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# Quasi-linear Theory of Energetic Particle Diffusion in Stationary 3-Dimensional Turbulence

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**Abstract**

A quasi-linear theory of cosmic ray diffusion in fully three-dimensional magnetic turbulence is developed, to lead to better understanding of perpendicular diffusion and the effects of imperfect symmetry in the magnetic turbulence. Calculation of the pitch angle scattering coefficient, in terms of the elements of the power spectrum tensor, in the case of turbulence symmetric about the direction of the mean field, is given as a demonstration of the method.

## 1. Introduction

Cosmic rays are coupled to cosmic plasmas, and share energy with them, because of the effect of the Lorentz force in the turbulent magnetic fields cosmic plasmas always contain. The basic ideas and expressions for the diffusion coefficients of energetic particles in a cosmic plasma, and their dependence on particle parameters and the power spectrum of the magnetic turbulence, were discovered over 30 years ago by Hasselmann and Wibberenz, Jokipii, and others. At the same time, correct equations for the transport of the cosmic ray gas in space and energy in a cosmic plasma were developed using these diffusion coefficients. Since these initial breakthroughs, cosmic-ray transport theory has moved into numerical calculations for spatial transport in the heliosphere, and for acceleration in the heliosphere, supernovae, clusters, and even hypernovae to make gamma ray bursts. But, it all depends still on the fundamentals of the coupling and the nature of the magnetic turbulence. We now realize that the first diffusion theories were a little too simple, and we know a lot more about magnetic turbulence in a variety of regions and behaviors in the solar wind. We also know how to describe this turbulence better (Matthaeus and Smith,1981; Oughton et al., 1997)

Two developments particularly draw fresh attention to this old problem: 1) the 3-dimensional nature of the magnetic turbulence, including helicity; and 2) the importance of diffusion perpendicular to the mean magnetic field in solar modulation in the distant heliosphere (where most seems to occur) and in shock acceleration.

The following is a re-start on a comprehensive cosmic ray diffusion theory from the quasi-linear approximation. The object is to develop the concepts and tools for an analytic theory to complement and elucidate the brute-force trajectory studies (Giacalone and Jokipii, 1994), and other approaches (Bieber and Matthaeus, 1997). The goal is a good theory for perpendicular diffusion and drift; along the way, the parallel diffusion, which is much easier theory, will be studied for corrections to some old expressions.

## 2. The basic scattering integral

Let f(**r**, **p**, t) be the particle distribution function in space, **r**, and momentum **p**; let v be the particle speed, Ze its charge, **B** the mean magnetic field, and <**b**(0)**b**(r)> the correlation tensor of the fluctuations in the magnetic field. Then in the quasi-linear approximation

$$\frac{\partial}{\partial t}f+\mathbf{v}\bullet\nabla f-\frac{Zev}{cp}\mathbf{B}\bullet\left(\mathbf{p}\times\frac{\partial}{\partial\mathbf{p}}f\right)=-\left(\frac{Zev}{cp}\right)^2\mathbf{p}\bullet\frac{\partial}{\partial\mathbf{p}}\times\int_0^\infty\langle\mathbf{b}(0)\mathbf{b}(\mathbf{r}(t))\rangle\bullet\left(\mathbf{p}\times\frac{\partial}{\partial\mathbf{p}}f(\mathbf{p}(t))\right)dt \qquad (1)$$

The right hand side of Eq.(1) is the QLA scattering operator on f, expressing the physics of how a stationary turbulent magnetic field with given **B** and statistics <**b**(0)**b**(**r**)> induces diffusive behavior in energetic particles in space. The integral path in Eq (1), is along the trajectory in the mean field **B**, of a particle which has momentum **p** at **r** = 0, t = 0. The f and the elements of the tensor <**b**(0)**b**(**r**(t))> are evaluated along the trajectory of the momentum, but the spatial trajectory determines where in space <**b**(0)**b**(**r**(t))> is evaluated. It is assumed that there are no significant spatial variations in f on these scales. In terms of the pitch angle θ between **p** and **B**, and unit vectors in momentum space of spherical coordinates whose polar axis is along **B**,

the anisotropy $\left(\mathbf{p}\times\frac{\partial}{\partial\mathbf{p}}\right)f=\left[\frac{\partial f}{\partial\theta}\hat{e}_\varphi-\frac{1}{\sin\theta}\frac{\partial f}{\partial\varphi}\hat{e}_\theta\right]$ appears on both sides of the basic

equation. The basic equation relates the spatial gradient in f, to a convolution of the correlation tensor of **b** with the anisotropy in f. We want to invert this to find the anisotropy in terms of the spatial gradient. Solution of (1) for f(**p**) requires careful attention to the trajectory in space and in momentum.

