Lagrangian Wave Mixing in Cosmic Ray Modified Flows

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Abstract

In this paper, a Lagrangian approach to wave mixing in cosmic-ray modified flows is developed. In particular, the wave mixing equations for short wavelength sound waves and entropy waves in a large scale background flow obtained by Webb et al. (1997) are cast in the form of two coupled equations for the Lagrangian fluid displacement $\xi$ and the Lagrangian entropy perturbation $\Delta S$. The fluid displacement vector $\xi(x,t)$ satisfies a second order wave equation, which describes the propagation of short wavelength sound waves in the large scale background flow, with source terms owing to the cosmic ray squeezing instability, and the entropy perturbation $\Delta S$. The source of the squeezing instability is identified with the non-adiabatic compression of the fluid element by the cosmic rays. The entropy perturbation $\Delta S$, is advected with the background flow. It is noted that these equations can be generalised to arbitrary wavelength waves.

1 Introduction

Webb, Brio, Zank and Story (1997) derived linear and weakly nonlinear evolution equations describing wave coupling and cosmic ray squeezing instabilities for short wavelength sound waves and entropy waves in a large scale background flow. The equations were used in Webb et al. (1999) to discuss instabilities in cosmic ray modified shocks with application to SNR shocks. The wave interaction equations were written in terms of the Eulerian density perturbations for the waves. In this paper, the equivalent Lagrangian form of the equations is obtained.

2 The Model

For the case of one dimensional flow along the $x$-axis, the two-fluid cosmic ray hydrodynamical equations may be written in the form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = 0,$$

$$\frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho u^2 + p_g + p_c) = 0,$$

$$\frac{\partial}{\partial t} (\rho S) + \frac{\partial}{\partial x} (\rho u S) = 0,$$

$$\frac{\partial p_c}{\partial t} + u \frac{\partial p_c}{\partial x} + \gamma p_c \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} \left( \kappa \frac{\partial p_c}{\partial x} \right) = 0,$$

where $S = C_v \ln(p_g/\rho^\gamma)$ is the gas entropy. In the above equations $\rho$, $u$, $p_g$ and $\gamma_g$ denote the density, fluid velocity, pressure and adiabatic index of the thermal gas; and $p_c$, $\gamma_c$ and $\kappa$ denote the cosmic ray pressure, adiabatic index and hydrodynamical diffusion coefficient respectively. $C_v$ is the gas specific heat at constant volume.

3 Eulerian Wave Mixing Equations

From Webb et al. (1997,1999), linear, short wavelength waves propagating through a non-uniform, large scale background flow governed by (1)-(4), satisfy wave mixing equations of the form:

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial}{\partial x} (\lambda_i \rho_i) + \sum_{j=1}^{3} A_{ij} \rho_j = 0, \quad i = 1, 2, 3,$$
where the \( \{ \rho_i : i = 1, 2, 3 \} \) are the Eulerian density perturbations for the backward sound wave (\( \rho_1 \)), entropy wave (\( \rho_2 \)), and forward sound wave (\( \rho_3 \)) respectively. The phase speeds of the waves are:

\[
\lambda_1 = u - a_g, \quad \lambda_2 = u, \quad \lambda_3 = u + a_g,
\]

where \( a_g = (\gamma g p_g / \rho)^{1/2} \) is the thermal gas sound speed. From the eigenvector relations the \( \{ \rho_j : j = 1, 2, 3 \} \) are given by the equations:

\[
\rho_1 = \frac{1}{2} \left( \frac{3 - \gamma g}{\gamma g C_v^2} - \frac{a_g S_x}{(\gamma g - 1) C_v^2} + \frac{a_g^2}{\kappa} + \frac{\zeta + 1}{\rho a_g} \right), \quad \rho_2 = -\frac{\rho \delta S}{\gamma g C_v}, \quad \rho_3 = \frac{1}{2} \left( \frac{\delta \rho + \rho \delta S}{\gamma g C_v} + \frac{\rho \delta u}{a_g} \right),
\]

where \( \delta \rho, \delta u \) and \( \delta S \) are the total Eulerian density, fluid velocity and entropy perturbations respectively.

The wave interaction coefficients \( \Lambda_{ij} \) in (5) depend on the gradients of the non-uniform background flow and have the form:

