$<\delta B \delta f>$: Practical Considerations for Measuring Particle–Field Correlations

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Abstract

The measurement and analysis of $<\delta B \delta f>$, the correlation between fluctuations in the magnetic field and fluctuations in the particle phase space density, has the potential to provide new information on the geometry of magnetic turbulence in the solar wind and on the process of particle scattering by magnetic turbulence. The theoretical description of $<\delta B \delta f>$ uses instantaneous values of δB and δf, but real world measurements only yield discrete or time-averaged values. Using a simulated slab magnetic field to represent the interplanetary magnetic field, we plan to determine what time resolution for δB and δf is required to yield an acceptable approximation to the theoretical continuous quantity. Interim results are reported here.

1 Introduction

A means of better understanding cosmic ray transport in the heliosphere may be provided by studies of the relationships between the fine scale structure of the interplanetary magnetic field (IMF) and the fine scale structure of the cosmic ray distribution function. Jokipii and Owens (1974) and Owens and Jokipii (1974) have emphasized the valuable information to be gained from studies of cosmic ray scintillations and the coherence between scintillations and magnetic field fluctuations. Bieber (1987) took a related approach, but with a different emphasis. He focused on the correlation between fluctuations of the magnetic field, δB, and fluctuations of the cosmic ray distribution function, δf. This quantity, $<\delta B \delta f>$, which occupies a central role in quasi-linear theory, is measurable and has the potential to provide new information on the geometry of magnetic turbulence in the solar wind and on the process of particle scattering by magnetic turbulence.

2 Computational Experiment

In our analysis, we use a slab magnetic field to represent the IMF. The magnetic field, $\vec{B}(\vec{r})$, is Fourier decomposed in terms of $\vec{b}(\vec{k})$, which is related to the power spectrum. The slab model assumes that the wavevector of each of the infinite Fourier components points in the same direction, say the z-direction. Therefore in the slab model, the magnetic field is written as $\vec{B}(\vec{r}) = (B_x(z), B_y(z), B_0)$.

Now we discretize the above results; therefore,

$$B_j(q) = \sum_{n=-N}^{N} b_{j_n} \, \mathrm{e}^{2\pi \imath n q/(2N+1)}, \tag{1}$$

where $B_j(q) = B_j(z_q)$, $b_{j_n} \equiv b_j(k_n)\Delta k$ and $z_q = q\Delta z$, $k_n = n\Delta k$. It is important to note that the choice of Δk, Δz, and N is not an arbitrary process; these quantities are related by the so-called reciprocity relations (Briggs and Henson, 1995):

$$(2K_d)(2L) = 2\pi(2N+1), \tag{2}$$

$$\Delta k \Delta z = \frac{2\pi}{2N+1}, \tag{3}$$

where $2L = (2N+1)\Delta z$, $2K_d = (2N+1)\Delta k$. The actual values used for these quantities depends on the minimum and maximum scale sizes of a given problem. We will return to this issue in the discussion, Sec. 3.

We have used the following two-sided power spectrum to describe the magnetic power spectrum in the IMF (Bieber et al., 1994):

$$P(k) = \begin{cases} 2\pi c\lambda\,(1+k^2\lambda^2)^{-5/6} & k < k_d \\ 0 & k > k_d \end{cases} \tag{4}$$

where $k_d \equiv 2\pi/\lambda_d$ and λ_d is the dissipation scale. The constants in the above equations are $c = 0.075$ nT^2, $\lambda = 4.55\times10^9$ m, and $k_d = 2.0\times10^{-5}$ m^{-1}. The modal power, P_n, is related to the magnetic power spectrum by $P_n = P(k_n)\Delta k$. The modal power is related to b_{j_n} by $P_{jj_n} = b_{j_n} b_{j_{-n}}$, where, for a real magnetic field, $\Im B(q) \equiv 0$, it can be shown that $b_{-n} = b_n^*$. Random fluctuations in the magnetic field are created via a random phase, ψ_{j_n}, in b_{j_n}; therefore,