## 3. The QLA scattering integral in three-dimensional turbulence.

The basic equation can be written (following Forman, et al., 1974)

$$\frac{\partial}{\partial t}f+\mathbf{v}\bullet\nabla f-\frac{Zev}{cp}\mathbf{B}\bullet\left(\mathbf{p}\times\frac{\partial}{\partial\mathbf{p}}f\right)=-\left(\frac{Zev}{cp}\right)^2\mathbf{p}\bullet\frac{\partial}{\partial\mathbf{p}}\times\mathbf{Q}$$

$$=-\left(\frac{Zev}{cp}\right)^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\mathbf{Q}_\varphi\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\left(\mathbf{Q}_\theta\right)\right] \qquad (2)$$

$$\text{where}\qquad \mathbf{Q}_{\varphi,\theta}=\hat{e}_{\varphi,\theta}\bullet\int_0^\infty\langle\mathbf{b}(0)\mathbf{b}(\mathbf{r}(t))\rangle\bullet\left[\hat{e}_\varphi(t)\frac{\partial f(t)}{\partial\theta}-\hat{e}_\theta(t)\frac{1}{\sin\theta}\frac{\partial f(t)}{\partial\varphi}\right]dt \qquad (3)$$

The direction of the momentum along the trajectory tells which elements of <**bb**> to use at each t; the spatial position of the particle at time t tells us where to evaluate that element.

The phase of the particle momentum evolves as φ(t) = φ - ωt and

$$\begin{aligned}\hat{e}_\varphi(t)&=\cos\varphi(t)\hat{e}_y-\sin\varphi(t)\hat{e}_x\\ \hat{e}_\theta(t)&=\cos\theta\left[\cos\varphi(t)\hat{e}_x+\sin\varphi(t)\hat{e}_y\right]-\sin\theta\hat{e}_z\end{aligned} \qquad (4)$$

Now we can evaluate the four combinations in the integrals for **Q**. For example, with the notation $\langle b_{xi}(0)b_{xj}(t')\rangle=\langle x_i,x_j\rangle$,

$$\begin{aligned}\langle\varphi,\varphi\rangle&=\frac{1}{2}\left[\langle x,x\rangle+\langle y,y\rangle\right]\cos\omega t-\frac{1}{2}\left[\langle x,x\rangle-\langle y,y\rangle\right]\cos(2\varphi-\omega t)\\&+\frac{1}{2}\left[\langle y,x\rangle-\langle x,y\rangle\right]\sin\omega t-\frac{1}{2}\left[\langle y,x\rangle+\langle x,y\rangle\right]\sin(2\varphi-\omega t)\end{aligned} \qquad (5)$$

and similarly for the other elements <φ,θ>,<θ,φ>, and <θ,θ>.

If there is any perpendicular flow at all, ∂f/∂θ and ∂f/∂φ also depend on φ(t) and therefore on ωt, so assumptions about the gyrophase distribution are necessary to do the integrations. Only in the case of a flow prepared so that the spatial gradient and the flow are both strictly parallel to the mean field, is f independent of φ; only <φ,φ> is involved and the ∂f/∂θ is constant on the trajectory of the integration. We consider this case in order to show how to do the integration along the trajectory.

## 4. Strictly parallel diffusion in three dimensional turbulence.

In this case, **Q** depends only on <φ,φ> and equation (1) reduces to the familiar forms (Jokipii, 1971; μ = cos(θ))

$$\mu v \frac{\partial f}{\partial z} = -\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right) D_{\mu\mu} \frac{\partial f}{\partial \mu}\right], \quad \text{resulting in } K_{\text{parallel}} = \frac{v^2}{8}\int_{-1}^{+1} \frac{\left(1-\mu^2\right)}{D_{\mu\mu}} d\mu \ .$$

$$\text{only now, } D_{\mu\mu} = \left(\frac{Zev}{cp}\right)^2 \int_0^\infty \langle \varphi,\varphi \rangle dt \, . \tag{6}$$

The integral can be evaluated if the elements of the correlation tensor are known as functions of **r**.

