Relations of Two Molière Theories: That Formulated by Kamata and Nishimura and That by Molière and Bethe

Takao Nakatsuka
Okayama Shoka University, Okayama 700-8601, Japan

Abstract

There exist two formulations of Molière theory, one proposed by Molière and Bethe in original and the other formulated by Kamata and Nishimura. Each theory has its advantage to the other. We show the higher terms of Kamata-Nishimura expansion explicitly up to the second order and compare the distributions derived from the both methods. The two formulations are equivalent in mathematics and there exists a translation formula. The Molière lateral distribution is also confirmed from the Kamata-Nishimura formulation.

1 Introduction:

Kamata and Nishimura founded [1, 2] another formulation of Molière theory [3, 4] in their construction of cascade shower theory. We can regard it as an extrapolation of Fermi-Yang theory of multiple Coulomb scattering [5, 6], very familiar to us cosmic-ray physicists, so that it is clear to understand and advantageous for other applications. By using the method, higher order term to include the effects of single and plural scatterings had been obtained in the cascade shower theory [1, 2] and in the excess-path-length problem [7], and Molière angular distribution is improved to take into account ionization [8]. It is also useful for investigations of Molière process. Mechanism of depth-variation of angular distribution is analyzed by the method [9].

Instead of its high applicability of Kamata-Nishimura formulation, the relation to the original Molière-Bethe formulation is not enough discussed yet. The angular distribution derived from Kamata-Nishimura formulation up to the second higher order is practically compared with that from Molière-Bethe. We show the translation formula between the two. As another application we confirm the derivation of Molière lateral distribution [10] from Kamata-Nishimura formulation.

2 Angular Distribution Expanded in Kamata-Nishimura Series:

Let $f(\theta)2\pi\theta d\theta$ be the angular distribution of charged particles after receiving multiple Coulomb scattering. According to the Kamata-Nishimura formulation, the diffusion equation in the frequency space for the angular distribution under the relativistic condition is expressed [8] as

$$\frac{\partial\tilde{f}}{\partial t} = -\frac{K^2\zeta^2}{4E^2}\tilde{f}\{1-\frac{1}{\Omega}\ln\frac{K^2\zeta^2}{4E^2}\}, \tag{1}$$

where K and Ω are constants specific to the material. This equation can be integrated as

$$\tilde{f} = \frac{1}{2\pi}\exp\{-\frac{K^2\zeta^2 t}{4E^2}(1-\frac{1}{\Omega}\ln\frac{K^2\zeta^2}{4E^2})\}, \tag{2}$$

If we introduce composite variables

$$\alpha = (K\sqrt{t}/E)\zeta \quad \text{and} \quad \phi = \theta/(K\sqrt{t}/E), \tag{3}$$

the integration can be expanded as

$$\begin{aligned}\tilde{f} &= \frac{1}{2\pi}\exp\{-\frac{\alpha^2}{4}(1-\frac{1}{\Omega}[\ln\frac{\alpha^2}{4}-\ln t])\} \\ &= \frac{1}{2\pi}\exp\{-\frac{\alpha^2}{4}\}\sum_{k=0}^{\infty}\frac{1}{k!}\{\frac{1}{\Omega}\frac{\alpha^2}{4}(\ln\frac{\alpha^2}{4}-\ln t)\}^k. \end{aligned} \tag{4}$$

So that, applying Hankel transforms we get

$$2\pi f(\phi) = f^{(0)}(\phi) + \frac{1}{\Omega}\{f^{(1)}(\phi)+f_1^{(1)}(\phi)\ln t\} + \frac{1}{\Omega^2}\{f^{(2)}(\phi)+f_1^{(2)}(\phi)\ln t + f_2^{(2)}(\phi)(\ln t)^2\} + \cdots. \tag{5}$$

The functions $f^{(0)}$, $f^{(1)}$, and $f^{(2)}$ are same as those indicated in Molière and Bethe [3, 4]. The others up to the second order are expressed as

$$\begin{aligned} f_1^{(1)}(\phi) &= -\int_0^{\infty}\alpha d\alpha J_0(\phi\alpha)(\alpha^2/4)e^{-\alpha^2/4} \\ &= 2e^{-\phi^2}(\phi^2-1), && (6) \\ f_1^{(2)}(\phi) &= -\int_0^{\infty}\alpha d\alpha J_0(\phi\alpha)(\alpha^2/4)^2 e^{-\alpha^2/4}\ln(\alpha^2/4) \\ &= 2e^{-\phi^2}(\phi^4-4\phi^2+2)[E_i(\phi^2)-\ln\phi^2] + 4e^{-\phi^2}(2\phi^2-3) - 2(\phi^2-3), && (7) \\ f_2^{(2)}(\phi) &= (1/2)\int_0^{\infty}\alpha d\alpha J_0(\phi\alpha)(\alpha^2/4)^2 e^{-\alpha^2/4} \\ &= e^{-\phi^2}(\phi^4-4\phi^2+2). && (8) \end{aligned}$$

We indicate these functions in Fig. 1.