$$b_{j_n} = \sqrt{P_n}\,\mathrm{e}^{\imath\psi_j}. \tag{5}$$

For charged particle scattering in slab turbulence, Bieber (1987) and Chen (1989) show that, under certain assumptions, the fluctuations in the particle phase space density, δf is given by

$$\delta f(z) = \delta f(0) - \frac{k_r}{B_0}\left.\frac{\partial <f>}{\partial\theta}\right|_z \int_{-z}^{0} [\delta B_x(z+\xi)\sin(\phi - k_r\xi) - \delta B_y(z+\xi)\cos(\phi - k_r\xi)]\,d\xi, \tag{6}$$

where (p,θ,ϕ) gives the particle momentum in spherical polar coordinates and $<f>$ is the ensemble average of the particle phase space density. The resonant wavenumber, $k_r \equiv 1/(R_L\mu)$, where, $\mu = \cos\theta$, is the cosine of the pitch angle and $R_L = P/(cB_0)$ is the Larmor radius in terms of the particle rigidity, P; c is the speed of light. We set $k_r \equiv r\Delta k$. The $d\xi$ integral is straightforward to do, if $B(z+\xi)$ is Fourier decomposed. For future reference, we also include the equation used to calculate the discrete analogue of δf; in the remainder of this paper it is assumed that all sums are from $n = -N$ to $n = N$, unless otherwise stated:

$$\begin{aligned}\delta f(q) = \delta f(0) \quad - \quad & \frac{k_r}{2B_0}\left.\frac{\partial <f>}{\partial\theta}\right|_z \Bigg\{ z_q \left(\Im\left[b_{x_r}\mathrm{e}^{\imath(k_r z_q+\phi)}\right] + \Re\left[b_{y_r}\mathrm{e}^{\imath(k_r z_q+\phi)}\right]\right) \\ & + \mathrm{e}^{\imath\phi}\sum_{n\neq r}\frac{b_{x_n} - \imath b_{y_n}}{k_r - k_n}\mathrm{e}^{2\pi\imath nq/(2N+1)} + \mathrm{e}^{-\imath\phi}\sum_{n\neq -r}\frac{b_{x_n} + \imath b_{y_n}}{k_r + k_n}\mathrm{e}^{2\pi\imath nq/(2N+1)} \\ & - \mathrm{e}^{\imath(k_r z_q+\phi)}\sum_{n\neq r}\frac{b_{x_n} - \imath b_{y_n}}{k_r - k_n} - \mathrm{e}^{-\imath(k_r z_q+\phi)}\sum_{n\neq -r}\frac{b_{x_n} + \imath b_{y_n}}{k_r + k_n}\Bigg\}\end{aligned} \tag{7}$$

A sample calculation of the particle phase space density is shown in Fig. 1 for a 10 GV particle with $\mu = 0.5$ moving in a magnetic field with an average magnitude of $B_0 = 4.0\times10^{-9}$ T. For this calculation we set $\delta f(0) = 0$ and $\partial <f>/\,\partial\theta = 1$.

Assuming that the data set is large enough and that conditions of ergodicity and stationarity are satisfied, the ensemble average, $<\delta B_j \delta f>$, is given by

$$<\delta B_j \delta f> = \frac{1}{2N+1}\sum_{q=-N}^{N}\delta B_j(q)\,\delta f(q). \tag{8}$$

Bieber (1987) showed that the continuous analogue of $<\delta B_j \delta f>$ approaches a constant value for $z \gg \lambda_c$, where λ_c is the magnetic field correlation length. This value depends only upon the power spectrum matrix and is independent of the wave phase. On the otherhand, for the discrete calculation we found that $<\delta B_j \delta f>$ is inherently random; that is, it retains an explicit dependence upon the wave phase. Therefore it varies according to the value of the random seed used to determine the wave phase.