\[
\begin{align*}
\Lambda_{11} &= \frac{1}{2} \left( \frac{3 - \gamma g}{\gamma g C_v^2} - \frac{a_g S_x}{(\gamma g - 1) C_v^2} + \frac{a_g^2}{\kappa} + \frac{\zeta + 1}{\rho a_g} \right), \\
\Lambda_{12} &= \frac{1}{2} \left( \frac{R_x^+ - R_x^-}{\gamma g C_v^2} - \frac{a_g S_x}{(\gamma g - 1) C_v^2} + \frac{\zeta + 1}{\rho a_g} \right) = \frac{1}{2} \frac{d u}{d t} + \frac{\zeta}{2 \rho a_g} \frac{d p_c}{d x}, \\
\Lambda_{13} &= \frac{1}{2} \left( \frac{\gamma g - 3 - R_x^+}{\gamma g C_v^2} + \frac{\zeta + 1}{\rho a_g} \right), \\
\Lambda_{21} &= -\Lambda_{23} = \frac{a_g S_x}{\gamma g C_v}, \quad \Lambda_{22} = 0, \\
\Lambda_{31} &= \frac{1}{2} \left( \frac{\gamma g - 3 - R_x^+}{\gamma g C_v^2} + \frac{\zeta + 1}{\rho a_g} \right), \\
\Lambda_{32} &= -\Lambda_{12}, \\
\Lambda_{33} &= \frac{1}{2} \left( \frac{3 - \gamma g}{\gamma g C_v^2} + \frac{a_g S_x}{(\gamma g - 1) C_v^2} + \frac{a_g^2}{\kappa} - \frac{\zeta + 1}{\rho a_g} \right),
\end{align*}
\]

where the parameter \( \zeta = \partial \ln \kappa / \partial \ln \rho \); \( d u / d t = d u / d t + u \partial u / \partial x \) is the acceleration vector of the fluid; and \( R^\pm = u \pm 2 a_g / (\gamma g - 1) \) are the Riemann invariants of isentropic gas dynamics. The \( \partial p_c / d x \) terms in (8) correspond to the cosmic ray squeezing instability investigated by Drury and Falle (1986) and Zank and McKenzie (1987). The \( a_c^2 / \kappa \) terms represent damping of the sound waves by the diffusing cosmic rays, and \( a_c = (\gamma g p_c / \rho)^{1/2} \) is the ‘cosmic ray sound speed’. Note that the character of the instability depends on whether \( \zeta < -1, \zeta = -1 \) or \( \zeta > -1 \), where \( \zeta = \partial \ln \kappa / \partial \ln \rho \). For quasi-parallel shocks in which the scattering wave field is due to a pre-existing Alfvén wave field upstream of the shock \( \kappa \propto \rho^{-3/2} \), and \( \zeta = -3/2 \), whereas for a quasi-perpendicular shock in which \( \kappa \sim \kappa_\perp \) is due to random walk of the field lines \( \kappa \sim \text{const.} \), and \( \zeta = 0 \) (e.g., Webb et al. 1999).

4 Lagrangian Wave Interaction Equations

In this section we generalize the Lagrangian wave interaction equations for adiabatic gas dynamics derived in Webb Brio and Zank (1998), to include the effect of cosmic rays. The resulting wave interaction equations can be shown to be equivalent to the Eulerian wave mixing equations (5). To proceed with the analysis, first note the perturbed mass continuity equation:

\[
\frac{\partial \delta \rho}{\partial t} + \frac{\partial}{\partial x}(\rho \delta u + u \delta \rho) = 0,
\]
is satisfied automatically by setting
\[
\delta \rho = -\frac{\partial}{\partial x}(\rho \xi), \quad \rho \delta u + u \delta \rho = \frac{\partial}{\partial t}(\rho \xi),
\]
where \(\xi\) is the Lagrangian fluid displacement. From (10), we obtain the formula
\[
\delta u = \xi_t + u \xi_x - \xi u_x,
\]
for the Eulerian fluid velocity perturbation \(\delta u\). The Lagrangian perturbation \(\Delta \psi\) of a physical quantity \(\psi\) and the corresponding Eulerian perturbation \(\delta \psi\) are related by
\[
\Delta \psi = \delta \psi + \xi \cdot \nabla \psi,
\]
(e.g. Newcomb, 1962). Using (10)-(12), we obtain the standard formulae:
\[
\Delta \rho = -\rho \frac{\partial \xi}{\partial x}, \quad \Delta u = \xi_t + u \xi_x \equiv \frac{d \xi}{d t},
\]
for the Lagrangian density and fluid velocity perturbations.

From (2), the linearized momentum equation may be written as:
\[
\frac{\partial}{\partial t} (\rho \delta u + u \delta \rho) + \frac{\partial}{\partial x} \left( u^2 \delta \rho + 2 p \delta u + \delta p_g + \delta p_e \right) = 0.
\]
In (14), \(\delta \rho\) and \(\delta u\) are given by (10) and (11). The gas pressure perturbation \(\delta p_g\):
\[
\delta p_g = a_g^2 \delta \rho + p_g \delta \tilde{S} \equiv a_g^2 \delta \rho + p_g (\Delta \tilde{S} - \xi \partial \tilde{S}/\partial x),
\]
can be written in terms of \(\xi\) and \(\Delta \tilde{S}\), where \(\tilde{S} = S/C_v\).