3 A Translation Formula to The Molière Series:

We show a translation formula between the two formulation. Let the image function by Kamata-Nishimura be of a form [11]

$$\tilde{f} = \frac{1}{2\pi}\exp\{-a\zeta^2 + b\zeta^2\ln(c\zeta^2)\}. \tag{9}$$

If we new define the expansion parameter B and the composite transform-variable u, as

$$B - \ln B = \frac{a}{b} - \ln\frac{c}{b} \qquad \text{and} \qquad u = 2\zeta\sqrt{bB}, \tag{10}$$

then we get the well known Molière form

$$\tilde{f} = \frac{1}{2\pi}\exp\{-\frac{u^2}{4}(1-\frac{1}{B}\ln\frac{u^2}{4})\}. \tag{11}$$

So that the probability density can be represented in the Molière series of

$$f(\vartheta) = f^{(0)}(\vartheta) + B^{-1}f^{(1)}(\vartheta) + B^{-2}f^{(2)}(\vartheta) + \ldots, \tag{12}$$

where the Molière angle is new defined as

$$\vartheta = \theta/(2\sqrt{bB}). \tag{13}$$

4 Comparison of Angular Distribution Expanded in The Two Series:

Image function (2) shows the form of Eq. (9). By putting

$$a = \frac{K^2 t}{4E^2}, \quad b = \frac{1}{\Omega}\frac{K^2 t}{4E^2}, \quad c = \frac{K^2}{4E^2}, \tag{14}$$

we can get the distribution in Molière series characterized with the expansion parameter B and the unit of Molière angle $\theta_{\rm M}$:

$$B - \ln B = \Omega - \ln\Omega + \ln t \tag{15}$$

and

$$\vartheta = \theta/\theta_{\rm M}, \quad \text{where} \quad \theta_{\rm M} = (K\sqrt{t}/E)\sqrt{B/\Omega}. \tag{16}$$

We compare the both angular distributions obtained through Kamata-Nishimura series of Eq. (5) and Molière-Bethe one of Eq. (12) in Fig. 2. Both agree well at moderate thicknesses of traverse. At very short thicknesses Molière-Bethe series indicates oscillating feature due to decrease of the expansion parameter B, as discussed in [12]. Kamata-Nishimura series also shows bad convergence in these thicknesses due to increase of the $\ln t$ term.

5 Other Molière Distributions Derived From Kamata-Nishimura Formulation:

Kamata-Nishimura equation [1, 2] was written under the relativistic condition. For charged particles with fixed finite energies, the equation becomes

$$\frac{\partial \tilde{f}}{\partial t} = -\frac{\zeta^2}{w^2}\tilde{f}\{1 - \frac{1}{\Omega}\ln\frac{\zeta^2}{w^2/\beta^2}\}. \tag{17}$$

This equation is integrated as

$$\tilde{f} = \frac{1}{2\pi}\exp\{-\frac{\zeta^2 t}{w^2}(1 - \frac{1}{\Omega}\ln\frac{\zeta^2}{w^2/\beta^2})\}. \tag{18}$$

So, using the translation formula we get the angular distribution characterized by B and $\theta_{\rm M}$ of

$$B - \ln B = \Omega - \ln\Omega + \ln(t/\beta^2) \quad \text{and} \quad \theta_{\rm M} = (2\sqrt{t}/w)\sqrt{B/\Omega}, \tag{19}$$

which agree with the Molière-Bethe's original result [3, 4].

Molière lateral distribution can also be confirmed. According to Scott's formula [13], Hankel transforms of the probability density can be derived from Eq. (18):

$$\begin{aligned} \ln(2\pi\tilde{f}) &= -\int_0^t \frac{\eta^2(t-t')^2}{w^2}\{1 - \frac{1}{\Omega}\ln\frac{\eta^2(t-t')^2}{w^2/\beta^2}\}dt' \\ &= -\frac{\eta^2 t^3}{3w^2}\{1 + \frac{1}{\Omega}(\frac{2}{3} - \ln\frac{\eta^2 t^2}{w^2/\beta^2})\}, \end{aligned} \tag{20}$$

where we took η as the transform variable of lateral displacement r. Applying our translation formula, we get the lateral distribution in Molière series with expansion parameter B and the unit of lateral displacement $r_{\rm M}$ of

$$B - \ln B = \Omega - \ln\Omega + (2/3) + \ln(t/(3\beta^2)) \quad \text{and} \quad r_{\rm M} = (2\sqrt{t^3/3}/w)\sqrt{B/\Omega}, \tag{21}$$

which agree with the Molière result with the combination angle of $\pi/2$ [10].