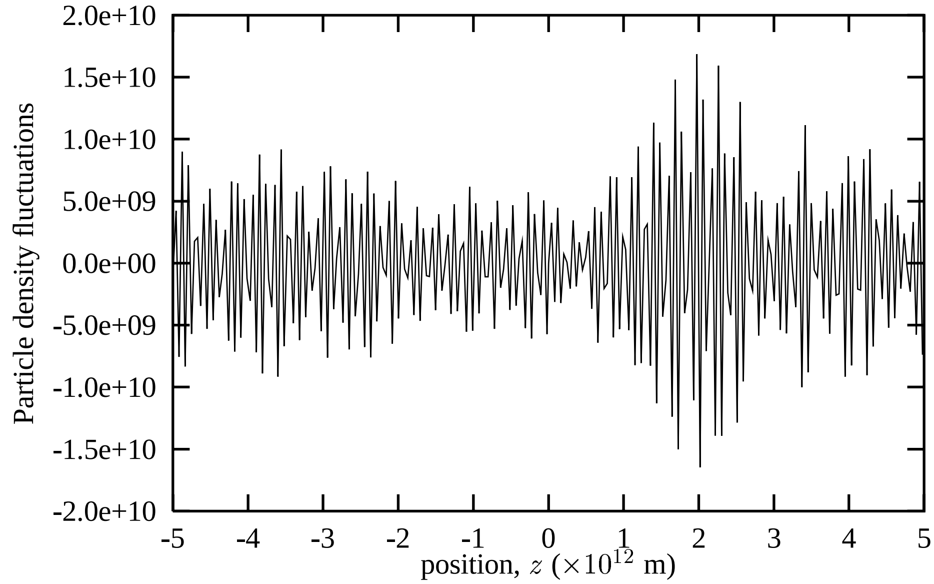

Figure 1: Particle phase space density fluctuations for a 10 GV particle. For further details see the text.

A look behind the scenes of Eq. 8 reveals how the dependence on the random seed is introduced. If we substitute Eq. 1 and Eq. 7 into Eq. 8 we obtain

$$\begin{aligned} <\delta B_x \delta f> \; = \; & \delta f(0) b_{x0} - \frac{k_r}{2B_0} \left. \frac{\partial <f>}{\partial \theta} \right|_z \left\{ \mathrm{e}^{\imath\phi} \sum_{n \neq r} \left(\frac{P_{xxn} - \imath P_{xy_n}}{k_r - k_n} - b^*_{xr} \frac{b_{xn} - \imath b_{y_n}}{k_r - k_n} \right) \right. \\ & + \mathrm{e}^{-\imath\phi} \sum_{n \neq -r} \left(\frac{P_{xxn} + \imath P_{xy_n}}{k_r + k_n} - b_{xr} \frac{b_{xn} + \imath b_{y_n}}{k_r + k_n} \right) \\ & \left. - \frac{z_q}{2} \left(\mathrm{e}^{\imath\phi} [P_{xxr} - P_{xy_r}] - \mathrm{e}^{-\imath\phi} [P_{xxr} + P_{yx_r}] \right) \right\} \end{aligned} \tag{9}$$

where r is the mode of the resonant wavenumber. Equation 9 reveals two ways that the random seed enters. The most obvious problem is the presence of the random terms b_{j_n}. A less obvious problem is the terms containing P_{jk_n}. For two uncorrelated sequences, $P_{jk_n} \rightarrow 0$ as $N \rightarrow \infty$, in agreement with the continuous analogue; however, as shown in Fig. 2, P_{jk_n} is finite, random, and large compared to P_{xxn} for finite N.