By using a multiple scales analysis of the cosmic ray energy equation (4) (see e.g. Webb et al. 1997), we find:
\[
\frac{\partial \delta p_e}{\partial x} = \frac{a_e^2}{\kappa} \rho \delta u - \frac{\xi}{\rho} \frac{\partial p_e}{\partial \rho} \delta \rho,
\]
for the cosmic ray pressure gradient perturbation term in (14). In (16), \(\partial(\delta p_e)/\partial x \equiv \partial(p_e^{(2)}/\partial x_1\), where \(\delta p_e = \varepsilon p_e^{(2)}, \delta u = \epsilon u^{(1)}, \delta \rho = \epsilon \rho^{(1)}; x_1 = x/\epsilon\) is the short scale space variable and \(\epsilon = \lambda/L\) is the ratio of the short scale \(\lambda\), characteristic of the wave to the scale length \(L\) of the background flow. Note that the perturbations depend on both the slow variables \(x = x/L\) and \(t = t/T\), characteristic of the background flow and also on the fast variables \(x_1 = x/\epsilon\) and \(t_1 = t/\epsilon\) characteristic of the waves.

Using (10), (11), (15) and (16) in the linearized momentum equation (14) yields the wave equation:
\[
\xi_{tt} + 2 u \xi_{xt} + (u^2 - a_g^2) \xi_{xx} - \frac{\gamma_g + 1}{\rho} \frac{\partial p_g}{\partial x} \frac{\partial \xi}{\partial x} + Q_{\Delta S} + Q_{\epsilon r} = 0,
\]
where
\[
Q_{\Delta S} = \frac{1}{\rho} \frac{\partial}{\partial x}(p_g \Delta \tilde{S}),
\]
\[
Q_{\epsilon r} = \frac{\xi}{\rho \kappa} \left[ \frac{\partial}{\partial x} \left( \kappa \frac{\partial p_e}{\partial x} \right) - \gamma_e p_e \frac{\partial u}{\partial x} \right] + a_e^2 \frac{\partial \xi}{\partial t} + \frac{\xi}{\rho} \frac{\partial p_e}{\partial \rho} \frac{\partial \xi}{\partial x},
\]
are effective source terms due to the Lagrangian entropy perturbations \((Q_{\Delta S})\), and the cosmic ray squeezing instability \((Q_{\epsilon r})\). The perturbed entropy conservation equation (3), yields the advection equation:
\[
\frac{\partial \Delta S}{\partial t} + u \frac{\partial \Delta S}{\partial x} = 0,
\]
for the Lagrangian entropy perturbation $\Delta S$. By using (4), we obtain the more suggestive form:

$$Q_{cr} = \frac{\xi_{1-\gamma_c}}{\rho \kappa} \frac{d}{dt} (p_c \xi^\gamma_c) + \frac{\zeta}{\rho} \frac{\partial p_c}{\partial x} \frac{\partial \xi}{\partial x}.$$  

(21)

for the cosmic ray squeezing instability term $Q_{cr}$. The first term on the righthand side of (21) suggests that it is the non-adiabatic squeezing of the fluid element by the cosmic rays ($d/dt(p_c \xi^\gamma_c) \neq 0$) that is primarily responsible for the squeezing instability (note $\zeta = 0$ if $\kappa = \text{const}.$).

### 4.1 Arbitrary Wavelength Waves.

In the more general case of arbitrary wavelength waves, $\xi(x,t)$ again satisfies the the wave equation (17), except that the cosmic ray source term $Q_{cr}$ has the form:

$$Q_{cr} = \frac{1}{\rho} \left( \xi \frac{\partial^2 p_c}{\partial x^2} + \frac{\partial \delta p_c}{\partial x} \right),$$  

(22)

where $\delta p_c$ satisfies the perturbed cosmic ray energy equation (4):

$$\mathcal{L}(\delta p_c) \equiv \left[ \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + \gamma_c \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} \left( \kappa(\rho) \frac{\partial}{\partial x} \right) \right] \delta p_c$$

$$= - \left( \delta u \frac{\partial p_c}{\partial x} + \gamma_c p_c \frac{\partial \delta u}{\partial x} \right) + \frac{\partial}{\partial x} \left( \kappa'(\rho) \frac{\partial p_c}{\partial x} \delta \rho \right),$$  

(23)

and $\Delta S$ satisfies the entropy advection equation (20). Equations (17), (18), (20), (22) and (23) can be combined to yield a single fourth order wave equation for $\xi(x,t)$, which results from simplifying the equation:

$$\mathcal{L} \frac{d}{dt} \left( \xi_{tt} + 2u \xi_{xt} + (u^2 - a_0^2) \xi_{xx} - \frac{\gamma_g + 1}{\rho} \frac{\partial p_g}{\partial x} \frac{\partial \xi}{\partial x} + Q_{\Delta S} + Q_{cr} \right) = 0,$$  

(24)

where $d/dt = \partial/\partial t + u \partial/\partial x$ is the Lagrangian time derivative.

### 5 Concluding Remarks

The Eulerian density perturbations (7) can be expressed in terms of $\xi$ and $\Delta S$. One can show that the Eulerian wave mixing equations (5) are satisfied if $\xi$ and $\Delta S$ satisfy the Lagrangian equations (17)-(20). It is also possible to derive more complicated wave interaction equations for $\xi$ and $\Delta S$ for arbitrary wavelength waves (see (22)-(24)).

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**References**

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