either a shock wave or a rarefaction propagates into each of the two constant initial states. In between these two waves the pressure and flow velocity are constant while the density contains a contact discontinuity. This contact discontinuity essentially separates the two initial states. In Figure 1a the solution consists of a shock moving into the left hand fluid and a rarefaction moving to the right. Figure 1b shows two shock waves, one moving into the left hand fluid and the other to the right while Figure 1c shows two rarefactions. Due to the high left hand flow velocity, the left shock in Figure 1b has a positive velocity even though it is actually moving into the left hand fluid. In both cases where shocks form, the shocks remain unchanged with time. The rarefactions, by contrast, are broader at $t = 5t_o$ than at $t = t_o$.

	ρ	u	P_g
Case 1	$\rho_1 > \rho_2$	$u_1 = u_2$	$P_{g1} < P_{g2}$
Case 2	$\rho_1 > \rho_2$	$u_1 > u_2$	$P_{g1} < P_{g2}$
Case 3	$\rho_1 > \rho_2$	$u_1 < u_2$	$P_{g1} < P_{g2}$

Table 1: Initial gas conditions.

2.2 Cosmic Rays:

Figure 2 illustrates the effect of adding cosmic rays to the three Riemann problems. The gas density, flow velocity and pressure are plotted together with the cosmic ray pressure. The cosmic rays are assumed to have the same pressure as the left hand fluid and to be initially constant across the box. As before, the dotted line represents the initial conditions, the dashed lines give the solution at $t = t_o$ and the solid lines give the solution at $t = 5t_o$. The initial gas conditions in each of these three cases are identical to those in the corresponding gasdynamic case.

The dashed lines in the density and pressure plots of Figure 2a show that, as in the gasdynamic case, a shock propagates to the left and a rarefaction to the right. The cosmic ray pressure gradient has, however, decelerated the flow upstream of the shock creating a foreshock. It has also modified the flow to the right of the rarefaction, smoothing its leading edge. At $t = 5t_o$ (solid line) both the shock and rarefaction have been completely smoothed. The effect of the cosmic ray pressure gradient is also evident in Figures 2b and 2c. In Figure 2b we have, as expected from the corresponding gasdynamic case, two shock waves. Once again, the cosmic ray pressure gradient has resulted in a foreshock forming upstream of both shocks. In this case, however, the shocks have not been completely smoothed by $t = 5t_o$. The cosmic ray pressure gradient may completely smooth these shocks at a later time, but that is not clear. As in the gasdynamic case, the velocity between the two outer waves is constant. In this case, however, it is the sum of gas and cosmic ray pressure that is constant and not the gas pressure alone. The cosmic ray pressure gradient across the contact discontinuity acts as a source, or sink, of momentum and manifests itself as an increase, or decrease, in density. This causes the peaks and troughs that can be seen on either side of the contact discontinuity in the density plots. In Figure 2c there are two rarefactions both of which have been completely smoothed by $t = 5t_o$.

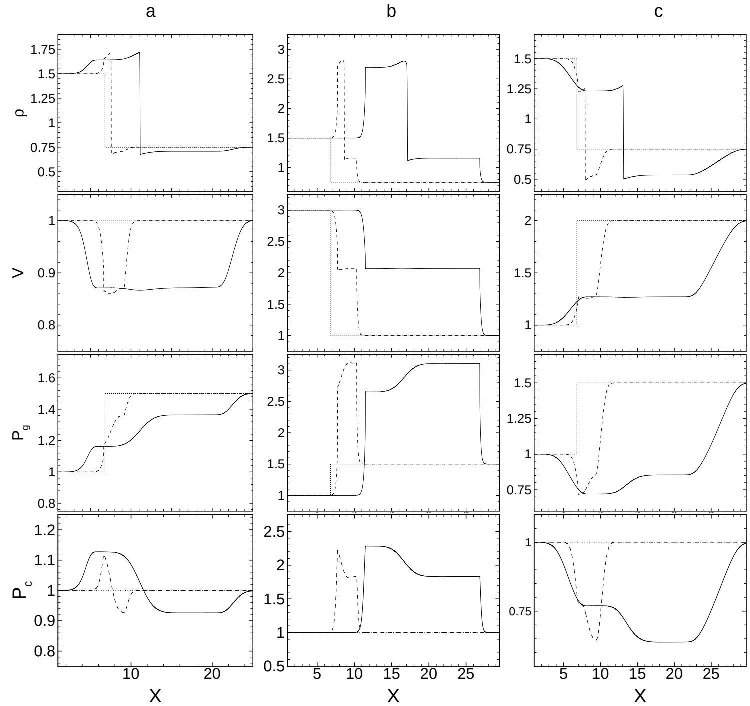

Figure 2: Cosmic ray Riemann problem solutions with the same initial gas conditions as in the gasdynamic case.

