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Wave Coupling in Oblique MHD Cosmic Ray Modified Shocks

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Abstract

Numerical simulations and multiple scales perturbation methods are used to study wave interactions in magnetohydrodynamics with application to oblique cosmic ray modified shocks. Coupled evolution equations for the Alfvén waves, fast and slow magneto-acoustic and entropy waves are solved using a spectral collocation method. Numerical simulations of the fully nonlinear cosmic ray MHD equations are compared to the solutions of the linear wave interaction equations.

1 Introduction

Wave interactions in magnetohydrodynamics (MHD) are an integral part of cosmic ray propagation problems. There is an extensive literature on the role of waves and wave coupling in astrophysical plasmas (e.g. Heinmann and Olbert, 1980; Zhou and Matthaeus, 1990; MacGregor and Charbonneau, 1990). The main aim of this paper is to explore MHD wave coupling processes in cosmic-ray modified shocks by using the formalism developed by Webb et al. (1999). Solutions of the equations describing wave mixing in a non-uniform MHD plasma modified by the comsic rays are compared to numerical solutions of the non-linear, two-fluid cosmic ray MHD equations in one Cartesian space dimension. We investigate the manner in which different MHD wave modes can be generated by wave coupling from a single wave mode initially present in the medium.

2 Model and Equations

Using the momentum averaged cosmic-ray transport equation, the two-fluid cosmic-ray modified MHD system can be written as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla (p_g + p_c)}{\rho} = \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{\mu \rho} \tag{2}$$

$$\frac{\partial p_g}{\partial t} + \mathbf{u} \cdot \nabla p_g + \gamma_g p_g \nabla \cdot \mathbf{u} = 0 \tag{3}$$

$$\frac{\partial p_c}{\partial t} + \mathbf{u} \cdot \nabla p_c + \gamma_c p_c \nabla \cdot \mathbf{u} = \nabla (\kappa \nabla p_c) \tag{4}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \tag{5}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{6}$$

where ρ, $\mathbf{u}$, p_g and γ_g denote the density, fluid velocity, pressure and adiabatic index of the thermal gas; $\mathbf{B}$ is the magnetic field induction and p_c, γ_c and κ denote the cosmic ray pressure, adiabatic index and hydrodynamical diffusion coefficient respectively.

2.1 Wave mixing equations The wave interaction equations obtained by Webb et al (1999) describe linear wave mixing of the different eigenmodes due to the gradients in the background flow; instability and damping terms due to the cosmic rays; and nonlinear interaction effects. For linear wave propagation in inhomogenous media in which nonlinear and second order terms are negligible, one obtains the wave interaction equations:

$$\frac{\partial a_j}{\partial t} + \frac{\partial}{\partial x}(\lambda_j a_j) + \sum_{s=1}^{7} \Lambda_{js} a_s = 0, \quad j = 1(1)7 \tag{7}$$

where a_j represents amplitude of the j^{th} MHD wave mode, and Λ_{js} denotes the wave mixing coefficients. To enumerate wave modes, we put them in the order of increasing wave speeds, λ_j:

$$\begin{aligned} &a_1 = a_f^-,\ a_2 = a_A^-,\ a_3 = a_s^-,\ a_4 = a_e, a_5 = a_s^+,\ a_6 = a_A^+,\ a_7 = a_f^+; \\ &\lambda_{1,7} = u_x \mp c_f,\ \lambda_{2,6} = u_x \mp c_A,\ \lambda_{3,5} = u_x \mp c_s,\ \lambda_4 = u_x. \end{aligned} \tag{8}$$

In (8) the subscripts f, s and A refer to the fast, slow magnetoacoustic and Alfvén waves, e denotes the entropy wave and $-$ and $+$ correspond to backward and forward propagating waves respectively. The wave mixing coefficients are given by

$$\Lambda_{js} = \mathbf{L}_j \cdot \frac{d\mathbf{R}_s}{dt_s} + L_j^2 \left((R_s^2 - u_x R_s^1) \frac{a_c^2}{\kappa} - \frac{\zeta}{\rho} \frac{\partial p_c}{\partial x} R_s^1 \right), \tag{9}$$