3 Discussion

Our computer model of the IMF contained some surprising elements that are not present in the continuous theory presented by Bieber (1987). The most surprising difference was the dependence of $<\delta B_j \delta f>$ on the random seed, as discussed above. This problem appears to be inherent in the discretization process. As an interim solution we set the r^{th} term of the modal power, $P_{jj_r} \equiv 0$; this also implies that $b_{xr} = 0$. In addition, setting $\psi_{xn} = \psi_{y_n}$ makes P_{jk_n} equal to the modal power, which is not random. Note that magnetic helicity is equal to zero when we set the random phase for both components equal. These interim measures introduce problems of their own, not the least of which is that it is artificial. In addition, if $\psi_{xn} = \psi_{y_n}$ then $P_{xy_n} = P_{yx_n}$, which violates one of the properties of Bieber's (1987) simple turbulence model; namely, the property of axisymmetric slab turbulence, which implies that $P_{xy_n} = -P_{yx_n}$.

We have thoroughly tested the program; for example, we checked that the magnetic field data we generated satisfied the Weiner-Khinchin Theorem. It should also be noted that N, L, and K_d have been chosen to provide

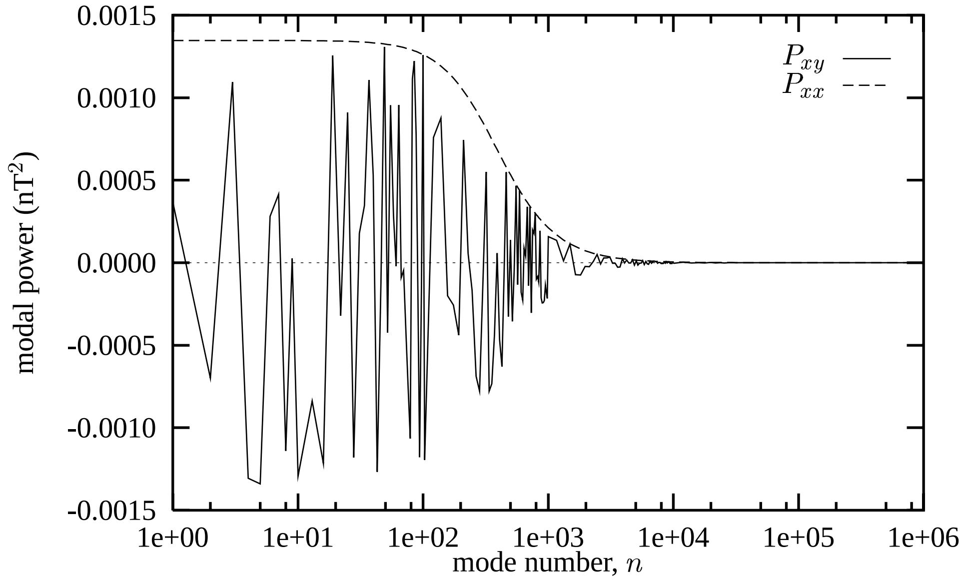

Figure 2: The power P_{xy_n} compared with P_{xxn}.

good resolution of all significant scales in the problem and to reduce edge effects. For example, for a 10 GV particle we choose $L = 5.0 \times 10^{12}$ m and $K_d = 5.0 \times 10^{-7}$ m^{-1}. These values ensure that both the resonant wave-number, $\lambda_r = 2\pi/k_r$, and the magnetic field correlation length, λ_c, are well resolved and smaller than the box size used in the simulation.

Our aim in performing this computational experiment is to answer two practical questions concerning the measurability of $<\delta B \delta f>$. Magnetic field data and cosmic ray intensity data do not exist with infinitesimal time resolution; rather, the data gives discrete or time-averaged values of δB and δf. Therefore we plan to determine the time resolution required in δB and δf to yield an acceptable approximation to the theoretical value of $<\delta B \delta f>$. At the opposite end of the scale, the theoretical calculation of $<\delta B \delta f>$ assumes that the average is taken over an infinite region, which is an experimental impossibility. Therefore a related question that we plan to study deals with the length scale that needs to be probed to yield an acceptable approximation of $<\delta B \delta f>$.

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