In Figure 2b one can also see that the separation between the two shocks is greater than in the corresponding gasdynamic case. This is due to the cosmic ray pressure changing the long wavelength sound speed and hence the shock propagation speed (Webb et al., 1995). This increased sound speed also applies to the other two cosmic ray Riemann problem solutions although the increased separation is not as easy to see.

3 Discussion

Although the cosmic ray Riemann problem looks somewhat complicated, it can be understood in a straightforward way. Initially the cosmic ray pressure is constant and hence, apart from changing the sound speed, does not effect the Riemann problem. The initial solution is therefore determined by the gasdynamic conditions. It is thus possible to determine analytically the flow velocity, gas pressure and density in the region between the two waves that propagate out into the two constant states. The pressure that is calculated determines whether the outer waves are shocks or rarefactions and the flow velocity gives the speed at which the contact discontinuity propagates. In the case of shocks, their Mach number and compression ratio can be determined. The technique of Axford, Leer, & McKenzie (1982) or Drury & Volk (1981) can then be used to determine how the cosmic rays will effect the shock waves. Depending on the shock Mach number, the cosmic rays will either completely smooth the shock (as in Figure 2a) or they will create a foreshock upstream of the actual shock discontinuity (as in Figure 2b). If the cosmic rays completely smooth the outer two waves, then these structures will propagate at the long wavelength sound speed determined by both the gas and cosmic ray pressures.

In the gasdynamic case, equations (1), (2) and (3) can be written in characteristic form (Whitham, 1974). This cannot, strictly speaking, be done in the cosmic ray case as equation (4) is parabolic. An approximate set of characteristic equations can be obtained if we consider the $\partial P_c/\partial x$ term in equation (3) to be a source term and assume that equation (4) is not part of the system. This ignores the fact that the cosmic ray pressure changes the sound speed but the characteristic curves that are obtained are still a useful illustration of the solution. Figure 3 shows the the forward and backward propagating sound speed characteristics for both the gasdynamic (Figure 3a) and cosmic ray (Figure 3b) Riemann problems of Figures 1a and 2a. In the top panel of Figure 3a one can see two characteristics that have intersected, so necessitating the formation of a shock. The characteristics on either side of these two are also converging and will intersect at a later time. In the lower panel a region exists where the characteristics are diverging, indicating the presence of a rarefaction.

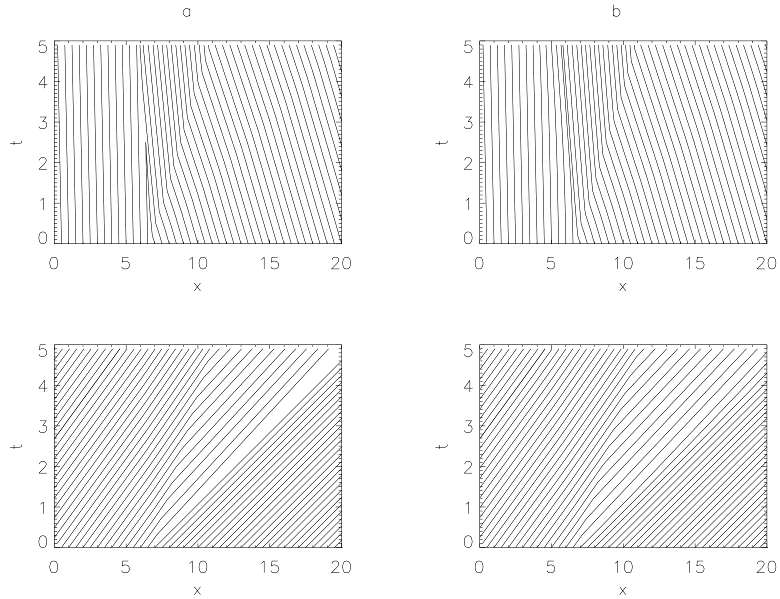

Figure 3: Gasdynamic sound speed characteristics for both the gasdynamic and cosmic ray Riemann problems.

In Figure 3b it appears that some characteristics should intersect. Before they do, however, they bend and become parallel. This is due to the cosmic ray smoothing of the shock. If our spatial resolution were better some characteristics would intersect at early times but beyond a certain time this would no longer occur and all points, after this time, can be traced back uniquely to the original data. In the lower panel one sees that characteristics that initially appear to be diverge bend and become parallel. This indicates that beyond a certain time all points can be traced back to the initial data, unlike the corresponding gasdynamic case.

4 Conclusion

The solution to the cosmic ray Riemann problems is given initially by the solution to the corresponding gasdynamic Riemann problem, except that the cosmic ray pressure modifies the long wavelength sound speed. Thus the rarefaction and shock wave properties can be determined initially. As the system evolves, the cosmic ray pressure will be modified by these waves and its gradient will start to smooth both the shocks and the rarefactions. Especially in the case of shocks, the extent to which these waves are smoothed depends on their initial properties (Axford, Leer, & McKenzie, 1982; Drury & Volk, 1981).

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