where $d/dt_s = \partial/\partial t + \lambda_s \partial/\partial x$ is the time derivative along the s^{th} wave mode characteristic, $\mathbf{L}_j$ and $\mathbf{R}_j$ are the left and right MHD eigenvectors corresponding to the conserved densities state vector $\Psi = (\rho, \rho u_x, \rho u_y, \rho u_z, B_y, B_z, \rho S)^T$ of the MHD fluid, S is the gas entropy. L_j^p and R_j^p denote the p^{th} component of $\mathbf{L}_j$ and $\mathbf{R}_j$. In (9) $a_c^2 = \gamma_c p_c/\rho$ and $\zeta = \partial \ln \kappa / \partial \ln \rho$. In order to assess more completely the role of the cosmic rays, one can eliminate d/dt_s terms in (9) using the normal momentum equation (2). For example, the Λ_{11} coefficient in eq. (7) for the backward propagating fast wave mode becomes:

$$\begin{aligned} \Lambda_{11} &= \frac{1}{2} \left\{ \frac{\alpha_f^2 c_f^2}{a_g^2} \left[\frac{a_c^2}{\kappa} + \frac{\zeta + 1}{\rho c_f} \frac{\partial p_c}{\partial x} + \frac{1}{\rho c_f} \frac{\partial}{\partial x} \left(p_g + \frac{B_\perp^2}{2\mu} \right) + \frac{\partial u_x}{\partial x} \right] \right. \\ &\left. + \frac{d}{dt_1} \ln a_g + \alpha_s^2 \frac{d}{dt_1} \ln \left(\frac{a_g}{\rho^{1/2}} \right) + \frac{\alpha_f^2}{\gamma_g} \frac{dS}{dt_1} + \frac{\alpha_s \alpha_f c_s}{a_g^2} \frac{d}{dt_1} u_y \right\}, \end{aligned} \tag{10}$$

where

$$\alpha_f^2 = \frac{a_g^2 - c_s^2}{c_f^2 - c_s^2}, \qquad \alpha_s^2 = 1 - \alpha_f^2, \qquad a_g^2 = \frac{\gamma_g p_g}{\rho}. \tag{11}$$

Therefore, if the cosmic ray pressure gradient is sufficiently large and negative, and $\zeta > -1$, then $a_c^2/\kappa + [(\zeta+1)/\rho c_f] \partial p_c/\partial x \ll 0$, and the backward fast mode wave can become unstable. Similar instability criteria were obtained by Drury and Falle, (1986) and Dorfi and Drury, (1985). For planar MHD flows the Alfvén waves decouple from other modes and the wave mixing equations (7) take the form:

$$\begin{aligned} &\frac{\partial a_2}{\partial t} + \frac{\partial}{\partial x}(\lambda_2 a_2) - \frac{1}{4} \frac{\partial}{\partial x} (u_x - 2c_A)\, a_2 + \frac{1}{4} \frac{\partial}{\partial x} (u_x + 2c_A)\, a_6 = 0 \\ &\frac{\partial a_6}{\partial t} + \frac{\partial}{\partial x}(\lambda_6 a_6) + \frac{1}{4} \frac{\partial}{\partial x} (u_x - 2c_A)\, a_2 - \frac{1}{4} \frac{\partial}{\partial x} (u_x + 2c_A)\, a_6 = 0. \end{aligned} \tag{12}$$

3 Numerical results

In this section we present examples of wave interactions in an oblique, cosmic ray modified shock. We choose a coordinate system in the rest frame of the shock with the x axis directed into the upstream region, and a planar MHD flow configuration in which magnetic field and fluid velocity are restricted to the xy-plane, with $\mathbf{B} = (B_x, B_y, 0)$ and $\mathbf{u} = (u_x, u_y, 0)$, so that $\mathbf{B} \cdot \nabla \times \mathbf{u} = 0$, $\mathbf{B} \cdot \nabla \times \mathbf{B} = 0$.

Fig. 1 shows a continuous cosmic-ray modified MHD shock transition in which $\gamma_g = 5/3$, $\gamma_c = 3/2$ and diffusion coefficient $\kappa = 0.1$. Far upstream $\rho = 1.0$, $p_g = 0.25$, $p_c = 6.5$, $u_x = 5.5$, $u_y = 0.0$, $B_x = 0.52$, $B_y = 0.9$, resulting in an Alfvén Mach number $M_A = u_x/c_A$=10.6. The shock transition is obtained numerically using explicit Eulerian multi dimensional MHD code ZEUS (Stone and Norman, 1992) coupled with the implicit Crank-Nicholson scheme for the cosmic-ray pressure equation. We specify inflow upstream state at the right boundary of the computational domain and reflecting left boundary. After the shock wave has formed and propagates with speed u_s, we change coordinate system to the shock frame, by setting $u_x \to u_x - u_s$. The wave interactions are studied by specifying a single mode perturbation in the flow, with a wave vector $\mathbf{k}$ parallel to the shock normal. We consider the Alfvén and magnetoacoustic wave interactions separately below.