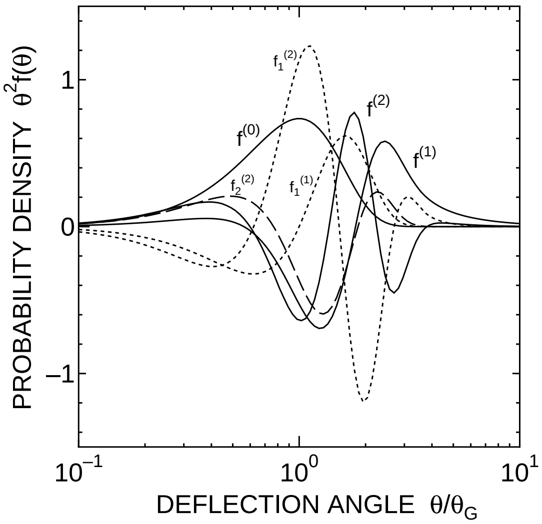

Figure 1: Kamata-Nishimura series functions multiplied by θ^2.

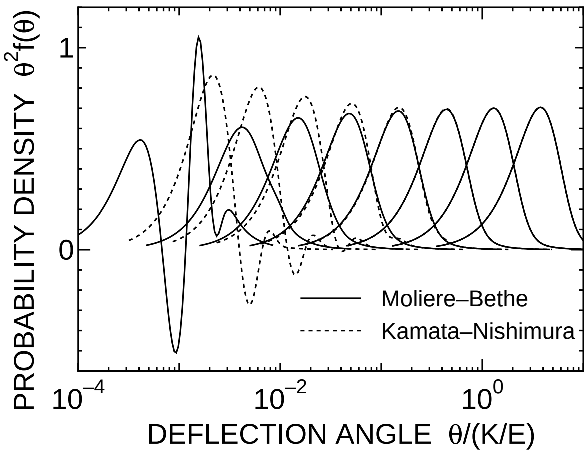

Figure 2: Comparison of angular distributions expanded in Molière-Bethe series and Kamata-Nishimura one at $t = e^{2k+1}\Omega e^{-\Omega}$ with $k = 0, 1, 2, \cdots, 7$, from left to right.

6 Conclusions:

We investigated relations of the two formulations of Molière theory. They are represented in slight different series. We compared angular distributions obtained through the both series. They, expanded up to the second higher order, agreed well within the expansion error. We found a translation formula between the two. Using it we confirmed angular distribution derived by Kamata-Nishimura formulation for finite-energy particles agree with Molière-Bethe's original one. We also confirmed derivation of Molière lateral distribution by Kamata-Nishimura formulation.

Acknoledgments

The author wishes to express his thanks to Prof. A. F. Bielajew for useful discussions about Molière works.

References

[1] K. Kamata and J. Nishimura, Prog. Theor. Phys. Suppl. **6**, 93(1958).

[2] J. Nishimura, in *Handbuch der Physik, Band 46*, edited by S. Flügge (Springer, Berlin, 1967), Teil **2**, p. 1.

[3] G. Molière, Z. Naturforsch. **3a**, 78(1948).

[4] H.A. Bethe, Phys. Rev. **89**, 1256(1953).

[5] B. Rossi and K. Greisen, Rev. Mod. Phys. **27**, 240(1941).

[6] C.N. Yang, Phys. Rev. **84**, 599(1951).

[7] T. Nakatsuka, Phys. Rev. **D35**, 210(1987).

[8] T. Nakatsuka, HE2.5.25 in this conference.

[9] T. Nakatsuka, HE2.5.16 in this conference.

[10] G. Molière, Z. Naturforsch. **10a**, 177(1955).

[11] Combination of coefficients a and c is not unique. We can put $a = 0$ or $c = 1$ without loss of generality, then the other is determined uniquely.

[12] A.F. Bielajew and R. Wang, Nucl. Instrum. Methods Phys. Res. **B82**, 503(1993).

[13] W.T. Scott, Rev. Mod. Phys. **35**, 231(1963).