Numerics of Interplanetary Transport

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Abstract

Interplanetary transport is governed by the processes of focusing, pitch-angle scattering, adiabatic deceleration and convection with the solar wind. The corresponding transport equation can be solved numerically by using a finite difference method. We will present a fast and accurate numerical method based on an explicit first-order upwind scheme for the spatial transport with a flux-limiter correction and an implicit centered difference scheme for the transport in pitch-angle. The transport of particles in momentum is described in an upwind scheme with flux-limiter correction. The convergence of the scheme will be discussed.

1. Introduction

Interplanetary transport is governed by a multitude of physical processes, affecting particles of different energy at different positions in space differently. A number of approximate solutions exist, such as the convection-diffusion equation, mostly applied to the modulation of galactic cosmic rays, or the equation of focused transport of Roelof (1969), in general applied to solar energetic particle events in the inner heliosphere. Ruffolo (1995) combined the effects of pitch-angle scattering, adiabatic focusing, convection with the solar wind, and adiabatic deceleration in a transport equation

$$\frac{\partial F}{\partial t} + \frac{\partial}{\partial s}\left(\left[\mu' v' + \left\{1 - \frac{(\mu' v')^2}{c^2}\right\} v_{\rm sw} \sec(\psi)\right] F\right) + \frac{\partial}{\partial \mu'}\left(v' \frac{1-\mu'^2}{2L} F - \kappa(s,\mu')\frac{\partial F}{\partial \mu'}\right)$$
$$-\frac{\partial}{\partial p'}\left(p' v_{\rm sw}\left[\frac{\sec(\psi)}{2L}(1-\mu'^2) + \cos(\psi)\frac{\mathrm{d}}{\mathrm{d}r}\sec(\psi)\mu'^2\right] F\right) = Q(t,s,\mu',p') \qquad (1)$$

and offered a numerical solution. In (1) the terms from left to right give: (a) the change in the distribution function $F(t,s,\mu',p')$, depending on time t, spatial distance s along the Archimedian magnetic field line, pitch-cosine μ' and particle momentum p', the latter two measured in the solar wind frame; (b) the field parallel motion of particles consisting of the direct propagation of particles with speed $\mu' v'$ along the field line and their convection with the solar wind speed along the field line, described by the solar wind speed $v_{\rm sw}$ and the spiral angle ψ; (c) changes in pitch-angle due to focusing with a focusing length $L(s) = -B/(\partial B/\partial s)$ and due to pitch-angle scattering with a pitch-angle diffusion coefficient $\kappa(\mu', s)$; and (d) changes in momentum due to the betatron effect (adiabatic cooling) and the "inverse Fermi-effect". The term on the right gives a sources function, e.g. a solar injection.

2. The numerical scheme

In this paper, Eq. (1) is solved with a numerical scheme based on an enhanced fractional time step and time splitting method (e.g. Marchuk, 1975). The basic idea of such a splitting scheme can be understood for the simplified example consisting of only spatial and pitch-angle transport. The arbitrary decision to transport first in s and then in μ' or vice versa gives two different numerical solutions. The differences are small for the omnidirectional intensity and more pronounced in the anisotropies. In general such an approximation is only of first order in Δt. A better result can be achieved by alternating the order of fractional time steps and hence get an approximation of second order in Δt:

$$L(t,s,\mu',p') \stackrel{\text{def}}{=} \frac{1}{2}L(s) + \frac{1}{2}L(\mu') + \frac{1}{2}L(p') + \frac{1}{2}L(p') + \frac{1}{2}L(\mu') + \frac{1}{2}L(s) + O((\Delta t)^2)$$

(1) *The spatial transport $L(s)$:* The spatial transport is described by a hyperbolic differential equation. It is discretized by a flux-limiter method, which is under optimal circumstances (low spatial gradient) of second order in space. Both a detailed discussion and a comparison to other methods can be found in Hatzky (1996) and Hatzky et al. (1997). The advantage to other methods (e.g. Ruffolo, 1995) is the good accuracy combined with a low computational effort.