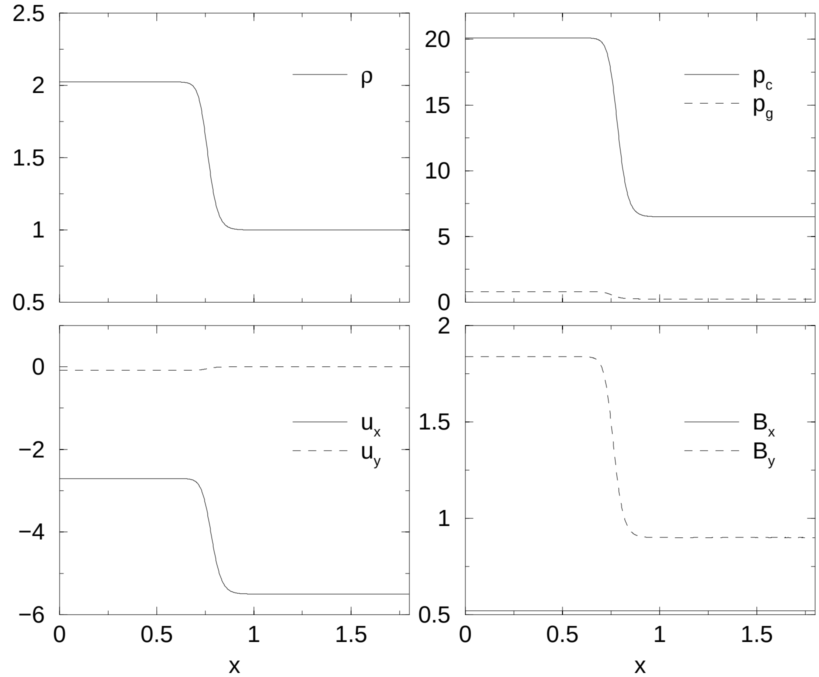

Figure 1: Cosmic ray modified oblique MHD shock wave.

3.1 Alfvén wave interactions In the case of the planar MHD flow, such as the shock wave in Fig. 1, the forward and backward Alfvén waves are coupled by (12).

Fig. 2 shows an example of the numerical solution of these wave mixing equations using Fourier spectral collocation method and with a shock wave from Fig.1 as a background. Dashed lines show for comparison wave amplitudes obtained by solving the nonlinear MHD equations (1)-(6) with the same initial conditions.

At time $t_0 = 0$ a forward Alfvén wave packet δB_z^+ is specified far upstream. The solutions for both forward and backward waves are shown at a later time $t_1 = 0.1$. The backward Alfvén wave δB_z^- is generated and the amplitudes of both waves increase, while the wavelengths decrease as they pass through the shock into the downstream region. Once the backward mode is generated, the waves separate, as they travel at speeds differing by $\lambda_6 - \lambda_2 = 2c_A$. In the case of the perpendicular shock $c_A = 0$ and both a_2 and a_6 travel at the speed of the background flow. The small difference in amplitudes between the nonlinear MHD solutions and the spectral code solutions of the wave mixing equations is due to the larger numerical diffusion present in the nonlinear solver. The two solutions are indistinguishable for smaller computational cell size.