Alternatively, the three-dimensional Fourier transform of the correlation tensor can be used, to see what wavevectors **k** are picked out by the integration over the trajectory, producing a theory independent of the details of the turbulence. It is convenient to use cylindrical coordinates for **k** ($k_z$, $k_\rho$, $\varphi_k$) and for the distance **r** so that (taking η=ωt) the first term in the expression for $D_{\mu\mu}$ is

$$\begin{aligned} \frac{1}{2\omega}\int_0^\infty \left(\langle x,x\rangle + \langle y,y\rangle\right) \cos\eta \, d\eta &= \iiint \left(P_{xx}(\mathbf{k}) + P_{yy}(\mathbf{k})\right) k_\rho dk_\rho dk_z d\varphi_k \int_0^\infty e^{i\mathbf{k}*r} \cos\eta \, d\eta \\ \text{with} \quad \mathbf{k}\bullet\mathbf{r} &= k_z \mu R_c \eta + 2k_\rho R_c \sqrt{\left(1-\mu^2\right)} \left|\sin\frac{\eta}{2}\right| \cos\left(\varphi - \frac{\eta}{2} - \varphi_k\right) \end{aligned} \tag{7}$$

Obviously, the integration over the trajectory will produce resonances in **k**. If all dependence on $k_\rho$ is ignored, the "slab" model (Jokipii, 1971) results. Retaining the $\varphi_k$ is essential at this point wether or not the turbulence is axi-symmetric. The integration over $\varphi_k$ should include the elements of the power spectrum tensor regarding $k_x$ and $k_y$. The integral over η in (7) is

$$\frac{\exp\left[iR_c k_\rho \sqrt{\left(1-\mu^2\right)} \sin(\varphi - \varphi_k)\right]}{2|\mu| R_c} \sum_{n=-\infty}^{n=\infty} \{\exp[-in(\varphi-\varphi_k)]\} J_n\left(R_c k_\rho \sqrt{\left(1-\mu^2\right)}\right)\left[\delta\left(k_z + \frac{n+1}{\mu R_c}\right) + \delta\left(k_z + \frac{n-1}{\mu R_c}\right)\right] \tag{8}$$

Then, if $P_{xx}$ abd $P_{yy}$ do not depend on $\varphi_k$, a second application of Jacobi's identity before integration over $\varphi_{k,}$ results in a sum of *squares* of Bessel functions. The three-dimensional Fourier transform of <φ,φ> is

$$\frac{1}{2}\left[\left[2E - \frac{k_\rho^{\ 2}}{k^2}\left(E + k_z^{\ 2}F\right)\right]\cos\eta + \frac{k_\rho^{\ 2}}{k^2}\left(E + k_z^{\ 2}F\right)\cos[2(\varphi_k - \varphi) + \eta] + 2iCk_z k_\rho^{\ 2} \sin[2(\varphi_k - \varphi) + \eta] - 2iHk_z \sin\eta\right] \tag{9}$$

using the notation E, F, C, H for the scalar functions of **k** as specified by Oughton, et al. (1997). Each term can be integrated over η as above, and then over $\varphi_k$. If the scalar functions are axi-symmetric, the result is

$$D_{\mu\mu} = \left(\frac{Ze}{cp}\right)^2 \frac{\pi v}{2|\mu|}\int_0^\infty k_\rho dk_\rho \int_{-\infty}^{\infty} dk_z \sum_{m=-\infty}^{\infty} \delta\left(k_z - \frac{m}{\mu R_c}\right) \bullet$$
$$\left\{2\left(J_{m-1}{}^2 + J_{m+1}{}^2\right)E - \frac{k_\rho{}^2}{k^2}\left(J_{m-1}{}^2 + J_{m+1}{}^2 - 2J_{m-1}J_{m+1}\right)\left(E + k_z{}^2 F\right) - Hk_z\left(J_{m-1}{}^2 - J_{m+1}{}^2\right)\right\} \quad (10)$$

The argument of the Bessel functions is $R_c k_\rho \sqrt{(1-\mu^2)}$. In isotropic turbulence, F=0. Fisk et al considered diffusion through isotropic turbulence, but did not handle $\varphi_k$ correctly. “2.5 dimensional turbulence” models (Bieber, et al., 1994) use the m=0 term only. “Slab” models use the $J_0$, m= ± 1 term only. Clearly there are many other terms whose importance needs to be evaluated.

## 5. Conclusions.

The helical motion of energetic particles in the mean magnetic field makes the magnetic power spectrum in a large range of **k** vectors at angles to the mean magnetic field, and all components of the spectral tensor, affect cosmic ray transport, especially perpendicular transport.

Further progress in cosmic-ray scattering theory needs coordination and comparison between analytic models like this one, and trajectory calculations, and other theories. We also need some guidance from plasma physics about the form and spectra of the elements of the magnetic power spectrum tensor in various locations where energetic particles are important. In particular, we need to work out how magnetic power *frequency* spectra observed by single spacecraft in the solar wind are best applied to cosmic ray diffusion theory.

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