(2) *The pitch-angle transport $L(\mu')$:* The transport in pitch angle is described by a parabolic-type convection-diffusion equation. The difficulty in solving numerically this equation comes from the two pitch-cosine terms which can be relative to each other of different magnitude. Close to the sun the focusing term is dominating while far away the pitch-angle diffusion becomes relatively dominating. The same problem occurs when cases with large and small scattering or a pitch-angle diffusion coefficient $D(\mu')$ with a pronounced shape shall be calculated. Usually in literature this is called a singular perturbed problem which needs special numerical treatment (e.g. Roos et al. 1996). A scheme constructed for this type of problem is the **Iljin scheme**. In this paper the following implicit scheme, which is under optimal circumstances of second order in pitch-cosine, is used:

$$\begin{aligned}
L_{\Delta\mu'} F_i \quad &\stackrel{\text{def}}{=} \quad \frac{a_{i+1/2}(F_{i+1}-F_i) - a_{i-1/2}(F_i + F_{i-1})}{\Delta\mu'} \\
&\quad - \frac{1}{(\Delta\mu')^2}\left[\chi_{i+1/2}\, b_{i+1/2}(F_{i+1}-F_i) - \chi_{i-1/2}\, b_{i-1/2}(F_i - F_{i-1})\right]
\end{aligned}$$

$$a_{i+1/2} \stackrel{\text{def}}{=} 1-\mu'^2_{i+1/2}\tilde{a} \qquad \tilde{a} \stackrel{\text{def}}{=} \frac{v'}{2L(s)} \qquad \chi_{i+1/2} \stackrel{\text{def}}{=} \frac{\tilde{a}\Delta\mu'}{2\tilde{b}_{i+1/2}}\coth(\frac{\tilde{a}\Delta\mu'}{2\tilde{b}_{i+1/2}})$$

$$b_{i+1/2} \stackrel{\text{def}}{=} 1-\mu'^2_{i+1/2}\tilde{b} \qquad \tilde{b}_{i+1/2} \stackrel{\text{def}}{=} \left(\frac{1}{\Delta\mu'}\int_{\mu'_i}^{\mu'_{i+1}} \frac{\mathrm{d}\mu'}{\tilde{D}(\mu')}\right)^{-1} \qquad D_{i+1/2} \stackrel{\text{def}}{=} 1-\mu'^2_{i+1/2}\tilde{D}(\mu')$$

If an isotropic pitch-angle coefficient $D(\mu') = A(1-\mu'^2)$ is considered, the scheme is a pure Iljin scheme. In case of a pronounced shape of $D(\mu')$, a so called "resonance gap" around $\mu' = 0$, the averaging of $\tilde{D}(\mu')^{-1}$ per $\Delta\mu'$-interval gives better results. It is motivated by an **integro-interpolation scheme** (e.g. Samarskii, Vabishchevich 1995), which leads to the definition of $\tilde{b}_{i+1/2}$. For a very pronounced $D(\mu')$ the following choice, where the focus is on the integro-interpolation scheme, gives even better results:

$$\chi_{i+1/2} \stackrel{\text{def}}{=} \frac{a_{i+1/2}\Delta\mu'}{2b_{i+1/2}} \coth(\frac{a_{i+1/2}\Delta\mu'}{2b_{i+1/2}}) \quad b_{i+1/2} \stackrel{\text{def}}{=} \left(\frac{1}{\Delta\mu'} \int_{\mu'_i}^{\mu'_{i+1}} \frac{\mathrm{d}\mu'}{D(\mu')} \right)^{-1}$$

(3) *The momentum transport $L(p')$:* The momentum transport also is described by a hyperbolic differential equation. As the spatial transport it is discretized by a flux-limiter method. The step size in $\Delta p'$ is chosen to be constant in a logarithmic scale of momentum p'. Compared to the geometric interpolation used by Ruffolo (1995), the flux-limiter method has the advantage of particle number conservation even if the momentum spectrum cannot be described by a power law – which will be the case during the course of a particle event, even if the initial spectrum is a power-law, cf. Fig. 2.