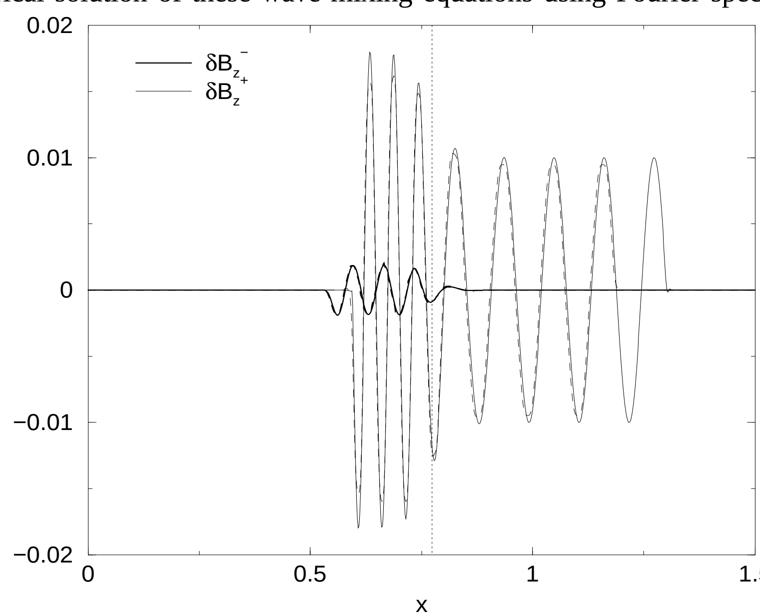

Figure 2: δB_z^-=$-a_2$, δB_z^+=$-a_6$ for interacting Alfvén waves.

3.2 Magnetoacoustic wave interactions Fig. 3 shows solutions of the wave mixing equations (7) without Alfvén waves ($s \neq 2, 6$) for the case where the initial data consists of a forward slow mode wave train specified initially throughout the shock structure.

The left panels show the density perturbations ρ_3 and ρ_5 for the backward and forward slow mode waves at times $t_1 = 0.025$ and $t_2 = 0.075$, whereas the right panels show the fast mode wave density perturbations ρ_1 and ρ_7 that have been generated by the wave mixing. The backward slow wave is amplified due to the cosmic ray squeezing instability and is approximately $\pi/2$ out of the phase with the forward slow wave ρ_5.

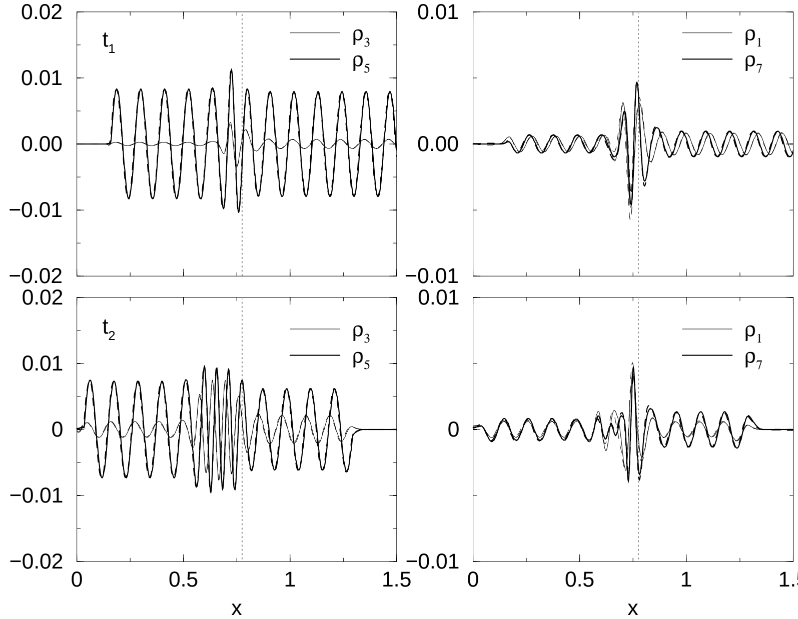

Figure 3: Slow mode (left panels) and fast mode (right panels) waves. Vertical lines mark position of the shock wave.

As expression for Λ_{11} in section 2.1 indicates, the wave coupling coefficients for the backward fast wave and the corresponding coefficients for the forward fast mode wave ρ_7, contain terms describing the cosmic ray squeezing instability, wave damping due to the diffusing cosmic rays, and MHD wave mixing. For example, if the entropy wave is specified as initial perturbation in the flow, it will generate all magnetoacoustic wave modes. In turn, magnetoacoustic waves can generate entropy wave perturbations, but only if the background flow contains a large scale entropy gradient.

Unlike the perpendicular shock case studied in Webb et al (1999), in the oblique shock the slow mode phase speed c_s in not zero, and the wave mixing coefficients contain terms proportional to the $c_s \partial u_y / \partial x$.

4 Summary

Numerical solutions of the fully nonlinear two-fluid MHD equations were compared with the solutions of the wave mixing equations obtained using Fourier collocation spectral method, for the case of a steady-state, oblique, cosmic ray modified shock. For this configuaration, the wave mixing equations for Alfvén waves are independent from the magnetoacoustic and entropy wave modes. Due to the gradients in the background thermal plasma state and cosmic ray pressure, a single mode perturbation creates different wave modes which can become linearly unstable if the cosmic ray pressure gradient is large.

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