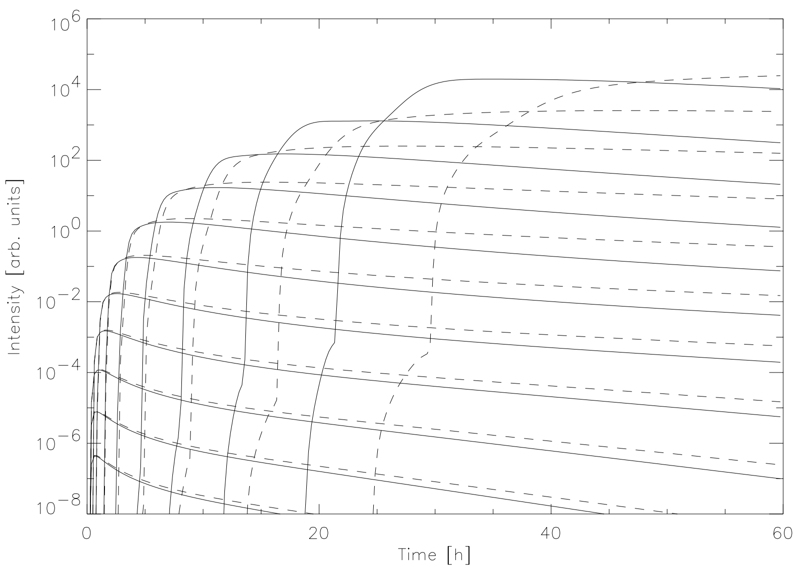

Fig. 1: Intensity time profiles with (solid) and without (dashed) solar wind effects for different energies (20, 66, 220, and 730 keV, 2.4, 8, 26, 85, 260, and 711 MeV, and 1.7 GeV), a δ-injection at $t = 0$, and a radial mean free path $\lambda_{\mathrm{r}} = 0.1$ AU.

3. Simulations

Figure 1 demonstrates the influence of solar wind effects at different energies (20 KeV, 66 KeV, 220 keV, 730 KeV, 2.4 MeV, 8 MeV, 26 MeV, 85 MeV, 260 MeV, 711 MeV, 1.7 GeV). The observer is located at 1 AU, the radial mean free path λ_{r} is 0.1 AU. Two effects can be separated:

(1) if solar wind effects are considered (solid lines), the intensities rise earlier and consequently show an earlier maximum. This becomes most obvious in the lowest energy bands where the average particle speed is comparable to the solar wind speed. The effect can be understood as mainly due to convection with the solar wind, adiabatic deceleration contributes only a small part to this because of the energy spectrum.

(2) the intensity decays faster because of adiabatic deceleration which, owing to the energy spectrum, adds a rather small number of particles from the higher energies while removing a larger number to lower energies. In the lower energy ranges, convection with the solar wind also contributes to the faster removal of particles from the observer's position. Because low energetic particles acquire maximum intensity at rather late times, this removal also leads to a lower maximum intensity.

In the anisotropies (not shown in the figure) consequently a faster decay towards isotropy can be seen, at late times anisotropies even can become negative as the – in the solar wind frame isotropic – particle distribution is convected across the observer, leading to an inward directed intensity gradient and thus a streaming of particles towards the Sun.

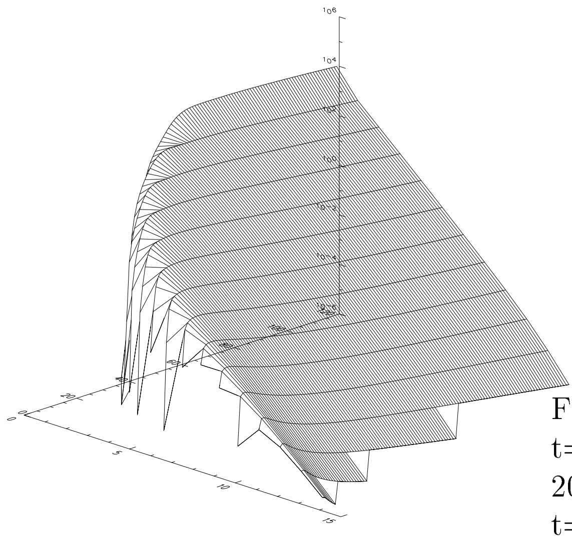

Fig. 2: Temporal evolution from t=0 h to t=60 h of the energy spectrum in the range 20 keV to 1.7 GeV after a solar injection at t=0.

Figure 2 shows the temporal evolution of the energy spectrum at the observer's site. The injection spectrum has a power-law index $\gamma = -2.5$ in energy. At early times, due to the late arrival of slow particles, the spectrum turns over at lower energies. Only with the arrival of the bulk of the slow particles, the spectrum turns to roughly a power-law, however, its slope is much steeper than the slope of the injection spectrum, even bending to a steeper slope at late times and high energies. Thus for most of the time of the event the description of the spectrum in interplanetary space by a power-law would be a crude simplification. Note that a similar behavior would be observed if solar wind effects were neglected, in particular the turn-over of the spectrum at low energies would be observed up to much later times, owing to the later arrival of the low energies. If a decreasing slope has been established it would even be steeper than under consideration of solar wind effects.

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